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Stability and error analysis of one-leg methods for nonlinear delay differential equations [1]

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Abstract

This paper is concerned with the numerical solution of delay differential equations (DDEs). We focus on the stability behaviour and error analysis of one-leg methods with respect to nonlinear DDEs. The new concepts of GR-stability, GAR-stability and weak GAR-stability are introduced. It is proved that a strongly A-stable one-leg method with linear interpolation is GAR-stable, and that an A-stable one-leg method with linear interpolation is GR-stable, weakly GAR-stable and D-convergent of order s, if it is consistent of order s in the classical sense. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, many papers discussed numerical methods for the solution of delay differential equations (DDEs) (see [1, 2, 9, 12–14, 19–24] and their references). They mainly focused on the stability of numerical methods for linear scalar model equation

$$\begin{aligned} &y'(t) = ay(t) + by(t-\tau), \quad t>0,\\ &y(t) = \phi(t), \quad t \leqslant 0, \end{aligned} \tag{1.1}$$

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where a, b are complex numbers which satisfy

$$|b| < -\mathrm{Re}(a),$$

$\tau(>0)$ is constant delay, and $\phi(t)$ is a continuous function. The concepts of P-stability and GP-stability were introduced and a significant number of results have already been found for both Runge–Kutta methods and linear multistep methods. Recently, the stability of RK methods has been studied in [13] based on the following test problem:

$$\begin{aligned} &y'(t) = Ly(t) + My(t-\tau), \quad t>0, \\ &y(t) = \phi(t), \quad t \leqslant 0, \end{aligned} \tag{1.2}$$

where L, M denote constant, complex matrices. However, we can not say that a stable method for (1.1) or (1.2) is also valid for a more general system of nonlinear DDEs. The stability results of numerical methods for nonlinear DDEs are much less. Up to now, we only see the stability analysis of some methods (cf. [2, 20, 23]). In this paper, we investigate the stability of one-leg methods with respect to nonlinear DDEs.

On the other hand, error analysis of numerical methods for DDEs is mostly based on Lipschitz conditions. For stiff DDEs, however, the Lipschitz constant will be very large, so that the classical convergence theory can not be applied. In this paper, in addition to stability analysis, we will also investigate the error behaviour and obtain the global error bounds independent on the stiffness of the underlying system. We will continue to analyse stability and error behaviour of Runge–Kutta methods for nonlinear DDEs in other papers.

This paper is structured as follows: In Section 2, we fix our attention on a particular class of DDEs, collecting several results from the literature. In Section 3, some new concepts of stability are introduced for nonlinear DDEs. They are reminiscent of that for the stiff ODEs field. In Section 4, we analyse the stability of A-stable one-leg methods with linear interpolation with respect to nonlinear DDEs. In Section 5, we investigate the error behaviour of A-stable one-leg methods with respect to nonlinear stiff DDEs. In Section 6, we briefly discuss the numerical solution of DDEs with several delays.

2. Test problems

Let $\langle \cdot, \cdot \rangle$ be an inner product on C^N and $\|\cdot\|$ the corresponding norm. Consider the following nonlinear equation:

$$\begin{aligned} &y'(t) = f(t, y(t), y(t-\tau)), \quad t \geqslant 0, \\ &y(t) = \phi_1(t), \quad t \leqslant 0, \end{aligned} \tag{2.1}$$

where τ is a positive delay term, ϕ_1 is a continuous function, and $f : [0, +\infty) \times C^N \times C^N \to C^N$, is a given mapping. In order to make the error analysis feasible, we always assume that the problem (2.1) have a unique solution $y(t)$ which is sufficiently differentiable and satisfies

$$\left\| \frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i} \right\| \leqslant M_i.$$

Definition 2.1. Let p,q be real constants. The class of all problems (2.1) with f satisfying the following conditions:

$$\mathrm{Re}\langle u_1-u_2, f(t,u_1,v)-f(t,u_2,v)\rangle \leqslant p\|u_1-u_2\|^2, \quad t\geqslant 0, u_1,u_2,\ v\in C^N, \tag{2.2}$$

$$\|f(t,u,v_1)-f(t,u,v_2)\| \leqslant q\|v_1-v_2\|, \quad t\geqslant 0, u, v_1,\ v_2\in C^N, \tag{2.3}$$

is denoted by $D_{p,q}$.

Remark 2.2. In the literature with respect to nonlinear stability and B-convergence of numerical methods for ODEs, the class $D_{p,0}$ has been used widely as the test problem class (cf. [3, 4, 6, 7, 9, 18]).

Proposition 2.3. *System* (1.1) *belongs to the class* $D_{p,q}$ *where* $p=\mathrm{Re}(a)$ *and* $q=|b|$.

Proposition 2.4. *System* (1.2) *belongs to the class* $D_{p,q}$ *where* $p=\mu(L), \mu(\cdot)$ *is the logarithmic matrix norm corresponding to the inner-product norm on* C^N, *and* $q=\sup_{\|x\|=1}\|Mx\|$.

For the nonlinear case, consider the following example (cf. [8]):

$$y'(t)=-ay(t)+\frac{by(t-\tau)}{1+[y(t-\tau)]^n},$$

where $a>0$ and b are real parameters and n is an even positive integer. This equation is a model for respiratory diseases, where $y(t)$ represents the concentration of carbon dioxide at time t. Obviously, this equation belongs to the class $D_{p,q}$with $p=-a$ and $q=|b|$ for $n=2$ or 4.

