J. Math. Anal. Appl. 356 (2009) 764–768

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Orthogonal rational functions with complex poles: The Favard theorem ☆

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ARTICLE INFO

Article history:
Received 21 February 2008
Available online 31 March 2009
Submitted by D. Khavinson

Keywords:
Orthogonal rational functions
Three-term recurrence relation
Favard theorem

ABSTRACT

Let $\{\varphi_n\}$ be a sequence of rational functions with arbitrary complex poles, generated by a certain three-term recurrence relation. In this paper we show that under some mild conditions, the rational functions φ_n form an orthonormal system with respect to a Hermitian positive-definite inner product.

1. Introduction

In [8] a Favard theorem was given for Laurent polynomials. Later, several Favard theorems were determined for classes of rational functions with restrictions on the poles: first, the restriction that the poles are complex and outside the extended real line (or, using an inverse Cayley transformation, outside the unit circle), see e.g. [1,2,4,7]; afterwards, the restriction that the poles are all on the extended real line (or on the unit circle), see e.g. [3,4]. Finally, in [5, Theorem 3.10] a Favard theorem was given for rational functions without restrictions on the poles.

The complete proof of this last Favard type theorem was omitted in [5] because at first it seemed that the outline of the proof would be similar to the proof given in [4, Chapter 11.9]. However, a detailed study in [6] revealed that Theorem 3.10 could not be proved as in [4]; hence, it is still unproved.

The aim of this paper is to give a complete proof for Theorem 3.10 in [5]. In Section 3 we will give this complete proof, but first we start with an overview of the theoretical preliminaries in the next section.

2. Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$, while the extended real line will be denoted by $\overline{\mathbb{R}}$. Let $c = a + \mathbf{i}b$, where $a, b \in \mathbb{R}$. Then we denote the real part of c by $\Re\{c\} = a$ and the imaginary part by $\Im\{c\} = b$.

Suppose a sequence of poles $\{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}} \setminus \{0\}$ is given. Define the factors

$$Z_k(x) = Z_{\alpha_k}(x) = x/(1 - x/\alpha_k), \quad k = 1, 2, \ldots,$$

☆ The work is partially supported by the Fund for Scientific Research (FWO), projects 'RAM: Rational modeling: optimal conditioning and stable algorithms,' grant #G.0423.05, and by the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with the authors.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.03.064

and the basis functions

$$b_0(x) \equiv 1, \qquad b_k(x) = b_{k-1}(x) Z_k(x), \quad k = 1, 2, \ldots.$$

Then the space of rational functions with poles in $\{\alpha_1, \ldots, \alpha_n\}$ is defined as

$$\mathcal{L}_n = \operatorname{span}\{b_0, b_1, \ldots, b_n\}.$$

We denote with $\mathcal{P}_n$ the space of polynomials of degree less than or equal to n. Let π_n be given by

$$\pi_0(x) \equiv 1, \qquad \pi_n(x) = \prod_{k=1}^{n} (1 - x/\alpha_k).$$

Then we may write equivalently $\mathcal{L}_n = \{p_n/\pi_n\colon\ p_n \in \mathcal{P}_n\}$. In the remainder, we will use the notation $\pi_{n\setminus j}$, with $0 \leqslant j \leqslant n$, to denote the polynomial $\pi_{n\setminus j} = \pi_n/\pi_j \in \mathcal{P}_{n-j}$.

Note that the value $\alpha_\emptyset = 0$ represents a forbidden value for the poles α_k. Since we consider only a countable number of poles α_k, we can always find a point $\alpha_\emptyset \in \mathbb{C}$ so that $\alpha_k \neq \alpha_\emptyset$ for every $k \geqslant 1$. A simple transformation can bring this $\alpha_\emptyset$ to any position that we would prefer. Therefore, this forbidden value $\alpha_\emptyset$ is not a real restriction on the sequence of poles, and we may assume it to be fixed by the value zero.

