On Generalized Inverses of Matrices over Integral Domains

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ABSTRACT

It is proved that a matrix A over an integral domain admits a 1-inverse if and only if a linear combination of all the $r \times r$ minors of A is equal to one, where r is the rank of A. Some results on the existence of Moore-Penrose inverses are also obtained.

1. INTRODUCTION

Let R be an integral domain, i.e., a commutative ring with no zero divisors and with 1. We consider matrices and vectors over R.

Let A be an $m \times n$ matrix. An $n \times m$ matrix G is called a *1-inverse* (also called a generalized inverse, as in [4]) if G satisfies the equation

$$AGA = A. \tag{1}$$

One can easily see that G is a 1-inverse of A if and only if whenever $Az = y$ has a solution, Gy is a solution of $Az = y$. A matrix A is said to be *regular* if it has a 1-inverse.

The theory of 1-inverses of matrices over fields is quite well developed in the literature. The ring of integers Z and the ring $\mathbb{R}[x]$ of all polynomials in a variable x over the field of real numbers $\mathbb{R}$ are important examples of integral domains which are not fields. But these two rings are principal ideal rings, and for matrices over such rings the theory of 1-inverses was studied in [2], [3], [4], [5] and [6].

The ring $Z[x]$ of polynomials in a variable x over the ring of integers Z and the ring $\mathbb{R}[x, y]$ of polynomials in variables x and y over the field of real numbers $\mathbb{R}$ are two more important examples (see [6]) of integral domains. However, these two rings are not principal ideal rings, and so the results of [4]

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are not applicable. In [10] some nice characterizations of matrices over these two rings which admit 1-inverses were given.

It is the purpose of this paper to give necessary and sufficient conditions for a matrix over a general integral domain to admit a 1-inverse. The nonavailability of the Smith normal form for matrices over integral domains necessitates a treatment entirely different from that given in [4]. We obtain our results by a careful and fine analysis of the minors of a given matrix. We also obtain some results about Moore-Penrose inverses of matrices over suitable integral domains.

Let us once for all fix an integral domain R. An element of R is called a unit if it has an inverse. The determinantal rank of a matrix A is defined as the size of the largest nonvanishing determinantal minor and is denoted by $\rho(A)$. From the Cauchy-Binet formula (see [9, Exercise 2.6, p. 33] or [1, p. 398]) one readily sees that $\rho(AB) \leqslant \rho(A), \rho(B)$. In case R is a field the determinantal rank coincides with the usual concept of rank.

We shall use all the usual properties of determinants in the sequel. If A is an $m \times n$ matrix, $\alpha = \{i_1, i_2, \dots, i_r\} \subset \{1,2,\dots,m\}$, and $\beta = \{j_1, j_2, \dots, j_r\} \subset \{1,2,\dots,n\}$, then A^α_β stands for the submatrix of A determined by the rows with indices in α and the columns with indices in β. In case $\alpha = \{1,2,\dots,m\}$ we shall denote A^α_β by A_β, and in case $\beta = \{1,2,\dots,n\}$ we shall denote A^α_β by A^α. For a square matrix B, $|B|$ stands for the determinant of B, and if b_{ij} is the (i, j)th element of B, then $\partial|B|/\partial b_{ij}$ is the coefficient of b_{ij} in the expansion of $|B|$. $\mathrm{Tr}(B)$ stands for the trace of B, and B^T stands for the transpose of B.

We shall also use some results about compound matrices (see [8]). If A is an $m \times n$ matrix and if $r \leqslant m, n$, then the rth compound matrix $C_r(A)$ is the matrix whose (α, β) element is A^α_β where α runs over all r-element subsets of $\{1,2,\dots,m\}$ and β runs over all r-element subsets of $\{1,2,\dots,n\}$. From the Cauchy-Binet formula one easily sees that $C_r(AB) = C_r(A)C_r(B)$ (see [8]).

2. RIGHT INVERSES AND LEFT INVERSES

It is well known that a square matrix A over R has an inverse if and only if $|A|$ is a unit of R.

We shall start our results by giving conditions for the existence of right (or left) inverses.

