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Weighted inequalities and pointwise estimates for the multilinear fractional integral and maximal operators ☆

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In this article we prove weighted norm inequalities and pointwise estimates between the multilinear fractional integral operator and the multilinear fractional maximal. As a consequence of these estimations we obtain weighted weak and strong inequalities for the multilinear fractional maximal operator or function. In particular, we extend some results given in Carro et al. (2005) [7] to the multilinear context. On the other hand we prove weighted pointwise estimates between the multilinear fractional maximal operator $\mathcal{M}_{\alpha,B}$ associated to a Young function B and the multilinear maximal operators $\mathcal{M}_{\psi}=\mathcal{M}_{0,\psi}$, $\psi(t)=B(t^{1-\alpha/(nm)})^{nm/(nm-\alpha)}$. As an application of these estimate we obtain a direct proof of the L^p-L^q boundedness results of $\mathcal{M}_{\alpha,B}$ for the case $B(t)=t$ and $B_k(t)=t(1+\log^+t)^k$ when $1/q=1/p-\alpha/n$. We also give sufficient conditions on the weights involved in the boundedness results of $\mathcal{M}_{\alpha,B}$ that generalizes those given in Moen (2009) [22] for $B(t)=t$. Finally, we prove some boundedness results in Banach function spaces for a generalized version of the multilinear fractional maximal operator.

1. Introduction and preliminaries

An important problem in Analysis is to control certain integral type operators by means of adequate maximal operators. This control is sometimes understood in the norm of the spaces where these operators act. For example, an interesting result due to Coifman [4] establishes that, if T is a Calderón–Zygmund integral operator, M is the Hardy–Littlewood maximal function and $0<p<\infty$, then the inequality

$$\int_{\mathbb{R}^n} |T(f)(x)|^p\,dx \leqslant C\int_{\mathbb{R}^n} |Mf(x)|^p\,dx$$

holds for some positive constant C. Thus, the maximal function M controls the singular integral in L^p-norm and the boundedness properties of M in L^p-spaces give the boundedness properties of T. The weighted version for A_∞ weights of inequality above is also true (see [5]).

For the fractional integral operator I_α, $0<\alpha<n$, $w\in A_\infty$ and $0<p<\infty$, Muckenhoupt and Wheeden [23] proved the following control-type inequalities involving the fractional maximal operator M_α

$$\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^p w(x)\,dx \leqslant C\int_{\mathbb{R}^n} |M_\alpha f(x)|^p w(x)\,dx,$$

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and

$$\sup_{\lambda>0}\lambda^q w\big(\{I_\alpha f>\lambda\}\big)\leqslant C\sup_{\lambda>0}\lambda^q w\big(\{M_\alpha f>\lambda\}\big),$$

where C depends on the A_∞-constant of w. Then, by the weighted boundedness results of M_α, they obtained the corresponding weighted boundedness results for I_α.

Similar problems for other operators such that commutators of singular and fractional integral operators, nonlinear commutators, potential operators, multilinear Calderón–Zygmund operators and multilinear fractional integrals, were studied by several authors (see, for example, [28–30,8,2,7,21,22,10]). Particularly, in [7], the authors obtain the boundedness of the fractional integral operator in term of the fractional maximal operator in weighted weak L^1-spaces, and then, by the weighted weak boundedness of M_α, they obtain weighted weak estimates for I_α.

Related to the control of commutators of singular and fractional integral operators appear the iterations of the Hardy–Littlewood maximal operator M and the composition of the fractional maximal operator with iterations of M. These types of maximal operators were proved to be equivalent to certain maximal operators associated to a given Young function (see, for example, [28,29,8,2]). Then, the study of the boundedness properties of these particular maximal operators seem to be an important tool because they enclose information about the behavior of the commutators that they control.

In the multilinear context, there were an increasing interest in investigate how to control integral operators by maximal functions. In [17] the authors proved that the multilinear Calderón–Zygmund operators are controlled in L^p-norms by the product of m Hardy–Littlewood maximal operators and they asked themselves if this product is optimal in some sense. This problem is then solved in [21], where the authors give a strictly smaller maximal operator and develop a corresponding weighted theory.

Later, in [22], a complete study of the weighted boundedness properties for the multilinear fractional integral operator is given, and the author proved that this operator is bounded in norm by the corresponding version of the fractional maximal operator which generalizes the maximal operator given in [21]. Again, the boundedness properties of the "maximal controller" gives the boundedness properties of the "controlled operator."

Pointwise estimates between operators are also of interest because they allow us to obtain boundedness properties of a given operator by means of the properties of others. For example, related to the fractional maximal operator and the Hardy–Littlewood maximal operator a pointwise estimate is given in [3]. Other known pointwise estimates between the fractional integral operator and maximal operators are due to Welland and Hedberg (see [32] and [18]).

In this paper we give "control type results" for the multilinear fractional maximal and integral operators. These results involved pointwise estimates and norm estimates between these operators, of the type described above. In particular, we extend some results given in [7] to the multilinear context. On the other hand we introduce the multilinear fractional maximal operator $\mathcal{M}_{\alpha,B}$ associated to a Young function B and we prove weighted pointwise estimates between these operators and the multilinear maximal operators $\mathcal{M}_\psi=\mathcal{M}_{0,\psi}$, where ψ is a given Young function that depends on B. As an application of these estimates we obtain a direct proof of the L^p–L^q boundedness results of $\mathcal{M}_{\alpha,B}$ for the case $B(t)=t$ and $B_k(t)=t(1+\log^+t)^k$ when $1/q=1/p-\alpha/n$. We also give sufficient conditions on the weights involved in the boundedness results of $\mathcal{M}_{\alpha,B}$ that generalizes those given in [22] for $B(t)=t$. The importance of a weighted theory for this maximal function is due to the fact that this operators are in intimate relation with the commutators of multisublinear fractional integral operators, as we shall see in a next paper.

On the other hand, we study boundedness results in Banach function spaces for a generalized version of the multilinear fractional maximal operator involving certain essentially nondecreasing function φ.

The paper is organized as follows. In Section 2 we state the main results of this article. We also include some corollaries and different proofs of results proved in [22]. The proof of the main results are in Section 4. In Section 3 we give some auxiliary lemmas and finally, in Section 5 we define a generalized version of the multilinear fractional maximal operator and we give some boundedness estimates in the setting of Banach function spaces.

Before stating the main results of this article, we give some standard notation. Throughout this paper Q will denote a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. With $\mathcal{D}$ we will denote the family of dyadic cubes in $\mathbb{R}^n$.

By a weight we understand a nonnegative measurable function.

We say that a weight w satisfies a Reverse Hölder's inequality with exponent s, $RH(s)$, if there exists a positive constant C such that

$$\left(\frac{1}{|Q|}\int_Q w^s\right)^{1/s}\leqslant\frac{w(Q)}{|Q|}.$$

By RH_∞ we mean the class of weights w such that the inequality

$$\sup_{x\in Q}w(x)\leqslant\frac{C}{|Q|}\int_Q w,$$

holds for every $Q\subset\mathbb{R}^n$ and some positive constant C. It is easy to check that $RH_\infty\subset A_\infty$.

Now we summarize a few facts about Orlicz spaces. For more information see [19] or [31].

We say that $B:[0,\infty)\to[0,\infty)$ is a Young function if there exists a nontrivial, nonnegative and increasing function b such that $B(t)=\int_0^t b(s)\,ds$. Then B is continuous, convex, increasing and satisfies $B(0)=0$ and $\lim_{t\to\infty}B(t)=\infty$. Moreover, it follows that $B(t)/t$ is increasing.

Let $B:[0,\infty)\to[0,\infty)$ be a Young function. The Orlicz space $L_B=L_B(\mathbb{R}^n)$ consists of all measurable functions f such that for some $\lambda>0$,

$$\int_{\mathbb{R}^n} B\big(|f|/\lambda\big)<\infty.$$

The space L_B is a Banach space endowed with the Luxemburg norm

$$\|f\|_B=\|f\|_{L_B}=\inf\Big\{\lambda>0\colon \int_{\mathbb{R}^n} B\big(|f|/\lambda\big)<\infty\Big\}.$$

The B-average of a function f over a cube Q is defined by

$$\|f\|_{B,Q}=\inf\Big\{\lambda>0\colon \frac{1}{|Q|}\int_Q B\big(|f|/\lambda\big)\leqslant 1\Big\}.$$

When $B(t)=t$, $\|f\|_{B,Q}=\frac{1}{|Q|}\int_Q|f|$.

