The Extended f-Vectors of 4-Polytopes

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For $P$ a $d$-dimensional convex polytope and $S = \{i_1, \ldots, i_s\} \subseteq \{0, 1, \ldots, d-1\}$, let $f_S(P)$ be the number of chains of faces $\emptyset \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_s \subseteq P$ with $\dim F_j = i_j$. By the generalized Dehn–Sommerville equations the dimension of the affine span of the extended $f$-vectors $(f_S(P))_{S \subseteq \{0, 1, \ldots, d-1\}}$ as $P$ ranges over all 4-polytopes is 4, and the extended $f$-vectors are determined by the values of $f_0, f_1, f_2,$ and $f_3$. Six linear and four nonlinear inequalities on extended $f$-vectors of 4-polytopes are given. The consequences for the basic $f$-vector, $(f_0, f_1, f_2, f_3)$, are derived. These include the inequality, $4f_2 \geq 3f_0 - 10 + 7f_3$, conjectured by Barnette.

1. INTRODUCTION

A polytope is the convex hull of finitely many points in Euclidean space. A $d$-dimensional polytope (or $d$-polytope) has in its boundary faces of dimension $0, 1, \ldots, d-1$. Faces of dimension 0, 1, and $d-1$ are called, respectively, vertices, edges, and facets. Write $f_i(P)$ for the number of $i$-dimensional faces (or $i$-faces) of a polytope $P$. The vector $f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ is called the $f$-vector of the polytope. For $S = \{i_1, i_2, \ldots, i_s\} \subseteq \{0, 1, \ldots, d-1\}$ let $f_S(P)$ be the number of chains of faces $\emptyset \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_s \subseteq P$, where $\dim F_j = i_j$; and call $(f_S(P))_{S \subseteq \{0, 1, \ldots, d-1\}}$ the extended $f$-vector of $P$. Any $d$-polytope $P$ has a dual $d$-polytope $P^*$ whose $f$-vector is $f(P^*) = (f_{d-1}(P), \ldots, f_1(P), f_0(P))$, and whose extended $f$-vector is given by $f_S(P^*) = f_T(P)$, where $T = \{d-1-s|s \in S\}$.

The big open question is to characterize the $f$-vectors and extended $f$-vectors of polytopes. This has been solved only up through dimension three. The set of $f$-vectors of 3-polytopes is $\{(f_0, f_0 + f_2 - 2, f_2) \mid 4 \leq f_0 \leq 2f_2 - 4$ and $4 \leq f_2 \leq 2f_0 - 4\}$ (see [8]); for extended $f$-vectors $f_{01} = f_{02} = f_{12} = 2f_1 = 2f_0 + 2f_2 - 4, f_{012} = 4f_1$ (see [6]). The answer is also known for the class of simplicial polytopes (those polytopes for which each boundary face is the

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convex hull of affinely independent points). The characterization of the $f$-vectors of simplicial polytopes was conjectured by McMullen [10] and proved by Stanley [12] and Billera and Lee [7]. This also gives a characterization of the extended $f$-vectors of simplicial polytopes, since these vectors depend only on the $f$-vectors. Also, the characterization can be reflected to give the characterization of $f$-vectors and extended $f$-vectors of simple polytopes ($d$-polytopes each of whose vertices lies on exactly $d$ facets).

We summarize here what is known about the $f$-vector and extended $f$-vector of arbitrary polytopes and, in particular, of 4-dimensional polytopes. This paper gives additional necessary conditions for a vector to be the $f$-vector or the extended $f$-vector of a 4-dimensional polytope.

