Minimax Inequalities of Ky Fan

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Abstract—in this paper, we found a new result by relaxing the condition of [1, Theorem 3]. As its direct consequence, we have obtained some new minimax inequalities of Ky Fan and minimax theorems. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

Since Fan has generalized KKM theorem, numerous applications of this theorem have been found. Fan’s theorem is now becoming a very versatile tool in nonlinear analysis, such as fixed point, variational inequalities, see [2-7]. Fan’s theorem was used in [3,8,9] to prove fixed point and minimax theorem in topological vector spaces. Ha [1,2] has given generalization of Fan’s theorem and Fan’s minimax inequality. In this paper, we obtain a new theorem by relaxing closed condition of sets of [1, Theorem 3]. From this, we give some new Fan’s minimax inequalities and minimax theorem. Our main result is the following Theorem 2.

Let $X$ and $Y$ be topological spaces. By a set-valued mapping $f$ defined on $X$ with values in $Y$, we mean that to each point $x \in X$, $f$ assigns a unique nonempty subset $f(x)$ of $Y$. $f$ is called upper semicontinuous if for each open subset $G$ of $Y$, the set \( \{x \in X: f(x) \subseteq G\} \) is open in $X$. It is easy to show (e.g., [10]) that if $Y$ is a compact Hausdorff and if $f(x)$ is closed for each $x \in X$, then $f$ is upper semicontinuous if and only if the graph \( \{(x,y) \in X \times Y: y \in f(x)\} \) of $f$ is closed in $X \times Y$. We first cite a lemma in [1] which will suit our purpose.

**Lemma 1.** Let $E$ be a Hausdorff topological vector space and $K \subseteq E$ be a compact convex subset. Let $Z$ be an $n$-simplex. If $q$ is an upper semicontinuous set-valued mapping defined on $Z$ such that $q(x)$ is a nonempty closed convex subset of $K$ for each $x \in Z$, and if $P: K \rightarrow Z$ is a (single-valued) continuous mapping, then there exists $x_0 \in Z$ such that $x_0 \in P(q(x_0))$.

Our result is as follows.
Theorem 2. Let $E$ and $F$ be Hausdorff topological vector spaces, let $X \subset E$ and $Y \subset F$ be nonempty convex subsets, and let $A \subset X \times Y$ be a subset such that

(a) for each $x \in X$, the set $\{y \in Y : (x, y) \notin A\}$ is convex, or empty,
(b) for each $y \in Y$, there exists a closed subset $X_y \subset X$ such that set $\{x \in X : (x, y) \in A\} \subset X_y$.

Suppose that there exists a subset $B$ of $A$ and a compact convex subset $K$ of $X$ such that $B$ is closed in $X \times Y$, and

(c) for each $y \in Y$, the set $\{z \in K : (z, y) \in B\}$ is nonempty and convex.

Then

$$\bigcap_{y \in Y} X_y \cap K \neq \emptyset.$$  

Remark 1. The above theorem has a weaken closed Condition (b) as compared with [1] and is a new result.

Proof. Suppose that the assertion of the theorem is false. That is, if $\exists x_0 \in X \setminus K$, we prove that there exists $z_0 \in K$ such that

$$[x_0] \times Y \subset A, \quad (1.1)$$

or else, for each $y \in Y$, let $A(y) = \{z \in X : (z, y) \notin A\}$. Then for each $x \in K$, there exists $y \in Y$ such that $(x, y) \notin A$, namely, $x \in A(y)$. By $A(y) = X \setminus \{x \in X : (x, y) \in A\}$ and Condition (b), $A(y) \supset X \setminus X_y = V_y$, $V_y \subset X$ can be an open subset, and $\cap_{y \in Y} V_y = X \setminus \cap_{y \in Y} X_y \supset K$. Thus, since $K$ is compact subset, there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ of $Y$ such that $K \subset \cup_{i=1}^n V_{y_i}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a continuous partition of unity on $K$ subordinated to the cover $V_{y_i}$ ($1 \leq i \leq n$), that is, $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a nonnegative real-valued continuous functions on $K$ such that for each $1 \leq i \leq n$, $\alpha_i$ vanishes on $K \setminus V_{y_i}$ and $\sum_{i=1}^n \alpha_i(x) = 1$, for all $x \in K$. Let us define a mapping $P : K \rightarrow Y$ by $P(z) = \sum_{i=1}^n \alpha_i(x) y_{i}$. For $x \in K$ and any index $i$, if $\alpha_i(x) > 0$, then $x \in A(y_i)$ and so $(x, y_i) \notin A$. By (a), we have $(x, \sum_{i=1}^n \alpha_i(x) y_i) \notin A$. Hence, for all $x \in K$,