THEOREM 1. *Let R be an integral domain, and let A be an $m \times n$ matrix over R. Then the following are equivalent*:

(i) *A has a right inverse.*

(ii) *A is regular and* $\rho(A)=m$.
(iii) $C_m(A)$ *has a right inverse.*

A similar result also holds for left inverses.

Proof. (i)$\Rightarrow$(ii): If B is a right inverse of A, $AB=I$. Since $\rho(I)=m$, we have that $\rho(A)=m$. Clearly B is a 1-inverse of A.

(ii)$\Rightarrow$(iii): If B is a 1-inverse of A, i.e., $ABA=A$, then we have that $C_m(A)C_m(B)C_m(A)=C_m(A)$. Observe that $C_m(A)$ is a row vector, $C_m(B)$ is a column vector, and since $\rho(A)=m$, $C_m(A)$ is a nonzero vector. This implies that (since R is an integral domain) $C_m(A)C_m(B)=1$, i.e., a linear combination of all the $m\times m$ minors of A is equal to one.

(iii)$\Rightarrow$(i): Suppose that $\Sigma|A_\beta|c_\beta=1$ for some elements c_β of R, with the summation taken over all subsets β of $\{1,2,\dots,n\}$ consisting of m indices. If we write $b_{ik}=\partial(\Sigma|A_\beta|c_\beta)/\partial a_{ki}$, then

$$\sum_{i=1}^{n} a_{ki}b_{ik}=\sum_{\beta}c_\beta\left\{\sum_{i\in\beta}a_{ki}\frac{\partial|A_\beta|}{\partial a_{ki}}\right\}=\sum_{\beta}c_\beta|A_\beta|=1$$

for any fixed k. Now for $l\neq k$, $\Sigma_{i=1}^{n}a_{li}b_{ik}$ can be expressed as $\Sigma_\beta|D_\beta|c_\beta$, where D_β stands for the β-columned minor of D, the matrix obtained from A by replacing the kth row of A with the lth row of A and keeping the rest of the rows as they were. Since $|D_\beta|=0$ for all β, we have that $\Sigma_{i=1}^{n}a_{li}b_{ik}=0$ if $l\neq k$. Thus if B is the $n\times m$ matrix whose (i,j)th element is b_{ij}, then B is a right inverse of A. ■

As a corollary we have

COROLLARY 2. *Let* A *be an* $m\times n$ *matrix with* $\rho(A)=m$. *Then the following are equivalent:*

(i) A *has a right inverse.*
(ii) A *is regular.*
(iii) *A linear combination of all the* $m\times m$ *minors of* A *is equal to one.*

REMARK 3. We in fact have that for a matrix A of full row rank a matrix G is a right inverse if and only if it is a 1-inverse.

The technique of the proof of Theorem 1 gives a criterion for the existence of 1-inverses for matrices which admit a *rank factorization*, i.e., for some B and C, $A=BC$ where B is of order $m\times r$, C is of order $r\times n$, and $r=\rho(A)$.

THEOREM 4. *Let A be a matrix of rank r which admits a rank factorization. Then A has a 1-inverse if and only if a linear combination of all the $r \times r$ minors is equal to one.*

Proof. Let $A = BC$ be a rank factorization of A. If G is a 1-inverse of A, then $BCGBC = BC$. Since B is a full column rank matrix, C is a full row rank matrix; and since R is an integral domain, we have that $CGB = I$. This implies that $C_r(C)C_r(G)C_r(B) = C_r(I) = 1$. Hence $\mathrm{Tr}(C_r(C)C_r(G)C_r(B)) = \mathrm{Tr}(C_r(B)C_r(C)C_r(G)) = \mathrm{Tr}(C_r(A)C_r(G)) = 1$, i.e., a linear combination of all the $r \times r$ minors of A is equal to one.

"If" part: If $A = BC$ is a rank factorization of A and if a linear combination of all the $r \times r$ minors of A is equal to one, by the formula $|A^\alpha_\beta| = |B^\alpha||C_\beta|$ we have that a linear combination of all the $r \times r$ minors of B is equal to one and also that a linear combination of all the $r \times r$ minors of C is equal to one. By Theorem 1, B has a left inverse B_L^{-1}, and C has a right inverse C_R^{-1}. Then clearly $C_R^{-1}B_L^{-1}$ is a 1-inverse of A. ∎

REMARK 5. Our task of characterizing all the matrices having 1-inverses would have been complete if every matrix over R had a rank factorization. However, this is not true. Let R be the ring generated by 1, x^2, xy, and y^2 in $\mathbb{R}[x, y]$. Then the matrix

$$\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$

has no rank factorization over R.

