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Global Geometry of the Stable Regions for Two Delay Differential Equations

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1. INTRODUCTION

The stable region for the two delay differential equations

$$\dot{x}(t)+ax(t)+bx(t-r)+cx(t-\sigma)=0, \qquad r\geqslant 0,\ \sigma\geqslant 0,\ t\geqslant 0 \quad (1.1)$$

has been investigated by many authors [1, 3–9], but has not yet been globally solved. The numerical results [3] show that the boundary of a stable region may be a very complicated curve which has infinitely many kinks. So an interesting question is: What is the asymptotical behavior of the boundary of a stable region? Can it become chaotic? In this paper, we will give a complete geometrical description of the stable region[1] for the equations (1.1) in the r–σ plane. In particular, we prove that:

(i) If a stable region is unbounded, then its boundary will approach a straight line parallel to the r-axis or σ-axis as $r+\sigma\to\infty$.

(ii) If a half line in the first quadrant of the r–σ plane contains an unstable point, then the intersection of this line and the boundary of the stable region contains at most finitely many points and eventually the half line leaves the stable region.

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[1] Here the stable region is defined as a maximal connected set $D\subset[0,\infty)\times[0,\infty)$ which contains the origin $(0,0)$ such that for each $(r,\sigma)\in D$ the zero solution of (1.1) is asymptotically stable. In this paper, we do not discuss the possibility of existence of any other stable region in $[0,\infty)\times[0,\infty)$ which is disjoint with D. We conjecture that there is only one connected stable region whenever it exists.

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2. Preliminary

It is clear that the investigation of the stable region will lead to studying the location of zeros of the characteristic equation of (1.1)

$$p(\lambda, r, \sigma) \stackrel{\text{def}}{=} \lambda + a + be^{-\lambda r} + ce^{-\lambda\sigma} = 0. \tag{2.1}$$

And, in particular, the study of the equation

$$p(iv, r, \sigma) = iv + a + be^{-ivr} + ce^{-iv\sigma} = 0,\ v \geqslant 0 \tag{2.2}$$

will play a key role.

Equation (2.2) implicitly defines a family of curves $(r(v), \sigma(v))$ in the r–σ plane which may have very complicated structures. But from the viewpoint of stability analysis, only those curves such that each point on these curves can be connected to the origin by a continuous path in the stable region will be of interest. Furthermore, since $e^{i\theta}$ $(\theta \in R)$ is a periodic function, as a first step, we will study Eq. (2.2) under the restriction of $0 \leqslant vr \leqslant 2\pi$, $0 \leqslant v\sigma \leqslant 2\pi$.

First, by using El'sgol'ts' D-partitions mathed it is not difficult to see that, if $b<0$, $c<0$ are fixed, then, for all $r \geqslant 0$, $\sigma \geqslant 0$, the zero solution of (1.1) is asymptotically stable if $a > |b+c|$, stable but not asymptotically stable if $a = |b+c|$ and unstable if $a < |b+c|$. Therefore, without loss of generality, we suppose $b>0$ and divide the relation of the coefficients a, b and c into the following cases according to their positions in the real line (see Fig. 2.1):

I. $b \pm a \geqslant |c|$ ($c \pm a \geqslant |b|$ can be considered as this case if we exchange r and σ).

II. $b + a \geqslant |c| > b - a$.

III. $-c \leqslant a + b < c$, $c > 0$.

IV. $a + b + c < 0$.

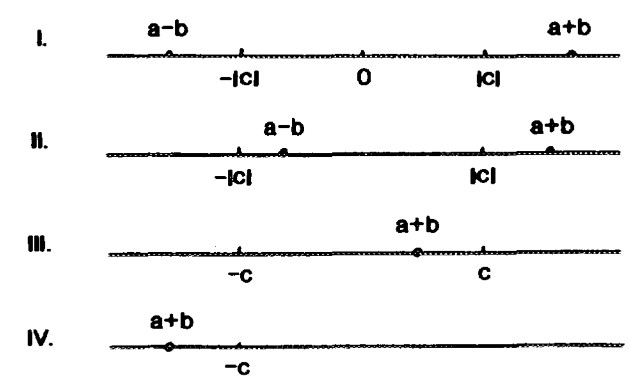


Figure 2.1

Case IV is simple. We have

PROPOSITION 2.1. *If* $a+b+c<0$, *then, for all delays* $r \geqslant 0$, $\sigma \geqslant 0$, *the zero solution of* (1.1) *is unstable.*

The proof of Proposition 2.1 is trivial.

In this paper, we will give a detailed analysis only for the case of $b \pm a > |c|$ and then, by using the idea developed in discussion of this case, we can describe the stable regions for the other cases.

3. Analysis of the Stable Region

Throughout this section we suppose $b \pm a > |c|$. Let $\mathbb{C}$ denote the complex plane and

$$\Gamma_{|c|} = \{z \in \mathbb{C} : |z| = |c|\}.$$

For $r > 0$, f_r: $[0, 2\pi] \to \mathbb{C}$ is defined by

$$f_r(s) = i\frac{s}{r} + a + be^{-is}$$

and for $r > 0$, $\sigma \geqslant 0$ let

$$\Delta(s, r, \sigma) = f_r(s) + ce^{-i(s/r)\sigma}, \quad s \in [0, 2\pi]. \tag{3.1}$$

It is apparent that (2.2) has a solution $v \in [0, 2\pi/r)$ for some $r > 0$, $\sigma \leqslant 0$ if and only if $\Delta(s, r, \sigma) = 0$ for some $s \in [0, 2\pi)$ and if and only if $f_r([0, 2\pi)) \cap \Gamma_{|c|} \neq \varnothing$. So we will study the equivalent $\Delta(s, r, \sigma) = 0$ rather than Eq. (2.2).