$$(x, P(x)) \notin A. \quad (1.2)$$

On the other hand, let $Z$ be the convex hull of the set $\{y_1, y_2, \ldots, y_n\}$ in $Y$. We define a set-valued mapping $q$ on $Z$ with values in $K$ by $q(z) = \{z \in K : (x, z) \notin B\}$. By (c), $q(z)$ is nonempty and convex for each $z \in Z$. Since $B$ is closed in $X \times Y$, each $q(z)$ is closed in $K$ and the graph of $q$ is closed in $Z \times K$; thus $q$ is an upper semicontinuous set-valued mapping defined on $Z$. Since $P$ is continuous by Lemma 1, there exists $z_0 \in Z$ such that $z_0 \in \hat{P}(q(z_0))$. If $z_0 \in q(z_0)$ is such that $P(x_0) = z_0$; then $(x_0, P(x_0)) \in B \subset A$, which contradicts (1.2), hence, (1.1) is true. But this contradicts $\cap_{y \in Y} X_y \subset X \setminus K$ again, therefore, we must have $\cap_{y \in Y} X_y \cap K \neq \emptyset$. This proves the theorem.

2. MAIN RESULTS

As an immediate consequence of Theorem 2, we obtain some new minimax theorems and some minimax inequalities of Ky Fan. Let $X$ be a convex set in a vector space and let $f$ be a real-valued function defined on $X$. We recall that $f$ is quasiconvex if for any real number the set $\{z \in X : f(z) < t\}$ is convex, $f$ is quasiconcave if $-f$ is quasiconvex. Clearly the quasiconvexity of $f$ implies that the set $\{z \in X : f(z) < t\}$ is convex for any real number $t$.

Theorem 3. Let $E$ and $F$ be Hausdorff topological vector spaces and let $X \subset E$ and $Y \subset F$ be nonempty convex subsets. If $f, g, h : X \times Y \rightarrow R$ are function such that

(i) $f(x, y) \leq g(x, y) \leq h(x, y)$, for all $(x, y) \in X \times Y$,
(ii) $f(x, y)$ is lower semicontinuous on $X$, for each $y \in Y$,
(iii) $g(x, y)$ is quasiconcave on $Y$, for each $x \in X$,
(iv) $h(x, y)$ is lower semicontinuous on $X \times Y$ and $h(x, y)$ is quasiconvex on $X$, for each $y \in Y$.
Then
\[
\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \inf_{K \in \mathcal{K}} \sup_{y \in Y} \min_{x \in K} h(x, y),
\] (2.1)
where \( \mathcal{K} = \{ K \subset X \mid K \text{ is compact convex subsets of } X \} \). If, in addition, \( X \) is compact, then
\[
\min_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} h(x, y).
\] (2.2)