REMARK 6. We would like to bring out an important point of the above proof. If $A = DE$ and if a linear combination of all the $s \times s$ minors of A is equal to one, then a linear combination of all the $s \times s$ minors of D (and of E) is also equal to one.

3. 1-INVERSES

The results of the previous section make one conjecture that a matrix A over R of rank r has a 1-inverse if and only if a linear combination of all the $r \times r$ minors of A is equal to one. This indeed is true, as will be shown in Theorem 8.

As a preliminary to Theorem 8, let us first consider a special case, namely, $\rho(A)=1$.

THEOREM 7. *Let A be an $m\times n$ matrix with $\rho(A)=1$. Let a_{ij} stand for the (i,j)th element of A. Then A is regular if and only if a linear combination of all the elements of A is equal to one. If $\sum_{i,j}a_{ij}g_{ji}=1$, then the matrix G whose (i,j)th element is g_{ij} is a 1-inverse of A. Indeed, this form gives all the 1-inverses of A.*

Proof. Suppose that G is an $n\times m$ matrix such that $AGA=A$. Since $\rho(A)=1$, there are indices k and l such that $a_{kl}\neq 0$. Then $a_{kl}=\sum_{i,j}a_{kj}g_{ji}a_{il}$. Again, since $\rho(A)=1$, every 2×2 minor of A vanishes. So for any k, l, j, and i, $a_{kj}a_{il}=a_{kl}a_{ij}$. Hence $a_{kl}=a_{kl}\sum_{i,j}a_{ij}g_{ji}$, i.e. $\sum_{i,j}a_{ij}g_{ji}=1$.
Retracing the steps, we get the proof of the "if" part also. ■

Now we present our main theorem.

THEOREM 8. *Let A be an $m\times n$ matrix with $\rho(A)=r$. Then the following are equivalent:*

(i) *A is regular.*
(ii) *$C_r(A)$ is regular.*
(iii) *A linear combination of all the $r\times r$ minors of A is equal to one.*

We need a result on compound matrices for the proof of this theorem. This result is known [8, p. 171], but we shall supply a simple proof here.

LEMMA 9. *Let A be an $m\times n$ matrix with $\rho(A)=r$. Then $\rho(C_r(A))=1$. In other words $C_r(A)$ is a nonzero matrix, and if α,γ are two subsets of $\{1,2,\dots,m\}$ and β,δ are two subsets of $\{1,2,\dots,n\}$ each containing r indices, then $|A^{\alpha}_{\beta}||A^{\gamma}_{\delta}|=|A^{\alpha}_{\delta}||A^{\gamma}_{\beta}|$.*

Proof. Without loss of generality let us assume that $|A^{\alpha}_{\beta}|\neq 0$. Let F be the field of quotients of R. Then A, considered as a matrix over F, is of rank r, and $\rho(A^{\alpha}_{\beta})=r$. Hence there is a matrix B over F such that $A^{\alpha}_{\delta}=A^{\alpha}_{\beta}B$. Similarly, since the r rows of A with the indices from α form an independent set and since $\rho(A)=r$, every row vector in A is a linear combination (with coefficients from F) of the rows of A with indices in α. Thus there is a matrix C over F such that $A^{\gamma}_{\beta}=CA^{\alpha}_{\beta}$ and $A^{\gamma}_{\delta}=CA^{\alpha}_{\delta}=CA^{\alpha}_{\beta}B$. Since all the matrices involved are square matrices, we have our result. ■

Proof of Theorem 8. (i) ⇒ (ii) is clear.

(ii) ⇒ (iii) follows from Theorem 7, because $\rho(C_r(A))=1$ by Lemma 9.

