Multiple solutions for resonant semilinear elliptic problems in \( \mathbb{R}^N \)

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Abstract

We prove the existence of multiple nontrivial solutions for the semilinear elliptic problem

\[ -\Delta u = h(\lambda u + g(u)) \quad \text{in} \quad \mathbb{R}^N, \quad u \in D^{1,2}, \]

where \( h \in L^1 \cap L^\alpha \) for \( \alpha > N/2 \), \( N \geq 3 \), \( g \) is a \( C^1(\mathbb{R}, \mathbb{R}) \) function that has at most linear growth at infinity, \( g(0) = 0 \), and \( \lambda \) is an eigenvalue of the corresponding linear problem

\[ -\Delta u = \lambda hu \quad \text{in} \quad \mathbb{R}^N, \quad u \in D^{1,2}. \]


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1. Introduction

Let \( D^{1,2} \) denote the Hilbert space obtained through completion of \( C_c^\infty(\mathbb{R}^N) \), where \( N \geq 3 \), via the method of Cauchy sequences, with respect to the norm
\[
\|u\| := \left( \int |\nabla u|^2 \right)^{\frac{1}{2}}.
\]
Then, \( D^{1,2} \) has real inner product \( \langle \cdot, \cdot \rangle \) defined by
\[
\langle u, v \rangle := \int \nabla u \cdot \nabla v \quad \text{for all } u, v \in D^{1,2}.
\]
Throughout this paper, we will always assume that \( N \) is an integer bigger than 2 and that all the integrals are computed over all of \( \mathbb{R}^N \), unless other domain is specified. The norm in the \( L^p \) spaces is denoted by \( |\cdot|_p \).

We study the existence of the nontrivial weak solutions of the problem
\[
-\Delta u = h(x)(\lambda u + g(u)), \quad x \in \mathbb{R}^N, \ u \in D^{1,2},
\]
where \( \lambda \) is a real parameter to be specified shortly, and \( h : \mathbb{R}^N \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are functions satisfying
\[
\begin{align*}
(h-1) & \quad h \in L^1 \cap L^\alpha \text{ for some } \alpha > \frac{N}{2}, \\
(h-2) & \quad h > 0 \text{ a.e. in } \mathbb{R}^N, \\
(g-1) & \quad g \in C^1(\mathbb{R}, \mathbb{R}) \text{ with } |g(s)| \leq c_1 + c_2|s| \text{ for all } s \in \mathbb{R}, \text{ and some positive constants } c_1 \text{ and } c_2, \\
(g-2) & \quad g(0) = 0.
\end{align*}
\]

The boundary value problem (1) is a nonlinear perturbation of the linear eigenvalue problem
\[
-\Delta u = \lambda h(x)u, \quad x \in \mathbb{R}^N, \ u \in D^{1,2}.
\]
We will see in the next section that, under the assumptions (\(h-1\)) and (\(h-2\)) on \( h \), the linear problem (2) has sequence of eigenvalues \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \), with \( \lambda_n \to \infty \) as \( n \to \infty \) and a corresponding sequence \((\phi_n)\) of eigenfunctions which forms a complete orthonormal system for \( D^{1,2} \). It can also be shown that \( \phi_1 \) may be chosen so that \( \phi_1 > 0 \) a.e. in \( \mathbb{R}^N \).

By a weak solution of problem (1) we mean a function \( u \in D^{1,2} \) for which
\[
\int \nabla u \cdot \nabla v - \lambda \int huv - \int hg(u)v = 0 \quad \text{for all } v \in D^{1,2}.
\]

The main result in this paper, which is proved in Section 4, can be stated as follows.
Theorem 1.1. Let \( h \) and \( g \) satisfy (\( h-1 \))–(\( h-2 \)) and (\( g-1 \))–(\( g-2 \)), respectively. Assume also that (\( g-3 \)) There exists \( s_1 > 0 \) and \( \gamma \in \mathbb{R} \) such that \( 0 < \gamma < \lambda_2 - \lambda_1 \), where \( \lambda_1 \) and \( \lambda_2 \) are the first eigenvalues of the problem (2), and

\[
0 \leq \frac{g(s)}{s} \leq \gamma \quad \text{for } |s| \geq s_1.
\]  

(\( L-L \)) (A Landesman–Lazer type condition) Let \( W \) denote the eigenspace corresponding to \( \lambda_1 \). For every \( u \in D^{1,2} \) write \( u_0 = Pu \), where \( P \) is the orthogonal projection onto \( W \). Suppose that whenever \( (u_n) \subset D^{1,2} \) is such that \( \|u_n\| \to \infty \) and \( \|u_n^0\| \to 1 \) as \( n \to \infty \), then

\[
\limsup_{n \to \infty} \int h g(u_n) \frac{u_n^0}{\|u_n^0\|} > 0.
\]  

In addition, suppose that either \( g'(0) < 0 \) or there exists \( m \geq 2 \) such that \( \lambda_m < \lambda_1 + g'(0) < \lambda_{m+1} \), then problem (1), for \( \lambda = \lambda_1 \), has at least two nontrivial solutions.

