Journal of Algebra **225**, 904–935 (2000)
doi:10.1006/jabr.1999.8179, available online at http://www.idealibrary.com on IDEAL®

Some Homological Invariants of Local PI Algebras

Q. S. Wu

Institute of Mathematics, Fudan University, Shanghai 200433, China

J. J. Zhang

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195
E-mail: zhang@math.washington.edu

Communicated by J. T. Stafford

Received August 20, 1998

Key Words: semilocal PI algebra; Auslander–Buchsbaum formula; depth; local cohomology; Morita duality.

1. INTRODUCTION

During the last several years, Artin, Jørgensen, Stafford, Yekutieli, and others have generalized many homological results from the commutative case to the noncommutative connected graded case [AZ, SZ1, Jo1, Jo2, Ye]. In this paper we generalize some of those to the noncommutative local case. Our main interest is on noetherian local PI (polynomial identity) algebras, though some results hold in a more general setting.

Throughout A is an algebra over a base field k, $\mathfrak{m}$ is its Jacobson radical, and $A_0 = A/\mathfrak{m}$. If A_0 is artinian, A is called *semilocal*. If A_0 is a simple artinian algebra (respectively, a division algebra), then A is called *local* (respectively, *scalar local*). Unless otherwise stated we are working with left modules and sometimes we use the term A^{op}-module for a right A-module where A^{op} is the opposite ring of A. A finite A-module means finitely generated over A. We use noetherian (respectively, artinian) for two-sided noetherian (respectively, two-sided artinian).

0021-8693/00 $35.00

For the convenience of the reader we review some basic notations of derived categories, the χ condition, and completion in Section 2. For more details about derived categories see [Ha2].

In Sections 3 and 4 we prove the Auslander–Buchsbaum formula, which relates the depth (see 3.2) and the projective dimension of a complex.

THEOREM 1.1. *Let A be either a noetherian local PI algebra or a noetherian scalar local algebra satisfying χ. Then the Auslander–Buchsbaum formula holds; namely, if $X \in D^b_{fg}(\text{Mod } A)$ with $\operatorname{pd} X < \infty$, then*

$$\operatorname{pd} X + \operatorname{depth} X = \operatorname{depth} A.$$

A version of this was proved by Nishida [Ni, 1.6] for Gorenstein orders, which are finite over their centers. The graded version was proved by Jørgensen [Jo2, 3.2].

Let lfPD be the left little finitistic dimension (see 6.9). It is well known that if A is a noetherian local ring with finite global dimension then $\operatorname{lfPD} A = \operatorname{gldim} A$. If moreover A is PI, then $\operatorname{gldim} A = \operatorname{Kdim} A$ where Kdim denotes the Krull dimension. An immediate consequence of the Auslander–Buchsbaum formula is that $\operatorname{lfPD} A$ is bounded by $\operatorname{depth} A$ without assuming finite global dimension.

COROLLARY 1.2. *Let A be a noetherian local PI algebra. Then*

$$\operatorname{lfPD} A \le \operatorname{depth} A \le \operatorname{Kdim} A = \operatorname{lcd}(A).$$

The proof of 1.2 is given in Section 6. We remark that the second inequality of 1.2 follows by local cohomology. Since every noetherian (semi)local PI algebra has finite Krull dimension [MR, 13.7.15 and 6.4.8], lfPD of a noetherian local PI algebra is finite.

By completion we always mean the completion with respect to the $\mathfrak{m}$-adic topology and write $\hat{M} = \varprojlim M/\mathfrak{m}^n M$. It is easy to show that $\operatorname{gldim} A = \operatorname{gldim} \hat{A}$. The following statements might be known but we could not find references for them, so proofs are given in Sections 5 and 8 (see 5.7(2) and 8.7(2)).

PROPOSITION 1.3. *Let A be a noetherian semilocal PI algebra and $\hat{A}$ its completion. Then*

(1) $\operatorname{id} X = \operatorname{id}_{\hat{A}} \hat{X}$ *for all* $X \in D^b_{fg}(\text{Mod } A)$.

(2) $\operatorname{Kdim} M = \operatorname{Kdim}_{\hat{A}} \hat{M}$ *for all finite A-modules M.*

Some basic properties of Morita duality are given in Section 7. Sections 7 and 8 are devoted to proving 1.4 and 1.5 below.

THEOREM 1.4. *Let A be a noetherian PI algebra. Suppose an (A, T)-bimodule ${}_A E_T$ defines a Morita duality between A and T. Then T is noetherian and PI.*

Vámos [Va] proved that if A is noetherian, complete, semilocal, and PI, then there exists an (A, T)-bimodule E, which induces a Morita duality, for some right noetherian, complete, and semilocal ring T. We use local cohomology to show that T is PI and left noetherian. If A is complete, local, and finite over its noetherian center, then A admits a Morita self-duality, which is induced by the Matlis duality of the center. It is unknown if A in 1.4 always admits a Morita self-duality.

THEOREM 1.5. *Let A be a noetherian semilocal PI algebra and M be a finite A-module. Then*

(1) $\operatorname{lcd}(M) = \operatorname{Kdim} M$.

(2) *If $d = \operatorname{Kdim} M$, then $\mathrm{H}^d_{\mathfrak{m}}(M)$ has dual Krull dimension d.*

(3) *For all i, $\mathrm{H}^i_{\mathfrak{m}}(M)$ has dual Krull dimension no more than i.*

The proof of 1.5 uses Morita duality and 1.4. In the commutative local case, 1.5(1) was first proved in [Gr, Proposition 6.4(4)]. An elementary proof was given in [MS] by using secondary representation of artinian modules which is not available in the noncommutative case. One way to view 1.5 is that the dualizing complex over (A, T) has the Auslander property [YZ, 2.1 and 2.14]. The existence of dualizing complexes is proved in [WZ]. The proof of 1.5 is very similar to the proof of [SZ1, 3.10] and in the case when A is noetherian local PI with finite global dimension, 1.5 and [SZ1, 3.10] are essentially equivalent.

Section 9 contains some examples. In particular, we modify an example of Stafford [SZ2] to show that noetherian, scalar local, complete algebras need not satisfy the weak χ condition nor the Auslander–Buchsbaum formula.

If an algebra is finite or integral over its noetherian center, then some versions of the above results are known to various authors [BH1, BH2, GN, Ni, SZ1]. We remark that not every complete, noetherian, local, PI algebra is integral over its center (see [SZ1, 5.13] or the completion of the ring in [SZ1, 5.10]). On the other hand, if A is a noetherian, local, PI algebra with finite global dimension, then A is integral over its center [SZ1, 1.4].

2. PRELIMINARY

First we recall a few definitions and notations about complexes and derived categories. We denote the homotopy categories by K and derived categories by D. Let A be an algebra and let $\operatorname{Mod} A$ be the category of left A-modules. We write $K(A)$ for $K(\operatorname{Mod} A)$ and $D(A)$ for $D(\operatorname{Mod} A)$. These categories $K(A)$ and $D(A)$ will be equipped with superscripts and/or subscripts in most cases. The superscript "$^+$" (respectively "$^-$",

respectively "^{b}") decorates the sign for categories of left-bounded (respectively right-bounded, respectively bounded) complexes, while the subscript "$_f$" decorates the sign for categories of complexes with finite (i.e., finitely generated) cohomological modules. For example, $D_f^-(A)$ is the derived category whose objects are right-bounded complexes with finite cohomological modules.

Let $X \in K^-(A)$. A free resolution of X is a complex $Y \in K^-(A)$ consisting of free modules such that there is a quasi-isomorphism $Y \xrightarrow{\cong} X$. A complex $Y \in K^-(A)$ consisting of free modules is called finite if it consists of finite free modules. A complex $Y \in K^-(A)$ consisting of free modules is called minimal if the image $B^i(Y)$ of the boundary map is in $\mathfrak{m}Y^i$ for each i. A projective, or flat, resolution of $X \in K^-(A)$ is similarly defined.

Let $X \in K^+(A)$. An injective resolution of X is a complex $Y \in K^+(A)$ consisting of injective modules such that there is a quasi-isomorphism $X \xrightarrow{\cong} Y$. A complex $Y \in K^+(A)$ consisting of injective modules is called minimal if the kernel $Z^i(Y)$ of the boundary map is essential in Y^i for each i.

Given any complex X, define

$$\sup X = \sup\{m \mid h^m(X) \neq 0\} \qquad \text{and} \qquad \inf X = \inf\{m \mid h^m(X) \neq 0\},$$

where $h^i(X)$ is the ith cohomology of X. Each $X \in K^-(A)$ has a free, in particular, a projective and a flat resolution. The resolution can be chosen to consist of modules vanishing above $\sup X$. If A is left noetherian then each $X \in K_f^-(A)$ has a finitely generated projective resolution and if, further, A is semiperfect, then each $X \in K_f^-(A)$ has a finitely generated minimal projective resolution. This resolution can also be chosen to consist of modules vanishing above $\sup X$. The projective dimension of X is

$$\operatorname{pd} X = \min_Y\{-\min\{i \mid Y^i \neq 0\}\},$$

where Y ranges over all projective resolutions of X. If X has a minimal projective resolution Y, then $\operatorname{pd} X = -\min\{i \mid Y^i \neq 0\}$. Each $X \in K^+(A)$ has a minimal injective resolution and the resolution can be chosen to consist of modules vanishing below $\inf X$. The injective dimension of X is

$$\operatorname{id} X = \max\{i \mid Y^i \neq 0\},$$

where Y is the minimal injective resolution of X.

The right derived functor of $\operatorname{Hom}: K(A)^{\mathrm{op}} \times K(A) \to K(Ab)$ is denoted by $R\operatorname{Hom}$ and the left derived functor of $\otimes$: $K(A) \times K(A) \to K(Ab)$ is denoted by ${}^L\otimes$. As usual,

$$\operatorname{Ext}_A^i(X, Y) := h^i R\operatorname{Hom}_A(X, Y) \qquad \text{and} \qquad \operatorname{Tor}_i^A(X, Y) := h^{-i}(X\, {}^L\otimes_A Y).$$

It follows from the standard arguments that

$$\operatorname{id} X = \max\{i \mid \operatorname{Ext}^i_A(M, X) \neq 0 \text{ for some } M \in \operatorname{Mod} A\}$$

and if A is left noetherian we only need to consider the finite A-modules M. Similarly

$$\operatorname{pd} X = \max\{i \mid \operatorname{Ext}^i_A(X, M) \neq 0 \text{ for all } M \in \operatorname{Mod} A\}$$

and if A is left noetherian and $\operatorname{pd} X < \infty$ then

$$\operatorname{pd} X = \max\{i \mid \operatorname{Ext}^i_A(X, A) \neq 0\}.$$

Next we introduce a version of the χ condition which is suitable for our purpose. The original χ condition is given in [AZ, 3.2 and 3.7] for graded algebras.