We shall say that B is doubling if there exists a positive constant C such that $B(2t)\leqslant CB(t)$ for every $t\geqslant 0$. Each Young function B has an associated complementary Young function $\tilde{B}$ satisfying

$$t\leqslant B^{-1}(t)\tilde{B}^{-1}(t)\leqslant 2t,$$

for all $t>0$. There is a generalization of Hölder's inequality

$$\frac{1}{|Q|}\int_Q|fg|\leqslant\|f\|_{B,Q}\|g\|_{\tilde{B},Q}. \tag{1.1}$$

A further generalization of Hölder's inequality (see [24]) is the following: If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are Young functions such that

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t)\leqslant\mathcal{C}^{-1}(t),$$

then

$$\|fg\|_{\mathcal{C},Q}\leqslant 2\|f\|_{\mathcal{A},Q}\|g\|_{\mathcal{B},Q}.$$

Definition 1.2. Let $0<\alpha<nm$ and $\vec{f}=(f_1,\dots,f_m)$. We define the multilinear fractional maximal operator $\mathcal{M}_{\alpha,B}$ associated to a Young function B by

$$\mathcal{M}_{\alpha,B}\vec{f}(x)=\sup_{Q\ni x}|Q|^{\alpha/n}\prod_{i=1}^m\|f_i\|_{B,Q} \tag{1.3}$$

where the supremum is taken over all cubes Q containing x.

Even though $\mathcal{M}_{\alpha,B}$ is sublinear in each entry, we shall refer to it as the multilinear fractional maximal operator.

For $\alpha=0$ we denote $\mathcal{M}_{0,B}=\mathcal{M}_B$. When $B(t)=t$, $\mathcal{M}_\alpha=\mathcal{M}_{\alpha,B}$ is the multilinear fractional maximal operator defined in [22] by

$$\mathcal{M}_\alpha\vec{f}(x)=\sup_{Q\ni x}|Q|^{\alpha/n}\prod_{i=1}^m\frac{1}{|Q|}\int_Q|f_i|. \tag{1.4}$$

$\mathcal{M}_0=\mathcal{M}$ is the multilinear maximal operator defined in [21]. When $m=1$ we write M and M_α to denote the Hardy–Littlewood and the fractional maximal operators defined, for a locally integrable function f and $0<\alpha<n$, by

$$Mf(x)=\sup_{Q\in x}\frac{1}{|Q|}\int_Q\big|f(y)\big|\,dy \tag{1.5}$$

and

$$M_\alpha f(x)=\sup_{Q\in x}\frac{1}{|Q|^{1-\alpha/n}}\int_Q\big|f(y)\big|\,dy \tag{1.6}$$

respectively.

If we take $g \equiv 1$ in inequality (1.1) it follows that for every Young function B, every α such that $0 \leqslant \alpha < nm$, the inequality

$$\mathcal{M}_\alpha(\vec{f})(x) \leqslant \mathcal{M}_{\alpha,B}(\vec{f})(x)$$

holds for every $x \in \mathbb{R}^n$.

The following class of weights was introduced in [21] and is a generalization of the Muckenhoupt A_p classes, $p > 1$. We use the notation $\vec{P} = (p_1, \ldots, p_m)$.

Definition 1.7. Let $1 \leqslant p_i < \infty$ for $i = 1, \ldots, m$, $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. For each $i = 1, \ldots, m$ let w_i be a weight and $\vec{w} = (w_1, \ldots, w_m)$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \left(\prod_{i=1}^m w_i^{p/p_i} \right) \right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{1/p_i'} < \infty. \tag{1.8}$$

When $p_i = 1$, $(\frac{1}{|Q|} \int_Q w_i^{1-p_i'})^{1/p_i'}$ is understood as $(\inf_Q w_i)^{-1}$.

Condition (1.8) is called the multilinear $A_{\vec{P}}$ condition.

2. Statement of the main results

In this section we establish the main results of this article. For a sake of completeness we consider four subsections.

2.1. Pointwise estimates for $\mathcal{M}_{\alpha,B}$

For $0 < \alpha < nm$ let B be a Young function such that $t^{\frac{\alpha}{nm}} B^{-1}(t^{1-\frac{\alpha}{nm}}) \leqslant B^{-1}(t)$. Let ψ be the function defined by $\psi(t) = B(t^{1-\alpha/(nm)})^{nm/(nm-\alpha)}$. From Lemma 3.1 below, ψ is a Young function. The following result gives a pointwise estimate between the multilinear fractional maximal associated to the Young function B, $\mathcal{M}_{\alpha,B}$ and the multilinear maximal operator $\mathcal{M}_\psi$ associated to the Young function ψ, and is an useful tool to obtain boundedness results for $\mathcal{M}_{\alpha,B}$.

Lemma 2.1. *Let $0 < \alpha < nm$. Let B be a Young function such that*

$$t^{\frac{\alpha}{nm}} B^{-1}\big(t^{1-\frac{\alpha}{nm}}\big) \leqslant B^{-1}(t) \tag{2.2}$$

and $\psi(t) = B(t^{1-\alpha/(nm)})^{nm/(nm-\alpha)}$.

For each $i = 1, \ldots, m$, let p_i, q_i and s_i be the real numbers defined, respectively, by $1 \leqslant p_i < nm/\alpha$, $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{nm}$ and $s_i = (1 - \alpha/(nm))q_i$ and p, q and s be the real numbers given by $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$, $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$ and $\frac{1}{s} = \sum_{i=1}^m \frac{1}{s_i}$. Let $w_1, \ldots, w_m$ be m weights. If $\vec{f}_w = (f_1/w_1, \ldots, f_m/w_m)$ and $\vec{g} = (f_1^{p_1/s_1} w_1^{-q_1/s_1}, \ldots, f_m^{p_m/s_m} w_m^{-q_m/s_m})$ then

$$\mathcal{M}_{\alpha,B} \vec{f}_w(x) \leqslant \mathcal{M}_\psi \vec{g}(x)^{1-\alpha/(nm)} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{p_i} \right)^{\frac{\alpha}{nm}}. \tag{2.3}$$

Remark 2.4. When $B(t) = t$ we have that $\psi(t) = t$. Then, from inequality (2.3), we get the following pointwise estimate between the multilinear fractional maximal operator $\mathcal{M}_\alpha$ and the multilinear maximal operator $\mathcal{M}$

$$\mathcal{M}_\alpha \vec{f}_w(x) \leqslant \mathcal{M}\vec{g}(x)^{1-\alpha/(nm)} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{p_i} \right)^{\frac{\alpha}{nm}}. \tag{2.5}$$

In the case $m = 1$ the result above was obtained in [14].

Remark 2.6. For $0 < \alpha < nm$ and $k \in \mathbb{N}$ let B_k be the Young function defined by $B_k(t) = t(1 + \log^+ t)^k$. Then B_k satisfies (2.2). Let $\psi_k(t) = B_k(t^{1-\alpha/(nm)})^{nm/(nm-\alpha)} \simeq t(1 + \log^+ t)^{knm/(nm-\alpha)}$. From the lemma above we get the following pointwise estimate

$$\mathcal{M}_{\alpha, L(\log L)^k} \vec{f}_w(x) \leqslant \mathcal{M}_{L(\log L)^{\frac{knm}{nm-\alpha}}} \vec{g}(x)^{1-\alpha/(nm)} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{p_i} \right)^{\frac{\alpha}{nm}}. \tag{2.7}$$

2.2. Weighted boundedness results for $\mathcal{M}_{\alpha,B}$

As an easy consequence of inequality (2.5) and the weighted boundedness results for the multilinear maximal operator $\mathcal{M}$ proved in [21] we obtain a direct proof of the weighted weak and strong boundedness of the multilinear fractional maximal operator $\mathcal{M}_\alpha$ proved in [22], when p and q satisfy $1/q = 1/p - \alpha/n$ and $1 < p_i < nm/\alpha$, $i = 1, \dots, m$. Actually, in [22] the author proves that the conditions on the weights are also necessary (see Theorems 2.7 and 3.6 in [22] applied to this case). These results are given in the following two theorems.

Theorem 2.8. *Let $0 < \alpha < nm$ and let p_i and q be defined as in Lemma 2.1. Let $\vec{f} = (f_1, \dots, f_m)$. If $(u, \vec{w})$ satisfy*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{1/p_i'} < \infty \tag{2.9}$$

then

$$\|\mathcal{M}_\alpha \vec{f}\|_{L^{q,\infty}(u)} \leqslant C \prod_{i=1}^m \|f_i w_i\|_{L^{p_i}}.$$

Theorem 2.10. *Let $0 < \alpha < nm$ and let $1 < p_i < nm/\alpha$ and q_i, s_i, p, q, and s be defined as in Lemma 2.1. Let $\vec{w^q} = (w_1^{q_1}, \dots, w_m^{q_m})$. If $\vec{f} = (f_1, \dots, f_m)$, $\vec{S} = (s_1, \dots, s_m)$ and $\vec{w^q} \in A_{\vec{S}}$, then*

$$\left\|\mathcal{M}_\alpha \vec{f}\left(\prod_{i=1}^m w_i\right)\right\|_{L^q} \leqslant C \prod_{i=1}^m \|f_i w_i\|_{L^{p_i}}.$$

Remark 2.11. It is easy to check that $\vec{w^q} \in A_{\vec{S}}$ if and only if $\vec{w} = (w_1, \dots, w_m)$ belongs to the $A_{\vec{P},q}$ classes introduced in [22]. This equivalence is a generalization to the multilinear case of that proved by Muckenhoupt and Wheeden in the linear case, which establishes that a weight $w \in A_{p,q}$ if and only if $w^q \in A_s$ with $1 \leqslant p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $s = 1 + q/p'$. For more details see [23].