Remark 1.2. Theorem 1.1 is based upon a multiplicity result in [11] for bounded domains in \( \mathbb{R}^N \). The multiplicity result in [11] is in turn an extension of a result of Ahmad in [1].

Remark 1.3. Condition (\( g-3 \)) makes problem (1) into a one-sided resonance problem. The term resonance for the equation in problem (1) usually refers to the case in which the ratio \( (\lambda s + g(s))/s \) can attain asymptotically, as \( |s| \to \infty \), any of the eigenvalues \( \lambda_j \), \( j \geq 1 \), of the linear problem (2). For the case in which \( \lambda = \lambda_1 \) we see, in view of (4) in condition (\( g-3 \)), that the above ratio could possibly attain the value \( \lambda_1 \) from the right as \( |s| \to \infty \). However, the restriction \( \gamma < \lambda_2 - \lambda_1 \) prevents the ratio from taking on any other eigenvalue of the linear problem (2) as \( |s| \to \infty \). This would be the case, for example, if \( g \) was bounded. The question of existence for this case, under more general assumptions, was treated by the authors in [12]. Thus, the present work also complements the authors’ results in [12] by answering the question of multiplicity of solutions in the case of resonance.

Remark 1.4. Condition (\( L-L \)) is a generalization of the original Landesman–Lazer condition found in [10] for the case in which \( g \) is bounded. Suppose, for example, that \( \lim s \to \infty g(s) \) and \( \lim s \to -\infty g(s) \) both exist, and denote them by \( g(+\infty) \) and \( g(-\infty) \), respectively. Then, the condition introduced in [10] takes the form

\[
g(-\infty) \int h \varphi_1 < 0 < g(+\infty) \int h \varphi_1.
\]  

It can be shown, as a consequence of the conditions (\( h-1 \))–(\( h-2 \)) and the Lebesgue dominated convergence theorem, that condition (6) implies the condition (\( L-L \)) used in this paper.

Remark 1.5. If the restriction \( \gamma < \lambda_2 - \lambda_1 \) in (4) is relaxed to include the possibility \( \gamma = \lambda_2 - \lambda_1 \), then problem (1) turns into a double resonance problem. For this case, Robinson
has shown in [15] that, in the bounded domain case, if more general Landesman–Lazer type conditions involving the eigenspaces of \( \lambda_1 \) and \( \lambda_2 \) are imposed, then one can obtain multiple solutions. Robinson’s result also applies to higher eigenvalues. In the last section of this paper we will outline how Robinson’s conditions in [15] can be used to obtain at least two nontrivial weak solutions for problem (1) in the double resonance case, and for higher eigenvalues.

2. Preliminary results

The arguments presented in this paper hinge on the following proposition regarding the eigenvalues and eigenfunctions of problem (2):

**Proposition 2.1.** Suppose that \( h \) satisfies \((h–1)\) and \((h–2)\). Then, there exists a sequence of eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \), of problem (2), with \( \lambda_n \to \infty \) as \( n \to \infty \), and a corresponding sequence \((\varphi_n)\) which forms a complete orthonormal system (CONS) for \( D^{1,2} \). Furthermore, the principal eigenvalue, \( \lambda_1 \), is simple and the corresponding eigenfunction, \( \varphi_1 \), may be chosen so that \( \varphi_1 > 0 \) a.e. in \( \mathbb{R}^N \).

Proposition 2.1 can be derived from the following compact embedding theorem through the use of spectral theory for compact self-adjoint operators.

**Proposition 2.2.** Let \( h \) satisfies \((h–1)\) and \((h–2)\) and define

\[
L^2_h := \left\{ u \in L \left| \int hu^2 < \infty \right. \right\},
\]

where \( L \) denotes the class of Lebesgue measurable functions in \( \mathbb{R}^N \). Then, the space \( D^{1,2} \) is compactly embedded in \( L^2_h \).

Proposition 2.2 is in turn a consequence of Lemma 2.1 in [8].

**Remark 2.3.** For the case in which \( h \) is assumed to be bounded and smooth, Proposition 2.1 can be obtained as an application of a result of Allegretto [3, Theorem 1]. See also a generalization of the Allegretto result in [7].

**Remark 2.4.** It can also be shown, as a consequence of the assumption \( h \in L^\alpha \) for some \( \alpha > \frac{N}{2} \) in \((h–2)\), that the eigenfunctions of problem (2) given by Proposition 2.1 satisfy the unique continuation property (see [5]); that is, for each \( \varphi_j, j \geq 1 \), the Lebesgue measure of the set in which \( \varphi_j \) vanishes is zero.