(iii) ⇒ (i): Suppose that $\Sigma_\alpha\Sigma_\beta|A^\alpha_\beta|c_{\alpha\beta}=1$ for some elements $c_{\alpha\beta}$ from R, where the summation is taken over all subsets α of $\{1,2,\dots,m\}$ and β of $\{1,2,\dots,n\}$ consisting of r indices. Then for any $1\leqslant k\leqslant m$ and $1\leqslant l\leqslant n$, we have

$$\sum_\alpha\sum_\beta a_{kl}|A^\alpha_\beta|c_{\alpha\beta}=a_{kl}, \tag{$*$}$$

where the summation is taken as before.

For any fixed $\alpha=\{i_1,i_2,\dots,i_r\}$ and $\beta=\{j_1,j_2,\dots,j_r\}$ consider the matrix

$$B=\left[\begin{array}{c|c} & a_{i_1l}\\ & a_{i_2l}\\ A^\alpha_\beta & \vdots\\ & a_{i_rl}\\ \hline a_{kj_1},a_{kj_2},\dots,a_{kj_r} & a_{kl}\end{array}\right].$$

Since $\rho(A)=r$ (irrespective of whether $k\in\alpha$ or $l\in\beta$ or not), $|B|=0$. Hence

$$a_{kl}|A^\alpha_\beta|=\sum_{i\in\alpha}\sum_{j\in\beta}a_{kj}a_{il}\frac{\partial|A^\alpha_\beta|}{\partial a_{ij}}.$$

The equation $(*)$ becomes

$$a_{kl}=\sum_\alpha\sum_\beta\left[\sum_{i\in\alpha}\sum_{j\in\beta}a_{kj}a_{il}\frac{\partial|A^\alpha_\beta|}{\partial a_{ij}}\right]c_{\alpha\beta}.$$

By interchanging the summations inside and outside the square brackets we obtain

$$a_{kl}=\sum_i\sum_j a_{kj}a_{il}\left[\sum_{\alpha:\,i\in\alpha}\;\sum_{\beta:\,j\in\beta}\frac{\partial|A^\alpha_\beta|c_{\alpha\beta}}{\partial a_{ij}}\right]$$

If we call the quantity inside the square brackets g_{ji}, then we have that the matrix G whose (i,j)th element is g_{ij} is a 1-inverse of A. ■

It is interesting to note that

$$g_{ji} = \frac{\partial}{\partial a_{ij}} \left(\sum_{\alpha} \sum_{\beta} |A^{\alpha}_{\beta}| c_{\alpha\beta} \right).$$

REMARK 10. Theorem 8 applied to matrices over the ring of polynomials in several variables with integer coefficients solves a problem posed in the last section of [6]. An earlier solution was obtained in [10].

REMARK 11. The absence of the Euclidean algorithm and the absence of the Smith normal form for matrices over $Z[x]$ and for matrices over $\mathbb{R}[x, y]$ forbid us to use the algorithm given in [4] to calculate the 1-inverses. It would be interesting and useful to obtain an algorithm to calculate the 1-inverses of matrices over $Z[x]$ and $\mathbb{R}[x, y]$ when they exist.

REMARK 12. If R is a commutative ring without zero divisors and without 1 (i.e., an integral domain without the multiplicative identity), no matrix over R has a 1-inverse. This follows from the proof of Theorem 8 and from the fact that for a, b from such an R, ab can never be equal to b. Thus to talk about the existence of 1-inverses of matrices over a commutative ring without zero divisors, it is essential to assume that there is a multiplicative identity in the ring.

REMARK 13. If R is a commutative ring with zero divisors, the problem of characterizing matrices over R admitting 1-inverses seems interesting, especially because the techniques of this paper break down in that case.

4. MOORE-PENROSE INVERSES

For matrices over an integral domain R consider the Moore-Penrose equations

$$AGA = A, \tag{1}$$

$$GAG = G, \tag{2}$$

$$(AG)^T = AG, \tag{3}$$

$$(GA)^T = GA. \tag{4}$$

For an $m \times n$ matrix A, an $n \times m$ matrix G which satisfies all the above equations is called a $\{1,2,3,4\}$-inverse or a Moore-Penrose inverse of A and is denoted by $A^{\dagger}$ (it is necessarily unique).