The following corollary is a consequence of Theorem 2.8 applied to the weights $u = \prod_{i=1}^m u_i^{q/q_i}$ and $w_i = M(u_i)^{1/q_i}$, where M is the Hardy–Littlewood maximal operator.

Corollary 2.12. *Let $0 < \alpha < nm$ and let p_i, q_i, s_i, p, q, and s be defined as in Lemma 2.1. Let $\vec{f} = (f_1, \dots, f_m)$ and $u = \prod_{i=1}^m u_i^{q/q_i}$. Then*

$$\|\mathcal{M}_\alpha \vec{f}\|_{L^{q,\infty}(u)} \leqslant C \prod_{i=1}^m \left\| f_i M(u_i)^{1/q_i} \right\|_{L^{p_i}}.$$

From the weak and strong characterizations obtained in [22, Theorems 2.7 and 2.8] applied to the case $p = q$, we obtain the following result.

Theorem 2.13. *Suppose that $0 < \alpha < nm$, $1 \leqslant p_1, \dots, p_m < mn/\alpha$ and $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. Let $u = \prod_{i=1}^m u_i^{p/p_i}$ and $v = \prod_{i=1}^m v_i^{1/p_i}$. Then*

$$\|\mathcal{M}_\alpha \vec{f}\|_{L^{p,\infty}(u)} \leqslant C \prod_{i=1}^m \|f_i\|_{L^{p_i}(M_{\alpha p_i/m}(u_i))},$$

and

$$\|\mathcal{M}_\alpha \vec{f} v\|_{L^p} \leqslant C \prod_{i=1}^m \left\| f_i M_{\alpha p_i/m}(v_i) \right\|_{L^{p_i}},$$

where $M_{\alpha p_i/m}$ denotes the fractional maximal operator defined in (1.6) with α replaced by $\alpha p_i/m$.

The proof of the first inequality above follows from the fact that the weights u and $w_i = M_{\alpha p_i/m}(u_i)$ satisfy the condition on the weights in [22, Theorem 2.7]. On the other hand, the weights v and $w_i = M_{\alpha p_i/m}(v_i)^{1/p_i}$ satisfy the hypotheses in [22, Theorem 2.8] and thus we obtain the second inequality.

Before state the next result, we introduce the following class of Young functions related to the boundedness of the sublinear maximal M_B between Lebesgue spaces. For more information see [25].

Definition 2.14. Let $1 < r < \infty$. A Young function B is said to satisfy the B_r condition if for some constant $c > 0$,

$$\int_c^\infty \frac{B(t)}{t^r}\frac{dt}{t} < \infty.$$

Theorem 2.15. *Let $0 \leqslant \alpha < nm$, $1 < p_i < \infty$, $i = 1, \dots, m$, $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. Let q be a real number such that $1/m < p \leqslant q < \infty$. Let B, A_i, and C_i, $i = 1, \dots, m$, be Young functions such that $A_i^{-1}(t)C_i^{-1}(t) \leqslant B^{-1}(t)$, $t > 0$ and C_i is doubling and satisfies the B_{p_i} condition for every $i = 1, \dots, m$. Let $(\nu, \vec{w})$ be weights that satisfy*

$$\sup_Q |Q|^{\alpha/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q \nu^q\right)^{1/q}\prod_{i=1}^m \|w_i^{-1}\|_{A_i,Q} < \infty. \tag{2.16}$$

Then

$$\|\mathcal{M}_{\alpha,B}\vec{f}\nu\|_{L^q} \leqslant C\prod_{i=1}^m \|f_i w_i\|_{L^{p_i}}$$

holds for every $\vec{f} \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$.

Remark 2.17. The linear case of theorem above was proved in [8], and in [9] for the case $\alpha = 0$ and $p = q$. For $B(t) = t$, Theorem 2.15 gives two weighted results proved in [22] for the multilinear fractional maximal operator $\mathcal{M}_\alpha$. The first one [22, Theorem 2.8] is obtained by considering $A_i = t^{rp_i'}$ and $C_i = t^{(rp_i')'}$ for some $r > 1$ and the second [22, Theorem 2.10] is obtained by taking $A_i = t^{p_i'}(1 + \log^+ t)^{p_i'-1+\delta}$ and $C_i = \frac{t^{p_i}}{(1+\log^+ t)^{1+\delta(p_i-1)}}$ for $\delta > 0$.

As a consequence of Theorem 2.15 and the pointwise estimate given in (2.7), we obtain the following result about the boundedness of $\mathcal{M}_{\alpha,B_k}$ for multilinear weights in the $A_{\vec{S}}$ class defined above, where $\vec{S} = (s_1, \dots, s_m)$ and $B_k(t) = t(1 + \log^+ t)^k$. In the proof, we also use the pointwise estimate given in (2.7).

Corollary 2.18. *Let $0 \leqslant \alpha < nm$ and let p_i, p, q_i, q, s_i and s be defined as in Lemma 2.1. For each $k \in \mathbb{N}$ let $B_k(t) = t(1 + \log^+ t)^k$. Let $\vec{w}^q = (w_1^{q_1}, \dots, w_m^{q_m})$. If $\vec{f} = (f_1, \dots, f_m)$ and $\vec{S} = (s_1, \dots, s_m)$ then the inequality*

$$\left\|\mathcal{M}_{\alpha,B_k}\vec{f}\left(\prod_{i=1}^m w_i\right)\right\|_{L^q} \leqslant C\prod_{i=1}^m \|f_i w_i\|_{L^{p_i}}$$

holds for every $\vec{f}$ if and only if $\vec{w}^q$ satisfies the $A_{\vec{S}}$ condition.

2.3. Weighted weak type inequalities for the multilinear fractional integral operator

In this section we obtain weighted estimates for the multilinear fractional maximal and integral operator.

The following definition of the multilinear fractional integral operator was considered by several authors (see, for example, [15,20,16,22]).

Definition 2.19. Let $0 < \alpha < nm$ and $\vec{f} = (f_1, \dots, f_m)$. The multilinear fractional integral is defined by

$$\mathcal{I}_\alpha\vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}}\,d\vec{y},$$

where the integral in convergent if $\vec{f} \in \mathcal{S} \times \cdots \times \mathcal{S}$.

Particularly, we study weighted weak type inequalities for the multilinear fractional maximal and integral operator. For the first one we obtain the following result.

Theorem 2.20. *Let $0 \leqslant \alpha < nm$, $\vec{w} = (w_1, \dots, w_m)$ and $u = \prod_{i=1}^m w_i^{1/m}$. Then*

$$u\left(\left\{x \in \mathbb{R}^n \colon \mathcal{M}_\alpha\vec{f}(x) > \lambda^m\right\}\right)^m \leqslant C\prod_{i=1}^m \int_{\mathbb{R}^n} \frac{|f_i|}{\lambda} M_{\alpha/m}w_i,$$

where $M_{\alpha/m}$ denotes the fractional maximal operator of order α/m defined in (1.6).

The case $\alpha = 0$ of the theorem above was proved in [21]. For $m = 1$ this is a well known result proved in [11].

In [7] the authors considered the problem of finding weights W such that

$$w\left(\left\{x \in \mathbb{R}^n:\ \left|I_\alpha f(x)\right| > \lambda\right\}\right) \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} \left|f(x)\right| W(x)\,dx$$

for a given weight w, for every $\lambda > 0$ and for suitable functions f. Particularly, they obtain that the weight $W = M_\alpha(M_{L(\log L)^\delta} w)$, $\delta > 0$, works. Motivated from the linear case, we study an analogous problem in the multilinear context and we obtain the following result.

Theorem 2.21. *Let $0 < \alpha < nm$, $\delta > 0$ and $u = \prod_{i=1}^m w_i^{1/m}$. Then*

$$\|\mathcal{I}_\alpha \vec{f}\|_{L^{1/m,\infty}(u)} \leqslant C \prod_{i=1}^m \int_{\mathbb{R}^n} |f_i| M_{\alpha/m} M_{L(\log L)^\delta}(w_i) \tag{2.22}$$

and, in particular

$$\|\mathcal{I}_\alpha \vec{f}\|_{L^{1/m,\infty}(u)} \leqslant C \prod_{i=1}^m \int_{\mathbb{R}^n} |f_i| M_{\alpha/m} M^2(w_i).$$

The result above is an immediate consequence of the next theorem.

