Extending a flexible unit-bar framework to a rigid one

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Abstract


We prove that (1) a flexible unit-bar framework $G$ in $\mathbb{R}^n$ can always be extended to a rigid unit-bar framework in $\mathbb{R}^n$, and (2) $G$ is 'congruent' to a subgraph of a rigid unit-bar framework in $\mathbb{R}^n$ if and only if the Euclidean distances between joints of $G$ are all algebraic numbers. Meanwhile, it is proved that a previous result on a framework in $\mathbb{R}^2$ [for any real algebraic number $r > 0$, there is a rigid unit-bar framework in $\mathbb{R}^2$ having two vertices with distance $r$ apart] extends to any dimension.

1. Introduction

A framework in Euclidean $n$ space $\mathbb{R}^n$ is a graph whose vertices are points in $\mathbb{R}^n$ and whose edges are line segments connecting two vertices. In a framework, vertices and edges are usually called joints and bars. If all bars of a framework are of unit-length, then it is called a unit-bar framework. A framework in $\mathbb{R}^n$ is called flexible if we can deform the framework in $\mathbb{R}^n$, that is, we can continuously move the joints in $\mathbb{R}^n$ preserving the length of bars so that at least a pair of joints change their mutual distance. A framework is rigid if it is not flexible.

Let $G$ be a flexible unit-bar framework in $\mathbb{R}^n$. Then, by adding some bars of appropriate lengths, we can always extend $G$ to a rigid framework. But, how about when only unit-bars are available? Can we always extend $G$ to a rigid unit-bar framework in $\mathbb{R}^n$? If necessary, we may deform $G$ as far as its graph structure remains unchanged.

Exercise. Extend the flexible unit-bar framework $F$ of Fig. 1 to a rigid unit-bar framework in the plane. (A solution will be given at the end of the paper.)
If we deform $F$ of Fig. 1 so that we can connect some two joints, (say, $x, y$) by a unit-bar, then some two joints (say, $u, v$) of $F$ coincide, which changes the graph structure of $F$. Hence, to extend $F$ to a rigid one, we need to add some extra joints, which may increase the degree of flexibility. Thus, even in the plane case, it is not obvious if a flexible unit-bar framework can always be extended to a rigid one.

**Theorem 1.** Any flexible unit-bar framework $F$ in $\mathbb{R}^n$ can be extended to a rigid unit-bar framework $G$ in $\mathbb{R}^n$.

This theorem is used to extend a result in [4] to higher dimension. The Euclidean norm is denoted by $| |$.

**Theorem 2.** For any $n \geq 2$ and any real algebraic number $r > 0$, there exists a rigid unit-bar framework $G(xy, r)$ in $\mathbb{R}^n$ which contains two joints $x, y$ satisfying $|x - y| = r$.

Two frameworks are said to be congruent if there is an isomorphism between them that preserves the Euclidean distances between joints. A subgraph $F$ of a framework $G$ is said to be rigid in $G$ if, in any deformation of $G$, $F$ always goes to a congruent one.

In Theorem 1, $F$ is not necessarily congruent to a subgraph of its extension $G$. Concerning a congruent embedding, we have the following.

**Theorem 3.** A unit-bar framework $F$ in $\mathbb{R}^n$ is congruent to a subgraph of a rigid unit-bar framework in $\mathbb{R}^n$ if and only if the Euclidean distances between joints of $F$ are all algebraic numbers.

A unit $n$-simplex in $\mathbb{R}^n$ is a regular $n$-simplex with unit side-length. Its 1-skeleton forms a rigid unit-bar framework of order $n + 1$, which is called simply a clique in $\mathbb{R}^n$. A clique-work $W$ in $\mathbb{R}^n$ is a unit-bar framework made from a finite sequence of cliques $Q_1, \ldots, Q_k$ by attaching in such a way that each $Q_i$, $i \geq 1$, shares a complete subgraph of order $n$ with some $Q_j$, $j < i$. Note that a clique work is a rigid unit-bar framework in $\mathbb{R}^n$.

The next theorem will be used to prove Theorem 1.
Theorem 4. Let \( Q \) be a fixed clique in \( \mathbb{R}^n \), \( n \geq 3 \). Then, for any open set \( U \neq \emptyset \) in \( \mathbb{R}^n \), \( Q \) can be extended to a clique-work which has a joint in the set \( U \).

