NO DENSE METRIZABLE $G_δ$-SUBSPACES IN BUTTERFLY SEMI-METRIZABLE Baire SPACES

Dennis K. BURKE and Eric K. van DOUWEN

Department of Mathematics, Miami University, Oxford, OH 45050, USA
Institute for Medicine and Mathematics, Ohio University, Athens, OH 45701, USA

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A class of Baire spaces, which contains many known examples and variations thereof, is described and it is shown that no space in this class contains a dense metrizable $G_δ$-subspace. This gives a class of semi-metrizable spaces which are not $σ$-spaces. We discuss the existence of Lindelöf semi-metrizable spaces which are not $σ$-spaces. This is of interest since the only known examples require the use of CH.

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1. Introduction

There is considerable interest in the question of when a space has a nice (usually meaning metrizable) dense subspace, see e.g., [1, 9, 11]. If the space is a Baire space one would hope for a nice dense $G_δ$-subspace. For example, Hodel showed that a Baire $p$-space has a dense $G_δ$-subspace which is a paracompact $p$-space [11, proof of Lemma 5.2], and used this to show that a quasi-complete Baire space with a $G_δ$-diagonal or with a point-countable point-separating open cover has a dense metrizable $G_δ$-subspace, [11, proof of Lemma 5.3], thereby generalizing the fact that a pseudo-complete Moore space has a dense metrizable $G_δ$-subspace, [1, Remark 2.4]. Also, in [6] it was noted that a Baire $σ$-space has a dense metrizable $G_δ$-subspace.

In this note we introduce butterfly spaces, which are of interest because many examples are butterfly spaces. We show that a separable Baire butterfly space cannot have a dense metrizable $G_δ$-subspace, hence cannot be a $σ$-space, which is of independent interest. We also discuss semi-metrizable spaces which are not $σ$-spaces; we give a construction of a Lindelöf such space from CH and also construct a normal non-Lindelöf such space from $MA + \neg CH$. 
The $T_1$ space $X$ is **semi-metrizable** if there is a function

$$U : X \times \omega \to \{\text{subsets of } X\}$$

such that $\{U(x, n) : n \in \omega\}$ is a neighborhood base at $x$ ($x \in X$), and $x \in U(y, n)$ if and only if $y \in U(x, n)$ ($x, y \in X, n \in \omega$). This is equivalent to the usual definition and is easier to use. A space is semi-metrizable if and only if it is first countable and semi-stratifiable (this we do not define), [5, Corollary 1.4].

A **network** for a space $X$ is a family $\mathcal{A}$ of sets such that if $x \in U$ and $U$ is open in $X$, then $x \in A \subseteq U$ for some $A \in \mathcal{A}$. A space is a **$\sigma$-space** if it has a $\sigma$-discrete closed network. (We add “closed”, which is not standard yet, because that is what everybody uses.) A $\sigma$-space is semi-stratifiable.

A subset of space $X$ is called **residual** if it includes the intersection $\cap$ some countable family of dense open subsets of $X$.

We define $c, d, \pi$ and $w$ (cellularity, density, $\pi$-weight and weight) as usual, cf. [13].

### 3. Butterfly spaces

**Definition.** A pair $(X_1, X_2)$ is a **butterfly pair** if:

1. $X_1$ and $X_2$ have the same underlying set, denoted by $X$, and the topology of $X_1$ is finer than the topology of $X_2$.
2. Each $x \in X$ has arbitrary small neighborhoods $U$ in $X_1$ such that $U - \{x\}$ is open in $X_2$.
3. Each $x \in X_1$ has a neighborhood $U$ in $X_1$ such that $x \in \text{Cl}_{X_1} \text{Int}_{X_1} (X_2 - U)$.

If $A \subseteq X$, then we write $A_i$ if we think of $A$ as a subspace of $X_i$ ($i = 0, 1$).

The space $X$ is a **butterfly space** if some space $Y$ makes $(X, Y)$ a butterfly pair.

#### 3.1. Examples.

(a) The Sorgenfrey line is a butterfly space.

(b) Let $K$ be the complex plane. For $x \in K$ and $n \in \omega$ define

$$U(x, n) = \{x\} \cup \{y \in K : |x - i \cdot 2^{-n} - y| < 2^{-n} \text{ or } |x - i \cdot 2^{-n} - y| < 2^{-n}\}.$$ 

Let $X$ be $K$, retopologized by declaring $\{U(x, n) : n \in \omega\}$ to be a neighborhood base of $x$ ($x \in X$). Then $(X, K)$ is a butterfly pair. $X$ is similar to Heath's "bow-tie" space in [10, Example 2.7]. Note that $X$ is semimetrizable.

