Asymptotic Properties of Solutions for First-Order Neutral Differential Equations with Distributed Deviating Arguments

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Abstract—In this paper, we study a class of first-order linear neutral differential equations with distributed deviating arguments. A number of lemmas and theorems are established to describe the asymptotic properties of nonoscillatory solutions to the equations. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The problem of oscillation and asymptotic properties of solutions to first-order neutral differential equations has received a great deal of attention over the last few years. Ladas and Sficas [1] and Grammatikopoulos, Grove and Ladas [2] studied the asymptotic properties of the delayed neutral differential equation of the form

\[ \frac{d}{dt} [y(t) + Py(t - \tau)] + Qy(t - \sigma) = 0, \quad t \geq t_0, \]  

(A)

and, as a result, established the sufficient conditions for oscillation of all the solutions to the equation in the case that \( Q, \tau, \sigma \) are all positive constants and \( P \) is a real parameter. Then, Grammatikopoulos, Grove and Ladas [3] extended the results of [1,2] to the case where \( \tau \) and \( \sigma \) are real numbers, and \( Q \) is a nonzero real number. Grammatikopoulos, Ladas and Sficas [4] further discussed the oscillation and asymptotic behavior of the equation with variable coefficients, that is,

\[ \frac{d}{dt} [y(t) + P(t)y(t - \tau)] + Q(t)y(t - \sigma) = 0, \quad t \geq t_0. \]  

(B)
Since then, many results regarding the asymptotic properties of solutions of first-order neutral differential equations have been obtained, and the reader is referred in particular to the monographs of Bainov and Mishev [5], Erbe, Kong and Zhang [6], and the papers of Grove, Ladas and Schultz [7], Graef, Grammatikopoulos and Spikes [8], Jaros and Kuşano [9], Yu and Fu [10], Shen and Yu [11], Wang, Lin and Yu [12], Wang [13] and the references cited therein. However, most of the works concerned only the case of discrete deviating arguments. To the best of our knowledge, very little has been done with distributed deviating arguments. Thus, in this paper, we study the asymptotic behavior of nonoscillatory solutions of the following neutral equation:

\[
\frac{d}{dt} \left[ y(t) + P(t)y(t - \tau) \right] + Q(t) \int_0^{\sigma(t)} y(t - s) \, d\eta(t, s) = 0, \quad t \geq t_0, \tag{1}
\]

where the integral of equation (1) is a Riemann-Stieltjes one.

2. PRELIMINARY NOTES

In this paper, we assume that the following conditions (H) hold:

(H1) \( P(t), Q(t) \in C([t_0, \infty), R) \), and \( 0 < Q_1 \leq Q(t) \leq Q_2 \), where \( Q_1 \) and \( Q_2 \) are some constants;

(H2) \( \sigma(t) \in C([t_0, \infty), R_+) \), and \( \lim_{t \to \infty} (t - \sigma(t)) = \infty \);

(H3) \( \eta(t, \sigma(t)) \in C([t_0, \infty), R) \), and \( \int_{t_0}^{\infty} \eta(t, \sigma(t)) \, dt = \infty \);

(H4) \( \eta(t, s) \) is nondecreasing with respect to \( s \) for \( s \in [0, \sigma(t)] \), and \( \eta(t, 0) = 0 \).

DEFINITION 1. The function \( f \) is said to eventually enjoy property A if there exists a \( t_\mu \) such that, for \( t \geq t_\mu \), the function \( f \) enjoys property A.

DEFINITION 2. The function \( y(t) \), defined for all sufficiently large \( t \), is said to be an eventual solution of equation (1), if for all sufficiently large \( t \), \( y(t) \) is a continuous function, \( y(t) + P(t)y(t - \tau) \) is a continuously differentiable function, and \( y(t) \) eventually satisfies equation (1).

DEFINITION 3. The solution \( y(t) \) of equation (1) is said to be oscillatory, if its set of zeros is unbounded above. Otherwise, the solution is said to be nonoscillatory.

We restrict our attention to proper solutions of equation (1), i.e., to nonconstant solutions in the interval \([T, \infty)\) for some \( T \geq t_0 \) satisfying \( \sup_{t \geq T} |x(t)| > 0 \).

The following lemmas will be useful to obtain our main results. Let

\[
z(t) = y(t) + P(t)y(t - \tau), \tag{2}
\]

then from (1), we have

\[
z'(t) = -Q(t) \int_0^{\sigma(t)} y(t - s) \, d\eta(t, s). \tag{3}
\]

LEMMA 1. If \( y(t) \) is an eventually positive solution of equation (1), then \( z(t) \) decreases eventually.