Theorem 2.23. *Let $0 < \alpha < nm$, $\delta > 0$ and let u be a weight. Then*

$$\|\mathcal{I}_\alpha \vec{f}\|_{L^{1/m,\infty}(u)} \leqslant C \|\mathcal{M}_\alpha \vec{f}\|_{L^{1/m,\infty}(M_{L(\log L)^\delta}(u))}.$$

Then, the proof of (2.22) follows by observing that

$$M_{L(\log L)^\delta}(u) = M_{L(\log L)^\delta}\left(\prod_{i=1}^m w_i^{1/m}\right) \leqslant \prod_{i=1}^m M_{L(\log L)^\delta}(w_i)^{1/m},$$

which is a consequence of the generalized Hölder's inequality in Orlicz spaces. Then, an application of Theorem 2.20 gives the desired result.

Recall that a weight v satisfies the RH_∞ condition if there exists a positive constant C such that the inequality

$$\sup_{x \in Q} v(x) \leqslant \frac{C}{|Q|} \int_Q v$$

holds for every $Q \subset \mathbb{R}^n$.

Lemma 2.24. *Let $0 < \alpha < nm$. Let v be a weight satisfying the RH_∞ condition. Then, there exists a positive constant C such that, if $u = \prod_{i=1}^m w_i^{1/m}$ and $\vec{f} = (f_1, \dots, f_m)$,*

$$\int_{\mathbb{R}^n} \mathcal{I}_\alpha \vec{f}(x) u(x) v(x)\,dx \leqslant C \int_{\mathbb{R}^n} \mathcal{M}_\alpha \vec{f}(x) M u(x) v(x)\,dx,$$

where M is the Hardy–Littlewood maximal function defined in (1.5)*.*

The following theorem establish some kind of control of the multilinear fractional integral operator by the multilinear fractional maximal in L^p, $0 < p \leqslant 1$.

Theorem 2.25. *Let $0 < p \leqslant 1$ and let u be a weight. Then*

$$\int_{\mathbb{R}^n} \left|\mathcal{I}_\alpha \vec{f}(x)\right|^p u(x)\,dx \leqslant C \int_{\mathbb{R}^n} \left|\mathcal{M}_\alpha \vec{f}(x)\right|^p M u(x)\,dx.$$

In the linear case, Lemma 2.24 and Theorem 2.25 were proved in [7].

2.4. Pointwise estimates between $\mathcal{I}_\alpha$ and $\mathcal{M}_\alpha$

A pointwise estimate relating both, the multilinear fractional and maximal operators is given in the next result.

Theorem 2.26 *(Welland's type inequality). Let $0 < \alpha < nm$ and $0 < \epsilon < \min\{\alpha, nm - \alpha\}$. Then, if $\vec{f} = (f_1, \ldots, f_m)$ where f_i's are bounded functions with compact support, then*

$$\left|\mathcal{I}_\alpha \vec{f}(x)\right| \leqslant C\big(\mathcal{M}_{\alpha+\epsilon}\vec{f}(x)\mathcal{M}_{\alpha-\epsilon}\vec{f}(x)\big)^{1/2},$$

where C depends only on n, m, α and ϵ.

The inequality in the theorem above was proved in [10] for multilinear fractional integral and maximal operators with m homogeneous kernels in order to obtain weighted boundedness results for the first operator. For a sake of completeness we include the proof in Section 4.

In [22], the author proves the following result.

Theorem 2.27. *(See [22, Theorem 2.2].) Suppose that $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$ and q is a number that satisfies $1/m < p \leqslant q < \infty$. Suppose that one of the two following conditions holds.*

(i) *$q > 1$ and $(\nu, \vec{w})$ are weights that satisfy*

$$\sup_Q |Q|^{\alpha/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q \nu^{qr}\right)^{1/(qr)}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q w_i^{-p_i'r}\right)^{1/(p_i'r)} < \infty$$

for some $r > 1$.

(ii) *$q \leqslant 1$ and $(\nu, \vec{w})$ are weights that satisfy*

$$\sup_Q |Q|^{\alpha/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q \nu^{q}\right)^{1/q}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q w_i^{-p_i'r}\right)^{1/(p_i'r)} < \infty$$

for some $r > 1$.

Then the inequality

$$\|\mathcal{I}_\alpha \vec{f}\nu\|_q \leqslant C\prod_{i=1}^m \|f_i w_i\|_{p_i}$$

holds for every $\vec{f} \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$.

A direct proof of theorem above for the case $q > 1$ can be given combining Theorem 2.26 with Theorem 2.15 applied to the case $A_i(t) = t^{rp_i'}$, and proceeding as in the corresponding result in [12, Theorem 6.5].

3. Auxiliary results

In this section we give some technical lemmas used in the proof of the main results in this paper.

Lemma 3.1. *Let B be a Young function and $0 < \gamma < 1$. Then $\psi(t) = B(t^\gamma)^{1/\gamma}$ is a Young function.*

Proof. It is enough to prove that there exists a nontrivial, nonnegative and increasing function g such that $\psi(t) = \int_0^t g(s)\,ds$. This function g is given by $g(s) = b(s^\gamma)(\frac{B(s^\gamma)}{s^\gamma})^{(1/\gamma)-1}$, where b is a nonnegative and increasing function such that $B(t) = \int_0^t b(s)\,ds$. The function g has the desired properties. □

The next lemma establishes the relation between the dyadic a nondyadic multilinear fractional maximal operators. Let $\mathcal{M}_{\alpha,B}^k$ be defined as $\mathcal{M}_{\alpha,B}$ but over cubes with side length less or equal than 2^k, $Q_k = Q(0, 2^{k+2})$, $\tau_t g(x) = g(x-t)$ and $\vec{\tau}_t(\vec{f}) = (\tau_t f_1, \ldots, \tau_t f_m)$.

Lemma 3.2. *For each k, $\vec{f}$ and every $x \in \mathbb{R}^n$ and $0 < q < \infty$, there exists a constant C, depending only on n, m, α and q such that*

$$\mathcal{M}_{\alpha,B}^k(\vec{f})(x)^q \leqslant \frac{C}{|Q_k|}\int_{Q_k}\big(\tau_{-t}\circ\mathcal{M}_{\alpha,B}^d\circ\vec{\tau}_t\big)(\vec{f})(x)^q\,dt. \tag{3.3}$$

For the linear case and $\alpha = 0$ this result was proved by Fefferman and Stein in [11] and can be also found in [13]. In the multilinear context and $\alpha = 0$ the result above is given in [21], and for $B(t) = t$ and $\alpha > 0$, in [22]. The proof of Lemma 3.2 is an easy modification of any of the mentioned results and we omit it.

In order to prove Theorem 2.23 we need the following results. The first of them was proved in [22] for the multilinear integral operator. For the linear case, a proof can be found in [26].

Lemma 3.4. *(See [22].) Let g and f_i, $i = 1, \dots, m$, be positive functions with compact support and let u be a weight. Then there exists a family of dyadic cubes $\{Q_{k,j}\}$ and a family of pairwise disjoint subsets $\{E_{k,j}\}$, $E_{k,j} \subset Q_{k,j}$ with*

$$|Q_{k,j}| \leqslant C|E_{k,j}|$$

for some positive constant C and for every k, j and such that

$$\int\limits_{\mathbb{R}^n} \mathcal{I}_\alpha \vec{f}(x)u(x)g(x)\,dx \leqslant C\sum_{k,j}|Q_{k,j}|^{\alpha/n}\left(\frac{1}{|Q_{k,j}|}\int\limits_{Q_{k,j}} u(x)g(x)\,dx\right)\left(\prod_{i=1}^m \frac{1}{|3Q_{k,j}|}\int\limits_{3Q_{k,j}} f_i(y_i)\,dy_i\right)|E_{k,j}|. \tag{3.5}$$

The following lemma was proved in [6] and gives examples of weights in the RH_∞ class.