The best-known result on $f$-vectors of polytopes is Euler's formula: for any $d$-polytope, $\sum_{i=0}^{d} (-1)^i f_i = 1 - (-1)^d$. It is easy to show that this is the only linear relation that holds for all $d$-polytopes. In particular, for 4-polytopes we have $f_0 - f_1 + f_2 - f_3 = 0$. The inequalities on $f$-vectors of 3-polytopes have easy generalizations to higher dimensions. In a $d$-polytope every $(d-2)$-face is contained in exactly two $(d-1)$-faces, and every $(d-1)$-face has at least $d$ $(d-2)$-faces. Hence $2f_{d-2}(P) \geq df_{d-1}(P)$. From the dual statement on vertex-edge incidences we have $2f_1(P) \geq df_0(P)$. For 4-polytopes, then, we have $f_2 \geq 2f_1$ and $f_1 \geq 2f_0$. Grünbaum, Barnette, and Reay have completely specified the projections of the $f$-vectors of 4-polytopes onto two coordinates [3, 4, 8].

Also important is the upper bound theorem, which says that the $f$-vector of any $d$-polytope with $n$ vertices is bounded above by the $f$-vector of some simplicial $d$-polytope with $n$ vertices [10]. This result holds also for the extended $f$-vector [5]. For 4-polytopes this gives

$$f_1 \leq \frac{1}{2} f_0 (f_0 - 1), \quad f_2 \leq f_0 (f_0 - 3), \quad f_3 \leq \frac{1}{2} f_0 (f_0 - 3), \quad f_{02} \leq 3f_0 (f_0 - 3).$$

The extended $f$-vector satisfies the generalized Dehn-Sommerville equations,

$$\sum_{i=1}^{k-1} (-1)^{i-1} f_{S \cup \{i+1\}} - (1 - (-1)^{k-i}) f_S,$$

where $\{i, k\} \subset S \cup \{-1, d\}$ and $\{i+1, \ldots, k-1\} \cap S = \emptyset$ [6]. In the case of 4-polytopes these yield the following relations: $f_{01} = 2f_1$, $f_{03} = 2f_0 - 2f_1 + f_{02}$, $f_{12} = f_{13} = f_{02}$, $f_{23} = 2f_2$, $f_{012} = f_{013} = f_{023} = f_{123} = 2f_{02}$, $f_{0133} = 4f_{03}$. So the problem of characterizing the extended $f$-vectors of 4-polytopes reduces to characterizing the vectors $(f_0(P), f_1(P), f_2(P), f_{02}(P))$, and in what follows we will usually mean this vector in $\mathbb{Z}^4$ when we say “the extended $f$-vector.” More generally, the generalized Dehn-Sommerville equations cut the dimension of the set of extended
f-vectors of $d$-polytopes to $c_d - 1$, where $c_d$ is the $d$th Fibonacci number initialized by $c_0 = c_1 = 1$.

In what follows we will need to refer to certain specific polytopes, and we introduce some symbols for them here. If $Q$ is any $d$-polytope, $PQ$ stands for the pyramid over $Q$, $BQ$ stands for the bipyramid over $Q$, and $IQ$ stands for the prism on $Q$. In particular, a $d$-simplex is represented by $P^d$. For any $k, 1 \leq k \leq \frac{1}{2}d$, let $T^d_k$ be the polytope obtained by stellar subdivision of a $k$-face in $P^d$. For $0 \leq r \leq d - 2$ and $1 \leq k \leq \frac{1}{2}(d - r)$, let \( T^d_k \) be the extended $f$-vectors of these polytopes can be computed easily. The computation of the $f$-vectors of pyramids, bipyramids, prisms, and the polytopes $T^d_k$ can be found in [8]. For dimension 4 it remains to give $f_0$. For any 3-polytope $Q$, $f_0(Q) = 2f_1(Q)$ by the generalized Dehn–Sommerville equations. Then

$$f_0(BQ) = f_0(Q) + f_0(Q) + f_1(Q) = 5f_1(Q)$$

Finally, $f_0(T^d_k) = 3f_2(T^d_k)$ for $d \geq 3$, so we can compute $f_0(T^d_k)$ by the rules above.