PROOF. From (3) and Conditions (H), we can easily obtain the result of Lemma 1.

LEMMA 2. If \( y(t) \) is an eventually positive solution of equation (1), and there exists a constant \( c \) such that \( \lim_{t \to \infty} z(t) = c \), then \( \liminf_{t \to \infty} y(t) = 0 \).

PROOF. Assume the contrary that \( \liminf_{t \to \infty} y(t) = d > 0 \). Then, there exists a \( t_1 \geq t_0 \), such that \( y(t) > d/2 \) for \( t \geq t_1 \). From (3), we have

\[
z'(t) \leq -Q(t) \min_{t - \sigma(t) \leq s \leq t} y(s)\eta(t, \sigma(t)) < -\frac{d}{2} Q_1 \eta(t, \sigma(t)), \tag{4}
\]

then, by integrating the above inequality from \( t_0 \) to \( t \), we have

\[
z(t) - z(t_0) < -\frac{d}{2} \int_{t_0}^{t} \eta(s, \sigma(s)) \, ds,
\]
that is,
\[ z(t_0) > z(t) + \frac{d}{2} Q_1 \int_{t_0}^{t} \eta(s, \sigma(s)) \, ds. \] (5)
From (H3), inequality (5) implies that \( \lim_{t \to -\infty} z(t) = -\infty \), which contradicts the assumption of \( z(t) \). The proof of Lemma 2 is completed.

**Lemma 3.** If \( y(t) \) is an eventually positive solution of equation (1), and
\[ -1 < P \leq P(t) \leq 0, \] (6)
then \( y(t) \) is a bounded function.

**Proof.** From Lemma 1, either \( z(t) > 0 \) eventually or \( z(t) < 0 \) eventually.

(i) Assume that \( z(t) < 0 \) eventually. Then, from (6), we have
\[ 0 > z(t) = y(t) + P(t)y(t - \tau) > y(t) - y(t - \tau). \] (7)
Thus, for sufficiently large \( t \), we have
\[ y(t) < y(t - \tau). \] (7)
Assume that \( y(t) \) is an unbounded function. Then, by choosing a sequence \( \{t_n\}^\infty_1 \) such that \( \lim_{n \to \infty} t_n = \infty \), it results that \( \lim_{n \to \infty} y(t_n) = \infty \), and \( \max_{t_1 \leq t \leq t_n} y(t) = y(t_n) \). If choosing \( N \) such that \( t_{N - \tau} > t_1 \), then \( \max_{t_{N - \tau} \leq t \leq t_N} y(t) = y(t_N) \), which contradicts (7).

(ii) Assume that \( z(t) > 0 \) eventually, then from Lemma 1, there exists a constant \( c \) such that \( \lim_{t \to -\infty} z(t) = c \geq 0 \), and thus, from Lemma 2, we have \( \liminf_{t \to -\infty} y(t) = 0 \). Hence, there exists a sequence \( \{T_k\}^\infty_1 \) such that \( \lim_{k \to \infty} T_k = \infty \), \( \liminf_{k \to \infty} y(T_k) = 0 \).

Assume that \( y(t) \) is an unbounded function. Then, there exists a sequence \( \{t_n\}^\infty_1 \) such that \( \lim_{n \to \infty} t_n = \infty \), \( \lim_{n \to \infty} y(t_n) = \infty \), \( \max_{t_1 \leq t \leq t_n} y(t) = y(t_n) \), and \( t_n - t_{n - 1} > \tau \) for any \( n \in \mathbb{N} \).

Let \( n, k \in \mathbb{N} \) be sufficiently large numbers and \( t_n > T_k \), then
\[
z(t_n) - z(T_k) = y(t_n) - y(T_k) + P(t_n)y(t_n - \tau) - P(T_k)y(T_k - \tau) \\
\geq y(t_n) - y(T_k) + P(T_k)y(T_k - \tau) \\
\geq y(t_n) - y(T_k) + P(T_k)y(T_k) \\
= (1 + P)(y(t_n) - y(T_k)). \] (8)
Noting that \( \lim_{n \to \infty} y(t_n) = \infty \), and \( \liminf_{k \to \infty} y(T_k) = 0 \), we can choose a sequence \( \{t_n\}^\infty_1 \) and \( \{T_k\}^\infty_1 \) in such a way that, for sufficiently large \( n \) and \( k \),
\[ (1 + P)(y(t_n) - y(T_k)) > 0. \] (9)
From (8) and (9), we have \( z(T_k) < z(t_n) \), that is, \( z(t) \) is increasing, which is in contradiction with Lemma 1. The proof of Lemma 3 is completed.