Lemma 3.6. *Let g be any function such that Mg is finite a.e. Then $(Mg)^{-\alpha} \in RH_\infty$, $\alpha > 0$.*

4. Proofs

Proof of Lemma 2.1. The proof is based in some ideas from Lemma 2.8 in [14]. Let g_i be a function such that $g_i^{s_i} w_i^{q_i} = f_i^{p_i}$. Then $f_i/w_i = g_i^{s_i/p_i} w_i^{q_i/p_i - 1} = g_i^{s_i/p_i + \alpha/(nm) - 1} g_i^{1-\alpha/(nm)} w_i^{(q_i/p_i - 1)}$. Let $r = nm/(nm - \alpha)$ and $r' = nm/\alpha$. If s and s_i are defined as in the hypotheses of the theorem we get

$$\left(\frac{q_i}{p_i} - 1\right)r' = \left(\frac{q_i}{p_i} - 1\right)\frac{nm}{\alpha} = q_i \tag{4.1}$$

and

$$\begin{aligned}
\left(\frac{s_i}{p_i} + \frac{\alpha}{nm} - 1\right)r' &= \left(\frac{s_i}{p_i} + \frac{\alpha}{nm} - 1\right)\frac{nm}{\alpha} \\
&= \left(\left(1 - \frac{\alpha}{nm}\right)\frac{q_i}{p_i} + \frac{\alpha}{nm} - 1\right)\frac{nm}{\alpha} \\
&= \left(1 - \frac{\alpha}{nm}\right)\left(\frac{q_i}{p_i} - 1\right)\frac{nm}{\alpha} \\
&= \left(1 - \frac{\alpha}{nm}\right)q_i \\
&= s_i.
\end{aligned} \tag{4.2}$$

Let B and ψ be the functions in the hypotheses of the theorem. From Lemma 3.1 ψ is a Young function. Let $\phi(t) = B(t)^{nm/(nm-\alpha)}$. Then, by the properties of the function B we obtain

$$\phi^{-1}(t)\,t^{\alpha/nm} \leqslant CB^{-1}(t).$$

By applying Hölder's inequality, and using (4.1) and (4.2) we obtain that

$$\begin{aligned}
\|f_i/w_i\|_{B,Q} &= \left\|g_i^{s_i/p_i} w_i^{q_i/p_i - 1}\right\|_{B,Q} \\
&= \left\|g_i^{1-\alpha/nm} g_i^{s_i/p_i + \alpha/nm - 1} w_i^{q_i/p_i - 1}\right\|_{B,Q} \\
&\leqslant \left\|g_i^{1-\alpha/nm}\right\|_{\phi,Q} \left\|g_i^{s_i/p_i + \alpha/nm - 1} w_i^{q_i/p_i - 1}\right\|_{nm/\alpha,Q} \\
&= \frac{1}{|Q|^{\alpha/nm}}\|g_i\|_{\psi,Q}^{1-\alpha/nm}\|f_i\|_{p_i}^{\alpha p_i/nm},
\end{aligned}$$

where we have used the fact that $\|g_i^{1-\alpha/nm}\|_{\phi,Q} = \|g_i\|_{\psi,Q}^{1-\alpha/nm}$. Then

$$|Q|^{\alpha/n}\prod_{i=1}^m \|f_i/w_i\|_{B,Q} \leqslant \prod_{i=1}^m \|g_i\|_{\psi,Q}^{1-\alpha/(nm)} \prod_{i=1}^m \|f_i\|_{p_i}^{\alpha p_i/(nm)} \leqslant \mathcal{M}_\psi \vec{g}(x)^{1-\alpha/(nm)}\left(\prod_{i=1}^m \|f_i\|_{p_i}^{p_i}\right)^{\alpha/(nm)},$$

and inequality (2.3) follows by taking supremum over the cubes Q in $\mathbb{R}^n$. $\square$

Proof of Theorem 2.8. We use the same notation as in the proof of Lemma 2.1. Thus, it is enough to prove that

$$\|\mathcal{M}_\alpha \vec{f}_w\|_{L^{q,\infty}(u)} \leqslant C \prod_{i=1}^m \|f_i\|_{p_i},$$

and then replace f_i by $f_i w_i$.

From the hypotheses on the weights and raising the quantity in (2.9) to the power $1-\alpha/(nm)$ we obtain that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u\right)^{1/s} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{1/s_i'} < \infty$$

or, equivalently

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u\right)^{1/s} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{q_i(1-s_i')}\right)^{1/s_i'} < \infty. \tag{4.3}$$

By inequality (2.5) and from (4.3) and the weighted weak boundedness result for $\mathcal{M}$ proved in [21] we obtain that

$$\begin{aligned}
\|\mathcal{M}_\alpha \vec{f}_w\|_{L^{q,\infty}(u)} &\leqslant C\left(\prod_{i=1}^m \|f_i\|_{p_i}^{p_i}\right)^{\alpha/nm} \|\mathcal{M}g\|_{L^{s,\infty}(u)}^{1-\alpha/nm} \\
&\leqslant C\left(\prod_{i=1}^m \|f_i\|_{p_i}^{p_i}\right)^{\alpha/nm} \left(\prod_{i=1}^m \|g_i\|_{L^{s_i}(w_i^{q_i})}\right)^{1-\alpha/nm} \\
&= C\left(\prod_{i=1}^m \|f_i\|_{p_i}^{p_i}\right)^{\alpha/nm} \left(\prod_{i=1}^m \|f_i\|_{p_i}^{p_i/s_i}\right)^{1-\alpha/nm} \\
&= C\prod_{i=1}^m \|f_i\|_{p_i},
\end{aligned} \tag{4.4}$$

where we have used the fact that $p_i\alpha/(nm) + (p_i/s_i)(1-\alpha/(nm)) = 1$. Thus the proof is done. □

Proof of Theorem 2.10. Let $\nu = \prod_{i=1}^m w_i$. As in the proof above, it is enough to show that

$$\|\mathcal{M}_\alpha \vec{f}_w \nu\|_q \leqslant C \prod_{i=1}^m \|f_i\|_{p_i},$$

but this inequality can be obtained in a similar way to that in (4.4) by replacing $\|\mathcal{M}g\|_{L^{s,\infty}(u)}$ by $\|\mathcal{M}g\|_{L^s(\nu^q)}$ and then using the weighted strong boundedness result proved in [21]. □

Proof of Theorem 2.15. We first consider the dyadic version $\mathcal{M}_{\alpha,B}^d$ of $\mathcal{M}_{\alpha,B}$ defined by

$$\mathcal{M}_{\alpha,B}^d = \sup_{Q\in\mathcal{D}:\ x\in Q} |Q|^{\alpha/n} \prod_{i=1}^m \|f_i\|_{B,Q},$$

where $\mathcal{D}$ denotes the set of dyadic cubes in $\mathbb{R}^n$. Let a be a constant satisfying $a > 2^{mn}$ and for each k let

$$\Omega_k = \left\{x \in \mathbb{R}^n:\ \mathcal{M}_{\alpha,B}^d(\vec{f})(x) > a^k\right\}.$$

It is easy to see that an analogue of the Calderón–Zygmund decomposition in Orlicz spaces holds for $\mathcal{M}_{\alpha,B}^d$ and, therefore there is a family of maximal nonoverlapping dyadic cubes $\{Q_{j,k}\}$ such that $\Omega_k = \bigcup_j Q_{j,k}$ and

$$a^k < |Q_{j,k}|^{\alpha/n} \prod_{i=1}^m \|f_i\|_{B,Q_{k,j}} \leqslant 2^{nm} a^k.$$

Moreover, each $\Omega_{k+1} \subset \Omega_k$ and the sets $E_{k,j} = Q_{k,j} \backslash (Q_{k,j} \cap \Omega_{k+1})$ are disjoint and satisfy

$$|Q_{k,j}| < \beta |E_{k,j}| \tag{4.5}$$

for some $\beta > 1$. Then, by the generalized Hölder's inequality and condition (2.16) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d_{\alpha,B}(\vec{f})^q \nu^q &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}^d_{\alpha,B}(\vec{f})^q \nu^q \\
&\leqslant a^q \sum_k a^{kq} \nu^q(\Omega_k) \\
&\leqslant a^q \sum_{k,j} a^{kq} \nu^q(Q_{k,j}) \\
&\leqslant C \sum_{k,j} \left(|Q_{k,j}|^{\alpha/n} \prod_{i=1}^m \|f_i\|_{B,Q_{k,j}} \right)^q \nu^q(Q_{k,j}) \\
&\leqslant C \sum_{k,j} \left(|Q_{k,j}|^{\alpha/n} \prod_{i=1}^m \|f_i w_i\|_{C_i,Q_{k,j}} \right)^q \left(\prod_{i=1}^m \|w_i^{-1}\|^q_{A_i,Q_{k,j}} \right) \nu^q(Q_{k,j}) \\
&\leqslant C \sum_{k,j} \left(\prod_{i=1}^m \|f_i w_i\|_{C_i,Q_{k,j}} \right)^q |Q_{k,j}|^{q/p}.
\end{aligned}$$

Now, from the fact that $p \leqslant q$ and using (4.5), the multilinear Hölder's inequality and the hypotheses on C_i we obtain that