The following theorem, which is an analogue of Theorem 6 of [4], helps us characterize all these matrices over $Z[x]$ which have Moore-Penrose inverses.

THEOREM 14. *Let the integral domain R satisfy the condition*

$$a_1 = a_1^2 + a_2^2 + \cdots + a_n^2 \quad \textit{implies that} \quad a_2 = a_3 = \cdots = a_n = 0.$$

Then for a matrix A, $A^{\dagger}$ exists if and only if there exist permutation matrices P and Q and a unit M (i.e., $|M|$ is a unit of R) such that

$$A = P\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}Q.$$

In this case

$$Q^T\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^T = A^{\dagger}.$$

Proof. Using the hypothesis on R, it is easily seen that if $E^2 = E = E^T$, then there is a permutation matrix P such that

$$P^TEP = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Now suppose $G = A^{\dagger}$ exists. Then by the above observation there are permutation matrices P and Q such that

$$P^TAGP = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad QGAQ^T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$P^TAQ^T = (P^TAG^T)(QGPP^TAQ^T) = P^TAQ^T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M & 0 \\ A_3 & 0 \end{bmatrix}$$

and

$$P^TAQ^T = (P^TAQ^TQGP)(P^TAQ^T) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P^TAQ^T = \begin{bmatrix} M & A_2 \\ 0 & 0 \end{bmatrix}$$

for some M, A_2, and A_3. This implies that

$$P^TAQ^T = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, since A has a 1-inverse, $\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ also has a 1-inverse, and by Theorem 10, a linear combination of all the $r \times r$ minors of $\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$ is equal to one. Since M is of order $r \times r$, where r is the rank of A, we have that $|M|$ is a unit of R. ■

We shall now show

COROLLARY 15. *For a matrix A over the ring $Z[x_1, x_2, \ldots, x_n]$ of polynomials in several variables with integral coefficients, $A^\dagger$ exists if and only if there are permutation matrices P and Q and a unit M such that*

$$A = P\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}Q.$$

Proof. Let us verify that the ring $Z[x_1, x_2, \ldots, x_n]$ satisfies the hypothesis of Theorem 14. For this, since Z satisfies that hypothesis, it is sufficient to verify that $R[x]$ satisfies the hypothesis of Theorem 14 whenever R does.

If $a_1 = a_1^2 + a_2^2 + \cdots + a_n^2$ where $a_1, a_2, \ldots, a_n$ are elements of $R[x]$, then by equating the coefficients of the maximum positive degree possible in the equation one sees that the degree of x in the equation is reduced. A repeated application of this procedure reduces the equation to an equation in R. Hence the result. ■

REMARK 16. The proof of Theorem 14 given here is considerably simpler than the proof of Theorem 6 of [4] and avoids the use of the Smith normal form.

Similar to Theorem 2 of [5], using the technique of the proof of Theorem 14, one can also obtain a characterization of all the matrices over $\mathbb{R}[x, y]$ which admit Moore-Penrose inverses.

THEOREM 17. *Let A be a matrix over* $\mathbb{R}[x, y]$. *Then* $A^\dagger$ *exists if and only if there exist (real) orthogonal matrices P and Q and a unit M (i.e.,* $|M|$ *is a nonzero real number) such that*

$$A = P\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}Q.$$

In this case

$$Q^T\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^T = A^\dagger.$$

REMARK 18. Concerning the existence of {1,3}-inverses and {1,4}-inverses, analogues of Theorem 7 of [4] for matrices over $Z[x, y]$ and Theorem 6 of [5] for matrices over $\mathbb{R}[x, y]$ can also be obtained.

Thanks are due to Dr. A. R. Rao for many valuable discussions during the preparation of the original version of this paper, which contained most of the present results in a different form. The use of compound matrices was suggested by the referee, and this led to the present greatly improved version of the original paper. Part (ii) of Theorem 8 and the availability of Lemma 9 in [8] were suggested by the referee. The author also expresses his appreciation to the referee for suggestions on improving the presentation and content of the paper. The original version of this paper was written in February 1980, and the author's attention was drawn at a later date to [10].

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Received 11 September 1981; revised 18 June 1982