**Lemma 4.** Assume that
\[ P_1 \leq P(t) \leq P_2 < -1. \] (10)
If \( y(t) \) is an eventually positive solution of equation (1), then \( z(t) < 0 \) eventually.

**Proof.** Assume the contrary that \( z(t) \geq 0 \) eventually. From (10), we have
\[ 0 \leq z(t) = y(t) + P(t)y(t - \tau) < y(t) - y(t - \tau), \] (11)
and thus, for any sufficiently large \( t \), we have
\[ y(t) > y(t - \tau). \] (12)
From Lemma 1, there exists a constant \( c \geq 0 \) such that \( \lim_{t \to \infty} z(t) = c \). Furthermore, from Lemma 2, we have \( \liminf_{t \to \infty} y(t) = 0 \). Thus, by choosing a sequence \( \{T_k\}^\infty_1 \) such that \( \lim_{k \to \infty} T_k = \infty \), we have \( \liminf_{k \to \infty} y(T_k) = 0 \) and \( \min_{t_1 \leq t \leq t_n} y(t) = y(t_n) \). By choosing a positive integer \( n \) such that \( t_n - t_1 > \tau \), it results that \( \min_{t_1 - \tau \leq t \leq t_n} y(t) = y(t_n) \). Hence, \( y(t_n) \leq y(t_n - \tau) \), which contradicts (12). The proof of Lemma 4 is completed.
3. MAIN RESULTS

THEOREM 1. Assume that (10) holds, then every nonoscillatory solution $y(t)$ of equation (1) is unbounded.

**Proof.** Assume that $y(t)$ is an eventually positive solution of equation (1). From Lemma 4, $z(t) < 0$ eventually, and we can assert that $\lim_{t \to -\infty} z(t) = -\infty$. In fact, if we assume that it is not true, then there exists a constant $c > 0$ such that

$$\lim_{t \to \infty} z(t) = -c > -\infty. \quad (13)$$

Noting that $z(t)$ decreases eventually, there exists $t_1 \geq t_0$ such that for $t \geq t_1$, we have

$$z(t) = y(t) + P(t)y(t - \tau) < -\frac{c}{2},$$

and consequently,

$$y(t) < -P(t)y(t - \tau) - \frac{c}{2} < -P_1y(t - \tau) - \frac{c}{2}. \quad (14)$$

From (13) and Lemma 2, we have $\lim_{t \to -\infty} y(t) = 0$, and thus, there exists a sequence $\{t_n\}_n$ such that $\lim_{t \to -\infty} t_n = \infty$ and $\lim_{t \to -\infty} y(t_n) = 0$. Hence, there exists a positive integer $N$ such that for $n \geq N$, $y(t_n) < -c/4P_1$. From (15), we have

$$y(t_n) < -P_1y(t_n) - \frac{c}{2} < \frac{c}{4} - \frac{c}{2} < 0,$$

which contradicts the assumption that $y(t) > 0$ eventually. Thus, we have $\lim_{t \to -\infty} z(t) = -\infty$. Using the inequality

$$P_1y(t - \tau) \leq P(t)y(t - \tau) < z(t),$$

we have $\lim_{t \to -\infty} y(t) = \infty$.

On the other hand, assume that $y(t)$ is an eventually negative solution. As equation (1) is a linear equation, we have $\lim_{t \to -\infty} y(t) = -\infty$. The proof of Theorem 1 is completed.

THEOREM 2. Assume that one of the following conditions hold:

$$-1 < P \leq P(t) \leq 1, \quad (15)$$

$$0 \leq P(t) \leq P < 1, \quad (16)$$

$$1 < P_1 \leq P(t) \leq P_2, \quad (17)$$

and $y(t)$ is a nonoscillatory solution of equation (1), then $\lim_{t \to -\infty} y(t) = 0$.

