NORTH-HOLLAND

A Conjecture on Permanents*

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ABSTRACT

We show, by a direct proof, that the $n \times n$ $(0,1)$ matrix with the last $n-1$ entries on the main diagonal equal to 0 and all the other entries equal to 1 is never barycentric for $n \geq 4$, which was a conjecture of R. A. Brualdi on permanents.

Let $D = [d_{ij}]$ be an n-square (0, 1) matrix, and let

$$\Omega(D) = \{X = [x_{ij}] \in \Omega_n \mid x_{ij} = 0 \textit{ whenever } d_{ij} = 0\}.$$

Then $\Omega(D)$ is a face of Ω_n, the polytope of $n \times n$ nonnegative doubly stochastic martices, and since it is compact, $\Omega(D)$ contains a minimizing matrix A such that $per\, A \leq per\, X$ for all $X \in \Omega(D)$.

Let R_n denote the $n \times n$ $(0,1)$ matrix with zero trace and all off-diagonal entries equal to 1, and $E_{1,1}$ denote the $n \times n$ matrix whose (1,1) entry is 1 and whose other entries are all zero. Let $C_n = R_n + E_{1,1}$.

Brualdi [1] defined an $n \times n$ $(0,1)$ matrix D to be *barycentric* if

$$per\, b(D) = min\{per\, X : X \in \Omega(D)\},$$

where the *barycenter* $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{per\, D} \sum_{p \leq D} P,$$

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where the summation extends over the set of all permutation matrices P with $P \le D$ and *per* D is their number.

Brualdi conjectured in [1] that C_n *is never barycentric for* $n \ge 4$ *and that*

$$X_n(\alpha) = \begin{bmatrix} \beta & \alpha & \alpha & \alpha & \cdot & \cdot & \cdot & \alpha \\ \alpha & 0 & \gamma & \gamma & \cdot & \cdot & \cdot & \gamma \\ \alpha & \gamma & 0 & \gamma & \cdot & \cdot & \cdot & \gamma \\ \vdots & \vdots & \cdot & \cdot & \vdots & \vdots & \vdots & \vdots \\ \alpha & \gamma & \gamma & \cdots & & \gamma & 0 & \gamma \\ \alpha & \gamma & \gamma & \cdots & & \gamma & \gamma & 0 \end{bmatrix} \qquad (1)$$

is the unique minimizing matrix in $\Omega(C_n)$ *for* $\alpha = \beta = 1/n$ *and* $\gamma = (n-1)/n(n-2)$. In [2], Minc showed that $X_n(\alpha)$ is not a minimizing matrix in $\Omega(C_n)$ for $\alpha = \beta = 1/n$ and $\gamma = (n-1)/n(n-2)$, but it is a minimizing matrix for $\alpha = (\mathit{per}\, R_{n-1})/d, \beta = (n-2)(\mathit{per}\, R_{n-2})/d$, and $\gamma = (\mathit{per}\, C_{n-1})/d$ with $d = \mathit{per}\, C_n - \mathit{per}\, C_{n-1}$ among the forms of (1), which implies that C_n is never barycentric for $n \ge 4$. Minc also claimed that $X_n(\alpha)$ in (1) becomes the barycenter of $\Omega(C_n)$ for $\alpha = (\mathit{per}\, R_{n-1})/(\mathit{per}\, C_n)$; but the barycenter is not correct.

In this note, we give the correct barycenter of $\Omega(C_n)$, and we show that C_n is never barycentric for $n \ge 3$ by a direct calculation from the correct barycenter.

LEMMA 1. *The barycenter of* $\Omega(C_n)$ *is the form* $X(\alpha)$ *in* (1) *with*

$$\alpha = \frac{\mathit{per}\, C_{n-1}}{\mathit{per}\, C_n}, \qquad \beta = \frac{\mathit{per}\, R_{n-1}}{\mathit{per}\, C_n}, \qquad \gamma = \frac{\mathit{per}\, C_{n-1} + \mathit{per}\, C_{n-2}}{\mathit{per}\, C_n}. \qquad (2)$$

PROOF. If we write the barycenter $b(C_n)$ of $\Omega(C_n)$ as

$$b(C_n) = \frac{1}{\mathit{per}\, C_n} \sum_{P \le C_n} P = \frac{1}{\mathit{per}\, C_n} [b_{ij}],$$

then b_{ij} is the number of permutations P such that the (i,j) position of P is 1 and $P \le C_n$. Thus $b_{ii} = 0$ for $i = 2, \ldots, n$, and for other i and j

$$b_{ij} = \mathit{per}\, C_n(i \mid j). \qquad (3)$$

That is, $b_{11} = \mathit{per}\, R_{n-1}$, $b_{j1} = b_{1j} = \mathit{per}\, C_{n-1}$, and $b_{ij} = \mathit{per}\, C_{n-1} + \mathit{per}\, C_{n-2}$ by changes of rows and columns, for $i, j = 2, \ldots, n$ with $i \ne j$.