2. INEQUALITIES ON THE EXTENDED f-VECTORS OF 4-POLYTOPE

Let $C \subset Q^4$ be the convex hull of the vectors $(f_0(P), f_1(P), f_2(P), f_0(P))$, as $P$ ranges over all 4-polytopes. Determining $C$ is not sufficient for a characterization of extended $f$-vectors of 4-polytopes, because an integer vector that is a convex combination of extended $f$-vectors is not necessarily an extended $f$-vector. For example, $f(BP^4) = (6, 14, 16, 48)$ and $f(IP^4) = (8, 16, 14, 48)$, but $(7, 15, 15, 48)$ is not the extended $f$-vector of a polytope (by the $(f_0, f_2)$-projection of $f$-vectors). Theorem 1 gives four facets, and two other bounding hyperplanes of $C$.

**Theorem 1.** If $(f_0, f_1, f_2, f_0) = (f_0(P), f_1(P), f_2(P), f_0(P))$ for some 4-polytope $P$, then

1. $f_0 - 3f_2 \geq 0$
2. $f_0 - 3f_1 \geq 0$
3. $f_0 - 3f_2 + f_1 \geq 0$
4. $6f_1 - 6f_0 - f_0 \geq 0$
5. $f_0 - 5 \geq 0$
6. $f_2 - f_1 + f_0 - 5 \geq 0$. 

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For \( i = 1, 2, 3, 4 \), we denote by \( A_i \) the hyperplane that bounds the halfspace given by the \( i \)-th inequality above; then \( A_i \cap C \) is a facet of \( C \).

**Proof.** (1) The first inequality says that each 2-face has at least 3 vertices, which is clear for all polytopes. The polytopes having \( f \)-vectors on \( A_1 \) are the 2-simplicial polytopes; \( \{ P^5, BP^4, PBP^3, T^3_4 \} \) is a set of 2-simplicial polytopes having affinely independent extended \( f \)-vectors. (2) This is the dual of (1) by the generalized Dehn–Sommerville equations \( f_{i+2} = f_i \). (3) This inequality is the lower bound theorem for arbitrary polytopes. (In the case of simplicial polytopes it reduces to \( f_1 - 4f_0 + 10 \geq 0 \).) It has been proved by Kalai [9] using rigidity; for polytopes with rational vertices, it also has a proof using algebraic geometry. \( A_3 \cap C \) is spanned by the extended \( f \)-vectors of the polytopes \( P^5, BP^4, PBP^3, \) and \( P^2B^2P^2 \), and these are affinely independent. (4) To prove this inequality, recall for any 3-polytope \( F \), \( 2f_{i+3}(F) \geq 3f_0(F) \). Summing this inequality over the 3-faces of a 4-polytope \( P \), we have

\[
2f_{i+3}(P) = \sum_{\dim F = 3} 2f_1(F) \geq \sum_{\dim F = 3} 3f_0(F) = 3f_0(P).
\]

Now by the generalized Dehn–Sommerville equations, \( f_{i+3} = f_{i+2} \) and \( f_{i+2} - 2f_0 - 2f_i + f_{i+1} \), the inequality \( 2f_{i+3} \geq 3f_0 \) becomes \( 2f_{i+2} \geq 6f_0 - 6f_1 + 3f_{i+2} \), or \( 6f_1 - 6f_0 - f_{i+2} \geq 0 \). Equality holds for a polytope \( P \) if and only if all the 3-faces of \( P \) are simple. \( A_4 \cap C \) is spanned by the extended \( f \)-vectors of \( P^5, BP^4, IP^4, \) and \( T^4_2 \), which are affinely independent. (5) is an obvious inequality, but it is not implied by the others. Equality holds in (5) only for the simplex, with vector \( (5, 10, 10, 30) \). (6) is the dual of (5).