DEFINITION 2.1. Let A be a left noetherian semilocal algebra.

(1) We say M satisfies the *χ condition*, or $\chi(M)$ holds, if $\operatorname{Ext}^i_A(A_0, M)$ is an A-module of finite length for any i. We say an algebra A satisfies the *χ condition*, or χ holds for A, if every finite A-module satisfies the χ condition.

(2) Let $d = \operatorname{depth} A$. We say A satisfies the *weak χ condition* if $\operatorname{Ext}^d_A(A_0, A)$ contains a simple A^{op}-submodule. If $\operatorname{depth} A = \infty$, the weak χ condition is vacuous.

The χ condition holds for several classes of rings [AZ, Sect. 8; SZ1, 3.5]. But not every noetherian local algebra satisfies χ or weak χ (see 9.4 and 9.6).

LEMMA 2.2. *Let A be a noetherian local algebra such that the $\chi(A)$ holds. Then A satisfies the weak χ condition.*

Proof. If $d = \operatorname{depth} A$ is finite, $\operatorname{Ext}^d_A(A_0, A)$ is nonzero. By $\chi(A)$, $\operatorname{Ext}^d_A(A_0, A)$ has finite length. Since it is right noetherian, by Lenagan's lemma [GW, 7.10], it is right artinian. Hence $\operatorname{Ext}^d_A(A_0, A)$ contains a simple right A-module. ∎

LEMMA 2.3. *Let A be a left noetherian semilocal algebra such that the χ condition holds. Then $\operatorname{Ext}^i_A(A/\mathfrak{m}^n, X)$ is of finite length for all $X \in D^b_f(A)$, all i and all n.*

Proof. Since $R\operatorname{Hom}_A(A_0, -): D(A) \to D(A)$ is a way-out right functor, it follows from the χ condition and [Ha2, I.7.3 (ii)] that $\operatorname{Ext}^i_A(A_0, X)$ is of finite length for all $X \in D^+_{fg}(A)$ and all i. Let $E(M)$ denote the injective hull of any A-module M. Let $T_n = \operatorname{Hom}_A(A/\mathfrak{m}^n, E(A_0))$. We claim that T_n is of finite length. It is trivial for $n = 1$ because $T_1 = A_0$. Now suppose

$n > 1$ and suppose T_{n-1} is of finite length. Consider the minimal injective resolution of T_{n-1},

$$0 \longrightarrow T_{n-1} \longrightarrow E(A_0) \longrightarrow E(E(A_0)/T_{n-1}) \longrightarrow \cdots;$$

one obtains $\mathrm{Ext}^1_A(A_0, T_{n-1}) = \mathrm{soc}(E(E(A_0)/T_{n-1})) = T_n/T_{n-1}$. By χ, T_n/T_{n-1} is of finite length. Therefore so is T_n.

Let Y be a minimal injective resolution of X and I the $\mathfrak{m}$-torsion subcomplex of Y. By the first paragraph, $\mathrm{soc}(I^i) = \mathrm{Ext}^i(A_0, X)$ is of finite length. This implies that each I^i is a submodule of some finite direct sum of $E(A_0)$. Let $I_n = \mathrm{Hom}_A(A/\mathfrak{m}^n, I)$. Then I_n^i is a submodule of some finite direct sum of T_n, which is of finite length by the last paragraph. Thus $\mathrm{Ext}^i_A(A/\mathfrak{m}^n, X) = h^i(I_n)$ is of finite length. ∎

In the rest of this section we recall some facts on the completion. Let $\hat{M}$ denote $\varprojlim M/\mathfrak{m}^n M$.

LEMMA 2.4. *Let $(A, \mathfrak{m})$ be a left noetherian semilocal algebra. Then*

(1) *There is a natural isomorphism $\hat{M} \cong \hat{A} \otimes_A M$ for all finite A-modules M.*

(2) *The completion functor $M \to \hat{M}$ is exact on finite A-modules if and only if $\hat{A}_A$ is flat.*

(3) *If $\hat{A}_A$ is flat then it is faithfully flat.*

(4) *If $\mathfrak{m}$ satisfies the AR (Artin–Rees) property* [CH, p. 140], *then the completion functor is exact on finite A-modules.*

(5) *The kernel of the map $M \to \hat{M}$ is $\bigcap_n \mathfrak{m}^n M$.*

(6) *If A is complete and $\bigcap_n \mathfrak{m}^n M = 0$, then M is finite if and only if $M/\mathfrak{m}M$ is.*

Proof. (2) follows immediately from (1). (3) holds because A is semilocal. We now prove (1). Let M be a finite A-module. Then $M/\mathfrak{m}^n M$ is artinian. Hence the inverse system $\{M/\mathfrak{m}^n M\}$ satisfies the Mittag–Leffler condition [We, 3.5.6]. Thus $\varprojlim$ is exact on such systems [We, 3.5.7]. Since $M \to M/\mathfrak{m}^n M$ is a right exact functor, the completion functor $M \to \hat{M}$ is right exact. By Watts' theorem [Rot, 3.34] $\hat{M} \cong \hat{A} \otimes_A M$ for all finite A-modules M.

(4) Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite A-modules. For each n,

$$0 \to L/(L \cap \mathfrak{m}^n M) \to M/\mathfrak{m}^n M \to N/\mathfrak{m}^n N \to 0$$

is an exact sequence of artinian A-modules. Hence $\varprojlim$ is exact and we have an exact sequence

$$0 \to \varprojlim L/(L \cap \mathfrak{m}^n M) \to \hat{M} \to \hat{N} \to 0.$$

If $\mathfrak{m}$ satisfies the AR property, then the induced filtration $\{L \cap \mathfrak{m}^n M\}$ and the adic filtration $\{\mathfrak{m}^n L\}$ are cofinal. Hence $\varprojlim L/(L \cap \mathfrak{m}^n M) \cong \varprojlim L/\mathfrak{m}^n L = \hat{L}$.

(5) is trivial. It remains to prove (6). One direction is clear. For the other direction we suppose $M/\mathfrak{m}M$ is finite. By (5) and the hypothesis $\bigcap_n \mathfrak{m}^n M = 0$ we only need to show $\hat{M}$ is finite. Let P be a finite free A-module and let $f_0: P \to M/\mathfrak{m}M$ be a surjective map. By projectivity of P the map f_0 can be lifted to $f_n: P \to M/\mathfrak{m}^n M$ inductively. By Nakayama's lemma, each f_n is surjective, which induces a surjective map $P/\mathfrak{m}^n P \to M/\mathfrak{m}^n M$. Thus there is an epimorphism from system $\{P/\mathfrak{m}^n P \mid n \geq 1\}$ to system $\{M/\mathfrak{m}^n M \mid n \geq 1\}$. These two systems and the kernel system satisfy the Mittag–Leffler condition. This implies the map $P = \varprojlim P/\mathfrak{m}^n P \to \varprojlim M/\mathfrak{m}^n M$ is surjective [We, 3.5.7]. Therefore $\hat{M}$ is finite. ∎

If A is a noetherian semilocal PI algebra, then $\mathfrak{m}$ satisfies the AR-property [CH, 11.3]. The following is due to Vámos [Va].

LEMMA 2.5 [Va]. *Let A be a noetherian semilocal PI algebra. Then $\hat{A}$ is noetherian, semilocal, and faithfully flat over A on both sides.*

3. THE AUSLANDER–BUCHSBAUM FORMULA

Let A^e be the enveloping algebra $A \otimes_k A^{\mathrm{op}}$. An A-bimodule can be viewed as a left A^e-module. The following lemma is easy to check directly from the definition of the derived functors $R\operatorname{Hom}_A(-,-)$ and $- {}^L\otimes_A -$. See [Jo2, 2.1] for the graded case.

LEMMA 3.1. *Let A be a left noetherian algebra and let $X \in D^b_f(A)$, $Y \in D^b(A^e)$, $Z \in D^b(A)$. Suppose either that X is quasi-isomorphic to a bounded complex consisting of finite free modules or that Z is quasi-isomorphic to a bounded complex of flat modules. Then*

$$R\operatorname{Hom}_A(X, Y {}^L\otimes_A Z) = R\operatorname{Hom}_A(X, Y) {}^L\otimes_A Z.$$

DEFINITION 3.2. Let A be an algebra and $A_0 = A/\mathfrak{m}$.

(1) For $X \in D^+(A)$, the *depth* of X is defined to be

$$\operatorname{depth} X = \inf R\operatorname{Hom}_A(A_0, X) = \inf\{ i \mid \operatorname{Ext}^i_A(A_0, X) \neq 0\} \quad \in \mathbb{Z} \cup \{\infty\}.$$

(2) For $X \in D^-(A)$, the *grade* or the *j-number* of X is defined to be

$$j(X) = \inf R\operatorname{Hom}_A(X, A) = \inf\{ i \mid \operatorname{Ext}^i_A(X, A) \neq 0\} \quad \in \mathbb{Z} \cup \{\infty\}.$$

Following the commutative proof, Jørgensen proved a graded version of the Auslander–Buchsbaum formula for noetherian connected graded algebras satisfying $\chi(A)$ [Jo2, 3.2]. The essential properties used in his proof are (a) existence of the minimal free (or projective) resolution of X and (b) the weak χ condition. The same idea works in the local case.

Recall that A is called *semiperfect* if $A/\mathfrak{m}$ is artinian and A has idempotent lifting property [AF, p. 303]. Scalar local algebras and left artinian algebras are semiperfect. If A is local and semiperfect, then $A \cong M_n(B)$ for some scalar local algebra B [Row, 2.7.21]. Hence A is Morita equivalent to B. It is easy to check that $M_n(B)$ satisfies the weak χ if and only if B does. If A is semiperfect and left noetherian, then every finite A-module M admits a minimal projective resolution.