Using the fact that \((\varphi_n)\) forms a CONS for \( D^{1,2} \), we obtain the following Poincaré inequality:

\[
\lambda_1 \int hu^2 \leq \int |\nabla u|^2 \quad \text{for all } u \in D^{1,2}. \tag{7}
\]
More generally, suppose that $\lambda_{k-1}$ and $\lambda_k$ are two eigenvalues of the linear problem (2) such that $\lambda_{k-1} < \lambda_k$, with $k \geq 2$, and let

$$W = \text{span}\{\varphi_j \mid \lambda_j \leq \lambda_{k-1}\}.$$ 

Then, for any $v \in \mathcal{D}^{1,2}$ which is orthogonal to $W$ (i.e., $\langle v, w \rangle = 0$ for all $w \in W$),

$$\lambda_k \int hv^2 \leq \int |\nabla v|^2.$$ (8)

In the next section we will use the following result which can be obtained as a consequence of the continuity of $g$ guaranteed by (g–1) and the assumption that $h \in L^1$ in (h–1)–(h–2).

**Lemma 2.5.** Let $g : \mathbb{R} \to \mathbb{R}$ satisfy (g–1), and $h : \mathbb{R}^N \to \mathbb{R}$ satisfy (h–1) and (h–2). Then, the map $u \mapsto g(u)$ is continuous from $L^2_h$ to $L^2_h$.

### 3. Variational setting

In this section we develop the variational formulation of problem (1).

Define a functional on $\mathcal{D}^{1,2}$ by

$$J(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{\lambda_1}{2} \int hu^2 - \int hG(u)$$ for all $u \in \mathcal{D}^{1,2}$, (9)

where $G(s) = \int_0^s g(t)dt$ for $s \in \mathbb{R}$. Conditions (g–1) on $g$ and (h–1) on $h$ imply that $J$ is well-defined on $\mathcal{D}^{1,2}$, continuous, and Fréchet differentiable, with continuous derivative $J'$ given by

$$\langle J'(u), v \rangle = \int \nabla u \cdot \nabla v - \lambda_1 \int hv - \int hg(u)v$$ (10)

for $u, v \in \mathcal{D}^{1,2}$. Thus, in view of (3), finding weak solutions of (1), with $\lambda = \lambda_1$, is equivalent to finding critical points of the functional $J$ defined in (9).

In order to apply minimax methods for finding critical points of $J$, we need to verify that $J$ satisfies the well-known Palais–Smale condition (PS): $J$ satisfies the (PS) condition iff every sequence $(u_n) \subset \mathcal{D}^{1,2}$ satisfying

(i) $J(u_n)$ is bounded, and
(ii) $\|J'(u_n)\|_\text{op} \to 0$ as $n \to \infty$, (11) (12)

has a strongly convergent subsequence. Any sequence $(u_n)$ in $\mathcal{D}^{1,2}$ satisfying (11) and (12) is said to be (PS)-sequence for $J$.

In this paper this condition comes about as a consequence of the following Landesman–Lazer type condition (L–L):

$$\lambda_{k-1} < \lambda_k$$ with $k \geq 2$. 

$W = \text{span}\{\varphi_j \mid \lambda_j \leq \lambda_{k-1}\}.$
Let $W$ denote the eigenspace corresponding to $\lambda_1$. For every $u \in D^{1,2}$ write $u^0 = Pu$, where $P$ is the orthogonal projection onto $W$. Suppose that whenever $(u_n) \subset D^{1,2}$ is such that $\|u_n\| \to \infty$ and $\frac{u_n^0}{\|u_n\|} \to 1$ as $n \to \infty$, then

$$\limsup_{n \to \infty} \int h g(u_n) \frac{u_n^0}{\|u_n\|} > 0. \quad (13)$$

Before establishing the fact that $J$ satisfies the (PS) condition, we require the following preliminary lemma.

**Lemma 3.1.** Suppose that $g$ satisfies (g–1) and (g–2), and $h$ satisfies (h–1) and (h–2). For $u \in D^{1,2}$, define

$$\Lambda_u(v) := \lambda_1 \int huv + \int h g(u) v \quad \text{for all } v \in D^{1,2}. \quad (14)$$

Then, there exists a continuous, compact operator $K : D^{1,2} \to D^{1,2}$ such that $\Lambda_u(v) = \langle K(u), v \rangle$ for all $v \in D^{1,2}$.