\( C \) is not a closed set. To see this let \( D \) be the closed convex set determined by the inequalities of Theorem 1. Then the closure of \( C, \overline{C} \), is contained in \( D \). The ray \( \{(5, f_1, 2f_1 - 10, 6f_1 - 30) \mid f_1 \geq 10 \} \) is an extreme ray of \( D \). It contains only one extended \( f \)-vector, that of the simplex, \( (5, 10, 10, 30) \). Yet it is an extreme ray of \( \overline{C} \). We exhibit a sequence of points in \( \overline{C} \) that converges to an interior point of the ray. For each integer \( n \geq 5 \) there exists a cyclic polytope, \( C(n, 4) \), with extended \( f \)-vector \( (f_0, f_1, f_2, f_0) = (n, (\frac{n}{2}), 2(\frac{n}{2}) - 2n, 6(\frac{n}{2}) - 6n) \). Thus, for each \( n \geq 6 \), \( C \) contains the point \( v_n = (5, 10, 10, 30) + (1/(\frac{n}{2}) - 10)) \( n - 5, (\frac{n}{2}) - 10, 2(\frac{n}{2}) - 2n - 10, 6(\frac{n}{2}) - 6n - 30) \). So \( \overline{C} \) contains \( \lim_{n \to \infty} v_n = (5, 11, 12, 36) \). The argument can be repeated using the segments from \( (5, 11, 12, 36) \) to \( (n, (\frac{n}{2}), 2(\frac{n}{2}) - 2n, 6(\frac{n}{2}) - 6n) \), and continuing, we see that the entire ray is in \( \overline{C} \).

If we could determine the facets of \( \overline{C} \) then we would know exactly what linear inequalities appear in the characterization of extended \( f \)-vectors. With this in mind, let us study the structure of \( D \), which could be \( \overline{C} \).
The set $D$ is a cone with vertex $(5, 10, 10, 30)$. Figure 1 is the Schlegel diagram of the 3-polytope that is the cross-section of $D$. The facets are marked with the numbers of the inequalities in Theorem 1. Referring to the vertex labels in Fig. 1, we can say the following about the extreme rays of $D$. Rays $l_3$ and $l_5$ are extreme rays of $C$; they each contain the extended $f$-vectors of an infinite sequence of polytopes. Rays $l_4$ and $l_6$ are extreme rays of $C$, not contained in $C$; this was shown for $l_4$ above, and is true for $l_6$ by duality. The ray $l_1$ contains an edge (at least) of $C$, for it contains the extended $f$-vector of the hypersimplex, $(10, 30, 30, 90)$, as well as that of the simplex. We do not know if $l_2$ and $l_7$ contain edges or rays of $C$, nor if $l_1$ is in fact an extreme ray of $C$ or $C$. An infinite family of 2-simplicial, 2-simple polytopes would be helpful here.

One might expect, however, that the inequality $f_0 - 5 \geq 0$ and its dual are not essential. This suggests the existence of a linear inequality that cuts off facets (5) and (6) from $D$. The hyperplane through rays $l_2$, $l_4$, and $l_6$ of $D$ is given by the equation $3f_2 + 3f_0 - f_{12} - 15 = 0$. There is some evidence that $3f_2 + 3f_0 - f_{02} - 15 \geq 0$ for all 4-polytopes $P$.

We turn now to nonlinear inequalities.

**Theorem 2.** If $(f_0, f_1, f_2, f_{02}) = (f_0(P), f_1(P), f_2(P), f_{02}(P))$ for some 4-polytope $P$, then

1. $2(f_{02} - 3f_2) + f_1 \leq \left(\frac{f_0}{2}\right)$
2. $2(f_{02} - 3f_1) + f_2 \leq \left(\frac{f_0 - f_1}{2} + f_0\right)$
3. $f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \left(\frac{f_2}{2}\right)$
4. $f_{02} + f_2 - 2f_1 - 2f_0 \leq \left(\frac{f_2 + f_0}{2}\right)$.

**Proof.** The right-hand side of (1), $\left(\frac{f_0}{2}\right)$, is the number of pairs of vertices in the polytope. The set of these pairs can be partitioned into three subsets: the set of pairs each determining an edge of $P$, the set of pairs each on some 2-face of $P$, but not on an edge of $P$, and the set of pairs not on any
2-face of $P$. The first set is counted by $f_1$. To count the second set, consider a 2-face $F$ with $n$ vertices. Then the number of nonedge pairs of vertices on $F$ is $\binom{n}{2} - n = \frac{1}{2}n(n-3) \geq 2(n-3)$ for $n \geq 3$. Summing over the 2-faces of $P$ we get that the second set has at least $\sum_{F \in P} 2(f_0(F) - 3) = 2(f_{02}(P) - 3f_2(P))$ pairs. So $2(f_{02} - 3f_2) + f_1 \leq \binom{n}{2}$. The second inequality, (2), is the dual of (1). To prove (3) and (4) we need the following lemma.