THEOREM 3.3. *Let A be a left noetherian semiperfect local algebra satisfying the weak χ condition. Then the Auslander–Buchsbaum formula holds. Namely, if $X \in D^b_f(A)$ with* $\operatorname{pd} X < \infty$, *then*

$$\operatorname{pd} X + \operatorname{depth} X = \operatorname{depth} A.$$

Proof. We first assume A is scalar local. View A as an object in $D(A^e)$. Let $X \in D^b_f(A)$ with $\operatorname{pd} X < \infty$. By 3.1, we have

$$R\operatorname{Hom}_A(A_0, X) = R\operatorname{Hom}_A(A_0, A\ {}^L\!\otimes_A X) = R\operatorname{Hom}_A(A_0, A)\ {}^L\!\otimes_A X.$$

Let Y be a minimal free resolution of X, which exists because A is scalar local. By definition,

$$R\operatorname{Hom}_A(A_0, A)\ {}^L\!\otimes_A X = R\operatorname{Hom}_A(A_0, A) \otimes_A Y.$$

If $\operatorname{depth} A = \inf R\operatorname{Hom}_A(A_0, A) = \infty$, then $R\operatorname{Hom}_A(A_0, A)$ is acyclic. This implies that $R\operatorname{Hom}_A(A_0, A) \otimes_A Y$ is acyclic. Hence $\operatorname{depth} X = \inf R\operatorname{Hom}_A(A_0, A) \otimes_A Y = \infty$. If $\operatorname{depth} A = d < \infty$, then $R\operatorname{Hom}_A(A_0, A) \to \sigma_{\geq d} R\operatorname{Hom}_A(A_0, A)$ is a quasi-isomorphism where $\sigma_{\geq d}$ is the truncation at d as in [Ha2, p. 69]. Let $T = \sigma_{\geq d} R\operatorname{Hom}_A(A_0, A)$. Hence

$$R\operatorname{Hom}_A(A_0, X) = T\ {}^L\!\otimes_A X = T \otimes_A Y.$$

The lowest nonzero module in the complex $T \otimes_A Y$ is the module $T^d \otimes_A Y^{-\operatorname{pd} X}$ at position $d - \operatorname{pd} X$. It follows that $\operatorname{depth} X = \inf R\operatorname{Hom}_A(A_0, X) \geq d - \operatorname{pd} X$ and $\operatorname{depth} X + \operatorname{pd} X = d$ if and only if the cohomology of $T \otimes_A Y$ at position $d - \operatorname{pd} X$ is nonzero.

Let $l = d - \operatorname{pd} X$. By the weak χ condition, $h^d(T) = \operatorname{Ext}^d_A(A_0, A)$ has a simple right A-submodule S. Since $h^d(T) \subset T^d$, $S \otimes_A Y^{-\operatorname{pd} X}$ is a nonzero submodule of $(T \otimes_A Y)^l$. By the minimality of Y, the image of $S \otimes_A Y^{-\operatorname{pd} X}$ in $(T \otimes_A Y)^{l+1}$ is in $S \otimes_A \mathfrak{m} Y^{-\operatorname{pd} X+1}$, which is zero because $S\mathfrak{m} = 0$. It follows that $h^l(T \otimes_A Y) \neq 0$. This completes the proof in the scalar local case.

Now we assume A is semiperfect. Then there is a left noetherian scalar local algebra B such that $A \cong M_n(B)$. The weak χ condition passes from A to B. Hence the Auslander–Buchsbaum formula holds for B. By the Morita equivalence between A and B, the Auslander–Buchsbaum formula holds for A. ∎

COROLLARY 3.4. *The Auslander–Buchsbaum formula holds for the following algebras*:

(1) *left noetherian scalar local algebras satisfying the weak χ condition*,

(2) *Noetherian semiperfect local algebras satisfying χ*.

The Auslander–Buchsbaum formula may not hold for a finite dimensional algebra nor a noetherian complete scalar local domain not satisfying χ (see 9.1 and 9.6).

There are many noetherian local, even PI, algebras which are not semiperfect and the proof of 3.3 does not work for non-semiperfect local algebras. In the case of PI algebras we use completion to overcome this difficulty. A complete local ring is semiperfect because it has idempotent lifting property.

4. PASSING TO COMPLETION

It is reasonable to conjecture that if A is noetherian and semilocal then $\hat{A}$ is noetherian and faithfully flat over A on both sides. This is true for noetherian semilocal PI rings [Va]. There are some partial results for FBN semilocal rings and for rings with enough normal elements [Br, CH]. However, very little is known in general.

Let A be a left noetherian semilocal algebra and $\hat{A}$ its completion. Then $\hat{A}$ is semilocal and $\hat{A}_0 = A_0$. By 2.4(1), if M is finite, then $\hat{M} \cong \hat{A} \otimes M$. If M is not finite, then the completion does not behave well. For convenience $\hat{M}$ denotes sometimes the tensor product $\hat{A} \otimes_A M$ even if M is not finite.

If $\hat{A}_A$ is flat, the exact functor $\hat{A} \otimes_A -: \operatorname{Mod} A \to \operatorname{Mod} \hat{A}$ can be extended to the derived categories directly; namely, if X is an object in $D(A)$ represented by a complex $X^\bullet$, then $\hat{A} \otimes_A X$ is defined to be the object in $D(\hat{A})$ represented by the complex $\hat{A} \otimes_A X^\bullet$.

The following lemma was proved in [BL, 1.6]. We need it in the language of derived category.

LEMMA 4.1. *Let A and B be left noetherian rings and $A \to B$ a ring homomorphism such that ${}_AB$ and B_A are flat. Suppose N is an (A, A)-bimodule* *such that the (A, B)-bimodule $N \otimes_A B$ has a natural (B, B)-bimodule structure. Then for all $X \in D_f^-(A)$ and all $i \geq 0$,*

$$\operatorname{Ext}_A^i(X, N) \otimes_A B \cong \operatorname{Ext}_B^i(B \otimes_A X, N \otimes_A B)$$

as right B-modules.

Proof. If ${}_AP$ is free and finite,

$$\operatorname{Hom}_A(P, N) \otimes_A B \xrightarrow{\theta} \operatorname{Hom}_B(B \otimes_A P, N \otimes_A B)$$

is an isomorphism where θ is the map defined by $\theta(f \otimes b): b' \otimes m \to b'(f(m) \otimes b)$. This θ is also natural on P. Hence the result follows by taking a finite free resolution of X. ∎

LEMMA 4.2. *Let A be a left noetherian semilocal algebra. Suppose that $\hat{A}$ is flat over A on both sides. Then the following hold.*

(1) $j(X) = j_{\hat{A}}(\hat{X})$ *for all $X \in D_f^-(A)$.*

(2) $\operatorname{depth} A = \operatorname{depth}_{\hat{A}} \hat{A}$.

(3) *If $X \in D_f^-(A)$ and* $\operatorname{pd} X < \infty$, *then* $\operatorname{pd} X = \operatorname{pd}_{\hat{A}} \hat{X}$.

Proof. (1) This follows from the definition of j-number, 4.1, and the fact that ${}_A\hat{A}$ is faithfully flat.

(2) This follows from (1) and the facts $\operatorname{depth} A = j(A_0)$ and $\operatorname{depth}_{\hat{A}} \hat{A} = j_{\hat{A}}(\hat{A}_0)$.

(3) This follows from 4.1 and the facts that $\operatorname{pd} X = \max\{i \mid \operatorname{Ext}_A^i(X, A) \neq 0\}$ and that $\hat{A}_A$ and ${}_A\hat{A}$ are faithfully flat. ∎

LEMMA 4.3. *Let A be a left noetherian semilocal algebra such that $\hat{A}_A$ is flat. Let $X \in D(A)$ and let $\hat{X} = \hat{A} \otimes_A X$. Then for any i,*

$$\operatorname{Ext}_A^i(A_0, \hat{X}) \cong \operatorname{Ext}_{\hat{A}}^i(\hat{A}_0, \hat{X}).$$

This isomorphism is natural with respect to X.

Proof. This follows from the flatness of $\hat{A}_A$ and the natural isomorphisms

$$R\operatorname{Hom}_A(Q, \hat{X}) \cong R\operatorname{Hom}_A(Q, \operatorname{Hom}_{\hat{A}}(\hat{A}, \hat{X})) \cong R\operatorname{Hom}_{\hat{A}}(\hat{A} \otimes_A Q, \hat{X})$$

for any $Q \in D^-(A)$. ∎

For any complex X, the canonical homomorphism $X \to \hat{X}$ induces a natural homomorphism $R\operatorname{Hom}_A(A_0, X) \to R\operatorname{Hom}_A(A_0, \hat{X})$. Combining this with the morphism given in 4.3, we have a natural homomorphism

$$\eta_X: R\operatorname{Hom}_A(A_0, X) \longrightarrow R\operatorname{Hom}_A(A_0, \hat{X}) \longrightarrow R\operatorname{Hom}_{\hat{A}}(\hat{A}_0, \hat{X})$$

and associated natural homomorphisms

$$\eta_X^i: \operatorname{Ext}_A^i(A_0, X) \longrightarrow \operatorname{Ext}_{\hat{A}}^i(\hat{A}_0, \hat{X}).$$

PROPOSITION 4.4. *Let A be a noetherian local algebra satisfying $\chi(A)$. Suppose $\hat{A}$ is flat over A on both sides. For every $X \in D_f^b(A)$ with* $\operatorname{pd} X < \infty$, *we have*

(1) η_X^i *is an isomorphism for all* i;

(2) $\operatorname{depth} X = \operatorname{depth}_{\hat{A}} \hat{X}$.

Proof. (1) If $X = A$, η_A^i is the natural map from $\operatorname{Ext}_A^i(A_0, A)$ to $\operatorname{Ext}_{\hat{A}}^i(\hat{A}_0, \hat{A})$. By $\chi(A)$ and Lenagan's lemma, $\operatorname{Ext}_A^i(A_0, A)$ is right artinian and hence $\operatorname{Ext}_A^i(A_0, A) \otimes \hat{A} = \operatorname{Ext}_A^i(A_0, A)$. Therefore 4.1 shows that η_A^i is an isomorphism for all i. Since η is natural, η_P^i is an isomorphism for every i and every finite projective A-module P. Now the statement follows by the induction argument on the length of projective resolution of X.

(2) follows from (1). ∎

As a consequence we have the following.

THEOREM 4.5. *Let A be a noetherian local algebra satisfying $\chi(A)$. Suppose $\hat{A}$ is left noetherian and flat over A on both sides. Then*

(1) *the Auslander–Buchsbaum formula holds for $\hat{A}$*;

(2) *the Auslander–Buchsbaum formula holds for A.*

Proof. (1) By 2.2, A satisfies the weak χ. By 4.4(1) η_A^d is an isomorphism; whence $\hat{A}$ satisfies the weak χ. Since the completion $\hat{A}$ is semiperfect and local, by 3.3, the Auslander–Buchsbaum formula holds for $\hat{A}$.