In order to discuss stability and asymptotic stability of DDEs (2.1) of the class $D_{p,q}$, we introduce another system, defined by the same function $f(t,u,v)$, but with another initial condition:

$$\begin{aligned} &z'(t)=f(t,z(t),z(t-\tau)), \quad t\geqslant 0,\\ &z(t)=\phi_2(t), \quad t\leqslant 0. \end{aligned} \tag{2.4}$$

Proposition 2.5. *Suppose systems* (2.1) *and* (2.4) *belong to the class* $D_{p,q}$ *with* $q\leqslant -p$. *Then the following is true*:

$$\|y(t)-z(t)\| \leqslant \max_{x\leqslant 0}\|\phi_1(x)-\phi_2(x)\|, \quad t\geqslant 0. \tag{2.5}$$

The proof of this proposition can be found in [20].

Proposition 2.6. *Suppose systems* (2.1) *and* (2.4) *belong to the class* $D_{p,q}$ *with* $q<-p$. *Then the following holds*:

$$\lim_{t\to+\infty}\|y(t)-z(t)\|=0. \tag{2.6}$$

The proof of this proposition can be found in [23], where a more general result was given.

3. Some concepts

We briefly recall the form of a one-leg method for the numerical solution of the ordinary differential equation

$$y'(t) = f(t, y(t)), \quad t \geq 0,$$
$$y(0) = y_0. \tag{3.1}$$

The one-leg k step method is the following:

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n), \tag{3.2}$$

where $h>0$ is the step size, E is the translation operator: $Ey_n = y_{n+1}$, each y_n is an approximation to the exact solution $y(t_n)$ with $t_n = nh$, and $\rho(x) = \sum_{j=0}^{k} \alpha_j x^j$ and $\sigma(x) = \sum_{j=0}^{k} \beta_j x^j$ are generating polynomials, which are assumed to have real cofficients, no common divisor. We also assume $\rho(1)= 0$, $\rho'(1) = \sigma(1) = 1$.

Apply the one-leg k-step method (ρ, σ) to DDE (2.1)

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n), \quad n = 0, 1, 2, \ldots, \tag{3.3}$$

where the argument $\bar{y}_n$ denotes an approximation to $y(\sigma(E)t_n - \tau)$ that is obtained by a specific interpolation at the point $t = \sigma(E)t_n - \tau$ using $\{y_i\}_{i \leqslant n+k}$.

Process (3.3) is defined completely by the one-leg method (ρ, σ) and the interpolation procedure for $\bar{y}_n$.

It is well known that any A-stable one-leg method for ODEs has order at most 2. So we can use the linear interpolation procedure for $\bar{y}_n$. Let $\tau = (m - \delta)h$ with integer $m \geqslant 1$ and $\delta \in [0, 1)$. We define

$$\bar{y}_n = \delta\sigma(E)y_{n-m+1} + (1 - \delta)\sigma(E)y_{n-m}, \tag{3.4}$$

where $y_l = \phi_1(lh)$ for $l \leqslant 0$.

Similarly, apply the same method (ρ, σ) to DDE (2.4)

$$\rho(E)z_n = hf(\sigma(E)t_n, \sigma(E)z_n, \bar{z}_n), \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

$$\bar{z}_n = \delta\sigma(E)z_{n-m+1} + (1 - \delta)\sigma(E)z_{n-m}, \tag{3.6}$$

where $z_l = \phi_2(lh)$ for $l \leqslant 0$.

Definition 3.1. A numerical method for DDEs is called R-stable if, under the condition that $q \leqslant -p$, there exists a constant C which depends only on the method, τ and q, such that the numerical approximations y_n and z_n to the solutions of any given problems (2.1) and (2.4) of the class $D_{p,q}$, respectively, satisfy the following inequality:

$$\|y_n - z_n\| \leqslant C \left(\max_{0 \leqslant j \leqslant k-1} \|y_j - z_j\| + \max_{t \leqslant 0} \|\phi_1(t) - \phi_2(t)\| \right) \tag{3.7}$$

for every $n \geqslant k$ and for every stepsize $h > 0$ under the constraint

$$hm = \tau, \tag{3.8}$$

where m is a positive integer.

GR-stability is defined by dropping the restriction (3.8).

Remark 3.2. Torelli [20] introduced RN- and GRN-stability for numerical methods applied to nonautonomous nonlinear DDEs. They require that the difference of two numerical solutions is bounded by the maximum difference of the initial values which means that the method is contractive. Here we relax their requirements. R- and GR-stability only require the difference to be controlled and uniformly bounded. Therefore, R-stability is a weaker concept than RN-stability. In fact, if a method is RN-stable, then we can choose $C = 1$ such that the method is R-stable.

Proposition 3.3. *Any R-stable one-leg method is A-stable.*

Definition 3.4. A numerical method for DDEs is called AR-stable if, under the condition that $q < -p$, the numerical approximations y_n and z_n to the solutions of any given problems (2.1) and (2.4) of the class $D_{p,q}$, respectively, satisfy the condition

$$\lim_{n\to\infty} \|y_n - z_n\| = 0 \tag{3.9}$$

for every stepsize $h > 0$ under the constraint (3.8).

GAR-stability is defined by dropping restriction (3.8).

Definition 3.5. A numerical method for DDEs is called weak AR-stable if, under the assumptions of Definition 3.4, (3.9) holds when f further satisfies

$$\|f(t,u_1,v) - f(t,u_2,v)\| \leqslant L\|u_1 - u_2\|^b, \ t \geqslant 0, \ u_1,u_2, \ v \in C^N, \tag{3.10}$$

where b is a positive real number and L is a nonnegative real number.

Weak GAR-stability is defined by dropping the restriction (3.8).

Remark 3.6. If the function $f(t,u,v)$ is uniformly Lipschitz continuous in variable u, then (3.10) holds.

Remark 3.7. Both AR-stability and weak AR-stability can be regarded as the nonlinear analogues of the concept of P-stability [1].

Up to now, error analyses of numerical methods for DDEs are mostly based on the function $f(t,u,v)$ satisfying Lipschitz conditions for u and v. For stiff DDEs, however, the Lipschitz constant with respect to u will be very large, so that the classical convergence theory cannot be applied. Now, we introduce the concept of D-convergence for stiff DDEs.