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} \mathcal{M}^d_{\alpha,B}(\vec{f})^q \, \nu^q \right)^{1/q} &\leqslant C \left(\sum_{k,j} \left(\prod_{i=1}^m \|f_i w_i\|_{C_i,Q_{k,j}} \right)^p |Q_{k,j}| \right)^{1/p} \\
&\leqslant C \left(\sum_{k,j} \left(\prod_{i=1}^m \|f_i w_i\|_{C_i,Q_{k,j}} \right)^p |E_{k,j}| \right)^{1/p} \\
&\leqslant C \prod_{i=1}^m \left(\sum_{k,j} \|f_i w_i\|^{p_i}_{C_i,Q_{k,j}} |E_{k,j}| \right)^{1/p_i} \\
&\leqslant C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} M_{C_i}(f_i w_i)^{p_i} \right)^{1/p_i} \\
&\leqslant \prod_{i=1}^m \|f_i w_i\|_{L^{p_i}}.
\end{aligned}$$

To prove the nondyadic case we use Lemma 3.2. Thus, from (3.3) it follows that

$$\left\| \mathcal{M}_{\alpha,B}(\vec{f})\nu \right\|_q \leqslant \sup_t \left\| \tau_{-t} \circ \mathcal{M}^d_{\alpha,B} \circ \vec{\tau}_t(\vec{f})\nu \right\|_q. \tag{4.6}$$

If the weights $(\nu, \vec{w})$ satisfy condition (2.16), then the weights $(\tau_t(\nu), \vec{\tau}_t \vec{w})$ satisfy the same condition with constant independent of t. Then, applying the dyadic case, we obtain

$$\begin{aligned}
\left\| \left(\tau_{-t} \circ \mathcal{M}^d_{\alpha,B} \circ \vec{\tau}_t\right)(\vec{f})\nu \right\|_q &= \left\| \left(\mathcal{M}^d_{\alpha,B} \circ \vec{\tau}_t\right)(\vec{f})\tau_t \nu \right\|_q \\
&\leqslant C \prod_{i=1}^m \|\tau_t f_i \tau_t w_i\|_{p_i} \\
&\leqslant C \prod_{i=1}^m \|f_i w_i\|_{p_i},
\end{aligned}$$

with C independent of t. Then, from (4.6) we obtain that

$$\left\| \mathcal{M}_{\alpha,B}(\vec{f})\nu \right\|_q \leqslant C \prod_{i=1}^m \|f_i w_i\|_{p_i}. \qquad \square$$

Proof of Corollary 2.18. We begin by proving the case $\alpha = 0$. If $\vec{w}^p \in A_{\vec{P}}$ then we have that $w_i^{p_i(1-p_i')} = w_i^{-p_i'} \in A_{mp_i'}$ (see [21]). Then, for each $i = 1, \dots, m$ there exist $s_i > 1$ such that $w_i^{-p_i'}$ satisfies a reverse Hölder inequality with exponent s_i.

Let $A_i(t) = t^{s_i p_i'}$ and $C_{i,k}(t) = (t(1+\log^+ t)^k)^{(s_i p_i')'}$. Then we have that $A_i^{-1}(t) C_{i,k}^{-1}(t) \cong B_k^{-1}(t)$ and $C_{i,k} \in B_{p_i}$. Thus, since $\vec{w}^p \in A_{\vec{P}}$ we obtain

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q \left(\prod_{i=1}^m w_i\right)^p\right)^{1/p} \prod_{i=1}^m \|w_i^{-1}\|_{A_i,Q} &\leqslant \left(\frac{1}{|Q|}\int_Q \left(\prod_{i=1}^m w_i\right)^p\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{-s_i p_i'}\right)^{1/(s_i p_i')} \\ &\leqslant \left(\frac{1}{|Q|}\int_Q \left(\prod_{i=1}^m w_i\right)^p\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{1/(p_i')} \\ &\leqslant C.\end{aligned}$$

Then by Theorem 2.15 applied to the case $\alpha = 0$ and $p = q$ we obtain that

$$\left\|\mathcal{M}_{B_k}(\vec{f})\left(\prod_{i=1}^m w_i\right)\right\|_{L^p} \leqslant \prod_{i=1}^m \|f_i w_i\|_{L_{p_i}}.$$

The other implication is a consequence of the inequality $\mathcal{M}(\vec{f}) \leqslant \mathcal{M}_{B_k}(\vec{f})$ and the boundedness results proved in [21] for the multilinear maximal operator $\mathcal{M}$.

Now we prove the case $\alpha > 0$. Let us first suppose that $\vec{w}^q \in A_{\vec{S}}$. It is enough to show that the following inequality

$$\left\|\mathcal{M}_{\alpha,B_k}(\vec{f}_w)\left(\prod_{i=1}^m w_i\right)\right\|_{L^q} \leqslant \prod_{i=1}^m \|f_i\|_{L_{p_i}} \tag{4.7}$$

holds for every $\vec{f} = (f_1, \ldots, f_m)$.

In order to prove (4.7) we use inequality (2.7). Let $\psi_k(t) = t(1+\log^+ t)^{knm/(nm-\alpha)}$. Then, from the case $\alpha = 0$ we obtain that

$$\begin{aligned}\left\|\mathcal{M}_{\alpha,B_k}(\vec{f}_w)\left(\prod_{i=1}^m w_i\right)\right\|_{L^q} &\leqslant C\left\|\mathcal{M}_{\psi_k}(\vec{g})^{1-\frac{\alpha}{nm}}\left(\prod_{i=1}^m w_i\right)\right\|_q \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{p_i\alpha}{nm}}\right) \\ &\leqslant C\left\|\mathcal{M}_{L(\log L)^{\frac{knm}{nm-\alpha}}}(\vec{g})\left(\prod_{i=1}^m w_i^{q/s}\right)\right\|_{L^s}^{s/q} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{p_i\alpha}{nm}}\right) \\ &\leqslant C\left\|\mathcal{M}_{L(\log L)^{[\frac{knm}{nm-\alpha}]+1}}(\vec{g})\left(\prod_{i=1}^m w_i^{q/s}\right)\right\|_{L^s}^{s/q} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{p_i\alpha}{nm}}\right) \\ &= C\left\|\mathcal{M}_{B_{[\frac{knm}{nm-\alpha}]+1}}(\vec{g})\left(\prod_{i=1}^m w_i^{q/s}\right)\right\|_{L^s}^{s/q} \left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{p_i\alpha}{nm}}\right) \\ &\leqslant C\left(\prod_{i=1}^m \|g_i w_i^{q/s}\|_{L^{s_i}}^{s/q}\right)\left(\prod_{i=1}^m \|f_i\|_{L^{p_i}}^{\frac{p_i\alpha}{nm}}\right),\end{aligned}$$

where in the last inequality we have used the fact that $\vec{w}^q \in A_{\vec{S}}$. We observe now that $\|g_i w_i^{q/s}\|_{L^{s_i}}^{s/q} = \|f_i\|_{p_i}^{p_i/q_i} = \|f_i\|_{p_i}^{1-\frac{\alpha p_i}{nm}}$ and inequality (4.7) follows immediately.

The other implication is a consequence of the inequality $\mathcal{M}_\alpha(\vec{f}) \leqslant \mathcal{M}_{\alpha,B}(\vec{f})$ and the boundedness result proved in [22]. □

Proof of Theorem 2.20. Let $\Omega_\lambda = \{x \in \mathbb{R}^n\colon \mathcal{M}_\alpha \vec{f}(x) > \lambda^m\}$. By homogeneity we may assume that $\lambda = 1$. Let K be a compact set contained in Ω_λ. Since K is a compact set and using Vitali's covering lemma we obtain a finite family of disjoint cubes $\{Q_j\}$ for which

$$1 < |Q_j|^{\alpha/n} \prod_{i=1}^m \frac{1}{|Q_j|}\int_{Q_j} |f_i|, \tag{4.8}$$

and $K \subset U_j 3Q_j$. Notice that, by Hölder's inequality we have that $\frac{u(Q)}{|Q|} \leqslant \prod_{i=1}^m (\frac{w_i(Q)}{|Q|})^{1/m}$. Then by (4.8) and Hölder's inequality at discrete level we obtain that

$$\begin{aligned}
u(K)^m &\leqslant C\left(\sum_j \frac{u(3Q_j)}{|3Q_j|}|Q_j|\right)^m \\
&\leqslant C\left(\sum_j \prod_{i=1}^m \left(\frac{1}{|3Q_j|}\int_{3Q_j} w_i\right)^{1/m} |Q_j|^{1/m}\left(\frac{|Q_j|^{\alpha/(nm)}}{|Q_j|}\int_{Q_j}|f_i|\right)^{1/m}\right)^m \\
&\leqslant C\left(\sum_j \prod_{i=1}^m \left(\frac{|3Q_j|^{\alpha/(nm)}}{|3Q_j|}\int_{3Q_j} w_i\right)^{1/m}\left(\int_{Q_j}|f_i|\right)^{1/m}\right)^m \\
&\leqslant C\left(\sum_j \prod_{i=1}^m \left(\int_{Q_j}|f_i|\, M_{\alpha/m} w_i\right)^{1/m}\right)^m \\
&\leqslant C\prod_{i=1}^m \int_{\mathbb{R}^n}|f_i|\, M_{\alpha/m} w_i,
\end{aligned}$$

and the proof concludes. □

Proof of Theorem 2.23. Let $p > 1$ to be chosen later. Thus, since $L^{p,\infty}$ and $L^{p',1}$ are associate spaces, we have that

$$\|\mathcal{I}_\alpha \vec{f}\|_{L^{1/m,\infty}(u)}^{1/(pm)} = \left\|(\mathcal{I}_\alpha \vec{f})^{1/(pm)}\right\|_{L^{p,\infty}(u)} = \sup_{\|g\|_{L^{p',1}(u)}\leqslant 1}\int_{\mathbb{R}^n}(\mathcal{I}_\alpha \vec{f})^{1/(pm)} g\, u.$$