(2) Follows from (1), 4.4(2), and 4.2(2,3). ∎

It is unclear if 3.3 holds for general non-semiperfect local algebra satisfying χ or weak χ because we do not know if $\hat{A}$ satisfies the hypotheses in 4.5. If A is PI, the hypotheses do hold. The following lemma is a special case of [SZ1, 3.5].

LEMMA 4.6. *Every noetherian semilocal PI algebra satisfies the χ condition.*

COROLLARY 4.7. *Let A be a noetherian local PI algebra. Then the Auslander–Buchsbaum formula holds.*

Proof. By 4.6, A satisfies χ and $\chi(A)$. By 2.5, $\hat{A}$ is flat over A on both sides. Hence the result follows from 4.5. ∎

Theorem 1.1 follows from 3.4(1) and 4.7.

5. GLOBAL DIMENSION AND INJECTIVE DIMENSION

In this section we show that homological invariants such as global dimension and injective dimension are preserved under completion.

LEMMA 5.1. *Let A be a noetherian semilocal PI algebra. If I is an ideal of A, then $\eta^i_{A/I}$ is an isomorphism for all i.*

Proof. (1) By [Br, 7], $\hat{I} = \hat{A}I = \hat{A}I\hat{A}$. Hence we have

$$\widehat{A/I} = A/I \otimes_A \hat{A} = \hat{A} \otimes_A A/I = \hat{A}/\hat{I}$$

as $\hat{A}$-bimodules. By 4.1, we have

$$\operatorname{Ext}^i_A(A_0, A/I) \otimes_A \hat{A} \cong \operatorname{Ext}^i_{\hat{A}}(\hat{A} \otimes_A A_0, A/I \otimes_A \hat{A}) = \operatorname{Ext}^i_{\hat{A}}(\hat{A}_0, \widehat{A/I}).$$

Since noetherian local PI algebras satisfy χ (see 4.6), $\operatorname{Ext}^i_A(A_0, A/I)$ is an A-module of finite length. By Lenagan's lemma, $\operatorname{Ext}^i_A(A_0, A/I)$ is of finite length as a right A-module. Hence

$$\operatorname{Ext}^i_A(A_0, A/I) \otimes_A \hat{A} = \operatorname{Ext}^i_A(A_0, A/I).$$

It follows that $\eta^i_{A/I}\colon \operatorname{Ext}^i_A(A_0, A/I) \to \operatorname{Ext}^i_{\hat{A}}(\hat{A}_0, \widehat{A/I})$ is an isomorphism. ∎

The following lemma is well known and see [SZ1, 2.1] for a proof.

LEMMA 5.2. *Let A be a noetherian PI ring and M a finite A-module. Then there exists a chain of submodules*

$$M = M_w \supset M_{w-1} \supset \cdots \supset M_0 = 0$$

such that the following hold.

(1) *For each i, set $\overline{M}_i = M_i/M_{i-1}$ and $P_i = \operatorname{ann}(\overline{M}_i)$. Then P_i is a prime ideal and $\overline{M}_i$ is a torsion-free, uniform A/P_i-module.*

(2) *Each $\overline{M}_i$ is isomorphic to a uniform left ideal of A/P_i. There exist an integer r_i and a short exact sequence*

$$0 \longrightarrow A/P_i \longrightarrow \overline{M}_i^{\oplus r_i} \longrightarrow H_i \longrightarrow 0,$$

where H_i is a torsion A/P_i-module.

(3) *If A is prime and M is a torsion A-module, then $zM = 0$ for some regular central element $z \in A$.*

Next is 4.4 for PI rings without the hypothesis $\operatorname{pd} X < \infty$.

PROPOSITION 5.3. *Let A be a noetherian semilocal PI algebra and $\hat{A}$ its completion. Let $X \in D^b_f(A)$. Then*

(1) η_X^i *is an isomorphism for all* i;

(2) $\operatorname{depth} X = \operatorname{depth}_{\hat{A}} \hat{X}$.

Proof. (2) follows directly from (1) and we now prove (1). By the truncation triangle argument as in [Ha2, I.7.1] we only need to show the statement when X is an A-module M.

In the rest of this proof we assume that M is a finite A-module and prove that η_M^i is an isomorphism for all i by induction on $\operatorname{Kdim} M$. If $\operatorname{Kdim} M = 0$, then $M = \hat{M}$. Hence η_M^i is an isomorphism for all i by 4.3. Now suppose that $\operatorname{Kdim} M = d > 0$. By noetherian induction and 5.2 we may further assume that (a) M is critical, (b) $P := \operatorname{ann}(M)$ is prime, (c) there is an integer r such that

$$0 \longrightarrow A/P \longrightarrow M^{\oplus r} \longrightarrow H \longrightarrow 0$$

is exact, and (d) H is A/P-torsion. This short exact sequence gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
E_A^{i-1}(A_0, H) & \longrightarrow & E_A^i(A_0, A/P) & \longrightarrow & E_A^i(A_0, M^{\oplus r}) & \longrightarrow & E_A^i(A_0, H) & \longrightarrow & E_A^{i+1}(A_0, A/P) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_{\hat{A}}^{i-1}(\hat{A}_0, \widehat{H}) & \longrightarrow & E_{\hat{A}}^i(\hat{A}_0, \widehat{A/P}) & \longrightarrow & E_{\hat{A}}^i(\hat{A}_0, \widehat{M^{\oplus r}}) & \longrightarrow & E_{\hat{A}}^i(\hat{A}_0, \widehat{H}) & \longrightarrow & E_{\hat{A}}^{i+1}(\hat{A}_0, \widehat{A/P})
\end{array}$$

where E stands for Ext. By [GW, 13.7], $\operatorname{Kdim} H < \operatorname{Kdim} M$. By 5.1 and induction assumption, the left two maps and right two maps in the above are isomorphisms. Hence the middle map $\eta_{M^{\oplus r}}^i$ is an isomorphism by the 5-lemma. Therefore η_M^i is an isomorphism as required. ∎

The first part of the following lemma can be proved easily by using Nakayama's lemma and the minimal projective resolution of a finite module, and the second part is a special case of [Ra, 8].

LEMMA 5.4. *Let A be a noetherian semilocal algebra.*

(1) *If A is semiperfect, then* $\operatorname{gldim} A = \operatorname{pd} A_0$.

(2) *If A is PI, then* $\operatorname{gldim} A = \operatorname{pd} A_0$.

PROPOSITION 5.5. *If A is a noetherian semilocal PI algebra, then* $\operatorname{gldim} A = \operatorname{gldim} \hat{A}$.

Proof. If $\operatorname{pd} A_0 < \infty$, then by 4.2(3), $\operatorname{pd} A_0 = \operatorname{pd}_{\hat{A}} \hat{A}_0$. If $\operatorname{pd}_{\hat{A}} \hat{A}_0 < \infty$, then by 5.3(1), $\operatorname{pd} A_0 < \infty$ and hence $\operatorname{pd} A_0 = \operatorname{pd}_{\hat{A}} \hat{A}_0$. Therefore $\operatorname{pd} A_0 = \operatorname{pd}_{\hat{A}} \hat{A}_0$ in all cases. Thus the result follows by 5.4. ∎

DEFINITION 5.6. Let A be a left noetherian semilocal algebra. For every $X \in D^+(A)$, we define the A_0-*injective dimension* of X to be

$$\operatorname{id}^0 X = \max\{i \mid \operatorname{Ext}^i(A_0, X) \neq 0\}.$$

It is clear that $\operatorname{id}^0 X \leq \operatorname{id} X$. We show next that the equality holds for noetherian semilocal PI algebras.

PROPOSITION 5.7. *Let A be a noetherian semilocal PI algebra and $X \in D^b_f(A)$. Then*

(1) $\operatorname{id} X = \operatorname{id}^0 X$.
(2) $\operatorname{id} X = \operatorname{id}_{\hat{A}} \hat{X}$.
(3) $\operatorname{id} A = \operatorname{id}_{\hat{A}} \hat{A}$.

Proof. It follows from 5.3(1) that $\operatorname{id}^0 X = \operatorname{id}^0_{\hat{A}} \hat{X}$. Hence (2) follows from (1). (3) is a special case of (2). So it remains to show (1). Let $w = \operatorname{id}^0 X$. It suffices to show that $\operatorname{Ext}^i(M, X) = 0$ for all $i > w$ and for all finite A-modules M. We use induction on $\operatorname{Kdim} M$. If $\operatorname{Kdim} M = 0$, the statement follows easily. If $d = \operatorname{Kdim} M > 0$. By 5.2 and noetherian induction, we only need to consider the case when $M = A/P$ for some prime ideal $P \subset A$. Since $B := A/P$ is PI, there is a regular central element x in the Jacobson radical of B. Hence we have a short exact sequence

$$0 \longrightarrow B \longrightarrow B \longrightarrow B/(x) \longrightarrow 0$$

and an associated long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i(B/(x), X) \longrightarrow \operatorname{Ext}^i(B, X) \longrightarrow \operatorname{Ext}^i(B, X) \\ \longrightarrow \operatorname{Ext}^{i+1}(B/(x), X) \longrightarrow \cdots.$$

If $i > w$, by induction hypothesis, $\operatorname{Ext}^i(B/(x), X) = \operatorname{Ext}^{i+1}(B/(x), X) = 0$. Note that the map $\operatorname{Ext}^i(B, X) \to \operatorname{Ext}^i(B, X)$ is the left multiplication by x. By [SZ1, 3.5] $\operatorname{Ext}^i(B, X)$ is a noetherian B-module. ([SZ1, 3.5] is stated for the case when X is a finite module M, but it can be easily extended to the case when X is a complex in $D^b_f(A)$.) Since x is in the Jacobson radical of B, $\operatorname{Ext}^i(B, X)$ is zero by Nakayama's lemma. Therefore

$$\operatorname{id} X = \max\bigl\{i \mid \operatorname{Ext}^i(M, X) \neq 0, \text{ for finite modules } M\bigr\} = w.$$

∎

6. LOCAL COHOMOLOGY AND FINITISTIC DIMENSION

A graded version of the following definition was given in [AZ, Ye].

DEFINITION 6.1. Let A be a left noetherian semilocal algebra with Jacobson radical $\mathfrak{m}$.