We define the substar conjugate of a function $f \in \mathcal{L}_\infty$ as

$$f_*(x) = \overline{f(\overline{x})}.$$

This way we have that $f(x)$ has a pole in $x = \alpha$ iff $f_*(x)$ has a pole in $x = \overline{\alpha}$. With $\mathcal{L}_{n*}$ we then denote the space of rational functions given by $\mathcal{L}_{n*} = \{f_*\colon\ f \in \mathcal{L}_n\}$.

Next, let us consider an inner product that is defined by a linear functional M:

$$\langle f, g \rangle = M\{f g_*\}, \quad f, g \in \mathcal{L}_\infty.$$

The functional M is called Hermitian positive-definite (HPD) iff $M\{f f_*\} > 0$ for all $f \in \mathcal{L}_\infty \setminus \{0\}$ and $M\{f_*\} = \overline{M\{f\}}$ for every $f \in \mathcal{L}_\infty \cdot \mathcal{L}_{\infty*}$.

Suppose there exists a sequence of rational functions $\{\varphi_n\}$, with $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, so that the φ_n form an orthonormal system with respect to the HPD linear functional M; i.e. $M\{\varphi_j \varphi_{k*}\} = \delta_{jk}$, where δ_{jk} denotes the Kronecker Delta. Further, assume that these rational functions are of the form $\varphi_n(x) = p_n(x)/\pi_n(x)$. We then call φ_n regular iff $p_n(\overline{\alpha}_{n-1}) \neq 0$ and $p_n(\alpha_{n-1}) \neq 0$. In the special case in which $\alpha_{n-1} = \infty$, $p_n(\infty) \neq 0$ means that $p_n \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$.

In case of a regular system of orthonormal rational functions (i.e., the orthonormal rational functions φ_n are regular for every $n \geqslant 1$), it follows from [5, Section 3] that the φ_n satisfy a three-term recurrence relation of the form:

$$\begin{aligned} &\varphi_n(x) = Z_n(x)\bigl\{E_n\bigl[1 + F_n/Z_{n-1}(x)\bigr]\varphi_{n-1}(x) + C_n \varphi_{n-2}(x)/Z_{n-2*}(x)\bigr\}, \\ &\quad E_n \neq 0,\ F_n \in \mathbb{C},\ C_n = -E_n\bigl[1 + F_n/Z_{n-1}(\overline{\alpha}_{n-1})\bigr]/\overline{E}_{n-1} \neq 0,\ n \geqslant 1. \end{aligned} \tag{1}$$

3. Favard theorem

In order to derive a Favard type theorem, we need to verify whether, starting from a regular system of rational functions $\{\varphi_n\}$ for which the φ_n are generated by the three-term recurrence relation (1), there exists a HPD inner product for which the φ_n form an orthonormal system. So, assume that $\{\varphi_n\}_{n=0}^{\infty}$ is a sequence of rational functions in $\mathcal{L}_\infty$ and that the following assumptions are satisfied:

(A1) $\alpha_{-1} \in \overline{\mathbb{R}} \setminus \{0\}$ and $\alpha_n \in \overline{\mathbb{C}} \setminus \{0\}$, $n = 0, 1, \ldots,$
(A2) φ_n is generated by the three-term recurrence relation (1),
(A3) $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $n = 1, 2, \ldots,$ and $\varphi_0 \in \mathbb{C} \setminus \{0\}$,
(A4) $|E_n|^{-2} = 2\Re\{\tau_n + \rho_n\}$, $n = 1, 2, \ldots,$ where

$$\tau_n = \frac{F_n[1 + \overline{F}_n A(\overline{\alpha}_{n-1}, \omega_n)]}{A(\alpha_n, \overline{\alpha}_n)}, \qquad \rho_n = \left| \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \right|^2 \frac{A(\alpha_{n-2}, \omega_n)}{A(\alpha_n, \overline{\alpha}_n)},$$

$A(\alpha, \beta) = 1/Z_\alpha(x) - 1/Z_\beta(x)$, $\omega_n = |\alpha_n|^2/\Re\{\alpha_n\}$, and $E_0 \in \mathbb{C} \setminus \{0\}$.