**Lemma 3.** If $F$ is a 3-polytope, $m_h(F)$ is the number of pairs of vertices contained in some 2-face of $F$, but not forming an edge of $F$, and $m_i(F)$ is the number of pairs of vertices not contained in the same 2-face of $F$, then

$$\frac{1}{2}m_h(F) + m_i(F) \geq \frac{1}{2}(f_2(F) - 4) + \frac{1}{2}(f_{02}(F) - 3f_2(F)).$$

**Proof.** First we assume $f_0(F) \geq 7$. Let $F$ be a simplicial polytope obtained from $F$ by adding diagonal edges on all nontriangular faces. Then $f(F') = (f_0(F), f_1(F) + f_{02}(F) - 3f_2(F), f_2(F) + f_{02}(F) - 3f_2(F))$. The number of pairs of vertices not forming edges of $F'$ (and hence not forming edges of $F$) is

$$\binom{f_0(F)}{2} - f_1(F') = \binom{f_0(F)}{2} - (3f_0(F) - 6) = \frac{1}{2}f_0^2 - \frac{7}{2}f_0 + 6,$$

because $F'$ is simplicial. Since $f_0 \geq 7$, this is at least $2f_0 - 8 - f_2(F') - 4 = f_2(F) - 4 + f_{02}(F) - 3f_2(F)$. Thus, if $f_0 \geq 7$, then

$$\frac{1}{2}m_h(F) + m_i(F) \geq \frac{1}{2}(m_h(F) + m_i(F)) \geq \frac{1}{2} \left[ f_2(F) - 4 + f_{02}(F) - 3f_2(F) \right].$$

To finish the proof, it is enough to check that the inequality holds for the ten combinatorial types of 3-polytopes with fewer than 7 vertices [8].

**Proof of Theorem 2 (continued).** For a 4-polytope $P$,

$$f_{02} - 4f_2 + 3f_1 - 2f_0 = f_1 + f_{02} - 3f_2 + f_2 - 2f_3$$

$$= f_1 + \frac{1}{2} \sum_{F \in P} (f_{02}(F) - 3f_2(F)) + \frac{1}{2} \sum_{F \in P} (f_2(F) - 4),$$

since each 2-face is contained in exactly two 3-faces. By the lemma, this is at most

$$f_1 + \frac{1}{2} \sum_{F \in P, \dim F = 3} m_h(F) + \sum_{F \in P, \dim F = 3} m_i(F).$$

Now $m_h(F)$ counts pairs of vertices contained in a 2-face but not forming an edge; as we sum over the 3-faces of $P$, such pairs will be counted twice.
So, letting $m_i(G)$ be the number of pairs of vertices contained in the 2-face $G$ but not forming an edge, we obtain

$$f_{02} - 4f_2 + 3f_1 - 2f_0 \leq f_1 + \sum_{G \in P, \dim G = 2} m_i(G) + \sum_{F \in P, \dim F = 3} m_i(F) \leq \binom{f_0}{2}.$$ 

The last inequality, (4), is the dual of (3).

The inequalities of Theorems 1 and 2 imply all the inequalities on extended $f$-vectors of 4-polytopes mentioned in Sect. 1. They do not, however, characterize extended $f$-vectors of 4-polytopes. The vector $(7, 17, 19, 59)$ satisfies all the inequalities but is not the extended $f$-vector of a 4-polytope (see [8] for a list of all 4-polytopes with 7 vertices).