Definition 3.8. The one-leg method (3.3) with interpolation procedure (3.4) is said to be D-convergent of order s for the problem class $D_{p,q}$ if this method when applied to any given problem (2.1) of the class $D_{p,q}$ with $y_i = y(t_i)$, $i < k$, produces an approximation sequence $\{y_n\}$, and the global error satisfies a bound of the form

$$\|y(t_n) - y_n\| \leqslant C(t_n)h^s, \quad n \geqslant k, \ h \in (0, h_0],$$

where the function $C(t)$ and the maximum stepsize h_0 depend only on the method, some of the bounds M_i, the delay τ, the characteristic parameters p and q of the problem class $D_{p,q}$.

Remark 3.9. D-convergence concept was firstly proposed by Zhang and Zhou [24] for Runge–Kutta methods. It was assumed that $q \leqslant -p$ in [24]. Here we drop this restriction. The D-convergence concept is wider than the well-known B-convergence concept (see [6, 7, 9, 15, 16]). D-convergence for the problem class $D_{p,0}$ is just B-convergence. B-convergence results of one-leg methods can be found in [11, 15].

4. Stability analysis

In this section, we focus on the stability analysis of A-stable one-leg methods with respect to the nonlinear test problem class $D_{p,q}$.

Let $y_n, z_n \in C^N$,

$$w_n = \begin{bmatrix} y_n - z_n \\ y_{n+1} - z_{n+1} \\ \vdots \\ y_{n+k-1} - z_{n+k-1} \end{bmatrix}$$

and for a real symmetric positive definite $k \times k$ matrix $G = [g_{ij}]$, the norm $\|\cdot\|_G$ is defined by

$$\|U\|_G = \left(\sum_{i,j=1}^{k} g_{ij}\langle u_i, u_j \rangle \right)^{1/2}, \quad U = (u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \ldots, u_k^{\mathrm{T}})^{\mathrm{T}} \in C^{kN}.$$

Theorem 4.1. *Assume that the one-leg k step method (ρ, σ) is G-stable for a real symmetric positive definite matrix G and that the solved problems* (2.1) *and* (2.4) *belong to the class $D_{p,q}$, then*

$$\|w_{n+1}\|_G^2 \leqslant \|w_0\|_G^2 + h\sum_{j=0}^{n} [(2p+q)\|\sigma(E)(y_j - z_j)\|^2 + \delta q\|\sigma(E)(y_{j-m+1} - z_{j-m+1})\|^2$$
$$+ (1-\delta)q\|\sigma(E)(y_{j-m} - z_{j-m})\|^2]. \tag{4.1}$$

Proof. Suppose the method is G-stable for G, then for all real $a_0, a_1, \ldots, a_k$,

$$A_1^{\mathrm{T}}GA_1 - A_0^{\mathrm{T}}GA_0 \leqslant 2\sigma(E)a_0\rho(E)a_0,$$

where $A_i=(a_i,a_{i+1},\dots,a_{i+k-1})^{\mathrm{T}}$, $i=0,1$. Therefore, we can easily obtain (cf. [4, 15, 17])

$$\|w_{n+1}\|_G^2-\|w_n\|_G^2\leqslant 2\mathrm{Re}\langle\sigma(E)(y_n-z_n),\rho(E)(y_n-z_n)\rangle. \tag{4.2}$$

Hence

$$\begin{aligned}\|w_{n+1}\|_G^2-\|w_n\|_G^2&\leqslant 2\mathrm{Re}\langle\sigma(E)(y_n-z_n),h(f(\sigma(E)t_n,\sigma(E)y_n,\bar{y}_n)\\&\quad-f(\sigma(E)t_n,\sigma(E)z_n,\bar{z}_n))\rangle\\&\leqslant 2\mathrm{Re}\langle\sigma(E)(y_n-z_n),h(f(\sigma(E)t_n,\sigma(E)y_n,\bar{y}_n)-f(\sigma(E)t_n,\sigma(E)z_n,\bar{y}_n))\rangle\\&\quad+2\mathrm{Re}\langle\sigma(E)(y_n-z_n),h(f(\sigma(E)t_n,\sigma(E)z_n,\bar{y}_n)-f(\sigma(E)t_n,\sigma(E)z_n,\bar{z}_n))\rangle\\&\leqslant 2h[p\|\sigma(E)(y_n-z_n)\|^2+q\|\sigma(E)(y_n-z_n)\|\cdot\|\bar{y}_n-\bar{z}_n\|]\\&\leqslant(2p+q)h\|\sigma(E)(y_n-z_n)\|^2+qh\|\bar{y}_n-\bar{z}_n\|^2.\end{aligned}$$

It follows from (3.4) and (3.6) that

$$\begin{aligned}\|\bar{y}_n-\bar{z}_n\|^2&\leqslant\delta^2\|\sigma(E)(y_{n-m+1}-z_{n-m+1})\|^2+(1-\delta)^2\|\sigma(E)(y_{n-m}-z_{n-m})\|^2\\&\quad+2\delta(1-\delta)\|\sigma(E)(y_{n-m+1}-z_{n-m+1})\|\cdot\|\sigma(E)(y_{n-m}-z_{n-m})\|\\&\leqslant\delta^2\|\sigma(E)(y_{n-m+1}-z_{n-m+1})\|+(1-\delta)^2\|\sigma(E)(y_{n-m}-z_{n-m})\|\\&\quad+\delta(1-\delta)(\|\sigma(E)(y_{n-m+1}-z_{n-m+1})\|^2+\|\sigma(E)(y_{n-m}-z_{n-m})\|^2)\\&=\delta\|\sigma(E)(y_{n-m+1}-z_{n-m+1})\|^2+(1-\delta)\|\sigma(E)(y_{n-m}-z_{n-m})\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|w_{n+1}\|_G^2-\|w_n\|_G^2&\leqslant(2p+q)h\|\sigma(E)(y_n-z_n)\|^2+\delta qh\|\sigma(E)(y_{n-m+1}-z_{n-m+1})\|^2\\&\quad+(1-\delta)qh\|\sigma(E)(y_{n-m}-z_{n-m})\|^2.\end{aligned}$$