(c) Let $R$ be the real line. For $x \in R$ and $n \in \omega$ define

$$U(x, n) = \{x\} \cup \bigcup_{m \geq n} \{y \in R : \left| x + \frac{1}{2^m} - y \right| < \frac{1}{n2^m + 2} \text{ or } \left| x - \frac{1}{2^m} - y \right| < \frac{1}{n2^m + 2}\}.$$
Let $Y$ be $R$, retopologized by declaring $\{U(x, n) : n \in \omega\}$ to be a neighborhood base $(x \in Y)$. Then $\langle Y, R \rangle$ is a butterfly pair, the line analogue of (b). Again $Y$ is semi-metrizable.

The usefulness of the notion of a butterfly pair is the interaction of the two topologies; if $\langle X_1, X_2 \rangle$ is a butterfly pair one can get considerable information about $X_1$ if one knows enough about $X_2$. This is illustrated by the following Lemma, the easy proof of which we omit.

3.2. Lemma. Let $\langle X_1, X_2 \rangle$ be a butterfly pair and let $X$ be the underlying set.

(a) If $A \subset X$, then $A$ is dense in $X_1$ if and only if $A$ is dense in $X_2$.

(b) If $A \subset X$ is dense in $X_1$, then $\langle A_1, A_2 \rangle$ is a butterfly pair.

(c) If $A \subset X$, then $A$ is nowhere dense/residual in $X_1$ iff $A$ is nowhere dense/residual in $X_2$, hence $X_1$ is Baire iff $X_2$ is Baire.

(d) $\phi(X_1) = \phi(X_2)$ if $\phi$ is one of $c$, $d$, $\pi$.

(e) If $A$ is closed discrete in $X_1$, then so is $\text{Cl}_{X_2} A$.

3.3. Lemma. If $X$ is a butterfly space, then $\omega(X) \geq |X|$.

Proof. Let $Y$ be a space making $\langle X, Y \rangle$ a butterfly pair. For each $x \in X$

$$\mathcal{B}_x = \{U \subset X : U \text{ open in } X, X \in U, U - \{x\} \text{ open in } Y, x \in \text{Cl}_Y \text{ Int}_Y(Y - U)\}$$

is a local base at $x$ in $X$, hence $\bigcup_{x \in X} \mathcal{B}_x$ is a base for $X$. As is well known, there must be a subfamily $\mathcal{A}$ of $\bigcup_{x \in X} \mathcal{B}_x$ with $|\mathcal{A}| = \omega(X)$ such that $\mathcal{A}$ is a base. Let $U \in \mathcal{A}$ be a neighborhood of $x$ in $X$ such that $x \in \text{Cl}_Y \text{ Int}_Y(Y - U)$. Then $U \notin \mathcal{B}_y$ if $y \neq x$, hence $|\mathcal{A}| \geq |X|$.

3.4. Theorem. (See also [19]). If $X$ is a Baire butterfly space with $c(X) = \omega$ (e.g., because $X$ is separable), then $X$ does not have a dense metrizable $G_\delta$-space.

Proof. Suppose $M$ is a dense metrizable subspace of $X$. Then $c(M) = c(X)$, so $M$ must be second countable. $M$ is a butterfly space by Lemma 3.2 (b), hence $|M| < w(M)$ by Lemma 3.3. Consequently $M$ is countable. Since $M$ is dense in $X$ and $X$ is Baire, $M$ cannot be a $G_\delta$.

3.5. Corollary. If $X$ is a Baire butterfly space with $c(X) = \omega$, then $X$ is not a $\sigma$-space.

Proof. As noted in [6], a Baire $\sigma$-space has a dense metrizable $G_\delta$-subspace.

Combining our results we get

3.6. Example. There is a semi-metrizable completely regular Baire space which has no dense metrizable $G_\delta$-subspace, hence is not a $\sigma$-space.
Proof. The space $X$ of Example 3.1(b) or the space $Y$ of Example 3.1(c). That these spaces are completely regular is known, at least for $X$, and is not hard to prove. The spaces are Baire spaces by Lemma 3.2(c).