2. The plane case

First we consider the plane case. The following theorem was proved in [4].

**Theorem A.** For any real algebraic number \( r > 0 \), there exists a rigid unit-bar framework \( G(xy, r) \) in \( \mathbb{R}^2 \) which contains two joints \( x, y \) satisfying \( |x - y| = r \).

Using this theorem, the plane case of Theorems 1 and 3 can be easily proved. Suppose that \( F \) is a flexible unit-bar framework in \( \mathbb{R}^2 \). Fix a unit-bar \( uv \) of \( F \) on the plane. Since \( F \) is flexible, there is a movable joint \( x \) of \( F \). Move \( x \) so that \( |x - u| \) becomes an algebraic number \( r \). Then, take a rigid unit-bar framework \( G(xu, r) \) as in Theorem A, and attach it to \( F \). If \( x \) is still movable, then make the distance \( |x - v| \) algebraic, and attach a rigid unit-bar framework \( G(xv, |x - v|) \). Then \( x \) becomes fixed. In this way we can fix all movable joints of \( F \), and get a rigid unit-bar framework \( G \) on the plane. This proves Theorem 1.

If the Euclidean distances between joints of \( F \) are all algebraic numbers, then we need not deform \( F \) to extend to a rigid one. This proves the 'if' part of Theorem 3. The 'only if' part of Theorem 3 follows from the next theorem which is a special case of [3, Theorem 2].

**Theorem B.** Let \( F \) be a rigid unit-bar framework in \( \mathbb{R}^n \). Then the Euclidean distances between joints of \( F \) are all algebraic numbers.

3. Proof of Theorem 1 for \( n \geq 3 \)

In the following, we assume Theorem 4, which will be proved in Section 7. We are going to construct a rigid unit-bar framework \( G \) in \( \mathbb{R}^n \) by deforming \( F \) and adding joints and unit bars to it.

Take a fixed clique \( Q \) in \( \mathbb{R}^n \), and attach \( F \) to \( Q \) so that they share the largest possible subgraph. Let \( H \) be the resulting framework. Note that the clique \( Q \) of \( H \) is fixed at some position in \( \mathbb{R}^n \). If \( H \) is rigid, then we are done. If \( H \) is flexible, take a joint \( x \) which can continuously move. Then it is possible to move \( x \) along a 'smooth' curve \( \Gamma \) in \( \mathbb{R}^n \) (see e.g. [1, 5]). By Theorem 4, we can build a clique-work \( W \) on \( Q \) such that a joint \( v_0 \) of \( W \) is sufficiently near the curve \( \Gamma \) (but not on \( \Gamma \)). Since \( Q \) is fixed, every joint of the clique-work \( W \) is also fixed. Let \( v_0, v_1, \ldots, v_n \) be the joints of a clique of \( W \). Denote by \( S_i \) the unit hypersphere centered at \( v_i, 1 \leq i \leq n \). Then any line through \( v_0 \) penetrates some \( S_i \). And since \( v_0 \) is sufficiently near \( \Gamma \), the curve \( \Gamma \) also penetrates some hypersphere \( S_i \). Now
move the joint \( x \) along \( \Gamma \) until it comes to the point on the hypersphere \( S_i \), and then connect \( x \) with \( v_i \) by a unit-bar. Then \( x \) is constrained on the hypersphere \( S_i \). If \( x \) can move along a smooth curve on \( S_i \), then we can build another appropriate clique-work \( W' \) on \( Q \) and take a hypersphere centered at a joint of \( W' \) which cuts the curve, and connect \( x \) with the center of the hypersphere by a unit-bar. Then \( x \) is constrained in the intersection of two hyperspheres, which is an \( (n - 2) \)-dimensional sphere. If \( x \) can still move, then repeat similar operations. Since each time the dimension of the sphere on which \( x \) is constrained decreases by one, after at most \( n \) times of such operations, the vertex \( x \) will be fixed.