(1) For $M \in \operatorname{Mod} A$, the $\mathfrak{m}$-*torsion functor* $\Gamma_{\mathfrak{m}}$ is defined to be

$$\Gamma_{\mathfrak{m}}(M) = \left\{x \in M \mid \mathfrak{m}^n x = 0, \text{ for } n \gg 0\right\}.$$

(2) The derived functor $R\Gamma_{\mathfrak{m}}$ is defined on the derived category $D^+(A)$. We define the *ith local cohomology* of $X \in D^+(A)$ to be

$$\mathrm{H}^i_{\mathfrak{m}}(X) = R^i\Gamma_{\mathfrak{m}}(X).$$

(3) For $X \in D^+(A)$, the *local cohomological dimension* of X is defined to be

$$\operatorname{lcd}(X) = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(X) \neq 0\}.$$

(4) The *cohomological dimension* of $\Gamma_{\mathfrak{m}}$ is defined to be

$$\operatorname{cd}(\Gamma_{\mathfrak{m}}) = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0, \text{ for some } A\text{-module } M\},$$

which is also called the *local cohomological dimension of A* and is also denoted by $\operatorname{cd}(A)$.

For any $X \in D^+(A)$, $\mathrm{H}^i_{\mathfrak{m}}(X)$ is always $\mathfrak{m}$-torsion. Clearly $\Gamma_{\mathfrak{m}}(M) = \varinjlim \operatorname{Hom}_A(A/\mathfrak{m}^n, M)$ for all $M \in \operatorname{Mod} A$. Hence

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \varinjlim_n \operatorname{Ext}^i_A(A/\mathfrak{m}^n, X).$$

If $\mathfrak{m}$ has the AR property, then the injective hull $E(A_0)$ of A_0 is $\mathfrak{m}$-torsion [MR, 4.2.2]. Hence the injective hull of any $\mathfrak{m}$-torsion module is $\mathfrak{m}$-torsion. In this case, if M is $\mathfrak{m}$-torsion, $\mathrm{H}^0_{\mathfrak{m}}(M) = M$ and $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i > 0$.

Since $\mathrm{H}^i_{\mathfrak{m}}$ commutes with direct limits, we have

$$\operatorname{cd}(A) = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0, \text{ for some finite } A\text{-module } M\}.$$

If $\operatorname{cd}(A)$ is finite, then $\operatorname{cd}(A) = \operatorname{lcd}(A)$ because $\mathrm{H}^{\operatorname{cd}(A)}_{\mathfrak{m}}(-)$ is a right exact functor.

PROPOSITION 6.2. *Let A be a left noetherian semilocal algebra. Then*

$$\operatorname{depth} X = \inf\{i \mid \mathrm{H}^i_{\mathfrak{m}}(X) \neq 0\}$$

for all $X \in D^+(A)$.

Proof. Let Y be a minimal injective resolution of X. By the minimality of Y,

$$\operatorname{depth} X = \inf\{i \mid Y^i \text{ contains a simple submodule}\}.$$

Since the boundary map sends simple submodules to zero, we also have

$$\begin{aligned}\inf\{i \mid \mathrm{H}^i_{\mathfrak{m}}(X) \neq 0\} &= \inf\{i \mid h^i\Gamma_{\mathfrak{m}}(Y) \neq 0\} \\ &= \inf\{i \mid Y^i \text{ contains a simple submodule}\}.\end{aligned}$$

Therefore the result follows. ∎

PROPOSITION 6.3. *Let A be a left noetherian semilocal algebra satisfying the χ condition. Suppose that the injective hull of A_0 is artinian. Then $\mathrm{H}^i_{\mathfrak{m}}(X)$ is artinian for all i and all $X \in D^+_f(A)$.*

Note that if A is noetherian, semilocal, and PI, then $E(A_0)$ is artinian [Va].

Proof. Let Y be a minimal injective resolution of X and I is the $\mathfrak{m}$-torsion part of Y. By definition, $\mathrm{H}^i_{\mathfrak{m}}(X) = h^i(I)$. By χ, $\operatorname{soc}(I^i)$ is artinian and hence of finite length. This implies that I^i is artinian. Therefore $\mathrm{H}^i_{\mathfrak{m}}(X) = h^i(I)$ is artinian. ∎

In the commutative and the graded case, local cohomology has the nice property that if M is a B-module for $B = A/I$ then $\mathrm{H}^i_{\mathfrak{m}}(M)$ can be computed as A-module or as B-module. We now prove the same statement in the noncommutative ungraded case.

LEMMA 6.4. *Let A be a right noetherian semilocal algebra such that ${}_A\hat{A}$ is flat. Then for any finite right A-module M and any i, the inverse system $\{\mathrm{Tor}^A_i(M, A/\mathfrak{m}^n)\}$ satisfies the* trivial Mittag–Leffler condition; *namely, for each n there is a $j > n$ such that the map $\mathrm{Tor}^A_i(M, A/\mathfrak{m}^j) \to \mathrm{Tor}^A_i(M, A/\mathfrak{m}^n)$ is zero.*

Proof. Define $F_i(M) = \varprojlim \mathrm{Tor}^A_i(M, A/\mathfrak{m}^n)$. Hence $F_0(M) = \hat{M}$. Since $\mathrm{Tor}^A_i(M, A/\mathfrak{m}^n)$ is artinian for all $i \geq 0$, the inverse system $\{\mathrm{Tor}^A_i(M, A/\mathfrak{m}^n) \mid n \geq 1\}$ satisfies the Mittag–Leffler condition [We, 3.5.6]. Therefore $\varprojlim \mathrm{Tor}^A_i(-, A/\mathfrak{m}^n)$ is exact when applied to finite modules [Ha1, II.9.1]. This implies that $\{F_i \mid n \geq 0\}$ is a covariant δ-functor [Ha1, p. 205]. By definition, $F_i(P) = 0$ for all $i > 0$ and for all projective modules P. By the right module version of 2.4(2), $F_0(-)$ is an exact functor on finite modules M. It follows from the long exact sequence and induction that $F_i(M) = 0$ for all $i > 0$. That is, $\varprojlim \mathrm{Tor}^A_i(M, A/\mathfrak{m}^n) = 0$ for all $i > 0$. Since the inverse system $\{\mathrm{Tor}^A_i(M, A/\mathfrak{m}^n)\}$ satisfies the Mittag–Leffler condition, it satisfies the trivial Mittag–Leffler condition. ∎

Let $A \to B$ be an algebra map such that the image of $\mathfrak{m}$, denoted by $\bar{\mathfrak{m}}$, is contained in $\mathfrak{m}_B$, the Jacobson radical of B. If A is noetherian, and if ${}_AB$ and B_A are finite, then $\{\mathfrak{m}_B^n \mid n \geq 0\}$ and $\{\bar{\mathfrak{m}}_n \mid n \geq 0\}$ are cofinal where $\bar{\mathfrak{m}}_n = B\bar{\mathfrak{m}}^n$. Hence $\mathrm{H}^i_{\mathfrak{m}_B}(-) = \varinjlim \mathrm{Ext}^i_B(B/\bar{\mathfrak{m}}_n, -)$.

PROPOSITION 6.5. *Let A be a noetherian semilocal algebra such that ${}_A\hat{A}$ is flat. Let $A \to B$ be an algebra map such that the image of $\mathfrak{m}$ is in $\mathfrak{m}_B$ and that ${}_AB$ and B_A are finite. Then, for every $X \in D^b(B)$, $\mathrm{H}^i_{\mathfrak{m}}({}_AX) = \mathrm{H}^i_{\mathfrak{m}_B}({}_BX)$.*

Proof. It suffices to prove the assertion for any B-module M. The restriction map ${}_BM \to {}_AM$ is an exact functor. It is clear that $\Gamma_{\mathfrak{m}}({}_AM) = \Gamma_{\mathfrak{m}_B}({}_BM)$. To prove the statement we only need to show that $\{\mathrm{H}^i_{\mathfrak{m}}({}_A-) \mid i \geq 0\}$ is a universal δ-functor on B-modules [Ha1, pp. 205–206]. This follows if $\mathrm{H}^i_{\mathfrak{m}}({}_A-)$ vanishes on injective B-modules for $i > 0$. Let E be an injective B-module; we have the standard isomorphism

$$\mathrm{Hom}_B(\mathrm{Tor}^A_q(B, A/\mathfrak{m}^n), E) \cong \mathrm{Ext}^q_A(A/\mathfrak{m}^n, E).$$

For every $q > 0$, by 6.4, $\{\mathrm{Tor}^A_q(B, A/\mathfrak{m}^n)\}$ satisfies the trivial Mittag–Leffler condition. This implies that $\varinjlim \mathrm{Hom}_B(\mathrm{Tor}^A_q(B, A/\mathfrak{m}^n), E) = 0$ for all $q > 0$. Therefore $\mathrm{H}^q_{\mathfrak{m}}(E) = \varinjlim \mathrm{Ext}^q_A(A/\mathfrak{m}^n, E) = 0$ for all $q > 0$ as required. ∎

The following lemma is not hard to see.

LEMMA 6.6. *Let A be a left noetherian semilocal algebra and x a central element in A. If x acts on bimodules N and M centrally, then x acts on $\mathrm{Ext}^i(N, M)$ and $\mathrm{H}^i_{\mathfrak{m}}(A)$ centrally.*

THEOREM 6.7. *Let A be a noetherian, semilocal, PI algebra. Then*

(1) $\mathrm{lcd}(M) \leq \mathrm{Kdim}\, M$ *for all finite A-modules M.*

(2) $0 \leq \mathrm{lcd}(A) = \mathrm{cd}(A) \leq \mathrm{Kdim}\, A$.

Proof. By [MR, 13.7.15 and 6.4.8], A has finite Krull dimension.

(1) We prove the statement by induction on $d = \mathrm{Kdim}\, M$. If $d = 0$, then M is artinian and hence $\mathrm{H}^0_{\mathfrak{m}}(M) = M$ and $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i > 0$. Thus $\mathrm{lcd}(M) = \mathrm{Kdim}\, M$. Suppose now $\mathrm{Kdim}\, M = d > 0$. By 5.2 and induction, we only need to consider the case when $M = A/P$ for some prime ideal $P \subset A$. By 6.5, we may assume $M = A$ and A is prime. Let x be a regular central element in the Jacobson radical of A. Applying $\mathrm{H}^i_{\mathfrak{m}}$ to the short exact sequence

$$0 \to A \to A \to A/(x) \to 0,$$

we obtain a long exact sequence

$$\cdots \mathrm{H}^{i-1}_{\mathfrak{m}}(A/(x)) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(A/(x)) \longrightarrow \cdots.$$

If $i > d$, then by induction hypothesis, $\mathrm{H}^{i-1}_{\mathfrak{m}}(A/(x)) = \mathrm{H}^{i}_{\mathfrak{m}}(A/(x)) = 0$. By 6.6, x acts on $\mathrm{H}^{i}_{\mathfrak{m}}(A)$ as a left multiplication. But $\mathrm{H}^{i}_{\mathfrak{m}}(A)$ is $\mathfrak{m}$-torsion; therefore $\mathrm{H}^{i}_{\mathfrak{m}}(A) = 0$ as required.