LEMMA 3.1. *For each fixed* $r > 0$, $f_r([0, 2\pi])$ *is a simple and smooth curve in* $\mathbb{C}$ *and*

$$f_r([\pi, 2\pi]) \cap \Gamma_{|c|} = \varnothing.$$

Proof. The first conclusion follows from the fact that f_r: $[0, 2\pi] \to \mathbb{C}$ is one-to-one and differentiable. Now for $s \in [\pi, 2\pi]$, we have

$$\begin{aligned}|f_r(s)|^2 &= \left(\frac{s}{r} - b\sin s\right)^2 + (a + b\cos s)^2 \\ &= \frac{s^2}{r^2} - \frac{2sb}{r}\sin s + a^2 + b^2 + 2ab\cos s \\ &> a^2 - 2|a|\, b + b^2 = (b - |a|)^2 \geqslant |c|^2.\end{aligned}$$

That is, $f_r([\pi, 2\pi]) \cap \Gamma_{|c|} = \varnothing$.

Now let $r' > r'' > 0$ and $s \in (0, \pi]$. It is clear that

$$\operatorname{Re} f_{r'}(s) = \operatorname{Re} f_{r''}(s) \qquad \text{and} \qquad \operatorname{Im} f_{r'}(s) < \operatorname{Im} f_{r''}(s).$$

This means that the curve $f_r(\cdot)$ moves downwards strictly as r increases (see Fig. 3.1). Note that

$$f_r([0, \pi]) \cap \Gamma_{|c|} = \varnothing$$

for sufficiently small r and large r, and, therefore, the following theorem is reasonable (see Fig. 3.2).

THEOREM 3.2. *If* $b + a > |c|$, $a - b < -|c|$, *then there are* $0 < r_0 < r_2 \leq \infty$ *and* $s_0^*, s_2^* \in (0, \pi)$ *such that*

(i) $f_{r_i}([0, \pi]) \cap \Gamma_{|c|} = \{f_{r_i}(s_i^*)\}$, $i = 0, 2$.

(ii) $f_r([0, \pi]) \cap \Gamma_{|c|} = \varnothing$, $r \in [0, r_0) \cup (r_2, \infty)$.

(iii) *There are continuous functions* $s_1, s_2 \colon [r_0, r_2] \to (0, \pi)$ *such that*

$$s_1(r_i) = s_2(r_i) = s_i^*, \qquad i = 0, 2$$
$$s_2(r) > s_1(r), \qquad r \in (r_0, r_2)$$

and

$$f_r([0, \pi]) \cap \Gamma_{|c|} = \{f_r(s_1(r)), f_r(s_2(r))\}, \qquad r \in (r_0, r_2).$$

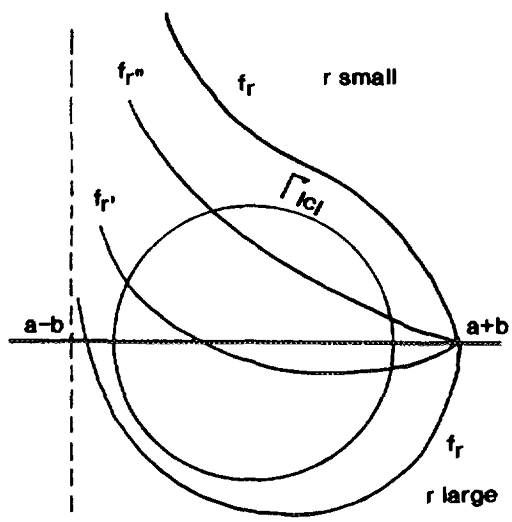


FIGURE 3.1

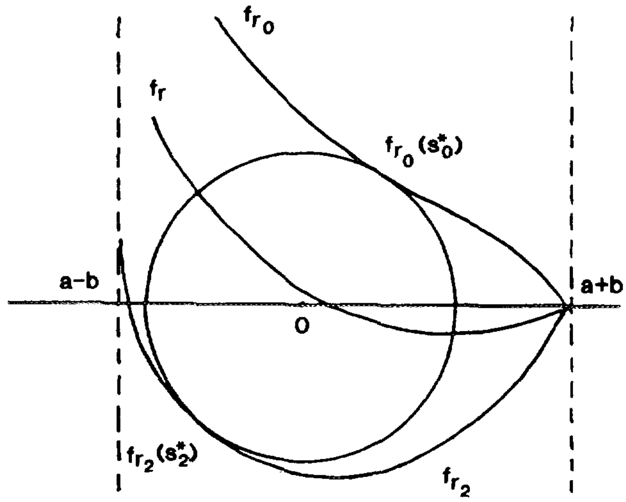


FIGURE 3.2

We prove this theorem in the Appendix.

With the aid of Theorem 3.2, we are able to characterize the curve (r, σ) determined by Eq. (3.1).

IA. Suppose $b \pm a > c > 0$.

It is clear that there is a unique $r_1 \in (r_0, r_2)$ $(r_1 = \tan^{-1}(-\sqrt{b^2-(a+c)^2}/(a+c))/\sqrt{b^2-(a+c)^2})$ such that

$$f_{r_1}(s_2(r_1)) = -c, \qquad (s_2(r_1) = r_1\sqrt{b^2-(a+c)^2}).$$

Now for each $r \in (r_0, r_2)$, let $p_i = f_r(s_i(r))$, $i = 1, 2$, and $\theta_i(r) \in [0, 2\pi)$ be the angles from the negative real axis to the rays starting at the origin and passing through p_i, $i = 1, 2$, in the clockwise sense respectively (see Fig. 3.3). It is clear that $\theta_1(r)$ is continuous on $[r_0, r_2]$, $\theta_2(r)$ is continuous on $[r_0, r_2]\backslash\{r_1\}$ and it has a jump at $r = r_1$ with