Further, let $\mathcal{S}_n$ and $\mathcal{S}_\infty$ be given by

$$\mathcal{S}_n = \operatorname{span}\{\varphi_k \varphi_{l*}\colon\ 0 \leqslant k, l \leqslant n \text{ and } k \neq l\} = \mathcal{S}_{n*}, \quad n > 0,$$

respectively $\mathcal{S}_\infty = \operatorname{span}\{\mathcal{S}_n\colon\ n = 1, 2, \ldots\}$. Then we define M on $\mathcal{S}_\infty$ by setting $M\{f\} = 0$ for every $f \in \mathcal{S}_\infty$. Note that it is always possible to define M in such a way, independent of whether the assumptions given by (A1)–(A4) are satisfied.

Next, let $\mathcal{T}_n$ and $\mathcal{T}_\infty$ be given by

$$\mathcal{T}_n = \text{span}\{\varphi_k\varphi_{k*}\colon\ 0 \leqslant k \leqslant n\} = \mathcal{T}_{n*}, \quad n > 0,$$

respectively $\mathcal{T}_\infty = \text{span}\{\mathcal{T}_n\colon\ n = 1, 2, \ldots\}$. Clearly we then have that $\mathcal{S}_n + \mathcal{T}_n = \mathcal{L}_n \cdot \mathcal{L}_{n*}$ for every $n > 0$. Hence, it remains to extend the definition of M to $\mathcal{T}_\infty$ in such a way that $M\{\varphi_k\varphi_{k*}\} = 1$ for every $k \geqslant 0$. Note that this extension is possible iff the following condition holds true:

$$\phi = \sum a_k\varphi_k\varphi_{k*} \in \mathcal{S}_\infty \quad \text{iff} \quad \sum a_k = 0. \tag{2}$$

We will now prove by induction that the condition given by (2) holds true under the assumptions given by (A1)–(A4).

Initialisation $n = 1$

Consider the subspace $\mathcal{L}_1 \cdot \mathcal{L}_{1*} = \mathcal{S}_1 + \mathcal{T}_1$. We then have the following theorems.

Theorem 1. *Under the assumptions given by* (A1)–(A4) *it holds that* $(\varphi_1\varphi_{1*} - \varphi_0\varphi_{0*}) \in \mathcal{S}_1$.

Proof. Starting from the three-term recurrence relation (1), and performing some computations similar to those in the proof of [5, Theorem 3.9] (but without taking inner products), we find that there exists a function $f_1 \in \mathcal{S}_1$ so that

$$|E_1|^{-2}\varphi_1\varphi_{1*} = 2\Re\{\tau_1\}\varphi_0\varphi_{0*} + f_1.$$

Finally, because $2\Re\{\rho_1\} = 0$ for $\alpha_{-1} \in \overline{\mathbb{R}} \setminus \{0\}$, it follows from assumption (A4) that $(\varphi_1\varphi_{1*} - \varphi_0\varphi_{0*}) = |E_1|^2 f_1 \in \mathcal{S}_1$. $\square$

Theorem 2. *Under the assumptions given by* (A1)–(A4) *it holds for* $j = 0, 1$ *that* $\varphi_j\varphi_{j*} \notin \mathcal{S}_1$.

Proof. First, consider the case that $\alpha_1 \in \overline{\mathbb{R}} \setminus \{0\}$, and suppose that $\varphi_1\varphi_{1*} = p_1p_{1*}/\pi_1^2 \in \mathcal{S}_1 = \text{span}\{\varphi_1\varphi_{0*}, \varphi_{1*}\varphi_0\}$. We then have that there exists a constant $c \neq 0$ so that $[cp_1(x) + \bar{c}p_{1*}(x)]/\pi_1(x) = p_1(x)p_{1*}(x)/\pi_1^2(x)$. Or, equivalently, $\pi_1(x)[cp_1(x) + \bar{c}p_{1*}(x)] = p_1(x)p_{1*}(x)$. Taking $x = \alpha_1$, it then follows that $p_1(\alpha_1) = 0$, contradicting our assumption given by (A3). Consequently, $\varphi_1\varphi_{1*} \notin \mathcal{S}_1$, and from Theorem 1 it then follows that $\varphi_0\varphi_{0*} \notin \mathcal{S}_1$.