(2) By (1), $0 \leq \mathrm{cd}(A) \leq \mathrm{Kdim}\, A$. Since $\mathrm{lcd}(A) = \mathrm{cd}(A)$, the result follows. ∎

COROLLARY 6.8. *Let A be a noetherian semilocal PI algebra. Then* $\mathrm{depth}\, A \leq \mathrm{Kdim}\, A$.

Proof. It is clear from 6.2 that $\mathrm{depth}\, A \leq \mathrm{lcd}(A)$. The result follows from 6.7(2). ∎

DEFINITION 6.9. Let A be an algebra. The *left little finitistic dimension* of A is defined to be

$$\mathrm{lfPD}\, A = \sup\{\mathrm{pd}\, M \mid M \text{ finite } A\text{-module with } \mathrm{pd}\, M < \infty\}.$$

The finitistic dimension conjecture states that the little finitistic dimension of a finite dimensional algebra is finite. One can even ask if the little finitistic dimension of a noetherian, semilocal, PI algebra is finite. First we have the following easy observation as in the commutative case.

PROPOSITION 6.10. *Let A be a left noetherian local algebra such that the Auslander–Buchsbaum formula holds. Then* $\mathrm{lfPD}\, A \leq \mathrm{depth}\, A$.

Proof. By the Auslander–Buchsbaum formula, $\mathrm{pd}\, M = \mathrm{depth}\, A - \mathrm{depth}\, M$, for any finite A-module M with $\mathrm{pd}\, M < \infty$. It follows from the definition that $\mathrm{depth}\, M \geq 0$. Therefore the result follows. ∎

COROLLARY 6.11. *Let A be a left artinian local algebra with* $\mathrm{soc}(A_A) \neq 0$. *Then* $\mathrm{lfPD}\, A = 0$.

Proof. The result follows by 3.3, 6.10, and the fact that $\mathrm{depth}\, A = 0$. ∎

COROLLARY 6.12. *Let A be a noetherian local PI algebra. Then*

$$\mathrm{lfPD}\, A \leq \mathrm{depth}\, A \leq \mathrm{Kdim}\, A < \infty.$$

Proof. The first inequality follows by 4.7 and 6.10 and the second by 6.8. ∎

This is basically 1.2 and the last equality of 1.2 follows from 8.7(1). There are other related invariants such as left big finitistic dimension $\mathrm{lFPD}\, A$ defined to be the supremum of the projective dimensions $\mathrm{pd}\, M$ of left A-module of finite projective dimension. Kirkman et al. proved that if A is a noetherian semiprime PI ring then $\mathrm{Kdim}\, A \leq \mathrm{lFPD}\, A + 1$ [KKS, 3.3].

7. MORITA DUALITY

We say an (A, T)-bimodule ${}_AE_T$ induces a *Morita duality* if (a) ${}_AE$ and E_T are injective cogenerators and (b) the right and left multiplication induces isomorphisms $\mathrm{End}_A({}_AE) \cong T$ and $\mathrm{End}_T(E_T) \cong A$. In this case we say A has a (*left*) *Morita duality* and T has a *right Morita duality*. If $T = A$, then we say A has a *Morita self-duality*. Basic properties about Morita duality can be found in [Xue].

Not every noetherian (artinian) semilocal algebra admits a Morita duality [AF, Ex 24.9] and not every local noetherian (artinian) ring having Morita duality admits a Morita self-duality [Xue, Ex 12.9]. Vámos proved the following theorem [Va].

THEOREM 7.1. *Let A be a complete noetherian semilocal PI ring and let ${}_AE$ be the minimal injective cogenerator. Then*

(1) $T := \mathrm{End}_A({}_AE)$ *is right noetherian and semiperfect*,

(2) $\mathrm{End}_T(E_T) = A$,

(3) E_T *and* ${}_AE$ *are both artinian*, *and*

(4) ${}_AE_T$ *induces a Morita duality.*

It is not clear from Vámos' work that T is PI and left noetherian, and one of the main results in the next section is to show that this is true. We list some facts about Morita duality, most of which are copied from [Xue].

LEMMA 7.2. *Let A and T be two rings. Suppose ${}_AE_T$ defines a Morita duality between A and T.*

(1) *If I is an ideal of A and $F = \mathrm{ann}_E(I)$, then F defines a Morita duality between A/I and T/J where $J = \mathrm{ann}_T(F)$.*

(2) *The lattice of ideals of A is isomorphic to the lattice of ideals of T via the map $\phi: I \to J$ described in* (1). *We denote J by $\phi(I)$.*

(3) *The centers of A and T are isomorphic and the isomorphism is compatible with ϕ in* (2). *We denote this isomorphism also by ϕ. Then ϕ maps the ideal (x) to the ideal $(\phi(x))$ for any x in the center of A.*

(4) *ϕ maps the Jacobson radical of A to the Jacobson radical of T.*

(5) *Suppose E is artinian on both sides. Then A is prime if and only if T is.*

(6) *Suppose E is artinian on both sides. Then ϕ defined in* (2) *maps the prime radical of A to the prime radical of T.*

(7) *Suppose A is a noetherian, semilocal, prime, PI algebra and T is right noetherian. Then every nonzero ideal of T contains a regular central element.*

Proof. (1) is [Xue, 2.5] and (2), (3) are [Xue, 2.6(8)].

(4) By [Xue, 2.7], both A and T are semiperfect and hence semilocal. In this case the Jacobson radical is the intersection of finitely many maximal ideals. Thus the statement follows from (2).

(5) By duality, A is left noetherian and T is right noetherian. Suppose T is prime and we need to show that A is prime. Let N be the prime radical of A. Suppose that $N \neq 0$. Then $\phi(N) \neq 0$ and $\operatorname{Kdim} T_T > \operatorname{Kdim}(T/\phi(N))_T$ because T is prime. Since A is left noetherian, there is an integer s such that $N^s = 0$. Hence, there is a chain of submodules

$$E = E_0 \supset E_1 \supset \cdots \supset E_{s-1} \supset E_s = 0,$$

where $E_i = N^i E$. Let $M_i = E_i/E_{i+1}$ and $L_i = E/E_i$. For each i the sequence

$$0 \longrightarrow M_i \longrightarrow L_{i+1} \longrightarrow L_i \longrightarrow 0$$

is exact. For any A-module M, let M^* denote the right T-module $\operatorname{Hom}_A(M, E)$. Applying * to E_i, M_i, and L_i we obtain a chain of submodules

$$T = E^* = L_s^* \supset L_{s-1}^* \supset \cdots \supset L_1^* \supset L_0^* = 0$$

and short exact sequences

$$0 \longrightarrow L_i^* \longrightarrow L_{i+1}^* \longrightarrow M_i^* \longrightarrow 0$$

for all i. Since M_i is an artinian A/N-module, $M_i^* = \operatorname{Hom}_{A/N}(M_i, \operatorname{ann}_E(N))$ is a noetherian right $T/\phi(N)$-module. Thus

$$\begin{aligned} \operatorname{Kdim} T_T &= \max_i\{\operatorname{Kdim} L_{i+1}^*/L_i^*\} = \max_i\{\operatorname{Kdim} M_i^*\} \\ &\leq \operatorname{Kdim}(T/\phi(N))_T < \operatorname{Kdim} T_T. \end{aligned}$$

This is a contradiction. Therefore $N = 0$ and A is semiprime. Since T is prime, by (2), the intersection of two nonzero ideals is nonzero. Hence the intersection of two nonzero ideals of A is nonzero, which implies that A is prime. By symmetry, if A is prime, so is T.

(6) Follows from parts (2) and (5).

(7) By (5) T is prime and hence every nonzero central element is regular. For every nonzero ideal $Q \subset T$, there is a nonzero ideal $P \subset A$ such that $Q = \phi(P)$. Since A is PI, there is a regular central element x in P. By (3) there is a central elememt $\phi(x)$ in Q. ∎

Let B be an artinian ring. We say B is *weakly symmetric* if

[WS1] for any bimodules ${}_C M_B$ and ${}_L N_B$ of finite length on both sides, where C and L are algebras, $\operatorname{Hom}_B(M, N)$ is of finite length on both sides, and

[WS2] the above condition holds after exchanging B and B^{op}.

Note that, in condition [WS1], it is automatic that $\mathrm{Hom}_B(M, N)$ is of finite length as a left L-module. It is easy to see that B is weakly symmetric if and only if $B/\mathfrak{m}_B$ is and that the matrix algebra $M_n(B)$ is weakly symmetric if and only if B is. There are division rings which are weakly symmetric [SZ1, 3.1].

PROPOSITION 7.3. *Every artinian PI algebra is weakly symmetric.*

Proof. By symmetry, we only prove [WS1]. Let B be artinian and PI. Suppose bimodules ${}_CM_B$ and ${}_LN_B$ are of finite length on both sides for some algebras C and L. By using exact sequences we may assume M and N are simple bimodules. Factoring by the annihilators, we may assume that M and N are faithful on both sides and B, C, and L are simple artinian. It is clear that $\mathrm{Hom}_B(M, N)$ is of finite length as a left L-module. To show $\mathrm{Hom}(M, N)$ is of finite length as a right C-module we may forget the L-module structure. Hence it suffices to show $\mathrm{Hom}_B({}_CM_B, B)$ is of finite length as a right C-module. Since B is PI, the result follows from [SZ1, 3.2]. ∎

Next we show that stratiform simple artinian rings introduced by Schofield [Sc] are weakly symmetric. A simple artinian ring B is called *stratiform* over the base field k if there is a chain of simple artinian rings

$$B = B_n \supset B_{n-1} \supset B_{n-1} \supset \cdots \supset B_1 \supset B_0 = k,$$

where either B_{i+1} is finite-dimensional over B_i on both sides or else B_{i+1} is isomorphic to $B_i(x_i, \sigma, \delta)$. The *stratiform length* of B is defined to be the number of steps in the chain that are infinite-dimensional. The Weyl skew fields and quotient division rings of skew polynomial rings are stratiform.

PROPOSITION 7.4. *Every stratiform simple artinian ring is weakly symmetric.*

Proof. If B is stratiform, then so is B^{op}. So it suffices to prove [WS1].