$$\lim_{r \nearrow r_1} \theta_2(r) = 0, \ \lim_{r \searrow r_1} \theta_2(r) = 2\pi \tag{3.2}$$

and

$$\theta_2(r) < \theta_1(r), \qquad r \in (r_0, r_1).$$

Now let

$$\sigma_i(r) = \frac{r\theta_i(r)}{s_i(r)}, \qquad i = 1, 2.$$

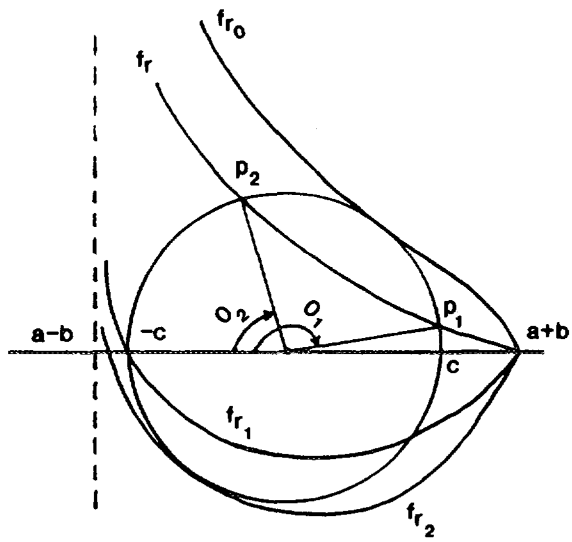


FIGURE 3.3

Then we have

$$\Delta(s_i(r), r, \sigma_i(r)) = f_r(s_i(r)) + ce^{-i(s_i(r)\,\sigma_i(r)/r)}$$
$$= -ce^{-i\theta_i(r)} + ce^{-i\theta_i(r)} = 0.$$

Conversely, if

$$\Delta(s, r, \sigma) = 0, \qquad 0 \leqslant s < 2\pi,\ 0 \leqslant \frac{s\sigma}{r} \leqslant 2\pi,$$

then necessarily

$$(s, r, \sigma) = (s_1(r), r, \sigma_1(r)) \qquad (\text{or } (s_2(r), r, \sigma_2(r)).$$

If we let r vary from r_0 to r_2, then there are two branches $\sigma_1(r)$, $\sigma_2(r)$ with

$$\sigma_1(r_i) = \sigma_2(r_i), \qquad i = 0, 2,$$

$$\sigma_2(r) = \frac{r\theta_2(r)}{s_2(r)} < \frac{r\theta_2(r)}{s_1(r)} < \frac{r\theta_1(r)}{s_1(r)} = \sigma_1(r), \qquad r \in (r_0, r_1). \tag{3.3}$$

It is obvious that $\sigma_1(r)$ is continuous on $[r_0, r_2]$ and from (3.2) it follows that $\sigma_2(r)$ is continuous on $[r_0, r_1) \cup (r_1, r_2]$ and

$$\lim_{r \nearrow r_1} \sigma_2(r) = 0, \ \lim_{r \searrow r_1} \sigma_2(r) = \lim_{r \searrow r_1} \frac{\theta_2(r)\, r}{s_2(r)} = \frac{2\pi r_1}{s_2(r_1)} = \frac{2\pi}{\sqrt{b^2 - (a+c)^2}}.$$

Thus, we have curves $\sigma_1(r)$ and $\sigma_2(r)$ shown in Fig. 3.4 (but we do not mean to imply that $\sigma_1(r) > \sigma_2(r)$ on (r_1, r_2)).

IB. Suppose $b \pm a > |c|$, $c < 0$.

In this case, if we again let $r_1 = \tan^{-1}(-\sqrt{b^2-(a+c)^2}/(a+c))/\sqrt{b^2-(a+c)^2}$, then we have $s_1(r_1) = r_1\sqrt{b^2-(a+c)^2}$ and

$$f_{r_1}(s_1(r_1)) = -c.$$

Furthermore, for each $r \in (r_0, r_2)$, let $\theta_i(r) \in [0, 2\pi]$ be the angles from the positive real axis to the rays starting at the origin and passing through $f_r(s_i(r))$, $i = 1, 2$ in the clockwise sense (see Fig. 3.5) and $\sigma_i(r) = r\theta_i(r)/s_i(r)$, $i = 1, 2$, respectively. By using analysis similar to case IA, the curves $\sigma_i(r)$ are shown as in Fig. 3.6 where

$$\sigma_1(r_i) = \sigma_2(r_i), \qquad i = 0, 2.$$

$\sigma_2(r)$ is continuous on $[r_0, r_2]$. $\sigma_1(r)$ is continuous on $[r_0, r_2]\setminus\{r_1\}$ and it has a jump at r_1 with

$$\lim_{r \searrow r_1} \sigma_1(r) = 0, \ \lim_{r \nearrow r_1} \sigma_1(r) = \frac{2\pi r_1}{s_1(r_1)} = \frac{2\pi}{\sqrt{b^2-(a+c)^2}}$$