By induction we have that (4.1) holds, which completes the proof of Theorem 4.1. □

Theorem 4.2. *Any A-stable one-leg method (ρ,σ) is GR-stable.*

Proof. Suppose the method is *A*-stable. Then $\beta_k/\alpha_k>0$ and the method is G-stable (cf. [5]). Application of Theorem 4.1 in combination with the condition that $0\leqslant q\leqslant -p$ yields

$$\begin{aligned}\|w_{n+1}\|_G^2&\leqslant\|w_0\|_G^2+ph\sum_{j=0}^{n}\|\sigma(E)(y_j-z_j)\|^2+\delta qh\sum_{j=-m+1}^{n-m+1}\|\sigma(E)(y_j-z_j)\|^2\\&\quad+(1-\delta)qh\sum_{j=-m}^{n-m}\|\sigma(E)(y_j-z_j)\|^2.\end{aligned}$$

When $m \geqslant 2$ we have

$$\begin{aligned}\|w_{n+1}\|_G^2 &\leqslant \|w_0\|_G^2 + \delta qh \sum_{j=-m+1}^{-1} \|\sigma(E)(y_j - z_j)\|^2 + (1-\delta)qh \sum_{j=-m}^{-1} \|\sigma(E)(y_j - z_j)\|^2 \\ &\leqslant \|w_0\|_G^2 + (m-1)\delta qh \max_{-m+1 \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2 \\ &\quad + (1-\delta)mqh \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2 \\ &\leqslant \|w_0\|_G^2 + (m-\delta)qh \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2 \\ &= \|w_0\|_G^2 + q\tau \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2. \end{aligned} \tag{4.4}$$

On the other hand, when $m = 1$ we have

$$\begin{aligned}\|w_{n+1}\|_G^2 &\leqslant \|w_0\|_G^2 + (1-\delta)qh\|\sigma(E)(y_{-1} - z_{-1})\|^2 \\ &= \|w_0\|_G^2 + q\tau\|\sigma(E)(y_{-1} - z_{-1})\|^2. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$\|w_{n+1}\|_G^2 \leqslant \|w_0\|_G^2 + q\tau \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2, \quad n \geqslant 0, \ m \geqslant 1. \tag{4.6}$$

Let λ_1 and λ_2 denote the maximum and minimum eigenvalue of the matrix G, respectively. Then we have

$$\lambda_2 \|y_{n+k} - z_{n+k}\|^2 \leqslant \lambda_1 \sum_{i=0}^{k-1} \|y_i - z_i\|^2 + q\tau \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2, \quad n \geqslant 0.$$

Hence,

$$\|y_{n+k} - z_{n+k}\|^2 \leqslant \frac{k\lambda_1}{\lambda_2} \max_{0 \leqslant i \leqslant k-1} \|y_i - z_i\|^2 + \frac{q\tau}{\lambda_2} \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2, \quad n \geqslant 0. \tag{4.7}$$

This shows that the method is GR-stable.

In the following, we further investigate the asymptotic stability of one-leg methods. A method is strongly A-stable if it is A-stable and the modulus of any root of $\sigma(x)$ is strictly less than 1.

Theorem 4.3. *A strongly A-stable one-leg method is GAR-stable.*

Proof. Analogous to Theorem 4.2, we can easily obtain

$$\begin{aligned}&\|w_{n+1}\|_G^2 + (-p-q)h \sum_{j=0}^{n} \|\sigma(E)(y_j - z_j)\|^2 \\ &\quad \leqslant \|w_0\|_G^2 + q\tau \max_{-m \leqslant j \leqslant -1} \|\sigma(E)(y_j - z_j)\|^2, \quad n \geqslant 0. \end{aligned} \tag{4.8}$$

Because $-p-q > 0$ and $h > 0$, we have

$$\lim_{n \to \infty} \|\sigma(E)(y_n - z_n)\| = 0. \tag{4.9}$$

On the other hand,

$$\sigma(E)(y_n - z_n) = \sum_{j=0}^{k} \beta_j (y_{n+j} - z_{n+j}), \tag{4.10}$$

which gives

$$w_{n+1} = Aw_n + B_n, \tag{4.11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{\beta_0}{\beta_k} & -\dfrac{\beta_1}{\beta_k} & -\dfrac{\beta_2}{\beta_k} & \cdots & -\dfrac{\beta_{k-1}}{\beta_k} \end{bmatrix} \otimes I_N, \qquad B_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{\beta_k}\sigma(E)(y_n - z_n) \end{bmatrix}.$$