Suppose bimodules ${}_CM_B$ and ${}_LN_B$ are of finite length on both sides and we may assume that C and L are simple artinian. Then B is a subring of $M_n(C)$ and C is a subring of $M_m(B)$ for some $n, m \in \mathbb{N}$. Hence we have injective homomorphisms $B \to M_n(C) \to M_{nm}(B)$. Since B and $M_{nm}(B)$ are stratiform of the same stratiform length, it follows from [Sc, Theorem 24] that $M_{nm}(B)$ is of finite length over B on both sides via the map above. Hence $M_n(C)$ is of finite length over B on both sides. Thus $M_n(C)$ is stratiform. Replace C by $M_n(C)$ and M by $M^{\oplus n}$; we may assume $B \hookrightarrow C$ and both are stratiform of the same length. Similarly, we may assume $B \hookrightarrow L$ and both are stratiform of the same length. Hence C and L are stratiform with the same length. Since it is automatic that $\mathrm{Hom}_B(M, N)$ is of finite length as a left L-module, by [Sc, Theorem 24], it is of finite length as a right C-module. ∎

The next proposition is used in the proof of 1.4.

PROPOSITION 7.5. *Suppose a two-sided artinian (A, T)-bimodule ${}_AE_T$ induces a Morita duality between A and T. Suppose that A_0 is weakly symmetric and that A/N is noetherian where N is the prime radical of A. Then A is noetherian.*

Proof. Let $E_0 = \operatorname{Hom}_A(A_0, E)$. Then E_0 induces a Morita duality between A_0 and $T_0 := T/\mathfrak{m}_T$. If we write $A_0 = \bigoplus_i M_{n_i}(D_i)$ for finite integers n_i and division rings D_i, then $T_0 \cong \oplus_i M_{m_i}(D_i)$ for some integer m_i. Hence A_0 is weakly symmetric if and only if T_0 is.

By duality, A is left noetherian and T is right noetherian and both are complete. Since A is complete and ${}_AE$ is artinian, $\bigcap_n \mathfrak{m}^n = 0$ by 1.4(4). If A is not noetherian, then N is not finite as a right A-module. Since A is left noetherian, there is an integer s such that $N^s = 0$. Hence N^i/N^{i+1} is not finite as a right A-module for some i. Factoring by N^{i+1} we may assume $N^{i+1} = 0$. By 7.2(1,2) we still have a Morita duality. Now $N^i = N^i/N^{i+1}$ is not a finite right A/N-module. Let $\bar{\mathfrak{m}} = \mathfrak{m}/N$. We have

$$\bigcap_n N^i \bar{\mathfrak{m}}^n = \bigcap_n N^i \mathfrak{m}^n \subset \bigcap_n \mathfrak{m}^n = 0.$$

With this equality 2.4(6) shows that $N^i/N^i\mathfrak{m}$ is not finite. Further, factoring by $N^i\mathfrak{m}$ we may assume $N^i\mathfrak{m} = 0$. Since N^i is a sub-bimodule of A, $Q := \operatorname{Hom}_A({}_AN^i, {}_AE)$ is a bimodule factor of E. Since $N^i\mathfrak{m} = 0$, $\mathfrak{m}Q = 0$; i.e., Q is a left A_0-module. By the hypothesis ${}_AE$ is artinian, so ${}_AQ$ is artinian and of finite length. This together with the fact that Q_T is artinian implies that Q_T is of finite length. Thus there is an integer s such that $Q\mathfrak{m}_T^s = 0$. By the first paragraph T_0 is weakly symmetric, which implies that $\operatorname{Hom}_T(Q_T, E_T^s)$ is a finite right A-module where $E^s = \operatorname{Hom}_T(T/\mathfrak{m}_T^s, E_T)$. Finally

$$N^i = \operatorname{Hom}_T(Q_T, E_T) = \operatorname{Hom}_T(Q_T, E_T^s),$$

which is finite on the right, a contradiction. ∎

Xue asked the following question [Xue, 11.18]: if A is a left artinian ring with a duality induced by ${}_AE_T$ and if T is right artinian, is A also right artinian? By 7.5, if A_0 (or equivalently T_0) is weakly symmetric (e.g., PI), then A and T are artinian. Hence we partially answer Xue's question.

PROPOSITION 7.6. *Let A be a left noetherian complete semilocal algebra such that $A_0 := A/\mathfrak{m}$ is weakly symmetric, and let D be a two-sided noetherian complete semilocal algebra. Suppose that a two-sided artinian*

(A,T)-bimodule ${}_AE_T$ induces a Morita duality. Then for any two-sided artinian (A,D)-bimodule M, $M^* := \mathrm{Hom}_A({}_AM, {}_AE)$ is a noetherian (D,T)-bimodule.

Proof. By [Xue, 2.6(12)] M^* is right noetherian. It remains to show it is left noetherian. Let $M_n = \mathrm{Hom}_D(D/\mathfrak{m}_D^n, M_D)$. Then M_n is a sub-bimodule of M. Because M is artinian, M_n is of finite length as a right D-module and $M = \bigcup_n M_n = \varinjlim_n M_n$. Thus

$$M^* = \mathrm{Hom}_A(M,E) = \mathrm{Hom}_A(\varinjlim M_n, E) = \varprojlim \mathrm{Hom}_A(M_n, E).$$

For any bimodule M and any injective left module E, for any finite right D-module F, there is a canonical isomorphism [Rot, 3.34]

$$\mathrm{Hom}_A(\mathrm{Hom}_D(F_D, {}_AM_D), {}_AE) \cong F \otimes_D \mathrm{Hom}_A({}_AM_D, {}_AE).$$

Hence we have

$$\begin{aligned} M_n^* &:= \mathrm{Hom}(M_n, E) = \mathrm{Hom}(\mathrm{Hom}(D/\mathfrak{m}_D^n, M), E) \\ &\cong D/\mathfrak{m}_D^n \otimes \mathrm{Hom}(M,E) \cong M^*/\mathfrak{m}_D^n M^*. \end{aligned}$$

Therefore $M^* = \varprojlim M^*/\mathfrak{m}_D^n M^*$. Since M_1 is of finite length as a right module and M is left artinian, the bimodule M_1 is of finite length as a left module. Hence there exists an s such that $\mathfrak{m}^s M_1 = 0$. Thus $M_1^* = \mathrm{Hom}_A(M_1, E_s)$ where $E_s = \mathrm{Hom}_A(A/\mathfrak{m}^s, E)$. Since E_s is of finite length on both sides, by the weak symmetry of A_0, M_1^* is of finite length as a D-module. It follows from 2.4(5,6) that M^* is a finite left D-module. ∎

8. LOCAL COHOMOLOGICAL DIMENSION

In this section we complete the proofs of 1.4 and 1.5.

PROPOSITION 8.1. *Let A and D be noetherian semilocal PI algebras and let M be an (A,D)-bimodule noetherian on both sides. Then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian on both sides for all i.*

Proof. By [Va, Theorem B and Corollary 3], the injective hull of a simple A-module is artinian. By 4.6 and 6.3, $\mathrm{H}^i_{\mathfrak{m}}(M)$ is left artinian. By the χ condition and 2.3, each $\mathrm{Ext}^i(A/\mathfrak{m}^n, M)$ is of finite length on the left. By Lenagan's lemma, $\mathrm{Ext}^i(A/\mathfrak{m}^n, M)$ is of finite length on the right. Therefore $\mathrm{H}^i_{\mathfrak{m}}(M) = \varinjlim \mathrm{Ext}^i(A/\mathfrak{m}^n, M)$ is $\mathfrak{m}_D$-torsion on the right. By [Va] the injective hull of a simple right D-module is right artinian. Hence, to prove that $\mathrm{H}^i_{\mathfrak{m}}(M)$ is right artinian, it suffices to show that the right socle of $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian.

We prove this assertion by induction on Kdim M. If Kdim $M = 0$, then M is left and right artinian and hence $\mathrm{H}^0_{\mathfrak{m}}(M) = M$ and $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i > 0$. Then the assertion holds. Now suppose Kdim $M = d > 0$. By noetherian induction and [GW, 7.6], we may assume

(a) M is critical as a bimodule,

(b) $\mathrm{l.ann}(M) = P$ is a prime ideal of A and $\mathrm{r.ann}(M) = Q$ is a prime ideal of D,

(c) ${}_{A/P}M$ and $M_{D/Q}$ are torsionfree.

By 6.5, we may further assume that $P = 0$ and $Q = 0$ and that A and D are prime. Since D is PI, there is a central regular element x in $\mathfrak{m}_D$. Hence the short exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M/Mx \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}_{\mathfrak{m}}(M/Mx) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M/Mx) \longrightarrow \cdots,$$

where the maps $M \to M$ and $\mathrm{H}^i_{\mathfrak{m}}(M) \to \mathrm{H}^i_{\mathfrak{m}}(M)$ are right multiplication by x. Since x is in $\mathfrak{m}_D$, the right socle of $\mathrm{H}^i_{\mathfrak{m}}(M)$ is in the image of $\mathrm{H}^{i-1}_{\mathfrak{m}}(M/Mx)$, which is artinian by induction on i. Therefore the right socle of $\mathrm{H}^i_{\mathfrak{m}}(M)$ is artinian as required. ∎

Suppose bimodule ${}_AE_T$ induces a Morita duality between a noetherian PI ring A and a ring T. By 7.1 and [Xue, 4.6(3)], T is right noetherian complete semilocal and ${}_AE_T$ is artinian. For every left A-module M we define

$$\Phi^i(M) = \mathrm{Hom}_A(\mathrm{H}^i_{\mathfrak{m}}(M), {}_AE).$$

Hence $\Phi^i(M)$ is a noetherian right T-module. Applying Φ^i to a short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

we obtain a long exact sequence

$$\cdots \longleftarrow \Phi^{i-1}(L) \longleftarrow \Phi^i(M) \longleftarrow \Phi^i(N) \longleftarrow \Phi^i(L) \longleftarrow \Phi^{i+1}(M) \longleftarrow \cdots. \tag{8.2}$$

We say a module M is *pure* of Krull dimension d if $d = \mathrm{Kdim}\, M = \mathrm{Kdim}\, N$ for all nonzero submodules $N \subset M$.

THEOREM 8.3. *Let A be a noetherian PI ring with a Morita duality induced by an artinian bimodule ${}_AE_T$. We use the notation Φ^i defined as above.*

(1) *Let M be a finite A-module. Then*

(a) $\operatorname{lcd}(M) = \operatorname{Kdim} M$.