Finally, consider the case that $\alpha_1 \notin \overline{\mathbb{R}}$, and suppose that $\varphi_0\varphi_{0*} \in \mathcal{S}_1 = \text{span}\{\varphi_1\varphi_{0*}, \varphi_{1*}\varphi_0\}$. We then have that there exists a constant $c \neq 0$ so that $cp_1(x)/\pi_1(x) + \bar{c}p_{1*}(x)/\pi_{1*}(x) = \varphi_0\varphi_{0*}$. Or, equivalently, $\pi_{1*}(x)cp_1(x) + \pi_1(x)\bar{c}p_{1*}(x) = \pi_1(x)\pi_{1*}(x)\varphi_0\varphi_{0*}$. Taking $x = \alpha_1$ or $x = \overline{\alpha}_1$, it then follows that $\pi_{1*}(\alpha_1)cp_1(\alpha_1) = 0$, respectively $\pi_1(\overline{\alpha}_1)\bar{c}p_{1*}(\overline{\alpha}_1) = 0$. But this is impossible due to our assumption given by (A3) and due to the fact that $c \neq 0$. Hence, $\varphi_0\varphi_{0*} \notin \mathcal{S}_1$, and from Theorem 1 it then follows that $\varphi_1\varphi_{1*} \notin \mathcal{S}_1$. $\square$

Hence, we now have proved that $\phi_1 = a_0\varphi_0\varphi_{0*} + a_1\varphi_1\varphi_{1*} \in \mathcal{S}_1$ iff $a_0 + a_1 = 0$.

Induction for $n > 1$

Consider the subspaces $\mathcal{L}_j \cdot \mathcal{L}_{j*} = \mathcal{S}_j + \mathcal{T}_j$, with $j = n-1, n$, and suppose that for $j = n-1$ it holds that $\phi_{n-1} = \sum_{k=0}^{n-1} a_k\varphi_k\varphi_{k*} \in \mathcal{S}_{n-1}$ iff $\sum_{k=0}^{n-1} a_k = 0$. We then have to prove for $j = n$ that $\phi_n = \sum_{k=0}^{n} a_k\varphi_k\varphi_{k*} \in \mathcal{S}_n$ iff $\sum_{k=0}^{n} a_k = 0$.

Theorem 3. *Under the assumptions given by* (A1)–(A4) *it holds for* $j = 0, \ldots, n-1$ *that* $(\varphi_n\varphi_{n*} - \varphi_j\varphi_{j*}) \in \mathcal{S}_n$.

Proof. Let us first consider the case in which $j = n-1$. Starting from the three-term recurrence relation (1), and performing some computations similar to those in the proof of [5, Theorem 3.9] (but without taking inner products), we find that there exists a function $f_n \in \mathcal{S}_n$ so that

$$|E_n|^{-2}\varphi_n\varphi_{n*} = 2\Re\{\tau_n\}\varphi_{n-1}\varphi_{n-1*} + 2\Re\{\rho_n\}\varphi_{n-2}\varphi_{n-2*} + f_n.$$

Due to our assumption (A4) it follows that $(\varphi_n\varphi_{n*} - \varphi_{n-1}\varphi_{n-1*}) = |E_n|^2k_n$, where $k_n = f_n - 2\Re\{\rho_n\}(\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2}\varphi_{n-2*})$. From the induction hypotheses it now follows that $(\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2}\varphi_{n-2*}) \in \mathcal{S}_{n-1} \subseteq \mathcal{S}_n$, so that $|E_n|^2k_n \in \mathcal{S}_n$.

Finally, for $j < n-1$ it holds that $(\varphi_n\varphi_{n*} - \varphi_j\varphi_{j*}) = (\varphi_n\varphi_{n*} - \varphi_{n-1}\varphi_{n-1*}) + (\varphi_{n-1}\varphi_{n-1*} - \varphi_j\varphi_{j*})$, where it follows from the induction hypotheses that $(\varphi_{n-1}\varphi_{n-1*} - \varphi_j\varphi_{j*}) \in \mathcal{S}_{n-1} \subseteq \mathcal{S}_n$. This concludes the proof. $\square$

It remains to prove that $\varphi_j\varphi_{j*} \notin \mathcal{S}_n$ for $j = 0, \ldots, n$. Therefore we first need the following lemma.