and

$$\sigma_1(r) > \sigma_2(r), \qquad r \in (r_0, r_1).$$

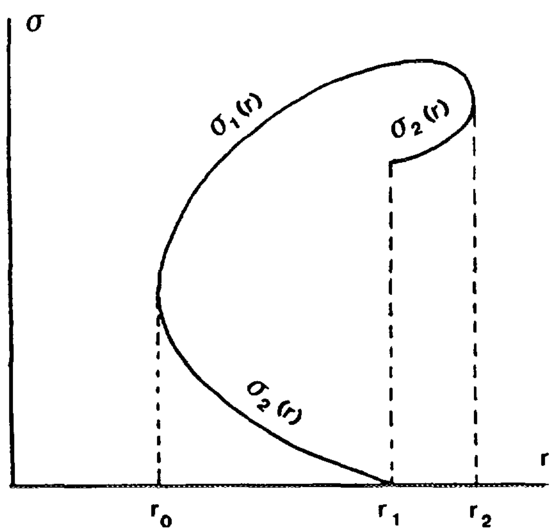


FIGURE 3.4

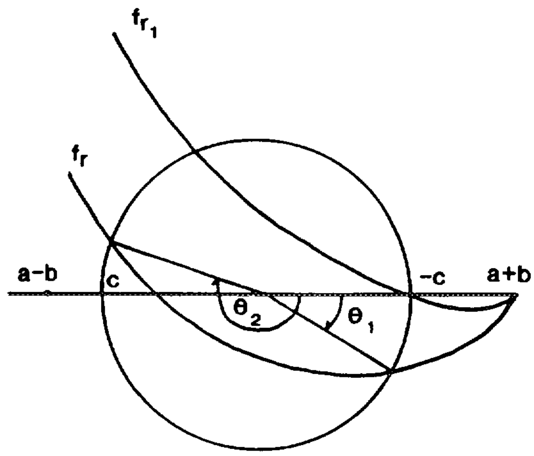


FIGURE 3.5

In both of the cases IA and IB, if we periodically extend $\sigma_i(r)$ by $\sigma_i^n(r) = \sigma_i(r) + 2n\pi r/s_i(r)$, $r \in [r_0, r_2]$, $i = 1, 2$, then the boundary of the stable region D (see first page for the definition of D) consists of the family $\{\sigma_i^n(r), i = 1, 2, r \in [r_0, r_2]\}_{n=0}^{\infty}$ which are bounded on the left by the line $r = r_0$ (see Figs. 3.7 and 3.8). Clearly, it is important to know how the curves $\sigma_i^n(r)$ change as r varies near r_0 because it might be possible that the curves $\sigma_i^n(r)$ near r_0 become chaotic as $n \to \infty$.

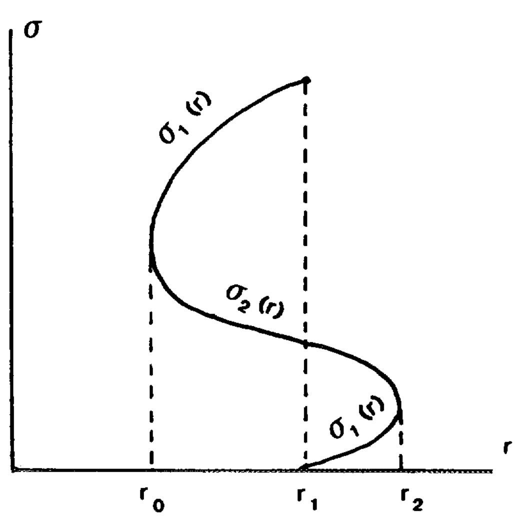


FIGURE 3.6

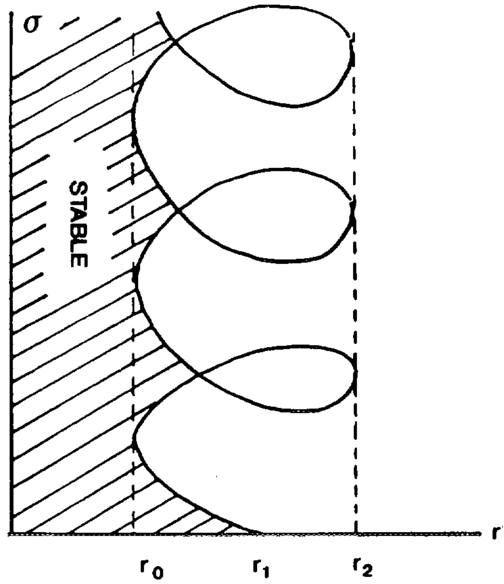


FIGURE 3.7

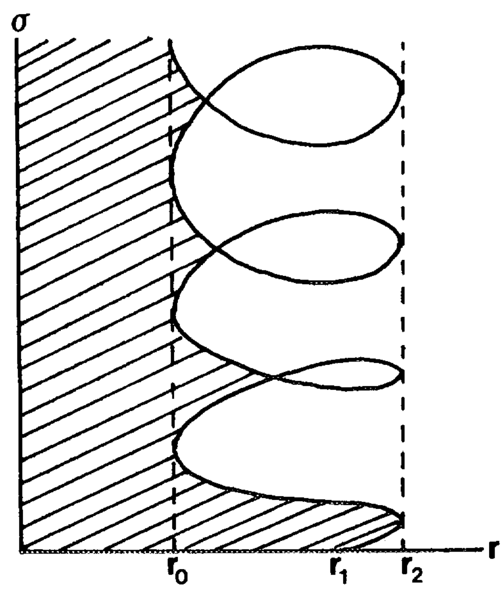


FIGURE 3.8

We now turn to study the properties of $\sigma_i^n(r)$ for r close to r_0.