Here $\otimes$ is the Kronecker product, and I_N is the $N \times N$-identity matrices. From the strong A-stability of the method, we have the spectral radius of the matrix A strictly less than 1. Therefore, there exists a norm $\| \cdot \|_*$ in C^{kN} such that the corresponding operator norm $\|A\|_* = \sup_{\|x\|=1} \|Ax\|_* < 1$. From $\|B_n\|_* \to 0$ we know that, for any $\varepsilon > 0$, there exists $l > 0$ such that $\|B_n\|_* < (1 - \|A\|_*)\varepsilon/2$ when $n \geqslant l$. Hence,

$$\|w_n\|_* = \|A^{n-l} w_l + \sum_{j=0}^{n-l-1} A^j B_{n-j-1}\|_* \leqslant \|A\|_*^{n-l} \|w_l\|_* + \varepsilon/2, \quad n \geqslant l, \tag{4.12}$$

which shows that there exists $N_0 > l$ such that $\|w_n\| < \varepsilon$ when $n > N_0$. Therefore, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0,$$

which completes the proof of Theorem 4.3. □

Lemma 4.4. *Suppose $\{\xi_i(x)\}_{i=1}^r$ are a basis of polynomials for P^{r-1}, the space of polynomials of degree strictly less than r, then there is always a unique solution $y_n, \ldots, y_{n+r-1}$ to the system of equations*

$$\xi_i(E) y_n = b_i, \quad b_i \in C^N, \ i = 1, \ldots, r \tag{4.13}$$

and there exists a constant D, independent of the b_i, such that

$$\max_{0 \leqslant i \leqslant r-1} |y_{n+i}| \leqslant D \max_{1 \leqslant i \leqslant r} |b_i|. \tag{4.14}$$

Theorem 4.5. *Any A-stable one-leg method (ρ, σ) is weak GAR-stable.*

Proof. It follows from $-p-q>0$ and $h>0$ that

$$\lim_{n\to\infty} \|\sigma(E)(y_n - z_n)\| = 0. \tag{4.15}$$

Considering (3.10), we further obtain

$$\lim_{n\to\infty} \|\rho(E)(y_n - z_n)\| = 0. \tag{4.16}$$

On the other hand, ρ and σ are coprime, and both are of degree k. Hence, $\{x^i\rho(x), x^i\sigma(x);\ i = 0, \ldots, k-1\}$ form a basis for P^{2k-1}. Considering (4.15), (4.16) and Lemma 4.4, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0,$$

which completes the proof of Theorem 4.5. □

Corollary 4.6. *Any A-stable one-leg method (ρ, σ) is GP-stable.*

5. Error analysis of A-stable one-leg methods

In this section, we focus on the error analysis of A-stable one-leg methods applied to stiff DDEs. For the sake of simplicity, we always assume that all constants h_i, c_i and d_i used later are dependent only on the method, some of the bounds M_i, the characteristic parameters p and q of the problem class $D_{p,q}$, and τ.

Now, we consider scheme (3.3) and the following scheme:

$$\rho(E)\hat{y}_n + \alpha_k e_n = hf(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n), \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

where

$$c_1 = -\frac{1}{2}\sum_{j=0}^{k-1}\left(\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j\right)j^2 - \frac{\beta_k}{\alpha_k}\sum_{j=0}^{k} j\beta_j + \frac{1}{2}\left(\sum_{j=0}^{k} j\beta_j\right)^2, \tag{5.2a}$$

$$\hat{y}_n = y(t_n) + c_1 h^2 y''(t_n), \tag{5.2b}$$

$$\bar{Y}_n = y(\sigma(E)t_n - \tau), \tag{5.2c}$$

where $y(t)$ is the exact solution of problem (2.1). Then for any $n \geqslant 0$, e_n is uniquely determined by Eq. (5.1).

Theorem 5.1. *Assume that the one-leg method (ρ, σ) is G-stable with respect to G and that the function $f(t, u, v)$ satisfies conditions (2.2) and (2.3). Then there exist constant h_1, d_1, d_2 and d_3 such that*

$$\|\varepsilon_{n+1}\|_G^2 \leqslant (1 + d_1 h)\|\varepsilon_n\|_G^2 + d_2 h\|\hat{y}_n - \bar{Y}_n\|^2 + d_3 h^{-1}\|e_n\|^2, \quad n = 0, 1, 2, \ldots, \tag{5.3}$$

where $\varepsilon_n = ((y_n - \hat{y}_n)^{\mathrm{T}}, (y_{n+1} - \hat{y}_{n+1})^{\mathrm{T}}, \ldots, (y_{n+k-1} - \hat{y}_{n+k-1})^{\mathrm{T}})^{\mathrm{T}}$.

Proof. In view of G-stability, analogous to Theorem 4.1, we have

$$\|\bar{\varepsilon}_{n+1}\|_G^2 \leqslant \|\varepsilon_n\|_G^2 + 2\mathrm{Re}\langle \sigma(E)(y_n - \hat{y}_n) - \beta_k e_n, \rho(E)(y_n - \hat{y}_n) - \alpha_k e_n\rangle, \tag{5.4}$$

where

$$\bar{\varepsilon}_{n+1} = ((y_{n+1} - \hat{y}_{n+1})^{\mathrm{T}}, (y_{n+2} - \hat{y}_{n+2})^{\mathrm{T}}, \ldots, (y_{n+k} - \hat{y}_{n+k} - e_n)^{\mathrm{T}})^{\mathrm{T}}.$$

It follows from conditions (2.2) and (2.3) that

$$\begin{aligned} \|\bar{\varepsilon}_{n+1}\|_G^2 &\leqslant \|\varepsilon_n\|_G^2 + 2h\mathrm{Re}\langle \sigma(E)(y_n - \hat{y}_n) - \beta_k e_n, f(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n) \\ &\quad - f(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n)\rangle \\ &\leqslant \|\varepsilon_n\|_G^2 + 2h[p\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\|^2 + q\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\| \cdot \|\bar{y}_n - \bar{Y}_n\|] \\ &\leqslant \|\varepsilon_n\|_G^2 + h[(2p+q)\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\|^2 + q\|\bar{y}_n - \bar{Y}_n\|^2]. \end{aligned} \tag{5.5}$$

Let

$$c_2 = \begin{cases} 0, & 2p+q \leqslant 0, \\ 2p+q, & 2p+q > 0, \end{cases} \tag{5.6}$$

then

$$\|\bar{\varepsilon}_{n+1}\|_G^2 \leqslant \|\varepsilon_n\|_G^2 + h[c_2\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\|^2 + q\|\bar{y}_n - \bar{Y}_n\|^2]. \tag{5.7}$$