Lemma 4. *Under the assumptions given by* (A1)–(A4) *it holds for every* $g_{n-2} \in \mathcal{L}_{n-2}$ *that* $Z_{n-1*}g_{n-2*}\varphi_n/Z_n \in \mathcal{S}_{n-1}$.

Proof. First, note that there exist coefficients $a_1, a_2, \ldots, a_{n-1}$ so that

$$Z_{n-1}(x)g_{n-2}(x) = \sum_{k=1}^{n-1} a_k b_k(x).$$

From the three-term recurrence relation (1) it now follows that

$$\frac{b_{k*}\varphi_n}{E_n Z_n} = \left[1 + \frac{F_n}{Z_{n-1}}\right]\varphi_{n-1} b_{k*} - \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2} \frac{b_{k*}}{Z_{n-2*}}. \tag{3}$$

It is easily verified that the right-hand side of (3) is in $\mathcal{S}_{n-1}$ for $k = 1, \ldots, n-2$. While for $k = n-1$ we have that

$$\left[1 + \frac{F_n}{Z_{n-1}}\right]\varphi_{n-1} b_{n-1*} = \left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right]\varphi_{n-1} b_{n-1*} + F_n \varphi_{n-1} b_{n-2*}$$

and $\varphi_{n-2} b_{n-1*}/Z_{n-2*} = A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})\varphi_{n-2} b_{n-1*} + \varphi_{n-2} b_{n-2*}$. Consequently,

$$\left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right]^{-1} b_{n-1*}\varphi_n / E_n Z_n = \left(\varphi_{n-1} - A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})\varphi_{n-2}/\overline{E}_{n-1}\right) b_{n-1*} - \varphi_{n-2} b_{n-2*}/\overline{E}_{n-1} + k_{n-1},$$

where $k_{n-1} = F_n[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})]^{-1}\varphi_{n-1} b_{n-2*} \in \mathcal{S}_{n-1}$. Suppose that $\varphi_{n-1} = \kappa_{n-1} b_{n-1} + \kappa'_{n-1} b_{n-2} + f_{n-3}$, where $\kappa_{n-1}, \kappa'_{n-1} \in \mathbb{C}$, $\kappa_{n-1} \neq 0$ and $f_{n-3} \in \mathcal{L}_{n-3}$. Then we get that

$$\begin{aligned}\left(\varphi_{n-1} - A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})\varphi_{n-2}/\overline{E}_{n-1}\right) b_{n-1*} &= \left(\varphi_{n-1} - A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})\varphi_{n-2}/\overline{E}_{n-1}\right)[\varphi_{n-1*} - \overline{\kappa}'_{n-1} b_{n-2*} - f_{n-3*}]/\overline{\kappa}_{n-1} \\ &= \left[\varphi_{n-1}\varphi_{n-1*} - \overline{\kappa}'_{n-1} A(\overline{\alpha}_{n-1}, \overline{\alpha}_{n-2})\varphi_{n-2} b_{n-2*}/\overline{E}_{n-1} - h_{n-1}\right]/\overline{\kappa}_{n-1},\end{aligned}$$

where

$$h_{n-1} = \varphi_{n-1}\left(\overline{\kappa}'_{n-1} b_{n-2*} + f_{n-3*}\right) + A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})\varphi_{n-2}(\varphi_{n-1*} - f_{n-3*})/\overline{E}_{n-1} \in \mathcal{S}_{n-1}.$$

Hence, with $h'_{n-1} = (\overline{\kappa}_{n-1} k_{n-1} - h_{n-1}) \in \mathcal{S}_{n-1}$ we have that

$$\left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right]^{-1} b_{n-1*}\varphi_n / E_n Z_n = \left\{\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2} b_{n-2*}\left[\overline{\kappa}_{n-1} + \overline{\kappa}'_{n-1} A(\overline{\alpha}_{n-1}, \overline{\alpha}_{n-2})\right]/\overline{E}_{n-1} + h'_{n-1}\right\}/\overline{\kappa}_{n-1}.$$