**Proof.** Fix $u \in D^{1,2}$. It follows from (g–1)–(g–2), (h–1) and (h–2), the Poincaré inequality (7), and the Cauchy–Schwartz inequality that

$$|\Lambda_u(v)| \leq \left( \|u\| + \left( 2e^1_1 |h|_1 + \frac{2e^1_2}{\lambda_1} \|u\| \right)^2 \right) \|v\| \quad \text{for all } v \in D^{1,2}.\]$$

Thus $\Lambda_u$ is a bounded linear functional on $D^{1,2}$. By the Riesz representation theorem, there exists $K(u) \in D^{1,2}$ such that $\Lambda_u(v) = \langle K(u), v \rangle$ for all $v \in D^{1,2}$. Now observe that for $u_1, u_2 \in D^{1,2}$ and all $v \in D^{1,2}$

$$\langle K(u_1) - K(u_2), v \rangle = \lambda_1 \int h((u_1 - u_2) + (g(u_1) - g(u_2))) v.$$

Using the Cauchy–Schwartz inequality and (7) again, we can prove that

$$\|K(u_1) - K(u_2)\| \leq \sqrt{\lambda_1} |u_1 - u_2|_{L^2_h} + \frac{1}{\sqrt{\lambda_1}} |g(u_1) - g(u_2)|_{L^2_h}. \quad (15)$$

The continuity of $K$ on $D^{1,2}$ then follows from Lemma 2.5 and the Poincaré inequality (7). Finally, the estimate in (15) implies that $K$ is compact in $D^{1,2}$. To see why this is the case, let $(u_n)$ be a bounded sequence in $D^{1,2}$, then it has a subsequence $(u_{n_k})$ which converges weakly in $D^{1,2}$; say $u_{n_k} \rightharpoonup u$ in $D^{1,2}$. Now, the compact embedding $D^{1,2} \subset L^2_h$, established in Proposition 2.2, implies $(u_{n_k})$ converges strongly in $L^2_h$. Thus, the estimate (15), Proposition 2.2 and Lemma 2.5 imply that $\|K(u_{n_k}) - K(u)\| \to 0$ as $k \to \infty$. Hence, $K$ is compact. □

**Proposition 3.2.** Assume that $g$ satisfies (g–1)–(g–3), $h$ satisfies (h–1)–(h–2), and that the ($L$–$L$) condition holds. Then $J$ satisfies the Palais–Smale condition.
Proof. In view of (10), we see that \( \langle J'(u), v \rangle = \langle u, v \rangle - \Lambda_u(v) \) for all \( u, v \in \mathcal{D}^{1,2} \), where \( \Lambda_u(v) \) is as defined in (14). It then follows from Lemma 3.1 that \( J'(u) = u - K(u) \), for all \( u \in \mathcal{D}^{1,2} \), where \( K \) is compact. Hence, by [16, Proposition 2.2, p. 71], it suffices to show that every (PS)-sequence for \( J \) in \( \mathcal{D}^{1,2} \) must also be a bounded sequence. Thus, let \( (u_n) \) be a sequence in \( \mathcal{D}^{1,2} \) for which (11) and (12) hold. Since \( \mathcal{D}^{1,2} \) is compactly embedded in \( L^2_h \), by Proposition 2.2, one can use the same argument as that used in [11] in the case of a bounded domain \( \Omega \) to show that \( (u_n) \) is bounded. In this case we would use \( L^2_h \) instead of \( L^2(\Omega) \). For the details of this argument we refer the reader to [11].

In what remains of this section we will show that the functional \( J \) defined in (9) has a critical point. This will be done through the use of the Ambrosetti–Rabinowitz saddle point theorem [14, Theorem 4.6]. In order to apply the Ambrosetti–Rabinowitz result, we will first show that \( J \) is anticoercive on the eigenspace corresponding to \( \lambda_1 \), and coercive on the orthogonal complement. The anticoercivity of the functional will be seen to be a consequence of the \( (L–L) \) condition.

Lemma 3.3. Assume that \( g \) satisfies \((g–1)–(g–2)\), \( h \) satisfies \((h–1)–(h–2)\), and that the \( (L–L) \) condition holds. Let \( W \) denote the eigenspace corresponding to \( \lambda_1 \). Then, \( J \) is anticoercive on \( W \); that is \( J(w) \to -\infty \) as \( \|w\| \to \infty \), \( w \in W \).

Proof. First, observe that the \( (L–L) \) condition implies that there exist \( \delta, R > 0 \) such that \( \int h g(u)u \geq \delta\|u\| \) for all \( u \in W \) with \( \|u\| \geq R \). If not, there is a sequence \( u_n \in W \) with \( \|u_n\| \to \infty \), and
\[
\int h g(u_n) \frac{u_n}{\|u_n\|} < \frac{1}{n}
\]
for all \( n \geq 1 \), which contradicts (13) in the \( (L–L) \) condition. Therefore, given \( u \in W \) with \( \|u\| = 1 \), we have
\[
\int h G(tu) = \int h G(Ru) + \frac{1}{s} \int h g(su) su \, dx \, ds.
\]
Hence,
\[
\int h G(tu) \geq \int h G(Ru) + \delta\|u\| \int \frac{1}{s} \, ds,
\]
which implies that \( \int h G(tu) \to \infty \) as \( t \to \infty \). Since \( W \) is one-dimensional, we therefore conclude that
\[
\int h G(u) \to \infty \quad \text{as} \quad \|u\| \to \infty \quad \text{for} \quad u \in W.
\]
Hence \( J(u) = -\int h G(u) \to -\infty \) as \( \|u\| \to \infty \) in \( W \). \( \square \)