On the other hand, it follows from (3.3) and (5.1)

$$\begin{aligned} \sigma(E)(y_n - \hat{y}_n) - \beta_k e_n &= \sum_{j=0}^{k-1} \left(\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j\right)(y_{n+j} - \hat{y}_{n+j}) \\ &\quad + \frac{\beta_k}{\alpha_k} h(f(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n) - f(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n)). \end{aligned}$$

From G-stability of the method we have $\beta_k/\alpha_k > 0$ (cf. [4, 5]). Then

$$\begin{aligned} &\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\|^2 \\ &= \mathrm{Re}\left\langle \sigma(E)(y_n - \hat{y}_n) - \beta_k e_n, \sum_{j=0}^{k-1} \left(\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j\right)(y_{n+j} - \hat{y}_{n+j})\right\rangle \\ &\quad + \frac{\beta_k}{\alpha_k} h\mathrm{Re}\langle \sigma(E)(y_n - \hat{y}_n) - \beta_k e_n, f(\sigma(E)t_n, \sigma(E)y_n, \bar{y}_n) \\ &\quad - f(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n)\rangle \\ &\leqslant \|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\| \left[\left\|\sum_{j=0}^{k-1} \left(\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j\right)(y_{n+j} - \hat{y}_{n+j})\right\|\right. \\ &\quad \left. + \frac{\beta_k}{\alpha_k} hp\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\| + \frac{\beta_k}{\alpha_k} hq\|\bar{y}_n - \bar{Y}_n\|\right]. \end{aligned}$$

When $hp\,\beta_k/\alpha_k < 1$, we have

$$\|\sigma(E)(y_n - \hat{y}_n) - \beta_k e_n\| \leqslant \frac{\alpha_k}{\alpha_k - \beta_k hp}\left[c_3\|\varepsilon_n\|_G + \frac{\beta_k}{\alpha_k} hq\|\bar{y}_n - \bar{Y}_n\|\right], \tag{5.8}$$

where

$$c_3 = \max_{0\leqslant j\leqslant k-1} \left|\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j\right| \Big/ \sqrt{\lambda_2} \tag{5.9}$$

and λ_2 denotes the minimum eigenvalue of the matrix G. Therefore, together with (5.7) and (5.8)

$$\begin{aligned}\|\bar{\varepsilon}_{n+1}\|_G^2 &\leqslant \|\varepsilon_n\|_G^2 + \frac{2c_2\alpha_k^2 h}{(\alpha_k - \beta_k hp)^2}\left[c_3^2\|\varepsilon_n\|_G^2 + \left(\frac{\beta_k}{\alpha_k}hq\right)^2 \|\bar{y}_n - \bar{Y}_n\|^2\right] + qh\|\bar{y}_n - \bar{Y}_n\|^2 \\ &\leqslant \left(1 + \frac{2c_2c_3^2\alpha_k^2 h}{(\alpha_k - \beta_k hp)^2}\right)\|\varepsilon_n\|_G^2 + \left(q + \frac{2c_2(\beta_k hq)^2}{(\alpha_k - \beta_k hp)^2}\right) h\|\bar{y}_n - \bar{Y}_n\|^2, \quad \beta_k hp/\alpha_k < 1. \end{aligned}\tag{5.10}$$

Let λ_1 denote the maximum eigenvalue of the matrix G, then

$$\begin{aligned}\|\varepsilon_{n+1}\|_G^2 &\leqslant \|\bar{\varepsilon}_{n+1}\|_G^2 + \lambda_1\|e_n\|^2 + 2\sqrt{\lambda_1}\|e_n\| \cdot \|\bar{\varepsilon}_{n+1}\|_G \\ &\leqslant (1+h)\|\bar{\varepsilon}_{n+1}\|_G^2 + \left(1 + \frac{1}{h}\right)\lambda_1\|e_n\|^2. \end{aligned}\tag{5.11}$$

Let

$$h_1 = \begin{cases} 1, & p \leqslant 0, \\ \min(1, \frac{\alpha_k}{2\beta_k p}), & p > 0, \end{cases} \tag{5.12}$$

$$d_1 = \sup_{h\in(0,h_1]} \left(1 + \frac{2c_2c_3^2\alpha_k^2(1+h)}{(\alpha_k - \beta_k hp)^2}\right), \tag{5.13a}$$

$$d_2 = \sup_{h\in(0,h_1]} \left(q + \frac{2c_2(\beta_k hq)^2}{(\alpha_k - \beta_k hp)^2}\right)(1+h), \tag{5.13b}$$

$$d_3 = \max_{h\in(0,h_1]} (1+h)\lambda_1, \tag{5.13c}$$

then (5.3) holds, which completes the proof of Theorem 5.1. □

Theorem 5.2. *Assume that the one-leg method* (ρ, σ) *applied to problem* (2.1) *of the class* $D_{p,q}$ *is consistent of order* $s \leqslant 2$ *in the classical sense for ODEs and that* $\beta_k/\alpha_k > 0$, *then there exists constant* d_4 *such that*

$$\|e_n\| \leqslant d_4 h^{s+1}, \quad h \in (0, h_1], \quad n = 0, 1, 2, \ldots, \tag{5.14}$$

where h_1 *is defined by* (5.12).