Finally, suppose that $\varphi_{n-2} = \kappa_{n-2} b_{n-2} + l_{n-3}$, where $\kappa_{n-2} \neq 0$ and $l_{n-3} \in \mathcal{L}_{n-3}$. Then it follows from [5, Theorem 3.2.4] that $[\kappa_{n-1} + \kappa'_{n-1} A(\alpha_{n-1}, \alpha_{n-2})]/E_{n-1} = \kappa_{n-2}$, so that

$$\left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right]^{-1} b_{n-1*}\varphi_n / E_n Z_n = \left[\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2}\varphi_{n-2*} + g'_{n-1}\right]/\overline{\kappa}_{n-1},$$

where $g'_{n-1} = (h'_{n-1} + \varphi_{n-2} l_{n-3*}) \in \mathcal{S}_{n-1}$, and $(\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2}\varphi_{n-2*}) \in \mathcal{S}_{n-1}$ as well due to the induction hypotheses. □

Theorem 5. *Under the assumptions given by* (A1)–(A4) *it holds for* $j = 0, \ldots, n$ *that* $\varphi_j \varphi_{j*} \notin \mathcal{S}_n$.

Proof. First, consider the case that $\alpha_n \in \overline{\mathbb{R}} \setminus \{0\}$, and suppose that $\varphi_n \varphi_{n*} = p_n p_{n*}/\pi_n \pi_{n*} \in \mathcal{S}_n = (\mathcal{S}_{n-1} + \varphi_n \cdot \mathcal{L}_{n-1*} + \varphi_{n*} \cdot \mathcal{L}_{n-1})$. We then have that there exist a polynomial $r_{n-1} \in \mathcal{P}_{n-1}$ and function $h_{n-1} \in \mathcal{S}_{n-1}$ so that

$$h_{n-1}(x) + \frac{p_n(x) r_{n-1}(x) + p_{n*}(x) r_{n-1*}(x)}{\pi_{n-1}(x)\pi_{n*}(x)} = \frac{p_n(x) p_{n*}(x)}{\pi_n(x)\pi_{n*}(x)}.$$

Or, equivalently,

$$\pi_n(x)\pi_{n*}(x) h_{n-1}(x) + \pi_{n\setminus(n-1)}(x)\left[p_n(x) r_{n-1}(x) + p_{n*}(x) r_{n-1*}(x)\right] = p_n(x) p_{n*}(x).$$

Taking $x = \alpha_n$, it then follows that $p_n(\alpha_n) = 0$, contradicting our assumption given by (A3). Consequently, $\varphi_n \varphi_{n*} \notin \mathcal{S}_n$, and from Theorem 3 it then follows that $\varphi_j \varphi_{j*} \notin \mathcal{S}_n$ for $j = 0, \ldots, n$.

Finally, consider the case that $\alpha_n \notin \overline{\mathbb{R}}$, and suppose that $\varphi_{n-1}\varphi_{n-1*} = \frac{p_{n-1} p_{n-1*}}{\pi_{n-1}\pi_{n-1*}} \in \mathcal{S}_n = (\mathcal{S}_{n-1} + \varphi_n \cdot \mathcal{L}_{n-1*} + \varphi_{n*} \cdot \mathcal{L}_{n-1})$. We then have that there exist a polynomial $r_{n-1} \in \mathcal{P}_{n-1}$ and function $h_{n-1} \in \mathcal{S}_{n-1}$ so that

$$h_{n-1}(x) + \frac{p_n(x) r_{n-1}(x)}{\pi_n(x)\pi_{n-1*}(x)} + \frac{p_{n*}(x) r_{n-1*}(x)}{\pi_{n*}(x)\pi_{n-1}(x)} = \frac{p_{n-1}(x) p_{n-1*}(x)}{\pi_{n-1}(x)\pi_{n-1*}(x)}.$$