A similar example of a semi-metrizable Baire space without a dense metrizable $G_δ$-subspace has been given independently by White [19], who also showed that a semi-metrizable Baire space does have a dense metrizable subspace. Kofner has given a similar example of a semi-metrizable, hence semi-stratifiable space that is not a $σ$-space [14].

We conclude this section with a result stating that if $(X_1, X_2)$ is a butterfly pair with $X_2$ completely metrizable and $X_1$ semi-metrizable, then $X_1$ is not Lindelöf. This proposition lays to rest the hope that a semi-metrizable Lindelöf space, which is not a $σ$-space, might be found by constructing a butterfly space on the real line or complex plane. Note that Example 3.1(b) is easily seen to be non-Lindelöf since every horizontal line is an uncountable closed discrete subset, however, it is not as easy to show that Example 3.1(c) is not Lindelöf without using the following proposition. There are also modifications of Example 3.1(b) for which it is not easy to show directly that the space is not Lindelöf.

3.7. Proposition. Let $(X_1, X_2)$ be a butterfly pair. If $X_1$ is semi-metrizable and $X_2$ is completely metrizable, then $X_1$ has a closed discrete subspace of cardinality $2^ω$ (hence, $X_1$ is not Lindelöf).

Proof. Let $U : X_1 \times ω → \{\text{subsets of } X_1\}$ be as in the definition of semi-metrizability, such that

(0) if $A ⊂ X_1$ and there is an $m ∈ ω$ with $a \notin U(b, m)$ for distinct $a, b ∈ A$, then $A$ is closed in $X_1$.

The existence of such a $U$ follows immediately from [3]. Let $d$ be a compatible metric for $X_2$. For $n ∈ ω$ put

$$Y_n = \{x ∈ X_2 : x ∈ Cl_{X_2}\{x : Int_{X_2}(X_2 - U(x, n))\}\}.$$

Then $X_2 = \bigcup_{n ∈ ω} Y_n$, hence there are $n, n' ∈ ω$ and open $W$ in $X_2$ such that $Y_n ∩ W$ is dense in $W$. With an easy recursion on $n ∈ ω$ construct $A = \{a_n : n ∈ ω\} ⊂ X_1$ such that

1. $a_{n+1} ∈ W \cap (\bigcap_{k < n} Int_{X_2}(X_2 - U(a_k, m)))$;
2. for all $n, k ∈ ω$ there is an $i ∈ ω$ with $d(a_n, a_m) < 1/2^k$.

Then $A$ is dense in itself as a subspace of $X_2$, hence

(3) $|Cl_{X_2}A| = 2^ω$, since $X_2$ is completely metrizable.

Since $x ∈ U(y, m)$ if and only if $y ∈ U(x, m)$ ($x, y ∈ X$), it follows from (1) that

4. if $a, b ∈ A$ are distinct, $a ∉ U(b, m)$.

This implies that $A$ is relatively discrete in $X_1$, and also that $A$ is closed in $X_1$ by (0).

The conclusion now follows from (3) and 3.2(e).
4. Remarks

E.S. Berney [3], E.A. Michael [16] and N.V. Veličko [18] constructed semi-metrizable Lindelöf spaces which are not $\sigma$-spaces from CH. These examples are butterfly spaces. Here we wish to comment on such spaces.

The technique of [2, Theorem 5] and [8, Theorem 1] can be used to construct from CH a dense Lusin subspace (=uncountable subspace, any nowhere dense subset of which is countable [8]) of the semi-metrizable spaces of Example (3.1(b) or (c)). This subspace is a Lindelöf Baire semi-metrizable space which is not a $\sigma$-space by 3.2(b) and 3.5 (cf. [18]). A modification of this technique yields a separable normal Baire subspace under MA, see [17]; this subspace need not be Lindelöf under MA + $\neg$CH. [Since under $2^\omega < 2^{\omega_1}$ every separable normal space is $\omega_1$-compact, [12], and $\omega_1$-compactness is equivalent to Lindelöfness for semi-metrizable spaces, MA + $\neg$ CH is essential.]

It is worth noting that in ZFC there probably is no semi-metrizable Lindelöf space which is not a $\sigma$-space, and which is the $X_1$ of a butterfly pair $\langle X_1, X_2 \rangle$, with $X_2$ a separable metrizable Baire space. Indeed, the argument of 3.7 suggests that the only way to ensure that $X_1$ is Lindelöf is for $X_2$ to be a Lusin space. However, under MA + $\neg$CH, there does not exist a Hausdorff Lusin space without isolated points [15].

References