Proof. Consider the following scheme:

$$y(\sigma(E)t_n) = \sum_{j=0}^{k-1}\left(\beta_j - \frac{\beta_k}{\alpha_k}\alpha_j\right)\hat{y}_{n+j} + \frac{\beta_k}{\alpha_k}hy'(\sigma(E)t_n) + R_1^{(n)}, \tag{5.15}$$

$$\rho(E)\hat{y}_n = hy'(\sigma(E)t_n) + R_2^{(n)}. \tag{5.16}$$

By Taylor expansion, there exists constant c_4 such that

$$\|R_1^{(n)}\| \leqslant c_4 h^3, \tag{5.17}$$

$$\|R_2^{(n)}\| \leqslant c_4 h^{s+1}. \tag{5.18}$$

From (5.1) and (5.15) we have

$$y(\sigma(E)t_n) - \sigma(E)\hat{y}_n - \beta_k e_n = \frac{\beta_k}{\alpha_k} h[f(\sigma(E)t_n, y(\sigma(E)t_n), y(\sigma(E)t_n - \tau)) - f(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n)] + R_1^{(n)}. \tag{5.19}$$

In view of (2.2) we can obtain further

$$\|y(\sigma(E)t_n) - \sigma(E)\hat{y}_n - \beta_k e_n\|^2 \leqslant \frac{\beta_k}{\alpha_k} hp\|y(\sigma(E)t_n) - \sigma(E)\hat{y}_n - \beta_k e_n\|^2 + \|R_1^{(n)}\| \cdot \|y(\sigma(E)t_n) - \sigma(E)\hat{y}_n - \beta_k e_n\|,$$

then

$$\|y(\sigma(E)t_n) - \sigma(E)\hat{y}_n - \beta_k e_n\| \leqslant 2\|R_1^{(n)}\|, \quad h \in (0, h_1], \tag{5.20}$$

where h_1 is defined by (5.12).

Substituting (5.19) with (5.20), we have

$$\|h[y'(\sigma(E)t_n) - f(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n)]\| \leqslant \frac{3\alpha_k}{\beta_k}\|R_1^{(n)}\|, \quad h \in (0, h_1]. \tag{5.21}$$

On the other hand, a combination of (5.1) and (5.16) yields

$$\alpha_k e_n = hf(\sigma(E)t_n, \sigma(E)\hat{y}_n + \beta_k e_n, \bar{Y}_n) - hy'(\sigma(E)t_n) - R_2^{(n)}.$$

Hence

$$\|e_n\| \leqslant \frac{3}{|\beta_k|}\|R_1^{(n)}\| + \frac{1}{|\alpha_k|}\|R_2^{(n)}\|, \quad h \in (0, h_1]. \tag{5.22}$$

In view of (5.17) and (5.18), (5.14) holds, which completes the proof of Theorem 5.2.

Theorem 5.3. *If an A-stable one-leg k step method (ρ, σ) is consistent of order s in the classical sense for ODEs, then it is D-convergent of order s, where $k \geqslant 1$, $s = 1, 2$.*

Proof. Suppose the method (ρ, σ) is A-stable. Then $\beta_k/\alpha_k > 0$ and the method is G-stable. From Theorems 5.1 and 5.2, we have

$$\|\varepsilon_{n+1}\|_G^2 \leqslant (1 + d_1 h)\|\varepsilon_n\|_G^2 + d_2 h\|\bar{y}_n - \bar{Y}_n\|^2 + d_3 d_4 h^{2s+1}, \quad h \in (0, h_1].$$

By induction, it is easily seen that

$$\|\varepsilon_{n+1}\|_G^2 \leqslant \|\varepsilon_0\|_G^2 + h\sum_{i=0}^{n}[d_2\|\bar{y}_i - \bar{Y}_i\|^2 + d_1\|\varepsilon_i\|_G^2 + d_3 d_4 h^{2s}], \quad h \in (0, h_1].$$

Therefore,

$$\|y_{n+k}-\hat{y}_{n+k}\|^2 \leqslant \frac{\lambda_1}{\lambda_2}\sum_{j=0}^{k-1}\|y_j-\hat{y}_j\|^2+\frac{h}{\lambda_2}\sum_{i=0}^{n}[d_2\|\bar{y}_i-\bar{Y}_i\|^2 + d_1\lambda_1\sum_{j=0}^{k-1}\|y_{i+j}-\hat{y}_{i+j}\|^2+d_3d_4h^{2s}],\quad h\in(0,h_1]. \tag{5.23}$$

On the other hand,

$$\begin{aligned}\|\bar{y}_i-\bar{Y}_i\| &= \|\bar{y}_i - y(\sigma(E)t_i-\tau)\|\\ &\leqslant \|\delta\sigma(E)(y_{i+1-m}-\hat{y}_{i+1-m})\|+(1-\delta)\|\sigma(E)(y_{i-m}-\hat{y}_{i-m})\|\\ &\quad+\|\delta\sigma(E)\hat{y}_{i+1-m}+(1-\delta)\sigma(E)\hat{y}_{i-m}-y(\sigma(E)t_i-\tau)\|\\ &\leqslant \delta\sum_{j=0}^{k}|\beta_j|\cdot\|y_{i+j+1-m}-\hat{y}_{i+j+1-m}\|+(1-\delta)\sum_{j=0}^{k}|\beta_j|\cdot\|y_{i+j-m}-\hat{y}_{i+j-m}\|\\ &\quad+\|\delta\sum_{j=0}^{k}\beta_j\hat{y}_{i+j+1-m}+(1-\delta)\sum_{j=0}^{k}\beta_j\hat{y}_{i+j-m}-y(\sigma(E)t_i-\tau)\|.\end{aligned}$$

By Taylor expansion, there exist constants h_2 and d_5 such that

$$\|\bar{y}_n-\bar{Y}_n\|^2\leqslant d_5\left[\sum_{j=0}^{k+1}\|y_{i+j-m}-\hat{y}_{i+j-m}\|^2+h^4\right],\quad h\in(0,h_2]. \tag{5.24}$$