First we try to reparameterize the curves $(r, \sigma_i(r))$ for r near r_0 as $(r(s), \sigma(s))$. We note that

$$\Delta(s, r, \sigma)=0$$

if and only if

$$f_r(s)+ce^{-i\theta}=0 \tag{3.4}$$

with $\sigma=\theta r/s$. We may separate the real and imaginary parts of (3.4) to obtain the equivalent system

$$F(s, r, \theta) \overset{\text{def}}{=} \frac{s}{r}-b\sin s-c\sin\theta=0$$

$$G(s, r, \theta) \overset{\text{def}}{=} a+b\cos s+c\cos\theta=0.$$

Let $\theta_0=s_0^*\sigma_0/r_0$ (here $\sigma_0=\sigma_1(r_0)=\sigma_2(r_0)$ and s_0^* is as defined in Theorem 3.2). It follows from the definition of s_0^*, r_0, and σ_0 that

$$F(s_0^*, r_0, \theta_0)=G(s_0^*, r_0, \theta_0)=0,$$

and

$$\left.\frac{\partial(F, G)}{\partial(r, \theta)}\right|_{(s_0^*, r_0, \theta_0)}=\begin{bmatrix} -\dfrac{s_0^*}{r_0^2} & -c\cos\theta_0 \\ 0 & -c\sin\theta_0 \end{bmatrix},$$

which is invertible since $s_0^*/r_0^2\neq 0$ and $c\sin\theta_0\neq 0$. Then the implicit function theorem yields that there are a neighborhood $I\times U\times V\subset R^3$ of (s_0^*, r_0, θ_0) and continuously differentiable functions $r: I\to U$, $\theta: I\to V$ such that

$$r(s_0^*)=r_0, \qquad \theta(s_0^*)=\theta_0, \tag{3.5}$$

$$F(s, r(s), \theta(s))=G(s, r(s), \theta(s))=0, \qquad s\in I, \tag{3.6}$$

and for $(s, r, \theta)\in I\times U\times V$,

$$F(s, r, \theta)=G(s, r, \theta)=0 \quad \text{if and only if} \quad r=r(s),\ \theta=\theta(s). \tag{3.7}$$

Now (3.6) is equivalent to

$$f_{r(s)}(s)=-ce^{-i\theta(s)}, \qquad s\in I.$$

(See Fig. 3.9).

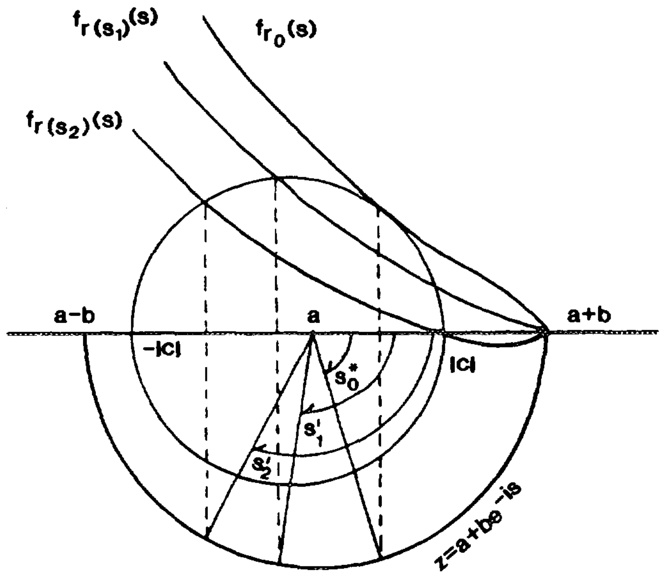


FIGURE 3.9

Since $f_{r_0}(\cdot)$ is tangent to the circle $\Gamma_{|c|}$ at $-ce^{i\theta_0}$ and $f_r(\cdot)$ moves upwards strictly as r decreases (see Fig. 3.10), thus for parameter values s_1', s_2', s_1'', s_2'',

$$r(s_2') > r(s_1') \geqslant r(s_0^*) = r_0 \qquad \text{if} \quad s_2' > s_1' \geqslant s_0^* \tag{3.8}$$

and

$$r(s_2'') > r(s_1'') \geqslant r(s_0^*) \qquad \text{if} \quad s_2'' < s_1'' \leqslant s_0^*. \tag{3.9}$$

This implies that $r(s)$ is monotone decreasing for $s \leqslant s_0^*$ and monotone increasing for $s \geqslant s_0^*$. So we have

$$\dot{r}(s) \begin{cases} \leqslant 0 & \text{if} \quad s \leqslant s_0^*, \\ \geqslant 0 & \text{if} \quad s \geqslant s_0^*. \end{cases} \tag{3.10}$$

Furthermore, let $s_1(r)$, $s_2(r)$, $\theta_1(r)$, $\theta_2(r)$ be the functions defined in Theorem 3.2 and IA. Then for $s \in I$ we have

$$f_{r(s)}(s_j(r(s))) = -ce^{-i\theta_j(r(s))}, \qquad j = 1, 2. \tag{3.11}$$

It is clear that $s_j(r(s))$, $j = 1, 2$ is continuous with respect to s and

$$s_j(r(s_0^*)) = s_j(r_0) = s_0^*, \; j = 1, 2.$$

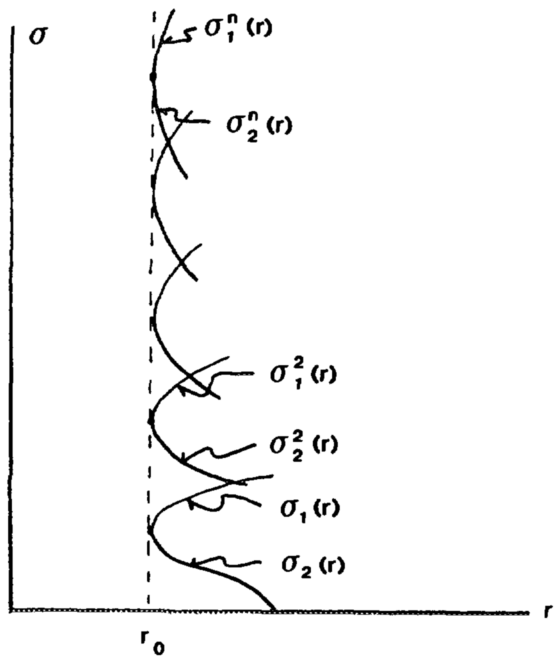


FIGURE 3.10

Therefore, there is a neighborhood $I' \subset I$ of s_0^* such that

$$s_j(r(s)) \in I, j = 1, 2, s \in I'.$$

It follows from (3.6), (3.11) that

$$r(s) = r(s_j(r(s))), \qquad s \in I', j = 1, 2. \tag{3.12}$$

Now from (3.8) and (3.9) we see that for $s \neq s_0^*$, $r(s) > r(s_0^*) = r_0$. So $s_2(r(s)) > s_1(r(s))$, $s \in I' \backslash \{s_0^*\}$. It now follows from (3.8), (3.9) and (3.12) that