**Proof.** We can assume that the right-hand side of (2.1) is not \(+\infty\). Choose a real number \( t \) such that
\[
t > \inf_{K \in \mathcal{K}} \sup_{y \in Y} \min_{x \in K} h(x, y).
\]
let \( A = \{(x, y) \in X \times Y : g(x, y) \leq t\} \) and \( B = \{(x, y) \in X \times Y : h(x, y) \leq t\} \). Then
(a) for each \( x \in X \), by (iii), set \( \{y \in Y : (x, y) \notin A\} \) is convex or empty and satisfies Condition (a) of Theorem 2,
(b) for each \( y \in Y \), by (i), set \( \{x \in X : (x, y) \in A\} \subseteq \{x \in X : f(x, y) \leq t\} = X_y \), by (ii), \( X_y \) is closed and satisfies Condition (b) of Theorem 2. It is easy to verify that \( B \) is closed in \( X \times Y \), \( B \subset A \), and for any \( y \in Y \), set \( \{x \in X : (x, y) \in B\} \) is convex,
(c) let \( K \) be a compact convex subset of \( X \) such that
\[
t > \sup_{y \in Y} \min_{x \in K} h(x, y).
\]
Then for any \( y \in Y \), the set \( \{x \in K : h(x, y) \leq t\} \) is nonempty and convex. Thus, by Theorem 2,
\[
\bigcap_{y \in Y} X_y \cap K \neq \emptyset; \tag{2.3}
\]
that is, there exists \( x_0 \in K \) such that
\[
f(x_0, y) \leq t, \quad \text{for all } y \in Y. \tag{2.4}
\]
This shows that
\[
\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq t, \tag{2.5}
\]
and hence, (2.1) is proved.

We shall establish the following similarities of the proof of Theorem 3.

**Theorem 4.** Let \( f, g, h : X \times Y \rightarrow R \) be as in Theorem 3. Then for each \( \lambda \in R \), one of the following situations holds:
(a) there exists \( x_0 \in X \) such that \( f(x_0, y) \leq \lambda \), for all \( y \in Y \),
(b) there exists \( y_0 \in Y \) such that \( h(x, y_0) > \lambda \), for all \( x \in X \).

**Remark 2.** The condition of Theorem 4 is different from [4, Theorem 6.2], where \( X \) or \( Y \) need not be compact.

The following three minimax theorems are obtain from Theorem 3 as special cases by taking \( f = g \), \( f = h \), \( f = g = h \).

**Corollary 5.** Let \( f, h : X \times Y \rightarrow R \) be two real-valued functions satisfying:
(i) \( f(x, y) \leq h(x, y) \), for all \( (x, y) \in X \times Y \),
(ii) \( f(x, y) \) is lower semicontinuous on \( X \), for each \( y \in Y \),
(iii) \( f(x, y) \) is quasiconcave on \( Y \), for each \( x \in X \),
(iv) \( h(x, y) : X \times Y \rightarrow R \) is lower continuous and \( h(x, y) \) is quasiconvex on \( X \), for each \( y \in Y \).
Then,
\[ \inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \inf_{K \subset R} \sup_{y \in Y} \min_{x \in K} h(x, y). \]

If \( X \) is compact, then
\[ \min_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} h(x, y). \]

**Corollary 6.** Let \( f, g : X \times Y \to \mathbb{R} \) be two real-valued functions satisfying:

(i) \( f(x, y) \leq g(x, y) \), for all \((x, y) \in X \times Y,\)
(ii) \( f(x, y) \) is lower continuous on \( X \), for each \( y \in Y,\)
(iii) \( g(x, y) \) is quasiconcave on \( Y \), for each \( x \in X,\)
(iv) \( g(x, y) : X \times Y \to \mathbb{R} \) is lower continuous and \( g(x, y) \) is quasiconvex on \( X \), for each \( y \in Y.\)

Then,
\[ \inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \inf_{K \subset R} \sup_{y \in Y} \min_{x \in K} g(x, y). \]

If \( X \) is compact, then
\[ \min_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} g(x, y). \]

**Corollary 7.** Let \( f : X \times Y \to \mathbb{R} \) be a real-valued function verifying:

(i) \( f(x, y) \) is quasiconcave on \( Y \), for each \( x \in X,\)
(ii) \( f(x, y) \) is quasiconvex on \( X \) for each \( y \in Y,\) and \( f(x, y) : X \times Y \to \mathbb{R} \) is lower continuous.

Then,
\[ \inf_{x \in X} \sup_{y \in Y} f(x, y) = \inf_{K \subset R} \sup_{y \in Y} \min_{x \in K} f(x, y). \]

If \( X \) is compact, then
\[ \min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y). \]

**Remark 3.** Corollary 7, that is, [3, Theorem 4], thus, Theorem 3, Corollary 5, and Corollary 6 are all the generalization of [1, Theorem 4].