Substituting (5.23) with (5.24), (5.2) and $y_j=y(t_j)$, $j\leqslant k-1$, leads to

$$\begin{aligned}\|y_{n+k}-\hat{y}_{n+k}\|^2 &\leqslant \frac{\lambda_1}{\lambda_2}kc_1^2M_2^2h^4+\frac{\lambda_1}{\lambda_2}d_1kh\sum_{i=0}^{n+k-1}\|y_i-\hat{y}_i\|^2\\ &\quad+\frac{h}{\lambda_2}\sum_{i=0}^{n}\left[d_2d_5\left(\sum_{j=0}^{k+1}\|y_{i+j-m}-\hat{y}_{i+j-m}\|^2+h^4\right)+d_3d_4h^{2s}\right]\\ &\leqslant\frac{1}{\lambda_2}\left[(\lambda_1kc_1^2M_2^2+d_2d_5nh)h^4+nhd_3d_4h^{2s}+\lambda_1d_1kh\sum_{i=0}^{n+k-1}\|y_i-\hat{y}_i\|^2\right.\\ &\quad\left.+d_2d_5(k+2)h\sum_{i=-m}^{n+k+1-m}\|y_i-\hat{y}_i\|^2\right],\quad h\in(0,h_3],\end{aligned} \tag{5.25}$$

where $h_3=\min(h_1,h_2)\leqslant 1$, $m\geqslant 1$, $n\geqslant 0$, $s=1,2$.

Therefore, whether $m=1$ or $m>1$, it is certain that there exist constants h_0, c_0 and d_0 such that

$$\|y_{n+k}-\hat{y}_{n+k}\|^2\leqslant c_0(1+t_{n+k})h^{2s}+hd_0\sum_{i=0}^{n+k-1}\|y_i-\hat{y}_i\|^2,\quad n=0,1,2,\ldots,\quad h\in(0,h_0]. \tag{5.26}$$

By induction we can obtain

$$\|y_{n+k}-\hat{y}_{n+k}\|^2 \leqslant \exp(d_0 t_{n+k})(c_0(1+t_{n+k})h^{2s}+hd_0k\max_{0\leqslant j<k}\|y_i-\hat{y}_i\|^2), n=0,1,2,\ldots,$$
$$h\in(0,h_0]. \tag{5.27}$$

In fact (5.27) obviously holds when $n=0$. If (5.27) holds for $n<l$, where l is a positive integer, then it follows from (5.26) that

$$\begin{aligned}\|y_{l+k}-\hat{y}_{l+k}\|^2 &\leqslant c_0(1+t_{l+k})h^{2s}+hd_0\sum_{i=0}^{l+k-1}\|y_i-\hat{y}_i\|^2\\ &\leqslant c_0(1+t_{l+k})h^{2s}+hd_0k\max_{0\leqslant j<k}\|y_i-\hat{y}_i\|^2+hd_0\sum_{i=k}^{l+k-1}[\exp(d_0t_i)(c_0(1+t_i)h^{2s}\\ &\quad +hd_0k\max_{0\leqslant j<k}\|y_i-\hat{y}_i\|^2)]\\ &\leqslant [c_0(1+t_{l+k})h^{2s}+hd_0k\max_{0\leqslant j<k}\|y_i-\hat{y}_i\|^2]\left(1+hd_0\sum_{i=k}^{l+k-1}\exp(d_0t_i)\right)\\ &\leqslant \exp(d_0t_{l+k})[c_0(1+t_{l+k})h^{2s}+hd_0k\max_{0\leqslant j<k}\|y_i-\hat{y}_i\|^2].\end{aligned}$$

This shows that (5.27) holds for every $n\geqslant 0$. From (5.2) we further obtain

$$\|y_{n+k}-y(t_{n+k})\| \leqslant |c_1|M_2h^2+\exp(\tfrac{1}{2}d_0t_{n+k})[h^s\sqrt{c_0(1+t_{n+k})}+\sqrt{d_0kh_0}|c_1|M_2h^2],$$
$$n=0,1,2,\ldots,\quad h\in(0,h_0]. \tag{5.28}$$

This shows that the method is D-convergent of order $s, s=1,2$, which completes the proof of Theorem 5.3.

Now, we review results from the literature for one-leg methods. For ODEs, Dahlquist [5] proved that A-stability is equivalent to G-stability. Li [15] proved that A-stability implies B-convergence, and Huang [10] further proved that B-convergence implies A-stability. For DDEs, from Definition 3.1 and Theorem 4.2, A-stability and GR-stability are equivalent. From Definition 3.8, D-convergence implies B-convergence. Theorem 5.3 shows that A-stability implies D-convergence. Therefore we have the following result.

Theorem 5.4. *For a one-leg k-step method* (3.3) *with linear interpolation* (3.4), *the following statements are equivalent:*
(1) (ρ,σ) *is A-stable.*
(2) (ρ,σ) *is G-stable.*
(3) (ρ,σ) *is B-convergent.*
(4) (ρ,σ) *is GR-stable.*
(5) (ρ,σ) *is D-convergent.*

For solvability of Eq. (3.3) with (3.4), we refer to [4] when $m>1$. When $m=1$, we have the following result whose proof is similar to the proof of Lemma 1.2 in [4].

Theorem 5.5. *Suppose that*

$$h(p+q) < \alpha_k/\beta_k.$$

If $t_n, y_{n-m}, y_{n-m+1}, \ldots, y_{n+k-1}$ are given, then y_{n+k} is uniquely determined by Eqs. (3.3) with (3.4).

6. Equations with several delays

Consider the following equation with several delays:

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_r)), \quad t \geqslant 0,$$
$$y(t) = \phi_1(t), \quad t \leqslant 0. \tag{6.1}$$

Because $\tau_1, \tau_2, \ldots, \tau_r$ are positive constants, there are no additional difficulties with respect to (6.1). We can similarly define the concepts of stability and convergence in this case. All results given in this paper can be modified easily to this more general situation. But we do not list them here for the sake of brevity.

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