$$s_2(r(s)) > s_0^* > s_1(r(s)), \ s \in I' \backslash \{s_0^*\}.$$

By using (3.12) again we arrive at

$$s = \begin{cases} s_1(r(s)) & \text{if} \quad s \leqslant s_0^*, \\ s_2(r(s)) & \text{if} \quad s \geqslant s_0^*. \end{cases} \tag{3.13}$$

Now if we set $\sigma(s)=\theta(s)\,r(s)/s$, then, for $s\in I'$, $s\leqslant s_0^*$, we have

$$\begin{aligned}\sigma(s)+\frac{2n\pi r(s)}{s}&=\frac{\theta(s)\,r(s)}{s_1(r(s))}+\frac{2n\pi r(s)}{s_1(r(s))}\\&=\sigma_1(r(s))+\frac{2n\pi r(s)}{s_1(r(s))}=\sigma_1^n(r(s))\end{aligned}\tag{3.14}$$

and, for $s\in I'$, $s\geqslant s_0^*$,

$$\sigma(s)+\frac{2n\pi r(s)}{s}=\sigma_2(r(s))+\frac{2n\pi r(s)}{s_2(r(s))}=\sigma_2^n(r(s)).\tag{3.15}$$

Note that $\dot r(s_0^*)=0$, $r(s_0^*)>0$ and $\dot\sigma(s)$ is continuous at s_0^*, so there are $\varepsilon>0$ and $\delta>0$, $m>0$ such that

$$|\dot r(s)\,s|<r(s)-\delta,\ |\dot\sigma(s)|\leqslant m,\ |\dot r(s)|\leqslant m,\ s\in[s_0^*-\varepsilon,s_0^*+\varepsilon]\subset I'.\tag{3.16}$$

Furthermore, for $s\in[s_0^*-\varepsilon,s_0^*)$, we have $r(s)\in(r_0,r(s_0^*-\varepsilon)]$. It follows from (3.14) that

$$\begin{aligned}\frac{d\sigma_1^n(r)}{dr}&=\left.\frac{d\left(\sigma(s)+\dfrac{2n\pi r(s)}{s}\right)\Big/ds}{dr(s)/ds}\right|_{s\in[s_0^*-\varepsilon,s_0^*)}\\&=\frac{\dot\sigma(s)+2n\pi(\dot r(s)\,s-r(s))/s^2}{\dot r(s)}\\&\geqslant\frac{\dfrac{2n\pi\delta}{s^2}-m}{m}\to+\infty\end{aligned}$$

as $n\to\infty$ uniformly for $r\in(r_0,r(s_0^*-\varepsilon)]$.
Similarly, we have

$$\frac{d\sigma_2^n(r)}{dr}\to-\infty\qquad\text{as}\quad n\to\infty$$

uniformly for $r\in(r_0,r(s_0^*+\varepsilon)]$ (see Fig. 3.10).

That is, the curves $\sigma_1^n(r)$ and $\sigma_2^n(r)$ for r near r_0 almost become a straight line as n becomes sufficiently large. As an immediate consequence we have

THEOREM 3.3. *Under the assumption of $b>0$, $b\pm a>|c|$, the boundary of stable region approaches the straight line $r=r_0$. Furthermore, the up half strip $\{(r,\sigma);\ 0\leqslant r<r_0,\ \sigma\geqslant 0\}$ belongs to the stable region.*

As a corollary of Theorem 3.3 we have

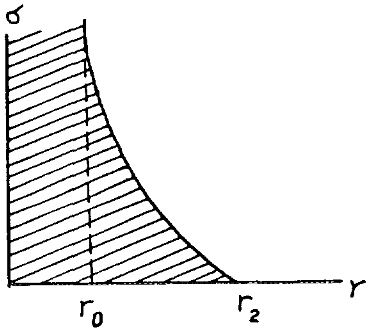
THEOREM 3.4. *Let $l \subset [0, \infty) \times [0, \infty)$ be a half line. If l contains a point $(r', \sigma') \in [0, \infty) \times [0, \infty) \backslash \bar{D}$ (where D is the stable region), then l intersects ∂D in at most finitely many points and eventually leaves the stable region D.*

Remark. Only the line $L: r = r_0$ intersects ∂D at infinitely many points.

By using the same argument as developed in the previous sections we are able to depict geometrically the stable region for all other cases as follows. The detailed analysis will be given in a separated paper.

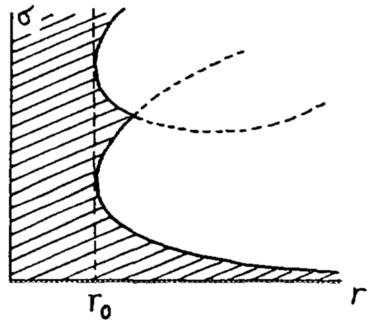
IC.