3. The Generalized $h$-Vector

Stanley's work on simplicial polytopes was based on the connection between ($f$-vectors of) polytopes and (Betti numbers of) certain algebraic varieties. In attempts to extend these methods to arbitrary polytopes, attention has focused on the generalized $h$-vector, which was originally calculated by algebraic geometers, but is of independent combinatorial interest [13]. The definition uses two polynomials given recursively. Let $g(\emptyset, x) = g(\text{point}, x) = h(\text{point}, x) = 1$. For $h(P, x) = \sum_{i=0}^{d} h_{d-i} x^i$ let $g(P, x) = h_{d} + \sum_{i=1}^{d-1} \binom{d+1}{i} (h_{d-i} - h_{d-i+1}) x_i$. Let $h(P, x) = \sum_{F \in P, F \neq P} g(F, x)$.

The generalized $h$-vector of the $d$-polytope $P$ is then $h(P) = (h_0, h_1, ..., h_d)$. It follows easily from the definition that the generalized $h$-vector is a linear function of the extended $f$-vector. It is also easy to check that $g(P^{d+1}, x) = 1$ for any $d$. For simplicial polytopes, then, the definition of the generalized $h$-vector reduces to that of the $h$-vector of the polytope (what McMullen introduced as the $g$-vector [10]); this can be obtained from the $f$-vector by a nonsingular linear transformation. The characterization of $f$-vectors of simplicial polytopes can thus be expressed in terms of the $h$-vector (here $\langle i \rangle$ represents a certain nonlinear operator [10, 12]).

**Theorem 4** (Billera and Lee, Stanley). A vector $(h_0, h_1, ..., h_d) \in N^{d+1}$ is the $h$-vector of a simplicial $d$-polytope if and only if

$$h_i = h_{d-i} \text{ for all } i; h_0 = 1; \text{ and }$$

$$0 \leq h_{i+1} - h_i \leq (h_i - h_{i-1})^\langle i \rangle \text{ for } 0 \leq i \leq \frac{1}{2} d - 1.$$
It is natural to ask, as Stanley and others have, whether this extends to
the generalized h-vectors of arbitrary polytopes.

Conjecture 5. A vector $(h_0, h_1, \ldots, h_d) \in \mathbb{N}^{d+1}$ is the generalized h-vector
of a d-polytope if and only if

\[ h_i = h_d - i \quad \text{for all } i; h_0 = 1; \text{ and} \]

\[ 0 \leq h_{i+1} - h_i \leq (h_i - h_{i-1})^{(i)} \quad \text{for } 0 \leq i \leq \frac{1}{2} d - 1. \]

The sufficiency part of the conjecture follows, of course, from Theorem 3. Necessity can be shown for $d \leq 5$, but is open for higher dimensions. For $d=4$ the generalized h-vector is $(1, f_0 - 4, f_0 - 3f_2 + f_1 - 3f_0 + 6, f_0 - 4f_3) = (f_0 - 3f_2 + f_1 - 4f_0 - 4, f_2 - 2f_1 + 3f_0 - 4, f_3 - f_2 + f_1 - f_0 + 1)$. The conditions are

\[ f_0 - 5 \geq 0 \]

\[ f_0 - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \]

and

\[ f_0 - 3f_2 + f_1 \leq \binom{f_0}{2}. \]

The second equation is a generalized Dehn–Sommerville equation. The inequalities are implied by, but are weaker than, the inequalities of Theorems 1 and 2. The extended f-vector cannot be recovered from the generalized h-vector, and the inequalities of Theorems 1 and 2 cannot be interpreted as inequalities on the generalized h-vector. Thus, the parameters suggested so far by the algebraic geometry will not be enough to give a good description of the combinatorics of polytopes.

4. The f-Vectors of 4-Polytopes

We turn now to the f-vectors of 4-polytopes.

Theorem 6. If $(f_0, f_1, f_2, f_3) = (f(P), f_1(P), f_2(P), f_3(P))$ for some
4-polytope P, then

1. $2f_1 - 2f_0 - f_2 \geq 0$
2. $f_1 - 2f_0 \geq 0$
3. \(-3f_2 + 7f_1 - 10f_0 + 10 \geq 0\)
4. \(f_0 - 5 \geq 0\)
5. \(f_2 - f_1 + f_0 - 5 \geq 0\).