Remark 3.4. Observe that (16) is the Ahmad–Lazer–Paul condition (cf. [2]) used by the authors in [12].
Lemma 3.5. Suppose \( g \) satisfies \((g-1)-(g-3)\) and \( h \) satisfies \((h-1)-(h-2)\). Let \( W \) denote the eigenspace corresponding to \( \lambda_1 \) and \( V \) be its orthogonal complement in \( D^{1,2} \). Then \( J \) is coercive in \( V \); that is, \( J(v) \to +\infty \) as \( \|v\| \to \infty \), \( v \in V \).

Proof. Using \((g-3)\), we can write \( g = g_1 + g_2 \), where \( g_1 \) and \( g_2 \) are \( C^1 \) functions on \( \mathbb{R} \) with

\[
g_1(s) = 0 \quad \text{for} \quad |s| \geq s_1, \tag{17}
\]

and

\[
0 \leq \frac{g_2(s)}{s} \leq \gamma \quad \text{for all} \quad s \in \mathbb{R}, \tag{18}
\]

where \( \gamma < \lambda_2 - \lambda_1 \). In \((17)\) and \((18)\), \( s_1 \) and \( \gamma \) are given by \((4)\) in the \((g-3)\) condition on \( g \). Then we can write \( G(s) = G_1(s) + G_2(s) \) for all \( s \in \mathbb{R} \), where \( G_1(s) = \int_0^s g_1(t) \, dt \) satisfies

\[
|G_1(s)| \leq C|s| \quad \text{for all} \quad s \in \mathbb{R}, \tag{19}
\]

and some constant \( C \) since \( g_1 \) is bounded by \((17)\), and \( G_2(s) = \int_0^s g_2(t) \, dt \) satisfies

\[
G_2(s) \leq \frac{\gamma}{2} s^2 \quad \text{for all} \quad s \in \mathbb{R}, \tag{20}
\]

by \((18)\).

We therefore obtain, as a consequence of \((h-1)-(h-2)\), \((9)\), \((19)\), \((20)\), and the Cauchy–Schwartz inequality,

\[
J(v) \geq \frac{\delta}{2} \|v\|^2 - C_2 \|v\| \quad \text{for all} \quad v \in V, \tag{21}
\]

and some constant \( C_2 \), where \( \delta = 1 - \frac{\lambda_1 + \gamma}{\lambda_2} > 0 \) since \( \gamma < \lambda_2 - \lambda_1 \). The coercivity of \( J \) on \( V \) now follows from \((21)\). \( \Box \)

Next, we show the existence of a saddle point for the functional \( J \).

Theorem 3.6. Assume that \( g \) satisfies \((g-1)-(g-3)\), \( h \) satisfies \((h-1)-(h-2)\), and that the \((L-L)\) condition holds. Then, problem \((1)\), with \( \lambda = \lambda_1 \), has a weak solution given by a saddle point of the functional \( J \).

Proof. Let \( W = \text{span}\{\varphi_1\} \) and \( V \) denote its orthogonal complement in \( D^{1,2} \). Using \((21)\), we conclude that \( J \) is bounded from below in \( V \). Hence, since \( J \) is weakly lower semi-continuous and coercive on \( V \) by Lemma 3.5, \( J \) attains a minimum value \( \beta := \inf_{v \in V} J(v) \).

Since \( J(w) \to -\infty \) as \( \|w\| \to \infty \) by Lemma 3.3, we can find \( s_0 > 0 \) so large that \( J(\pm s_0 \varphi_1) < \beta \). By Proposition 3.2, \( J \) satisfies \((PS)\), thus we can apply the saddle point theorem of Ambrosetti and Rabinowitz [14, Theorem 4.6] to conclude that \( J \) has a critical point \( u_0 \). In fact, put \( w_0 = -s_0 \varphi_1 \) and \( w_1 = +s_0 \varphi_1 \), and let \( \Gamma \) consist of all continuous maps \( \eta : [0,1] \to D^{1,2} \) such that \( \eta(0) = w_0 \) and \( \eta(1) = w_1 \), then the critical value \( J(u_0) \) can be characterized by

\[
J(u_0) = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} J(\eta(t)) \geq \beta. \tag{22}
\]

\( \Box \)
4. Multiple solutions

We are now in a position to prove our main result, Theorem 1.1, which we state here as

**Theorem 4.1.** Assume that $g$ satisfies $(g-1)-(g-3)$, $h$ satisfies $(h-1)-(h-2)$, and that the $(L-L)$ condition holds. In addition, suppose that either $g'(0) < 0$ or there exists $m \geq 2$ such that $\lambda_m < \lambda_1 + g'(0) < \lambda_{m+1}$, then problem (1), with $\lambda = \lambda_1$, has at least two nontrivial solutions.