**Proof.**

(i) Assume that (15) holds, and $y(t)$ is an eventually positive solution of equation (1). From Lemma 1, $z(t)$ decreases eventually and from Lemma 3, $y(t)$ is a bounded function. Letting

$$\lim_{t \to -\infty} y(t) = b, \quad (18)$$

we can assert that $b = 0$. In fact, if we assume that it is not true, then there exists $\mu > 0$ such that $\mu < b(1 + P)/(2 - P)$. From (18), there exists a sequence $\{t_n\}_n$ such that $\lim_{n \to \infty} t_n = \infty$, and $\lim_{n \to \infty} y(t_n) = b$. By the definition of upper limits, there exist $T$ and $N$ such that, for $n > N$ and $t > T$,

$$|y(t_n) - b| < \mu, \quad y(t) - b < \mu.$$  

Since $y(t)$ is a bounded solution of equation (1), it follows from (2) that $z(t)$ is bounded. From Lemma 1, there exists a constant $c$ such that $\lim_{t \to -\infty} z(t) = c$. From Lemma 2, we
have \( \liminf_{t \to \infty} y(t) = 0 \). Thus, we can choose a sequence \( \{T_k\}_1^\infty \) such that \( \lim_{k \to \infty} T_k = \infty \), \( \lim_{k \to \infty} y(T_k) = 0 \). Furthermore, there exists an \( N_1 \) such that \( y(T_k) < \eta \) for \( k > N_1 \).

Choosing \( t_i \) and \( T_j \) from sequences \( \{t_n\}_1^\infty \) and \( \{T_k\}_1^\infty \) such that \( i > N, j > N, t_i > T_j, \) and \( t_i - \tau > t_0 \), we have

\[
z(t_i) - z(T_j) = y(t_i) - y(T_j) + P(t_i)y(t_i - \tau) - P(T_j)y(T_j - \tau) \\
\geq y(t_i) - y(T_j) + P(t_i)y(t_i - \tau) \\
\geq y(t_i) - y(T_j) + P(y(t_i - \tau)) \\
> b - \mu - \mu + P(b + \mu) \\
= (1 + P)b - (2 - P)\mu > 0,
\]

which is in contradiction with the fact that \( z(t) \) decreases eventually.

From \( b = 0 \), i.e., \( \limsup_{t \to \infty} y(t) = 0 \), we have \( \lim_{t \to \infty} y(t) = 0 \).

As equation (1) is a linear equation, if \( y(t) \) is an eventually negative solution of equation (1), then \( -y(t) \) is an eventually positive solution of equation (1), and consequently, we also have \( \lim_{t \to \infty} y(t) = 0 \).

(ii) If condition (16) holds, then proceeding with the same arguments as in the case of (i), we can obtain the result and the details are omitted here.

(iii) If condition (17) holds, without loss of generality, assume that \( y(t) \) is an eventually positive solution of equation (1). From (2), \( z(t) > 0 \) eventually. From Lemma 1, there exists \( \lim_{t \to \infty} z(t) = c \geq 0 \), and we can assert that \( c = 0 \). In fact, assume that \( c \neq 0 \), then \( c > 0 \).

From Lemma 2, we have \( \liminf_{t \to \infty} y(t) = 0 \). Thus, there exists a sequence \( \{t_n\}_1^\infty \) such that \( \lim_{n \to \infty} t_n = c \) and \( \lim_{n \to \infty} y(t_n) = 0 \).

Consider sequence \( \{T_n\}_1^\infty \) and \( T_n = t_n + \tau \). From (2), we have

\[
z(T_n) = y(T_n) + P(T_n)y(t_n). \tag{19}
\]

Letting \( n \to \infty \), we have \( \lim_{t \to \infty} y(T_n) = c \). Furthermore, consider sequence \( \{t_n\}_1^\infty \) and \( T_n = t_n + \tau \), we have

\[
z(T_n) = y(T_n) + P(T_n)y(T_n) > P(T_n)y(T_n) \geq P_1y(T_n). \tag{20}
\]

Letting \( n \to \infty \) and noting that \( P_1 > 1 \), we have \( c \geq P_1c > c \), which is not possible. Thus, \( \lim_{t \to \infty} z(t) = 0 \).

Noting that \( y(t) \) is an eventually positive solution of equation (1), from (2), we have \( z(t) > y(t) \). Thus, \( \lim_{t \to \infty} y(t) = 0 \). The proof of Theorem 2 is completed.

REMARK 1. The results presented above generalize some results in the literature [3-5,12].

REMARK 2. The work presented can be extended to study the following nonlinear equation:

\[
\frac{d}{dt} [y(t) + P(t)y(t - \tau)] + Q(t) \int_0^{\sigma(t)} f(y(t - s)) \, ds = 0,
\]

in which \( f(u) \in C(R,R), uf(u) > 0 \) for \( u \neq 0 \); \( f(u) \neq 0 \) is bounded when \( u \neq 0 \) is bounded.

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