*Proof.* These are simply the projections of the inequalities of Theorem 1. \(\blacksquare\)

The inequalities of Theorem 6 imply all known linear inequalities on \(f\)-vectors of 4-polytopes. Among them (4) and (5) are trivial, and (1) and (2) are well known (see Sec. 1). Inequality (3) was conjectured by Barnette [2] and is proved for the first time here. Barnette also conjectured that this inequality is sharp for most values of \(f_3\). This is true, as we see in the following proposition.

**Proposition 7.** For each \(n \geq 5\) there exists a 4-polytope with \(f_3 = n\) and \(-3f_2 + 7f_1 - 10f_0 + 10 = 0\).

*Proof.* We define a sequence of simple 4-polytopes as follows. Let \(Q_n\) be the 4-simplex. Construct \(Q_{n+1}\) by truncating \(Q_n\) with a hyperplane that separates one vertex of \(Q_n\) from all other vertices. The effect of this operation on the \(f\)-vector is to add \((3, 6, 4, 1)\). Thus, \(f(Q_n) = (3n - 10, 6n - 20, 4n - 10, n)\), and the equation holds. The polytopes constructed here are the duals of the simplicial stacked polytopes [11]. \(\blacksquare\)

The projection of Fig. 1 gives Fig. 2, which represents the cross section of the cone \(K\) with vertex \((5, 10, 10)\) and facets given by the inequalities of Theorem 6. If we let \(J\) be the convex hull of the set \(\{(f_0(P), f_1(P), f_2(P))| P\ \text{is a 4-polytope}\}\), then by the analysis of Section 2, \(J\) is not closed, \(J \subset K\), the rays \(m_1\) and \(m_4\) are extreme rays of \(K\) contained in \(J\), and the rays \(m_3\) and \(m_5\) are extreme rays of \(J\) not contained in \(J\). As before we may look for inequalities cutting off the ray \(m_3\) from \(K\). We can generate these by modifying half spaces (4) and (5) by the equations defining hyperplanes (1)–(3). In this way we get the inequality, \(5f_0 - 2f_1 + f_2 - 15 \geq 0\), and its dual, \(5f_0 - 4f_1 + 3f_2 - 15 \geq 0\), which might hold for 4-polytopes.

![Fig. 2. Cross section of polyhedron given by inequalities of Theorem 6.](image-url)
If we use the linear inequalities of Theorem 1 to eliminate \( f_{02} \) from the inequalities of Theorem 2, we get the following, originally proved by Barnette [2]. (The other projected inequalities are implied by these two.)

**Theorem 8.** If \( (f_0, f_1, f_2, f_3) = (f_0(P), f_1(P), f_2(P), f_3(P)) \) for some 4-polytope \( P \), then

1. \(-f_2 + 3f_1 - 2f_0 \leq (f_0)\)
2. \(f_2 + f_1 - 2f_0 \leq (f_0 + f_0)\).

As in the case of extended \( f \)-vectors, Theorems 6 and 8 do not characterize the \( f \)-vectors of 4-polytopes. The vector \((8, 22, 22, 8)\) satisfies the inequalities of these theorems, but it is not the \( f \)-vector of any 4-polytope (see [1] for a complete list of 4-polytopes with 8 vertices).

**5. CONCLUSION**

The technique of counting incidences of faces in polytopes is not new; it was used extensively in earlier work on \( f \)-vectors. The introduction of the language of extended \( f \)-vectors facilitates the technique. It enabled us to prove a new result on \( f \)-vectors of 4-polytopes, and can easily be used to generate inequalities for higher dimensional polytopes.

The extended \( f \)-vectors of polytopes are not simply a tool for studying \( f \)-vectors, however. They provide a better description of the combinatorial types of polytopes. They are also necessary to extend Stanley's techniques involving algebraic geometry from simplicial to arbitrary polytopes.

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**REFERENCES**