**Proof.** The proof uses arguments similar to those in [11], but we include here an outline for the convenience of the reader.

Let $J$ be the functional defined by (9). Then $J$ is a $C^2$ functional with Fréchet derivative given by (10), and second derivative

$$J''(u)(w, v) = \int \nabla w \cdot \nabla v - \lambda_1 \int hwv - \int hg'(u)wv$$

(22)

for $u, v, w \in D^{1,2}$.

We shall prove the theorem by showing that $J$ has at least two nontrivial critical points.

By Theorem 3.6, $J$ has a critical point $u_0$ of mountain pass type (mp-type); see [4] and [9, Definition 1] for the definition of mp-type critical point. Put $c_0 = J(u_0)$. If $u_0$ is the only critical point of $J$ corresponding to the critical value $c_0$, and if $u_0$ is nondegenerate, then Theorem 4.2 in [4] shows that the Morse index of $u_0$ is 1. Using (22), we have

$$J''(0)(w, v) = \int \nabla w \cdot \nabla v - \lambda_1 \int hwv - \int hg'(0)wv$$

(23)

for all $v, w \in D^{1,2}$. Thus since $\lambda_1 + g'(0)$ is not an eigenvalue of the linear problem (2), 0 is a nondegenerate critical point of $J$. Now, if $g'(0) < 0$, using (23), we obtain for $v \neq 0$

$$J''(0)(v, v) = \|v\|^2 - (\lambda_1 + g'(0)) \int hv^2 \geq -g'(0) \int hv^2 > 0,$$

where we have used the Poincaré inequality (7). Thus, the Morse index of 0 is 0. On the other hand, if $\lambda_m < \lambda_1 + g'(0) < \lambda_{m+1}$ for some $m \geq 2$, let $Y$ be the span of the eigenfunctions corresponding to eigenvalues $\lambda_k$ of problem (2) with $k \leq m$, then $J''(0)(v, v) \leq (\lambda_m - (\lambda_1 + g'(0))) \int hv^2 < 0$ for all $v \in Y$ with $v \neq 0$. Thus, in this case, the Morse index of 0 is at least 2. Hence by Ambrosetti’s Theorem [4] there must be a critical point $u_0 \in J^{-1}(c_0)$ of $J$ with $u_0 \neq 0$.

We next show that there must be another critical point of $J$ distinct from 0 and $u_0$. To this end we use a result of Hofer in [9] about the Leray–Schauder index of critical points of mp-type.

By Lemma 3.1, we can write $\langle J'(u), v \rangle = \langle u - K(u), v \rangle$ for all $u, v \in D^{1,2}$, where $K$ is a compact map. The local degree of $J' = I - K$ at 0 can be determined by computing the Leray–Schauder degree of $I - K'(0)$ at 0 with respect to a sufficiently small ball $B_{\epsilon}$, where $K'(u)$ is defined by

$$\langle K'(u)w, v \rangle = \int h(\lambda_1 + g'(u))wv$$

for every $v, w \in D^{1,2}$. 

and each \( u \in D^{1,2} \). This is known as the Leray–Schauder index of \( I - K \) at 0. If 0 and \( u_0 \) are the only critical points of \( J \), then \( u_0 \) is an isolated critical point of mp-type. We can therefore apply [9, Theorem 2] to conclude that \( \text{LS-index}(I - K, u_0) = -1 \). Furthermore, if \( R > 0 \) is so large that \( u_0 \in B_R \) then the addition property of the Leray–Schauder degree implies

\[
\text{deg}_{\text{LS}}(I - K, B_R, 0) = \text{LS-index}(I - K, 0) + \text{LS-index}(I - K, u_0),
\]

so that \( \text{deg}_{\text{LS}}(I - K, B_R, 0) = \text{LS-index}(I - K, 0) - 1 \). A degree theoretic argument based on the homotopy invariance of the Leray–Schauder degree can now be used, as outlined in [11], to get that \( \text{deg}_{\text{LS}}(I - K, B_R, 0) = -1 \) for a sufficiently large \( R \). We therefore obtain

\[
\text{LS-index}(I - K, 0) = 0. \tag{24}
\]