$a + b = c$, $a - b < -c$, $\quad c > 0$

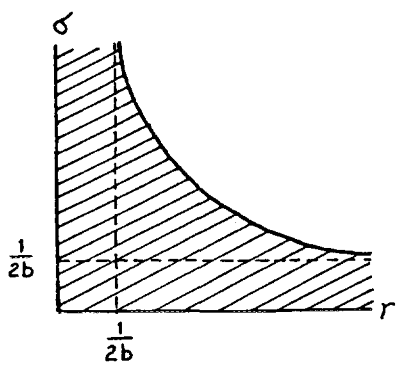


$$r_0 = \frac{b - \sqrt{b(a+b)}}{-2ab}$$

$a + b > c$, $a - b = -c$, $\quad c > 0$

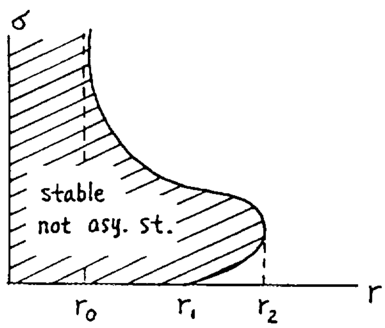


$a = 0$, $b = c > 0$

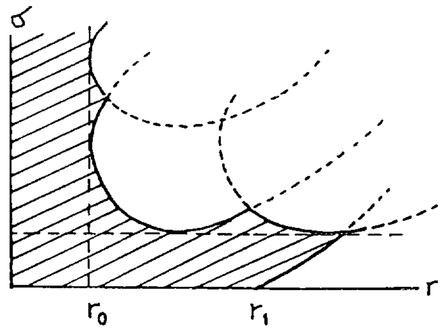


ID.

$a+b=|c|,\ a-b<-|c|, \qquad c<0$

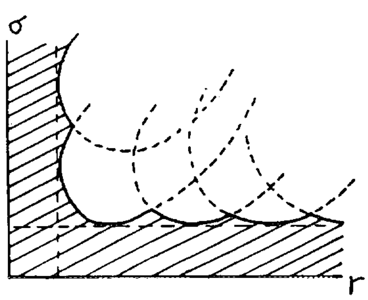


$a+b>|c|,\ a-b=-|c|, \qquad c<0$

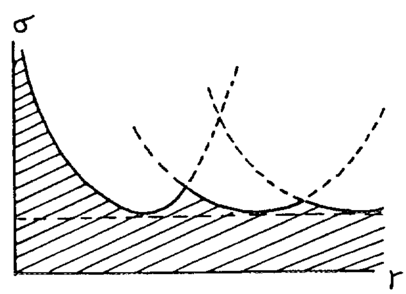


IIA.

$a+b>c>a-b, \qquad c>0$

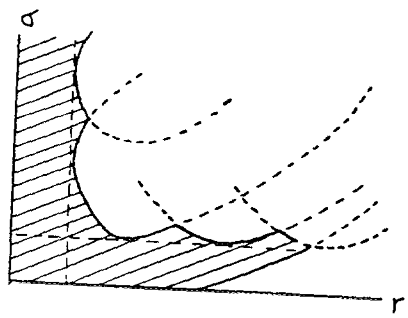


$a+b=c>a-b, \qquad c>0$

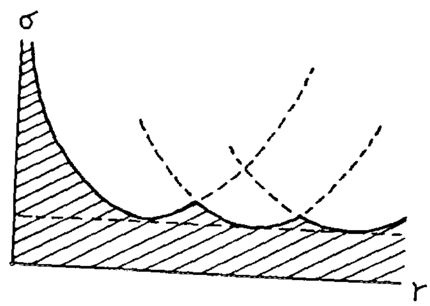


IIB.

$a+b>|c|>a-b, \quad c<0$

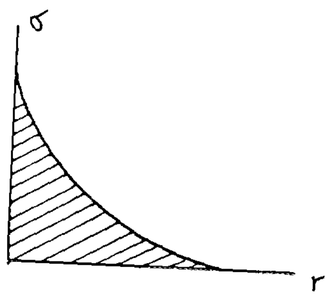


$a+b=|c|>a-b, \quad c<0$

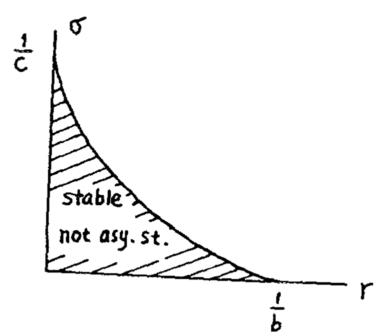


III.

$c>a+b>-c, \quad c>0$



$c>a+b=-c, \quad c>0$



Appendix

This appendix is devoted to proving Theorem 3.2.

Let f_r be defined as in Section 3 and let

$$g(r, y) \overset{\text{def}}{=} r^2(|f_r(y)|^2 - c^2)$$
$$= y^2 - 2bry \sin y + 2abr^2 \cos y + r^2(a^2 + b^2 - c^2). \tag{A1}$$

Then

$f_r([0, \pi]) \cap \Gamma_{|c|} \neq \varnothing$ if and only if $g(r, y) = 0$ has a solution $y \in [0, \pi]$.

Before proceeding to the proof, we need a few lemmas.

LEMMA 1. *Suppose* $H\colon [0, \pi] \to R$ *is twice differentiable and* $H(0) > 0$, $H(\pi) > 0$. *If* $H''(y)$ *has at most one zero in* $(0, \pi)$, *then* $H(y)$ *has at most two zeros in* $[0, \pi]$.

Proof. Since $H''(y)$ has at most one zero in $(0, \pi)$, $H'(y)$ has at most two zeros in $(0, \pi)$. If the lemma were false, then $H(y)$ would have at least three zeros in $(0, \pi)$. Let $y_1 < y_2 < y_3$ be the first three successive zeros, then $H'(y)$ must have two zeros y_1^*, y_2^* such that $y_1^* \in (y_1, y_2)$, $y_2^* \in (y_2, y_3)$. Thus $H'(y) \neq 0$, $y \in (0, y_1^*) \cup (y_2^*, \pi)$, for $H'(y)$ has at most two zeros in $(0, \pi)$. That is, $H(y)$ is strictly monotone on $(0, y_1^*)$ and (y_2^*, y), respectively. Since $H(0) > 0$, $H(y_1) = 0$ and $y_1 < y_1^*$ it follows that $H(y_1^*) < 0$. Similarly, we have $H(y_2^*) < 0$. Note that $H(y_2) = 0$ and $y_2 \in (y_1^*, y_2^*)$, so $H(y)$ is not monotone in (y_1^*, y_2^*). That is, $H'(y)$ has a zero in (y_1^*, y_2^*). This is a contradiction and Lemma 1 follows.