By Theorem 2.25 we obtain that

$$\int_{\mathbb{R}^n}(\mathcal{I}_\alpha \vec{f})^{1/(pm)} g\, u \leqslant \int_{\mathbb{R}^n}(\mathcal{M}_\alpha \vec{f})^{1/(pm)} M(gu) = \int_{\mathbb{R}^n}(\mathcal{M}_\alpha \vec{f})^{1/(pm)}\frac{M(gu)}{M_{L(\log L)^\delta}(u)} M_{L(\log L)^\delta}(u),$$

for $\delta > 0$.

By applying Hölder's inequality in Lorentz spaces we obtain that

$$\int_{\mathbb{R}^n}(\mathcal{I}_\alpha \vec{f})^{1/(pm)} g\, u \leqslant \left\|(\mathcal{M}_\alpha \vec{f})^{1/(pm)}\right\|_{L^{p,\infty}(M_{L(\log L)^\delta}(u))}\left\|\frac{M(gu)}{M_{L(\log L)^\delta}(u)}\right\|_{L^{p',1}(M_{L(\log L)^\delta}(u))}.$$

Now we proceed as in the linear case (see [7]) by taking $p = 1+\delta-2\epsilon$ with $0 < 2\epsilon < \delta$ which allows us to obtain that

$$\left\|\frac{M(gu)}{M_{L(\log L)^\delta}(u)}\right\|_{L^{p',1}(M(\log L)^\delta(u))} \leqslant C\|g\|_{L^{p',1}(u)}$$

and taking supremum over $\|g\|_{L^{p',1}(u)} \leqslant 1$. □

Proof of Lemma 2.24. From inequality (3.5) with g replaced by v and the RH_∞ condition on v we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n}\mathcal{I}_\alpha \vec{f}(x)u(x)v(x)\,dx &\leqslant C\sum_{k,j}|Q_{k,j}|^{\alpha/n}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} uv\right)\left(\prod_{i=1}^m\left(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i\right)\right)|E_{k,j}| \\
&\leqslant C\sum_{k,j}|Q_{k,j}|^{\alpha/n}\int_{Q_{k,j}} u\left(\prod_{i=1}^m\left(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i\right)\right)\sup_{Q_{k,j}} v \\
&\leqslant C\sum_{k,j}|Q_{k,j}|^{\alpha/n}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} u\right)\left(\prod_{i=1}^m\left(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i\right)\right)v(Q_{k,j}).
\end{aligned}$$

Since $v \in A_\infty$ and by the properties of the sets $E_{k,j}$ we obtain that

$$\int_{\mathbb{R}^n}\mathcal{I}_\alpha \vec{f}(x)u(x)v(x)\,dx \leqslant C\sum_{k,j}|3Q_{k,j}|^{\alpha/n}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} u\right)\left(\prod_{i=1}^m\left(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i\right)\right)v(E_{k,j})$$

$$\leqslant C\sum_{k,j}\int_{E_{k,j}}\mathcal{M}_\alpha\vec{f}(x)Mu(x)\,v(x)\,dx$$
$$\leqslant C\int_{\mathbb{R}^n}\mathcal{M}_\alpha\vec{f}(x)Mu(x)v(x)\,dx. \qquad\square$$

Proof of Theorem 2.25. We proceed as in the linear case (see [7]). We use the duality for L^p spaces for $p<1$: if $f\geqslant 0$

$$\|f\|_p=\inf\bigl\{fu^{-1}:\ \|u^{-1}\|_{p'}=1\bigr\}=\int fu^{-1}$$

for some $u\geqslant 0$ such that $\|u^{-1}\|_{p'}=1$, with $p'=\frac{p}{p-1}<0$. This follows from the following reverse Hölder's inequality, which is a consequence of the Hölder's inequality,

$$\int fg\geqslant\|f\|_p\|g\|_{p'}. \tag{4.9}$$

We choose a nonnegative function g such that $\|g^{-1}\|_{L^{p'}(Mu)}=1$, and such that

$$\|\mathcal{M}_\alpha\vec{f}\|_{L^p(Mu)}=\int\mathcal{M}_\alpha\vec{f}\,\frac{Mu}{g}.$$

Let $\delta>0$. By Lebesgue differentiation theorem we get

$$\|\mathcal{M}_\alpha\vec{f}\|_{L^p(Mu)}\geqslant\int\mathcal{M}_\alpha\vec{f}\,\frac{Mu}{M_\delta(g)},$$

where $M_\delta(g)=M(g^\delta)^{1/\delta}$. Then applying Lemmas 2.24 and 3.6 to the weight $M_\delta(g)^{-1}$ and the reverse Hölder's inequality (4.9), we obtain that

$$\|\mathcal{M}_\alpha\vec{f}\|_{L^p(Mu)}\geqslant\int\mathcal{I}_\alpha\vec{f}\,\frac{u}{M_\delta(g)}\geqslant\|\mathcal{I}_\alpha\vec{f}\|_{L^p(u)}\bigl\|M_\delta(g)^{-1}\bigr\|_{L^{p'}(u)},$$

and everything is reduced to proving

$$\bigl\|M_\delta(g)^{-1}\bigr\|_{L^{p'}(u)}\geqslant\bigl\|g^{-1}\bigr\|_{L^{p'}(Mu)}=1.$$

Now, the proof follows as in the linear case (see [7]). Since $p'<0$, this is equivalent to prove that

$$\int_{\mathbb{R}^n}M_\delta(g)^{-p'}(x)u(x)\,dx\leqslant C\int g^{-p'}(x)Mu(x)\,dx.$$

By choosing δ such that $0<\delta<\frac{p}{1-p}$, we have that $-p'/\delta>1$ and the above inequality follows from the classical weighted Fefferman–Stein norm inequality (see [11]). $\square$

Proof of Theorem 2.26. Let s be a positive number. We split $\mathcal{I}_\alpha$ as follows

$$\bigl|\mathcal{I}_\alpha\vec{f}(x)\bigr|\leqslant\int_{\sum_{i=1}^m|x-y_i|<s}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}\,d\vec{y}+\int_{\sum_{i=1}^m|x-y_i|\geqslant s}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}\,d\vec{y}$$
$$=I_1+I_2.$$