Therefore the barycenter $b(C_n)$ of $\Omega(C_n)$ is the form $X_n(\alpha)$ in (1) with the required values α, β, and γ in (2). ■

The following lemma is a known result (see [2]).

LEMMA 2. *If $A = (a_{ij})$ is a minimizing matrix in $\Omega(C_n)$ and $a_{pq} > 0$, then $per\, A(p \mid q) = per\, A$.*

Let C_n, R_n, and $b(C_n)$ be the matrices defined above. Clearly,

$$per\, C_n = per\, R_n + per\, R_{n-1}. \tag{4}$$

And we have (from §3.4 in [3])

$$per\, R_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right). \tag{5}$$

THEOREM. *For $n \geq 4$, C_n is never barycentric.*

PROOF. Suppose to the contrary that C_n is barycentric. Then Lemma 2 implies that

$$per\, b(C_n)(1 \mid 1) = per\, b(C_n)(1 \mid n). \tag{6}$$

Using (2) and (4), we calculate

$$\begin{aligned} per\, b(C_n)(1 \mid 1) &= \gamma^{n-1} per\, R_{n-1} \\ &= \gamma^{n-2} \cdot \frac{per\, C_{n-1} + per\, C_{n-2}}{per\, C_n} \cdot per\, R_{n-1} \\ &= \frac{\gamma^{n-2}(per\, C_{n-1} + per\, R_{n-2} + per\, R_{n-3}) per\, R_{n-1}}{per\, C_n} \end{aligned} \tag{7}$$

and

$$\begin{aligned} per\, b(C_n)(1 \mid n) &= \alpha\gamma^{n-2} per\, C_{n-1} \\ &= \gamma^{n-2} \cdot \frac{per\, C_{n-1}}{per\, C_n} \cdot per\, C_{n-1} \\ &= \frac{\gamma^{n-2}(per\, R_{n-1} + per\, R_{n-2}) per\, C_{n-1}}{per\, C_n}. \end{aligned} \tag{8}$$

Substituting (7) and (8) into (6), we have

$$\begin{aligned} (per\, R_{n-2} + per\, R_{n-3}) per\, R_{n-1} &= per\, R_{n-2} per\, C_{n-1} \\ &= per\, R_{n-2} \cdot (per\, R_{n-1} + per\, R_{n-2}). \end{aligned}$$

That is,

$$per\, R_{n-3} per\, R_{n-1} = (\, per\, R_{n-2})^2. \tag{9}$$

Now, we consider three cases;

Case 1. For $n = 4$, we have $per\, R_1 = 0, per\, R_2 = 1$, and $per\, R_3 = 2$. Thus

$$per\, R_{n-3} per\, R_{n-1} = 0 < 1 = (\, per\, R_{n-2})^2,$$

which contradicts (9). That is, C_4 is not barycentric.

Case 2. Let n be any odd integer greater than 4. Using (5), we have

$$\begin{aligned} per\, R_{n-3} per\, R_{n-1} &= (n-3)!(n-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(n-3)!}\right) \\ &\quad \times\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{(n-2)!} + \frac{1}{(n-1)!}\right) \\ &> (n-3)!(n-2)!(n-2) \\ &\quad \times\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(n-3)!} - \frac{1}{(n-2)!}\right) \\ &\quad \times\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{(n-2)!}\right) \\ &= (\, per\, R_{n-2})^2. \end{aligned}$$

This inequality contradicts the equation (9), and hence C_n is never barycentric for any odd integer n greater than 4.

Case 3. Let n be any even integer greater than 4. From (5), we have

$$\begin{aligned} per\, R_{n-2} - n &= (n-2)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{(n-2)!}\right) - n \\ &> (n-2)!\left(\frac{1}{2!} - \frac{1}{3!}\right) - n \\ &> 0, \qquad (10) \\ per\, R_{n-1} &= (n-1) per\, R_{n-2} - 1, \qquad (11) \\ per\, R_{n-3} &= \frac{per\, R_{n-2} - 1}{n-2}. \qquad (12) \end{aligned}$$

Using (10), (11), and (12), we have

$$\begin{aligned} & per\, R_{n-3} per\, R_{n-1} - (\,per\, R_{n-2})^2 \\ &= \frac{per\, R_{n-2} - 1}{n-2}\{(n-1) per\, R_{n-2} - 1\} - (\,per\, R_{n-2})^2 \\ &= \frac{n-1}{n-2}(\,per\, R_{n-2})^2 - \frac{n}{n-2} per\, R_{n-2} + \frac{1}{n-2} - (\,per\, R_{n-2})^2 \\ &= \frac{1}{n-2} per\, R_{n-2}(\,per\, R_{n-2} - n) + \frac{1}{n-2} \\ &> 0. \end{aligned}$$

This implies (9) does not hold for this case. That is, C_n is never barycentric for any even integer n greater than 4. ■

We remark that C_3 is also not barycentric, because $\frac{1}{2}R_3$ is the unique minimizing matrix on $\Omega(C_3)$ (by Theorem 5 in [1]) and $b(C_3)$ is

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

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