We let $g_y(r, y) = \partial g(r, y)/\partial y$, and for integer $k \geqslant 2$, let

$$g_y^{(k)}(r, y) = \frac{\partial^k g(r, y)}{\partial y^k}.$$

LEMMA 2. *For each fixed* $r > 0$, $g_y^{(2)}(r, y)$ *has at most one zero in* $(0, \pi)$.

Proof. We have

$$g_y(r, y) = 2y - (2br + 2abr^2) \sin y - 2bry \cos y$$
$$g_y^{(2)}(r, y) = 2 - 2br(2 + ar) \cos y + 2bry \sin y \tag{A2}$$
$$g_y^{(3)}(r, y) = br(6 + 2ar) \sin y + 2bry \cos y \tag{A3}$$

(i) Suppose $2 + ar \geqslant 0$, then $g_y^{(2)}(r, y) > 0$, $y \in [\pi/2, \pi]$. Since $2 + ar \geqslant 0$ implies $6 + 2ar > 0$, it follows from (A3) that $g_y^{(3)}(r, y) > 0$,

$y \in (0, \pi/2]$. That is, $g_y^{(2)}(r, y)$ is strictly increasing on $[0, \pi/2]$. Therefore, $g_y^{(2)}(r, y)$ has at most one zero in $(0, \pi)$.

(ii) Suppose $2 + ar < 0$, $6 + 2ar > 0$. In this case (A2) yields that $g_y^{(2)}(r, y) > 0$, $y \in [0, \pi/2]$ and

$$g_y^{(4)}(r, y) = br(8 + 2ar)\cos y - 2bry \sin y < 0, \qquad y \in \left(\frac{\pi}{2}, \pi\right].$$

It follows that $g_y^{(3)}(r, y)$ is strictly decreasing on $(\pi/2, \pi]$. Since $g_y^{(3)}(r, \pi/2) = rb(6 + 2ar) > 0$ and $g_y^{(3)}(r, \pi) = -2br\pi < 0$, there is a unique $y^* \in (\pi/2, \pi)$ such that

$$g_y^{(3)}(r, y) > 0, \qquad y \in \left[\frac{\pi}{2}, y^*\right)$$

$$g_y^{(3)}(r, y) < 0, \qquad y \in (y^*, \pi].$$

Hence $g_y^{(2)}(r, y) > 0$ for $y \in [0, y^*]$ and $g_y^{(2)}(r, y)$ is strictly decreasing for $y \in (y^*, \pi]$. Therefore, $g_y^{(2)}(r, y)$ has at most one zero in $[0, \pi]$.

(iii) Suppose $6 + 2ar \leqslant 0$. In this case we have $g_y^{(2)}(r, y) > 0$, $y \in [0, \pi/2]$ and $g_y^{(3)}(r, y) < 0$, $y \in (\pi/2, \pi)$. So it is clear that $g_y^{(2)}(r, y)$ has at most one zero in $(0, \pi)$.

Proof of Theorem 3.2. First, by the assumption of Theorem 3.2, we can easily check that, for each $r > 0$,

$$g(r, 0) > 0, \; g(r, \pi) > 0.$$

So it follows from Lemma 1 and Lemma 2 that $g(r, \cdot)$ has at most two zeros in $[0, \pi]$. If we look at the function $f_r(\cdot)$, it is clear that $f_r(\cdot)$ is above $\Gamma_{|c|}$ if r is sufficiently small and below $\Gamma_{|c|}$ if r is large enough. Since $f_r(\cdot)$ moves downward strictly as r increases, there are exactly two values r_0 and r_2 such that $f_r([0, \pi]) \cap \Gamma_{|c|} = \emptyset$, $r \in (0, r_0) \cup (r_2, \infty)$ and $f_{r_i}(\cdot)$ $(i = 0, 2)$ is tangential to $\Gamma_{|c|}$. That is, $g(r_i, \cdot)$ is tangential to the y-axis. Since $g_y(r_i, y)$ has at most two zeros in $(0, \pi)$, $g(r_i, \cdot)$ is tangential to the y-axis at exactly one point. Hence, there is unique $s_i \in (0, \pi)$ such that

$$f_{r_i}([0, \pi]) \cap \Gamma_{|c|} = \{f_{r_i}(s_i)\}, \; i = 0, 2.$$

Furthermore, by the definition of r_0, r_2 and the properties of $f_r(\cdot)$, it is obvious that for each $r \in (r_0, r_2)$, $f_r([0, \pi])$ intersects $\Gamma_{|c|}$ at least at two points. That is, $g(r, y)$ has at least two zeros in $[0, \pi]$. But, since $g(r, y)$ has at most two zeros in $(0, \pi]$ for each $r > 0$, we deduce that $g(r, y)$ has exactly two zeros in $[0, \pi]$. Equivalently, there are $s_1(r)$, $s_2(r)$, $s_2(r) > s_1(r)$ such that

$$f_r([0, \pi]) \cap \Gamma_{|c|} = \{f_r(s_1(r)), f_r(s_2(r))\}, \; r \in (r_0, r_2).$$

The continuity of $s_i(r)$ easily follows from the continuity of $g(r, y)$.

The proof of Theorem 3.2 is completed.

Acknowledgment

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