Now, \( K'(0)w = \lambda w \) for some \( \lambda \in \mathbb{R} \) and \( w \neq 0 \) if and only if

\[
\lambda \int \nabla w \cdot \nabla v = (\lambda_1 + g'(0)) \int hvw \quad \text{for every } v \in D^{1,2};
\]

that is, \( \frac{\lambda_1 + g'(0)}{\lambda} \) is an eigenvalue of the linear problem (2). Thus if \( g'(0) < 0 \), then \( \lambda < \frac{1}{\lambda_k} \)

for all \( k \geq 1 \), so that all eigenvalues of \( K'(0) \) are strictly less than 1, this implies by [13, Theorem 2.8.1] that the Leray–Schauder index of \( I - K \) at 0 is 1. On the other hand, if \( \lambda_m < \lambda_1 + g'(0) < \lambda_{m+1} \) for some \( m \geq 2 \), then \( \lambda = \frac{\lambda_1 + g'(0)}{\lambda_k} > 1 \) for \( k \leq m \). Hence, by the same result in [13], \( \text{LS-index}(I - K, 0) = (-1)^m \). In either case we contradict (24), and this contradiction yields the result. \( \square \)

5. Double resonance and higher eigenvalues

In this section we consider extensions of the result in Theorem 1.1 to include the case of double resonance and higher eigenvalues. These results are based upon previous results by Robinson [15] in the case of bounded domains.

As mentioned in the introduction, if \( \gamma = \lambda_2 - \lambda_1 \) in (4) of condition (g–3), then problem (1) could be at resonance with the second eigenvalue, as well as the first. For this case, following Robinson in [15], we may impose the following generalized Landesman–Lazer type conditions on the eigenspaces of \( \lambda_1 \) and \( \lambda_2 \); we shall refer to it as (GLL1):

Let \( W_i \) denote the eigenspace corresponding to \( \lambda_i \), \( i = 1, 2 \), and let \( P_i : D^{1,2} \to D^{1,2} \) denote orthogonal projections onto \( W_i \) for \( i = 1, 2 \). For every \( u \in D^{1,2} \) write \( u^{(i)} = P_i u \) for \( i = 1, 2 \). Suppose that for every \( (u_n) \subset D^{1,2} \) such that \( \|u_n\| \to \infty \) the following conditions hold:

(i) if \( \frac{\|u_n^{(1)}\|}{\|u_n\|} \to 1 \) as \( n \to \infty \), then there exist \( \delta_1 > 0 \) and \( n_1 > 0 \) such that

\[
\int h(g(u_n))u_n^{(1)} \geq \delta_1 \quad \text{for all } n > n_1; \tag{25}
\]

(ii) if \( \frac{\|u_n^{(2)}\|}{\|u_n\|} \to 1 \) as \( n \to \infty \), then there exist \( \delta_2 > 0 \) and \( n_2 > 0 \) such that

\[
(\lambda_2 - \lambda_1) \int h(u_n)u_n^{(2)} \geq \delta_2 \quad \text{for all } n > n_2. \tag{26}
\]
Replacing \((L-L)\) with \((GLL_1)\) and allowing \(\gamma\) to equal \(\lambda_2 - \lambda_1\), we obtain the following multiplicity result for the doubly resonant problem (1) with \(\lambda = \lambda_1\).

**Theorem 5.1.** Let \(h\) and \(g\) satisfy \((h-1)-(h-2)\) and \((g-1)-(g-2)\), respectively. Assume also that there exists \(s_1 > 0\) such that \(0 \leq \frac{g(s)}{s} \leq \lambda_2 - \lambda_1\) for \(|s| \geq s_1\), and that the generalized Landesman–Lazer type condition \((GLL_1)\) holds. In addition, suppose that \(g'(0) < 0\) or there exists \(m \geq 2\) such that \(\lambda_m < \lambda_1 + g'(0) < \lambda_{m+1}\), then problem (1), for \(\lambda = \lambda_1\), has at least two nontrivial solutions.

**Proof.** By virtue of the compact embedding \(D^{1,2} \hookrightarrow L^2_h\) guaranteed by Proposition 2.2, the arguments in [15] for bounded domains carry over to our situation. We shall therefore provide an outline of the proof, and refer the reader to [15] for details.

As in the proof of Theorem 1.1, we let \(W\) be the span of \(\varphi_1\) and \(V\) be its orthogonal complement in \(D^{1,2}\). Then, by (27) in part (i) of \((GLL_1)\), the functional \(J\) defined in (9) is anticoercive on \(W\), and by (28) in part (ii) of \((GLL_1)\) and \((g-4)\), it is coercive on \(V\). However, in this case, \(J\) does not necessarily satisfy the Palais–Smale condition. Instead, a special case of a compactness condition used in [6] can be proved:

\[(C)\] If \(\{u_n\} \subset D^{1,2}\) is such that \((1 + \|u_n\|)\|J'(u_n)\|_{op} \to 0\) as \(n \to \infty\), then \(\{u_n\}\) has a subsequence which converges strongly in \(D^{1,2}\).