In the same way we can fix all movable joints, whence we can get a rigid unit-bar framework \( G \) in \( \mathbb{R}^n \) which contains a subgraph isomorphic to \( F \). \( \Box \)

4. A generalized octahedron

A unit-bar framework in the plane isomorphic to the bipartite graph \( K(2, 2) \) is clearly flexible. However, any unit-bar framework in \( \mathbb{R}^3 \) isomorphic to the complete tripartite graph \( K(2, 2, 2) \) is rigid since it is congruent to the 1-skeleton of a regular octahedron with unit side-length. For \( n \geq 3 \), a unit-bar framework in \( \mathbb{R}^n \) isomorphic to the complete \( n \)-partite graph \( K(2, 2, \ldots, 2) \) is called a generalized octahedron, and is denoted by \( O_n \).

**Lemma 1.** For \( n \geq 3 \), (1) the joints of \( O_n \) span \( \mathbb{R}^n \), and (2) \( O_n \) is rigid in \( \mathbb{R}^n \).

**Proof.** If \( n = 3 \), the lemma is clearly true. Suppose that the lemma is true for \( n - 1 \), and consider the case \( n \). Let \( x, y \) be a pair of non-adjacent joints of \( O_n \). Then the other \( 2(n - 1) \) joints always lie on the hyperplane which perpendicularly bisects the line segment \( xy \). Hence the subgraph of \( O_n \) induced by these \( 2(n - 1) \) joints is considered as an \( O_{n-1} \) in \( (n - 1) \)-space. Then it spans \( (n - 1) \)-space and is rigid in the \( (n - 1) \)-space by inductive assumption. Therefore, the \( 2(n - 1) \) joints and \( x, y \) span \( \mathbb{R}^n \), and since \( x \) and \( y \) are at unit distance from the \( 2(n - 1) \) joints that span a hyperplane, the whole \( O_n \) must be rigid in \( \mathbb{R}^n \). \( \Box \)

Note that every chordless 4-cycle of \( O_n \), \( n \geq 3 \), is a square.

5. Proof of Theorems 2 and 3

**Proof of Theorem 2.** By Theorem A, it will be enough to show that for any rigid unit-bar framework \( F \) in the plane, there is a rigid unit-bar framework in \( \mathbb{R}^n \) which contains a subgraph congruent to \( F \). And hence it is enough to show the next lemma.
Lemma 2. Let $F$ be a rigid unit-bar framework in $\mathbb{R}^n$, $n \geq 2$. Then there exists a rigid unit-bar framework in $\mathbb{R}^{n+1}$ which contains a subgraph congruent to $F$.

Proof. We regard $F$ as a framework on a hyperplane $\Pi$ in $\mathbb{R}^{n+1}$. Let $F'$ be the translation of $F$ by a unit vector perpendicular to $\Pi$. For a joint $v$ of $F$, the translation of $v$ is denoted by $v'$. Now connect each pair $v, u'$ by a unit-bar. Then we get a realization $F \times I$ of the Cartesian product of $F$ and the complete graph $K_2$. For any unit-bar $uv$ of $F$, the 4-cycle $uv'u'v$ of $F \times I$ is a square. To each such square, attach a generalized octahedron $O_{n+1}$ so that the square becomes rigid. The resulting framework is denoted by $H$.

First we show that $F$ is rigid in $H$. Since $O_{n+1}$ is rigid in $\mathbb{R}^{n+1}$, for any bar $uv$ of $F$, the square $uv'u'$ is rigid in $H$. And since $F$ is connected, we can deduce that for any joints $u, v$ of $F$, the two unit-bars $uu', vuv'$ are parallel, and hence all bars of $F$ are perpendicular to $uu'$. Hence, under any deformation of $H$ that fixes the bar $uu'$, $F$ remains in the hyperplane $\Pi$. Therefore $F$ cannot change its shape, and hence $F$ is rigid in $H$.

Now by Theorem 1, we can extend $H$ to a rigid unit-bar framework $G$. Then $G$ contains a subgraph congruent to $F$. $\square$

Proof of Theorem 3. Since the 'only if' part follows form Theorem B, we show the 'if' part. Suppose that $F$ is a flexible unit-bar framework in $\mathbb{R}^n$ in which distances between joints are all algebraic numbers. For each non-adjacent pair of joints $x, y$ in $F$, we attach a rigid unit-bar framework $G(xy, |x - y|)$ as in Theorem 2. Let $H$ be the resulting graph. Then it is clear that $F$ is rigid in $H$. Now, extend $H$ to a rigid unit-bar framework $G$. Then $G$ contains a subgraph congruent to $F$. $\square$

6. Some lemmas

To prove Theorem 4, we use the fact that the dihedral angle between two facets of a regular simplex of dimension $\geq 3$ is irrational when measured by degree. This fact was also used by Dehn to solve Hilbert's third problem.