Let us first estimate I_1. Thus, if Q_k is a cube centered at x with side length $2^{-k}s$, $k\in\mathbb{N}\cup\{0\}$, we obtain

$$I_1=\sum_{k=0}^\infty\int_{2^{-k-1}s<\sum_{i=1}^m|x-y_i|\leqslant 2^{-k}s}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}\,d\vec{y}$$
$$\leqslant C\sum_{k=0}^\infty\frac{1}{(2^{-k}s)^{mn-\alpha}}\int_{\sum_{i=1}^m|x-y_i|\leqslant 2^{-k}s}\left(\prod_{i=1}^m\bigl|f_i(y_i)\bigr|\right)d\vec{y}$$
$$\leqslant C\sum_{k=0}^\infty\frac{1}{(2^{-k}s)^{-\alpha}}\prod_{i=1}^m\frac{1}{|Q_k|}\int_{Q_k}\bigl|f_i(y_i)\bigr|\,dy_i$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{1}{(2^{-k}s)^{-\alpha}}\frac{(2^{-k}s)^{\alpha-\epsilon}}{(2^{-k}s)^{\alpha-\epsilon}}\prod_{i=1}^{m}\frac{1}{|Q_k|}\int\limits_{Q_k}\big|f_i(y_i)\big|\,dy_i$$

$$\leqslant Cs^{\epsilon}\mathcal{M}_{\alpha-\epsilon}\vec{f}(x).$$

Now, we proceed to estimate I_2. Let P_k be the cube centered at x with side length $2^k s$. Then we obtain

$$I_2=\sum_{k=0}^{\infty}\int\limits_{2^k s<\sum_{i=1}^{m}|x-y_i|\leqslant 2^{k+1}s}\frac{\prod_{i=1}^{m}|f_i(y_i)|}{(\sum_{i=1}^{m}|x-y_i|)^{mn-\alpha}}\,d\vec{y}$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{1}{(2^k s)^{mn-\alpha}}\int\limits_{\sum_{i=1}^{m}|x-y_i|\leqslant 2^{k+1}s}\left(\prod_{i=1}^{m}\big|f_i(y_i)\big|\right)d\vec{y}$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{1}{(2^k s)^{-\alpha}}\prod_{i=1}^{m}\frac{1}{|P_{k+1}|}\int\limits_{P_{k+1}}\big|f_i(y_i)\big|\,dy_i$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{1}{(2^k s)^{-\alpha}}\frac{(2^k s)^{\alpha+\epsilon}}{(2^k s)^{\alpha+\epsilon}}\prod_{i=1}^{m}\frac{1}{|P_{k+1}|}\int\limits_{P_{k+1}}\big|f_i(y_i)\big|\,dy_i$$

$$\leqslant C\frac{1}{s^{\epsilon}}\mathcal{M}_{\alpha+\epsilon}\vec{f}(x).$$

Collecting both estimates we obtain

$$\mathcal{I}_\alpha\vec{f}(x)\leqslant C\big(s^{\epsilon}\mathcal{M}_{\alpha-\epsilon}\vec{f}(x)+s^{-\epsilon}\mathcal{M}_{\alpha+\epsilon}\vec{f}(x)\big),$$

for any $s>0$. Then, to complete the proof, we just have to minimize the expression above in the variable s. □

5. Banach function spaces

We introduce now some basic facts about the theory of Banach function spaces. For more information about these spaces we refer the reader to [1].

Let X be a Banach function space over $\mathbb{R}^n$ with respect to the Lebesgue measure. X has an associate Banach function space X' for which the generalized Hölder inequality,

$$\int\limits_{\mathbb{R}^n}\big|f(x)g(x)\big|\,dx\leqslant\|f\|_X\|g\|_{X'},$$

holds. Examples of Banach functions spaces are given by the Lebesgue L^p spaces, Lorentz spaces and Orlicz spaces. The Orlicz spaces are one of the most relevant Banach function spaces, and a brief description was given in Section 1.

Given any measurable function $f\in X$ and a cube $Q\subset\mathbb{R}^n$, we define the X average of f over Q to be

$$\|f\|_{X,Q}=\big\|\delta_{l(Q)}(f\chi_Q)\big\|_X,$$

where $\delta_a f(x)=f(ax)$ for $a>0$ and χ_A denotes the characteristic function of the set A. In particular, when $X=L^r$, $r\geqslant 1$, we have that

$$\|f\|_{X,Q}=\left(\frac{1}{|Q|}\int\limits_{Q}\big|f(y)\big|^r\right)^{1/r},$$

and if $X=L^B$, the Orlicz space associated to a Young function B, then

$$\|f\|_{X,Q}=\|f\|_{B,Q}.$$

For a given Banach function space X, we associate the following maximal operator defined for each locally integrable function f by

$$M_X f(x)=\sup_{Q\ni x}\|f\|_{X,Q}.$$

If $Y_1,\dots,Y_m$ are Banach function spaces, the multilinear version of the maximal function above is given by

$$\mathcal{M}_{\vec{Y}}\vec{f}(x)=\sup_{Q\ni x}\prod_{i=1}^{m}\|f_i\|_{Y_i,Q}.$$

Let $1 < p_1, \dots, p_m < \infty$ and suppose that $M_{Y_i}: L^{p_i} \to L^{p_i}$. From the fact that $\mathcal{M}_{\vec{Y}} \vec{f}(x) \leqslant \prod_{i=1}^m M_{Y_I} f_i(x)$ and applying Hölder's inequality we obtain that

$$\mathcal{M}_{\vec{Y}}: L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

We define now the multilinear maximal operator associate to certain function φ that generalizes the multilinear fractional maximal operator $\mathcal{M}_\alpha$. We shall assume that the function $\varphi: (0,\infty) \to (0,\infty)$ is essentially nondecreasing, that is, there exists a positive constant ρ such that, if $t \leqslant s$ then $\varphi(t) \leqslant \rho\varphi(s)$. We shall also suppose that $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$. The linear case of the operator below was study in [27].

Definition 5.1. Let $\vec{f} = (f_1, \dots, f_m)$. The multilinear maximal operator $\mathcal{M}_\varphi$ associated to the function φ is defined by

$$\mathcal{M}_\varphi \vec{f}(x) = \sup_{Q \ni x} \varphi(|Q|) \prod_{i=1}^m \frac{1}{|Q|} \int_Q f_i.$$

When $m = 1$ we simply write $\mathcal{M}_\varphi = M_\varphi$.

The following result is a generalized version of Theorem 2.15 when $B(t) = t$. The case $m = 1$ was proved in [27].

Theorem 5.2. *Let $1/m < p \leqslant q < \infty$, $1 < p_i < \infty$, $i = 1, \dots, m$, $1/p = \sum_{i=1}^m 1/p_i$. Let φ be a function as in Definition 5.1. Let Y_i, $i = 1, \dots, m$, be m Banach function spaces such that $\mathcal{M}_{Y'}: L^{p_i} \to L^{p_i}$. Suppose that $\nu, w_1, \dots, w_m$ are weights such that, for some positive constant C and for every cube Q*

$$\varphi(|Q|)|Q|^{1/q-1/p} \left(\frac{1}{|Q|} \int_Q \nu^q\right)^{1/q} \prod_{i=1}^m \|w_i^{-1}\|_{Y_i,Q} \leqslant C. \tag{5.3}$$

Then

$$\|\mathcal{M}_\varphi \vec{f} \nu\|_q \leqslant C \prod_{i=1}^m \|f_i w_i\|_{p_i} \tag{5.4}$$

holds for every $\vec{f} \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$.

When $\varphi(t) = t^{\alpha/n}$ and $Y_i = L_{A_i}$, $i = 1, \dots, m$, are the Orlicz spaces associated to the Young functions A_i, then we obtain Theorem 2.15 for the case $B(t) = t$.

The proof of Theorem 5.2 follows similar arguments to those in the proof of Theorem 2.15. The main tools used are an analogue of the Calderón–Zygmund decomposition for $\mathcal{M}_\varphi^d$ adapted to the essentially nondecreasing function φ, the generalized Hölder's inequality and the boundedness of $\mathcal{M}_{Y'}$ in the right places.

Corollary 5.5. *Let $1/m < p < \infty$, $1 < p_i < \infty$, $i = 1, \dots, m$, $1/p = \sum_{i=1}^m 1/p_i$. Let φ be a function as in Definition 5.1. Then*

(i) *There exists a positive constant C such that, for every $\vec{f} = (f_1, \dots, f_m)$, and every positive functions u_i*

$$\left(\int_{\mathbb{R}^n} \mathcal{M}_\varphi \vec{f}(y)^p \left(\prod_{i=1}^m u_i(y)^{1/p_i}\right)^p dy\right)^{1/p} \leqslant C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i(y)|^{p_i} M_{\varphi^p}(u_i)\right)^{1/p_i}.$$

(ii) *If $s_i > p_i' - 1$, there exists a positive constant C such that, for every $\vec{f} = (f_1, \dots, f_m)$, and every positive functions u_i*

$$\left(\int_{\mathbb{R}^n} \mathcal{M}_\varphi \vec{f}(y)^p \frac{dy}{(\prod_{i=1}^m M_{\varphi^{ps_i}}(u_i^{s_i})(y)^{1/(p_i s_i)})^p}\right)^{1/p} \leqslant C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i(y)|^{p_i} \frac{dy}{u_i(y)}\right)^{1/p_i}.$$

The proof of (i) follows by applying Theorem 5.2 to the weights $\nu = \prod_{i=1}^m u_i^{1/p_i}$, $w_i = M_{\varphi^p}(u_i)^{1/p_i}$ and $Y_i = L^{p_i' r}$, $1 < r < \infty$.

To prove (ii) we apply Theorem 5.2 to the weights $\nu = \prod_{i=1}^m M_{\varphi^{ps_i}}(u_i^{s_i})(y)^{1/(p_i s_i)}$, $w_i = u_i^{-1/p_i}$ and $Y_i = L^{p_i' r_i}$, $r_i = (p_i - 1)s_i$.

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