This condition allows one to derive all the critical point theory results needed to make the arguments in the proof of Theorem 1.1 work in this case as well; namely, the saddle point theorem of Ambrosetti and Rabinowitz, the Morse-index result of Ambrosetti, and the Leray–Schauder index calculation of Hofer for mp-type critical points.

We next consider the case in which \(\lambda = \lambda_k\) in problem (1), where \(\lambda_k\) is an eigenvalue of problem (2) with \(k \geq 2\) and \(\lambda_k < \lambda_{k+1}\). If we replace \((g-4)\) by

\[(g-5)\] there exists \(s_1 > 0\) such that \(0 \leq \frac{g(s)}{s} \leq \lambda_{k+1} - \lambda_k\) for \(|s| \geq s_1\),

then problem (1), with \(\lambda = \lambda_k\), yields a doubly resonant problem, with resonance between \(\lambda_k\) and \(\lambda_{k+1}\). To obtain existence and multiplicity of solutions, we therefore require the following generalized Landesman–Lazer type condition (cf. [15]) which we shall denote by \((GLL_k)\):

Let \(W_i\) denote the eigenspace corresponding to \(\lambda_i\), \(i = k, k+1\), and let \(P_i : D^{1,2} \to D^{1,2}\) denote orthogonal projections onto \(W_i\) for \(i = k, k+1\). For every \(u \in D^{1,2}\) write \(u^{(i)} = P_i u\) for \(i = k, k+1\). Suppose that for every \(\{u_n\} \subset D^{1,2}\) such that \(\|u_n\| \to \infty\) the following conditions hold:
(i) if \( \frac{\|u(i)\|}{\|u\|} \to 1 \) as \( n \to \infty \), then there exist \( \delta_1 > 0 \) and \( n_1 > 0 \) such that
\[
\int hg(u_n)u^{(i)}_n \geq \delta_1 \quad \text{for all } n > n_1;
\] (27)

(ii) if \( \frac{\|u(i+1)\|}{\|u\|} \to 1 \) as \( n \to \infty \), then there exist \( \delta_2 > 0 \) and \( n_2 > 0 \) such that
\[
(\lambda_{k+1} - \lambda_k) \int hu_n u^{(i+1)}_n - \int hg(u_n)u^{(i+1)}_n \geq \delta_2 \quad \text{for all } n > n_2.
\] (28)

With conditions (g–5) and (GLLk), we are able to establish the following version of [15, Theorem 2, p. 11] for our problem (1) with \( \lambda = \lambda_k \).

**Theorem 5.2.** Let \( h \) and \( g \) satisfy (h–1)–(h–2) and (g–1)–(g–2), respectively. Assume also that \( g \) satisfies (g–5) and that the generalized Landesman–Lazer type condition (GLLk) holds. In addition, suppose that \( \lambda_k + g'(0) < \lambda_1 \), then problem (1), for \( \lambda = \lambda_k \), has at least two nontrivial solutions.

**Outline of the proof.** As in the proof of the previous result, Proposition 2.2 makes it possible for the arguments in [15] for bounded domains to carry over to this case. In particular, if we let \( W = \text{span}\{\varphi_j \mid \lambda_j \leq \lambda_k\} \) and \( V \) denote the orthogonal complement to \( W \) in \( D^{1,2} \), then it follows from (g–5) and (GLLk) that the functional \( J_k \) defined by
\[
J_k(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{\lambda_k}{2} \int hu^2 - \int hG(u) \quad \text{for all } u \in D^{1,2},
\]
is coercive on \( V \) and anticoercive on \( W \). Also, the compactness condition (C) holds for \( J_k \).

The assumption \( \lambda_k + g'(0) < \lambda_1 \) implies that 0 is a strict local minimum of \( J_k \), thus there exist \( r > 0 \) and \( c > 0 \) such that \( J_k|_{\partial B_R(0)} \geq c \). Since \( J_k \) is anticoercive on \( W \), we can find \( w \in W \) such that \( \|w\| > r \) and \( J_k(w) < 0 \). This is precisely the setup needed for the application of the mountain pass theorem [14, Theorem 2.2, p. 7], whose use in this case is justified by condition (C). Let \( u_0 \) be a critical point of \( J_k \) given by the mountain pass theorem, then \( u_0 \) is an mp-type critical point and \( J_k(u_0) \geq c > 0 \), so that \( u_0 \) is nontrivial.

To find a second nontrivial solution proceed now as in the proof of Theorem 1.1 by observing that
\[
\text{deg}_{LS}(J'_k, B_R, 0) = (-1)^v, \quad \text{where } v = \text{dim } W,
\]
for a sufficiently large \( R \), and LS-index\( (J'_k, 0) = 1 \). \( \square \)

**References**