Lemma 3. Suppose that $0 < \theta < \pi/2$, $\theta \neq \pi/3$, and $\cos \theta$ is a rational number, and let $\rho$ be the rotation of $\mathbb{R}^2$ around a point $z$ through the angle $\theta$. Then for any point $p \neq z$ in $\mathbb{R}^2$, the point set

$$\{p, \rho(p), \rho^2(p), \rho^3(p), \ldots\}$$

is dense on the circle of radius $|p - z|$ centered at $z$.

Proof. It will be enough to show that $\rho$ is not a cyclic transformation, that is, $\theta/\pi$ is not rational. Hence we show $\cos(k\theta) \neq \pm 1$ for $k = 1, 2, 3, \ldots$.
Let \( \cos \theta = m/n \) (an irreducible fraction). Since \( \theta \neq \pi/3 \), we must have \( n > 2 \). Applying the additive formula for cosine, we can get

\[
\cos(k + 1)\theta + \cos(k - 1)\theta = 2 \cos(k\theta)\cos \theta.
\]

Hence we have

\[
\cos(k + 1)\theta = 2 \cos(k\theta)\cos \theta - \cos(k - 1)\theta.
\]

Using this formula, it can be easily proved by induction on \( k \) that (1) if \( n \) is odd, then \( \cos(k\theta) = a/n^k \) for some integer \( a \) relatively prime to \( n \), and (2) if \( n \) is even \((=2s)\) then \( \cos(k\theta) = b/(2s^k) \) for some integer \( b \) relatively prime to \( s \).

In [2], the irrationality of \( \theta/\pi \) was proved by using a rectangular lattice.

**Lemma 4.** Let \( \theta \) be the angle between two facets of a unit \( n \)-simplex in \( \mathbb{R}^n \) (see Fig. 2). Then \( \cos \theta = 1/n \).

**Proof.** Let \( h \) be the ‘altitude’ of a unit \( (n - 1) \)-simplex. Then by the cosine law, we have

\[
1 = 2h^2 - 2h^2 \cos \theta.
\]

Hence

\[
\cos \theta = 1 - 1/(2h^2).
\]

To compute \( h \), consider a unit \( (n - 1) \)-simplex in \( \mathbb{R}^{n-1} \) with vertices \( o, x_1, \ldots, x_{n-1} \), where \( o \) is the origin. Then

\[
h = \|(x_1 + \cdots + x_{n-1})/(n - 1)\|.
\]

Since \( o, x_1, x_2 \) form a equilateral triangle of unit side, \( (x_i, x_j) = 1 \) and \( (x_i, x_j) = \frac{1}{2} \) for \( i \neq j \). Therefore,

\[
h^2 = (x_1 + \cdots + x_{n-1}, x_1 + \cdots + x_{n-1})/(n - 1)^2 = n/(2(n - 1)).
\]

Thus,

\[
\cos \theta = 1 - (n - 1)/n = 1/n.
\]

Let \( \Delta \) be a fixed unit \( n \)-simplex in \( \mathbb{R}^n \). If we rotate \( \Delta \) around the \( (n - 2) \)-space spanned by some \( n - 1 \) vertices of \( \Delta \), then the remaining two vertices would draw one and the same circle. Such a circle is called an associated circle of the unit.

![Fig. 2.](image-url)
$n$-simplex $\Delta$. For each vertex $x$ of $\Delta$, exactly $n$ associated circles of $\Delta$ pass through the vertex $x$.

**Lemma 5.** Let $\Delta = x_0x_1 \cdots x_n$ be a fixed unit $n$-simplex in $\mathbb{R}^n$. Then the tangent vectors at $x_0$ of the $n$-associated circles through $x_0$ are linearly independent.

**Proof.** Let $C_i$ be the associated circle through $x_0$ and $x_i$. Then the center $z_i$ of $C_i$ is the barycenter of the $(n-2)$-face opposite to the edge $x_0x_i$. Let $z$ be the barycenter of $\Delta$. Then the line $x_iz$ is perpendicular to the facet opposite to the vertex $x_i$. And since the plane $x_0x_iz$ contains $z$, the line $x_iz$ meets the line $x_0z_i$ perpendicularly. Hence the vector $\overline{x_iz}$ is parallel to the tangent line of $C_i$ at $x_0$.

