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Lyapunov exponent analyses of chaotic oscillations in rf-biased Josephson junctions

Y. Tamura, H. Shimakage*

Graduate School of Science and Engineering, Ibaraki University, Hitachi, Ibaraki 316-8511, Japan

Abstract

We have been studying chaos in Josephson junctions for the purpose of applying it to physical random number generators. In this paper, parameter maps, under which chaotic oscillation occurred, were evaluated with RCSJ simulations. Lyapunov exponents were calculated to judge whether their dynamics were chaotic or not. The mappings of chaos occurrence with different system parameters were reported by calculating the exponent for each system parameters. The conditions to generate chaos in Josephson junction toward the random number generator were discussed.

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1. Introduction

Random numbers are essential elements in various applications, such as cryptography [1], simulations [2], amusements, and so on. Among them, high-quality random numbers, in particular, have been required for the security applications. Up to now, pseudo random numbers which are created by computer algorithms have been in use for it. However, these numbers are not at all truly random practically because of their properties with periodicity and reproducibility. On the other hand, physical random numbers which are based on physical phenomena, such as thermal noise [3] and radiation [4], have properties with non-periodicity and non-reproducibility. Thus, physical random number generator is promising for the security applications. One of the promising effects with property of phenomena is chaos, which is found in various physical systems. Although a high-speed physical random number

* Corresponding author. Tel.: +81-294-38-5101; fax: +81-294-38-5375.
E-mail address: simakage@mx.ibaraki.ac.jp
A generator using chaotic lasers was proposed and its speed was obtained of up to 12.5 Gbps [5], it had some periodic properties because of an effect of its return light in the cavity.

Up to now, chaotic oscillations in the rf-biased Josephson junctions have widely studied [6-8]. Their researches were mainly focused on understanding the chaos theory [6, 7] because of its simple non-linear equation, as well as avoiding the generating chaos for the Josephson voltage standards [8]. However, it is expected that chaos oscillations in Josephson junctions could be applied to generate high quality random numbers. In fact, it has been demonstrated the almost perfect white noise generation from the rf-biased Josephson junction by numerical simulation [9]. In addition, it has been shown that the chaos states of Josephson junctions induced unpredictable dynamics [10].

We also have been studying chaos in Josephson junctions using resistively and capacitively shunted Josephson junction (RCSJ) model, and proposed the usage of the chaos states for the purpose of developing physical random number generators. Our final target is to develop physical random number generator using Josephson junctions but not with simulation. For realizing our purpose, it is necessary to know the detailed parameters, though the condition of chaos occurrence in junction has been reported previously. In this paper, experimental parameters, under which chaotic oscillation occurs, are evaluated with RCSJ simulations. The capability of the random number generator will also be discussed.

2. Numerical simulation

2.1. RF-biased Josephson junction

We use the equivalent circuit model of rf-biased Josephson junctions. A total current, which consists of dc current source and rf current source, are divided to Josephson current $I_{c}$, a shunted current flowing through resistance $R$ and current flowing through capacitance $C$. Here $\phi(t)$ is a phase difference of the superconducting wave function across the junction. Since the junction voltage $V$ is $\frac{h}{2e}\phi(t)$, the normalization equation is given by

$$\beta \phi'(\tau) + \phi(\tau) + \sin\phi(\tau) = i_0 + i_1 \sin\Omega \tau,$$

where $\beta = 2eI_c R^2 C / h$ is McCumber parameter, $\Omega = \hbar \omega_0 / (2eI_c R)$ is the normalized angular frequency, $\tau = (2eI_c R / \hbar)$ is the normalized time, the dot denotes a derivative from the normalized time, $i_0$ is the normalized dc current source, and $i_1$ is the normalized rf bias term. The voltage $V$ across the junction is corresponding to normalized junction voltage $\phi(\tau)$. Eq. (1) is calculated by using the fourth-order Runge-Kutta method and time evolution of $\phi(\tau)$ is obtained. Then, the behavior of the oscillations of $\phi(\tau)$ for certain value of the system parameters, $\beta$, $\Omega$, $i_0$, and $i_1$ are evaluated whether the junctions are in chaotic state or not. In this calculation, no noise is taken into account. In general measurements, the current voltage characteristics of the Josephson junctions are represented by dc current $I_0$ and junction voltage $\langle V \rangle$, which is time averaged value of the oscillated voltage. The $I$-$\langle V \rangle$ characteristic curves with different parameters are shown in Fig. 1(a) and (b). Although typical Shapiro steps are observed in Fig. 1(a), fluctuations, which could be due to the chaos, are observed in Fig. 1(b).

2.2. Lyapunov exponent

In general, chaotic signals, in spite of being deterministic, are characterized by random-like behavior, with broadband power spectrum and sensitive dependence on initial conditions. If we have two identical chaotic systems which start with two initial conditions which have very small difference, the behavior of their systems will behave quite differently. Due to this nature, a long-term prediction of a chaotic system is practically impossible.