**Theorem 8.** Let \( E \) and \( F \) be Hausdorff topological vector spaces, \( X \subset E, \ Y \subset F \) be nonempty convex subsets, and \( Y \) be compact. Let \( f, g : X \times Y \to \mathbb{R} \) be two real-valued function such that

(i) \( f(x, y) \leq g(x, y) \), for all \((x, y) \in X \times Y,\)
(ii) \( f(x, y) \) is quasiconvex on \( X \), for each \( y \in Y,\)
(iii) \( g(x, y) \) is upper continuous on \( Y \), for each \( x \in X.\)

If \( T \) is an upper semicontinuous set-valued mapping defined on \( X \) such that \( T \) is a nonempty closed convex subset of \( Y \) for each \( x \in X, \) then
\[ \inf_{y \in T_x} f(x, y) \leq \max_{y \in Y} \min_{x \in X} g(x, y). \tag{2.6} \]

**Remark 4.** By taking Theorem 8, \( f = g, \) one gets [2, Theorem 1].

**Proof.** We can choose a real number \( t \) such that
\[ \inf_{y \in T_x} f(x, y) > t, \]

let \( A = \{(x, y) \in X \times Y : f(x, y) \geq t\}, \ B = \{(x, y) \in X \times Y : y \in T_x\}, \) and \( Y_x = \{y \in Y : g(x, y) \geq t\} \) for each \( x \in X. \) It is easy to verify that \( A \) and \( Y_x \) satisfy Conditions (a) and (b) of Theorem 2, and that \( B \) is closed in \( X \times Y \) and satisfies Condition (c) of Theorem 2 by taking \( K = Y. \) Thus, by Theorem 2, \( \cap_{x \in X} Y_x \cap Y \neq \emptyset, \) that is, there exists \( y_0 \in K \) such that
\[ g(x, y_0) \geq t, \tag{2.7} \]
for all \( x \in X \), this shows that
\[
\max_{y \in Y} \inf_{x \in X} g(x, y) \geq t,
\]
and therefore, (2.6) is proved.

By Theorem 8, we can obtain the following corollaries and theorems.

**Corollary 9.** Let \( f, g, T \) be as in Theorem 8. Assume further, that given \( \lambda \in \mathbb{R} \), we have
\[
\min_{x \in T x} f(x, y) \geq \lambda,
\]
for all \( x \in X \). Then there exists \( y_0 \in Y \) such that \( g(x, y_0) \geq \lambda \), for all \( x \in X \).

**Remark 5.** Corollary 9 is similar to the result of [4, Theorem 13.41.

**Theorem 10.** Let \( E \) be a Hausdorff topological vector space, \( X \subset Y, Y \subset E \) be nonempty convex subsets, and \( X \) be compact. Let \( f, g : X \times Y \rightarrow \mathbb{R} \) be two real-valued function satisfying Conditions (i)–(iii) of Theorem 8. Then,
\[
\inf_{x \in X} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).
\]

**Theorem 11.** Let \( E \) be a Hausdorff topological vector space, \( X \subset Y, Y \subset E \) be nonempty convex subsets, and \( X \) be compact. Let \( f : X \times Y \rightarrow \mathbb{R} \) be a real-valued function such that
\begin{enumerate}
\item \( f(x, y) \) is quasiconvex on \( X \), for each \( y \in Y \),
\item \( f(x, y) \) is upper continuous on \( Y \), for each \( x \in X \).
\end{enumerate}
Then,
\[
\inf_{x \in X} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} f(x, y).
\]

**Corollary 12.** Let \( f, g : X \times Y \rightarrow \mathbb{R} \) be two real-valued function verifying
\begin{enumerate}
\item \( f(x, y) \leq g(x, y) \), for all \( (x, y) \in X \times Y \),
\item \( f(x, y) \) is lower continuous on \( Y \), for each \( x \in X \),
\item \( g(x, y) \) is quasiconcave on \( X \), for each \( y \in Y \).
\end{enumerate}
Then,
\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, y).
\]

**References**