Or, equivalently,

$$\begin{aligned}&\pi_n(x)\pi_{n*}(x) h_{n-1}(x) + \pi_{n*\setminus(n-1)*}(x) p_n(x) r_{n-1}(x) + \pi_{n\setminus(n-1)}(x) p_{n*}(x) r_{n-1*}(x) \\ &\quad = \pi_{n\setminus(n-1)}(x)\pi_{n*\setminus(n-1)*}(x) p_{n-1}(x) p_{n-1*}(x).\end{aligned}$$

Taking $x = \alpha_n$ or $x = \overline{\alpha}_n$, it then follows that $\pi_{n*\setminus(n-1)*}(\alpha_n)p_n(\alpha_n)r_{n-1}(\alpha_n) = 0$, respectively $\pi_{n\setminus(n-1)}(\overline{\alpha}_n)p_{n*}(\overline{\alpha}_n)r_{n-1*}(\overline{\alpha}_n) = 0$. Consequently, $r_{n-1}(\alpha_n) = r_{n-1*}(\overline{\alpha}_n) = 0$ due to our assumption given by (A3). Hence, there exists a function $g_{n-2} \in \mathcal{L}_{n-2}$ so that

$$p_n(x)r_{n-1}(x)/\pi_n(x)\pi_{n-1*}(x) + p_{n*}(x)r_{n-1*}(x)/\pi_{n*}(x)\pi_{n-1}(x)$$
$$= Z_{n-1*}(x)g_{n-2*}(x)\varphi_n(x)/Z_n(x) + Z_{n-1}(x)g_{n-2}(x)\varphi_{n*}(x)/Z_{n*}(x).$$

From Lemma 4 it now follows that $(h_{n-1} + p_n r_{n-1}/\pi_n\pi_{n-1*} + p_{n*}r_{n-1*}/\pi_{n*}\pi_{n-1}) \in \mathcal{S}_{n-1}$, while it follows from the induction hypotheses that $\varphi_{n-1}\varphi_{n-1*} \notin \mathcal{S}_{n-1}$. Hence, $\varphi_{n-1}\varphi_{n-1*} \notin \mathcal{S}_n$, and from Theorem 3 it then follows that $\varphi_j\varphi_{j*} \notin \mathcal{S}_n$ for $j = 0, \ldots, n$. □

Thus, we now have proved the following theorem.

Theorem 6. *Under the assumptions given by* (A1)–(A4) *it holds that* $\phi = \sum a_k\varphi_k\varphi_{k*} \in \mathcal{S}_\infty$ *iff* $\sum a_k = 0$.

Finally we have the following Favard type theorem. The proof is the same as the proof of [4, Theorem 11.9.4], and hence, we omit it. (For the reformulation of assumption (A4), we refer to [6, p. 13].)

Theorem 7 *(Favard). Let* $\{\varphi_n\}$ *be a sequence of rational functions, and assume that the assumptions given by* (A1)–(A3) *are satisfied, together with the assumption that*

(A4) $\Im\{F_n\} = \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{1}{|E_n|^2} - \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} \cdot \frac{1}{|E_{n-1}|^2}$, *if* $\alpha_{n-1} \in \overline{\mathbb{R}} \setminus \{0\}$, *respectively*

$$\Re\{F_n\}^2 + \big(\Im\{F_n\} - \mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})\big)^2 = \big[\mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})\big]^2 \frac{|E_{n-1}|^2}{|E_n|^2} \cdot \frac{\Delta_n}{\Delta_{n-1}},$$

if $\alpha_{n-1} \notin \overline{\mathbb{R}}$, $n = 1, 2, \ldots$, *where* $\Delta_n = |E_n|^2 - 4\frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} > 0$, *with* $E_0 \in \mathbb{C} \setminus \{0\}$.

Then there exists a functional M *on* $\mathcal{L}_\infty \cdot \mathcal{L}_{\infty*}$ *so that* $\langle f, g\rangle = M\{fg_*\}$ *defines a HPD inner product on* $\mathcal{L}_\infty$ *for which the* φ_n *form an orthonormal system.*

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