One of the techniques to judge the presence of chaotic behavior is the Lyapunov exponent $\lambda$, which measures dependence of the sensitivity of initial conditions. Specifically, when two trajectories have an initial separation $\delta$, they diverge as $|\delta| \approx |\delta_0| \exp(\lambda t)$ where $\lambda$ is the Lyapunov exponent. If the $\lambda$ is positive, then the system is chaotic, and if negative, then non-chaotic.
In this research, Lyapunov exponents are computed using an algorithm of Shimada and Nagashima [11] which is allowed to avoid numerical overflow for rapidly growing of length of the vector. The vector is renormalized to unit length using a Gram-Schmidt procedure at every time interval. Thus, the Lyapunov exponents are given by the average of growth rates of the each vector. Lyapunov exponent corresponding to the DC current is shown in Fig. 1(c) and (d), which are derived from $I - \langle V \rangle$ curves in Fig. 1(a) and 1(b), respectively. Lyapunov exponents in Fig. 1(c) do not exceed 0.00, thus, it is judged that chaos does not occurred in $I - \langle V \rangle$ characteristic of Fig. 1(a). On the other hand, in Fig. 1(d), the exponents exceed 0.00 around $i_0 = 0.2$. Consequently, it is realized that chaos occurs in $I - \langle V \rangle$ curve of Fig. 1(b). When dc current is $i_0 = 0.75$ in Fig. 1(b) and (d), Lyapunov exponent is $\lambda = -0.071$, and the power spectrum of voltage waveforms at this parameter is shown in Fig. 2(a). The observed peaks are discrete and corresponded to the irradiated microwave frequency and its harmonics. On the other hand, when dc current is $i_0 = 0.0$, Lyapunov exponent is $\lambda = 0.11$, which indicates that the system is in chaos state. The voltage waveform is irregular-like and in fact, a power spectrum is consecutive over broadband frequency (Fig. 2(b)). As a result, the results indicate that the chaos estimation by the plus or minus of the Lyapunov exponent would be functioned effectively.

![Fig. 1](image1.png)

**Fig. 1.** (a) Average voltage as a function of dc bias with $\beta=0.2, \Omega=0.5, i_1=2.14$, (b) Average voltage as a function of dc bias with $\beta=2.0, \Omega=0.5, i_1=2.14$, (c) and (d) Lyapunov exponent corresponding to each parameters from (a) and (b).

![Fig. 2](image2.png)

**Fig. 2** Power spectra with $\beta=2.0, \Omega=0.5, i_1=2.14$ (a) at $i_0=0.75$ and (b) at and $i_0=0.0$.

### 3. Chaos-generating region

The conditions, in which chaos occurs, depend on system parameters. The chaos occurrence mappings were calculated by Lyapunov exponent, which has been described in previous section. Fig. 3(a), (b), and (c) are the mapping with different system parameters. Here, the values of a McCumber parameter and normalized angular frequency are fixed to $\beta = 2.0$ and $\Omega = 0.5$, respectively. The normalized dc current and normalized rf bias current are changed from 0 to 10 with steps of 0.01. If a point is judged that chaos occurred, then, the point corresponding to the parameter is dotted, but, if not, no point is dotted on the graph. In all figures in Fig. 3, within the calculated parameter range, chaos appears in the lower right hand side only. Therefore, it is suggested that normalized rf current have to be greater than normalized dc bias current for the chaos generation condition. Moreover, when $\Omega$ is
0.1, parameters of occurring chaos are most densely plotted within our calculated parameters. That is, it indicated that there is the $\Omega$ value, at which the chaos condition are likely to be obtained. Furthermore, the results indicate that chaos also occurs with no dc bias in wide range, and for the practical random number creation it would be useful.

Fig. 3 Chaos state mappings with $E=2.0$ at (a) $\Omega=0.3$, (b) $\Omega=0.1$ and (c) $\Omega=0.03$. The system is chaos state at the condition of the dotted point.

4. Discussion and Conclusion

Finally, the capability of the physical random number generator with the Josephson junction is discussed. Here, superconducting material is supposed to be $\text{YBa}_2\text{Cu}_3\text{O}_7$ (YBCO). The typical value of $I_cR_n$ products of the YBCO grain boundary Josephson junctions is about 2.0 mV [12]. In our simulation, when the $I_cR_n$ products are fixed to 2.0 mV, chaos appears at the capacitance of about 1.0 pF, and chaos no longer occurs at less $C=10$ fF with rf current of 0 to 5 mA and rf-frequency of 0 to 500 GHz. The capacitance shunt structure could be fabricated using by the insulator thin films as the capacitance insulator. As we have been studying CeO$_2$ thin film fabrications [13], the capacitance shunt could be relatively easily fabricated. In our simulation, rf-frequency is required roughly from 50 to 400 GHz and rf current from 1.5 to 5 mA. The rf irradiation could be utilized by using the commercial oscillators, such as Gunn oscillators and IMPATT diodes, our simulated value is thought to be feasible.

In conclusion, we have studied the parameters of chaos occurrence in Josephson junctions toward physical random number generators. Lyapunov exponents were calculated to judge whether the dynamics is chaotic or not. In our simulation, it was revealed that the fine control of the parameter was required as shown in Fig. 3. In addition, the parameters, in which the chaos signal from Josephson junctions could be observed experimentally, were obtained.

References