Since the $n$ vectors $\overline{x_iz}$, $i = 1, \ldots, n$ are linearly independent, the lemma follows. \qed

For a point $p$ of $\mathbb{R}^n$ and a non-empty set $Y$ in $\mathbb{R}^n$, let $d(p, Y) = \inf \{|p - y|: y \in Y\}$.

**Lemma 6.** Let $x$ be a vertex of a unit $n$-simplex $\Delta$ in $\mathbb{R}^n$. For any real $r > 0$, there is an $\epsilon = \epsilon(r) > 0$ such that if $|x - p| < r + \epsilon$ then $d(p, C) < r$ for some associated circle $C$ of $\Delta$.

**Proof.** Let $x = x_0, x_1, \ldots, x_n$ be the vertices of $\Delta$. Denote by $C_i$ the associated circle of $\Delta$ passing through $x_0, x_i$, and let $Y$ be the union of the $n$ associated circles $C_1, \ldots, C_n$. Since the tangent vectors of $C_i$s at $x$ are linearly independent by Lemma 5, any hyperplane passing through $x = x_0$ cuts one of the circles $C_1, \ldots, C_n$.

Hence any hypersphere through $x$ cuts one of the circles $C_1, \ldots, C_n$. Therefore,

$$d(w, Y) < |w - x| \quad \text{for every } w \neq x.$$ 

Let $\delta$ be the supremum of $d(w, Y)$ for $w$ with $|w - x| = r$. Since the set of points $w$ satisfying $|w - x| = r$ is compact, and $d(w, Y)$ is continuous on $w$, the sup $\delta$ is attained at some point $w$. Hence $\delta < r$.

Now, put $\epsilon = r - \delta$, and suppose that $|x - p| < r + \epsilon$. Let $q$ be the point on the half line $xp$ such that $|x - q| = r$. Then $d(q, Y) \leq \delta$. Hence

$$d(p, Y) \leq |p - q| + d(q, Y) < \epsilon + \delta = r.$$ 

And since $Y$ is compact, $d(p, Y) = |p - y|$ for some $y$ of $Y$. \qed

7. **Proof of Theorem 4**

Suppose there is an open ball with center $p$ in $\mathbb{R}^n$ such that no clique-work containing $Q$ has a joint in the open ball. Let $r$ be the sup of radius of such balls
centered at \( p \). Then for any \( \varepsilon > 0 \), there is a clique-work \( W \) containing \( Q \) which has a joint \( x \) within distance \( r + \varepsilon \) from \( p \). We may suppose that \( \varepsilon \) was chosen as in Lemma 6. Let \( \Delta \) be the unit \( n \)-simplex spanned by a clique \( Q' \) in \( W \) that contains the joint \( x \). Then by Lemma 6, there is an associated circle \( C \) of \( \Delta \) through \( x \) such that \( d(p, C) < r \). Let \( z \) be the center of this circle \( C \). Fix an orientation on the circle \( C \), and consider the sequence of points

\[
x = x_0, x_1, x_2, x_3, \ldots
\]

on \( C \) such that the directed angle \( \angle x_i x_{i+1} = \theta \), where \( \theta \) is the angle between two facets of \( \Delta \). Then by Lemmas 3, 4, these points are dense on \( C \). Hence there is some integer \( m \) such that \( |x_m - p| < r \). Now, in the point set

\[
\{ \text{vertices of } \Delta \} \cup \{ x_0, x_1, x_2, x_3, \ldots, x_m \},
\]

connect every pair of points with unit-distance apart by a unit-bar. Then we get a unit-bar framework \( W' \), and it is not difficult to see that \( W' \) is a clique-work. Since \( W \) and \( W' \) share the clique \( Q' \), we can attach them at \( Q' \), and get a bigger clique-work, which has a joint \( x_m \) such that \( |x_m - p| < r \). This contradicts the choice of \( r \). \( \square \)

8. Solution of exercise

Fig. 3 shows a solution of the exercise from Section 1.

Fig. 3. Solution of exercise.

References