(b) *If $d = \operatorname{Kdim} M$, then the right T-module $\Phi^d(M)$ is pure of Krull dimension d.*

(c) *The right T-module $\Phi^i(M)$ has Krull dimension no more than i for all i.*

(2) *T is PI.*

The ideas of the following proof are from [SZ1, 3.10].

Proof. (1) By [MR, 13.7.15 and 6.4.8], A has finite Krull dimension. We prove (a), (b), and (c) by the induction on $d = \operatorname{Kdim} M$.

If $d = 0$, then M is artinian and hence $\Phi^0(M) = M^*$ is artinian and $\Phi^i(M) = 0$ for all $i > 0$. It is not hard to see (1) holds for M. If A is prime with Kdim 0, then A is simple artinian and $T = \operatorname{End}_A(E)$. Therefore T is simple artinian and PI.

We now assume $\operatorname{Kdim} M = d > 0$. We first look at the case where $M = A/P$ for some prime P. By 6.5 we may assume that $M = A$ and A is prime. By 7.2(5), T is prime. For every regular central element $x \in \mathfrak{m}$, $\operatorname{Kdim} A = \operatorname{Kdim} A/(x) + 1$ by the principal ideal theorem [MR, 4.1.12]. Applying 8.2 to the short exact sequence

$$0 \longrightarrow A \longrightarrow A \longrightarrow A/(x) \longrightarrow 0$$

we obtain a long exact sequence

$$\begin{aligned} \cdots \longleftarrow \Phi^{i-1}(A/(x)) \longleftarrow \Phi^i(A) \longleftarrow \Phi^i(A) \\ \longleftarrow \Phi^i(A/(x)) \longleftarrow \Phi^{i+1}(A) \longleftarrow \cdots. \end{aligned} \tag{8.4}$$

By 6.7(1), $\Phi^i(A) = 0$ for all $i > d$. Next we show $\Phi^d(A) \neq 0$ by contradiction. If $\Phi^d(A) = 0$, then by 8.4 at $i = d-1$, $\Phi^{d-1}(A/(x))$ is a submodule of $\Phi^{d-1}(A)$. Replacing x by x^n, $\Phi^{d-1}(A/(x^n))$ is a submodule of $\Phi^{d-1}(A)$. Therefore $\{\Phi^{d-1}(A/(x^n))\}$ is an ascending chain of submodules of the noetherian right T-module $\Phi^{d-1}(A)$. By the noetherian property, $\Phi^{d-1}(A/(x^n)) \to \Phi^{d-1}(A/(x^{n+1}))$ is an isomorphism for $n \gg 0$. Applying 8.2 to the exact sequence

$$0 \longrightarrow A/(x) \xrightarrow{\cdot x^n} A/(x^{n+1}) \longrightarrow A/(x^n) \longrightarrow 0$$

and noting that $\Phi^d(A/(x)) = 0$ by 6.7(1) we obtain

$$\begin{aligned} &\longleftarrow \Phi^{d-2}(A/(x^n)) \longleftarrow \Phi^{d-1}(A/(x)) \\ &\longleftarrow \Phi^{d-1}(A/(x^{n+1})) \longleftarrow \Phi^{d-1}(A/(x^n)) \longleftarrow 0. \end{aligned}$$

This implies that $\Phi^{d-1}(A/(x))$ is a submodule of $\Phi^{d-2}(A/(x^n))$ for $n \gg 0$. But by the induction hypothesis, $\operatorname{Kdim} \Phi^{d-1}(A/(x)) = d-1$ and $\operatorname{Kdim} \Phi^{d-2}(A/(x^n)) \leq d-2$. This yields a contradiction. Therefore $\Phi^d(A) \neq 0$ and we have proved (a). Note that x induces the map $\phi(x)$ (see 7.2(3)) on $\Phi^d(A)$. By Nakayama's lemma, $\Phi^d(A)/\Phi^d(A)\phi(x) \neq 0$ for all regular central elements $x \in \mathfrak{m}$. By 8.4 at $i = d$ and the induction hypothesis that $\Phi^{d-1}(A/(x))$ is pure of Krull dimension $d-1$, we have

$$\begin{aligned}\operatorname{Kdim} \Phi^d(A) &\geq \operatorname{Kdim} \Phi^d(A)/\Phi^d(A)\phi(x) + 1 \\ &= \operatorname{Kdim} \Phi^{d-1}(A/(x)) + 1 = (d-1) + 1 = d.\end{aligned}$$

By 8.1 and 7.6, $\Phi^d(A)$ is a left noetherian A-module and a right noetherian T-module. Let L be the T-torsion submodule of $\Phi^d(A)$. Then L is a sub-bimodule and $\operatorname{ann}_T(L)$ is a nonzero ideal of T. By 7.2(6), there is a central regular element $\phi(x) \in \operatorname{ann}_T(L) \cap \mathfrak{m}_T$ where x is a central regular element of A. Since x induces the map $\phi(x): \Phi^d(A) \to \Phi^d(A)$, the kernel of $\phi(x)$ is L. By 8.4 at $i = d$, and using the induction hypothesis $\Phi^d(A/(x)) = 0$, the kernel L is zero. Namely, $\Phi^d(A)$ is right T-torsionfree. It is similar to show that $\Phi^d(A)$ is left A-torsionfree. Let Q_A be the left Goldie quotient ring of A and Q_T be the right Goldie quotient rings of T. Then

$$Q_A \otimes_A \Phi^d(A) \cong \Phi^d(A) \otimes_T Q_T \cong Q_A \otimes_A \Phi^d(A) \otimes_T Q_T$$

and it is artinian as a left Q_A-module and as a right Q_T-module. Therefore Q_T is a subring of a matrix algebra over Q_A. Since A is PI, so are Q_A and Q_T and T. Thus by [MR, 13.6.15] T is a noetherian prime PI ring and we have proved (2) in the case when A is prime. By [MR, 6.4.12 and 6.4.13],

$$\operatorname{Kdim} T = \operatorname{Kdim} \Phi^d(A)_T = \operatorname{Kdim} {}_A\Phi^d(A) = \operatorname{Kdim} A = d.$$

The purity of $\Phi^d(A)_T$ follows from the torsionfree property. Thus we have proved (b). Next we consider $\Phi^i(A)$ for $i < d$. If $\operatorname{Kdim} \Phi^i(A) = \alpha > i$, then there is a submodule $V \subset \Phi^i(A)$ such that $\operatorname{Kdim} \Phi^i(A)/V = \alpha - 1 \leq d-1$. Since T is PI, $J := \operatorname{ann}_T(\Phi^i(A)/V)$ is nonzero. Let $\phi(x)$ be a central regular element in $J \cap \mathfrak{m}_T$ which corresponds to a central element x in A. Thus

$$\operatorname{Kdim} \Phi^i(A)/\Phi^i(A)\phi(x) \geq \operatorname{Kdim} \Phi^i(A)/V > i-1.$$

By 8.4, $\Phi^i(A)/\Phi^i(A)\phi(x)$ is a submodule of $\Phi^{i-1}(A/(x))$. This contradicts the inductive hypothesis. Therefore we proved (c).

For the general case, by 5.2 and long exact sequence 8.2 we may reduce to the case when M is a uniform left ideal of A/P and there is a short exact sequence

$$0 \longrightarrow A/P \longrightarrow M^{\oplus r} \longrightarrow H \longrightarrow 0,$$

where $\operatorname{Kdim} H < d$. Then the statement follows by 8.2, the induction hypothesis, and the case $M = A/P$. Therefore we proved (1).

(2) In the proof of (1) we see that, for every prime ideal $P \subset T$, T/P is PI. Therefore $T/\phi(N)$ is PI where $\phi(N)$ is the prime radical of T. Since T is right noetherian, $\phi(N)^s = 0$ for some integer s. Therefore T is PI [MR, 13.1.7]. ∎

We are now ready to prove 1.4.

THEOREM 8.5. *Let A be a PI ring and let T be a ring. If an artinian bimodule ${}_AE_T$ induces a Morita duality between A and T, then both A and T are noetherian and T is PI.*

Proof. By duality, A is left noetherian, T is right noetherian, and both are complete and semilocal. Let N be the prime radical of A. By [MR, 13.6.15] A/N is noetherian since A is left noetherian. Obviously A_0 is PI. By 7.3 and 7.5, A is noetherian. Since A is PI, by 8.3, T is PI. Exchanging A and T, the above argument shows that T is noetherian. ∎

If A is noetherian but not PI, it is still an open question if T is two-sided noetherian [Xue, 17.7]. To prove 1.5 we need the following lemma.

LEMMA 8.6. *Let A be a noetherian semilocal PI algebra and $\hat{A}$ its completion. Let $\mathrm{H}^i_{\hat{\mathfrak{m}}}(Y) = \varinjlim \mathrm{Ext}^i_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}}^n, Y)$ for $Y \in D^+(\hat{A})$. Then for $X \in D^b_f(A)$,*

$$\mathrm{H}^i_{\mathfrak{m}}(X) \cong \mathrm{H}^i_{\hat{\mathfrak{m}}}(\hat{X}),$$

where $\hat{X} = \hat{A} \otimes X$.

Proof. By 5.3(1), $\mathrm{Ext}^i_A(A_0, X) \cong \mathrm{Ext}^i_{\hat{A}}(\hat{A}_0, \hat{X})$ for all i and all $X \in D^b_f(A)$. By the proof of 5.3 one can also show that there is a natural isomorphism

$$\mathrm{Ext}^i_A(A/\mathfrak{m}^n, X) \cong \mathrm{Ext}^i_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}}^n, \hat{X})$$

for all i and all n and all $X \in D^b_f(A)$. The assertion follows by taking $\varinjlim$. ∎

The *dual Krull dimension* is defined to be the Krull dimension of the object in the dual category. The Krull dimension is defined for noetherian modules; the dual Krull dimension is defined for artinian modules. Next are 1.5 and 1.3(2).

THEOREM 8.7. *Let A be a noetherian semilocal PI algebra and let M be a finite A-module. Then*

(1) $\mathrm{lcd}(M) = \mathrm{Kdim}\, M$.
(2) *If $d = \mathrm{Kdim}\, M$, then $\mathrm{H}^d_{\mathfrak{m}}(M)$ has dual Krull dimension d.*
(3) *For all i, $\mathrm{H}^i_{\mathfrak{m}}(M)$ has dual Krull dimension no more than i.*
(4) $\mathrm{Kdim}\, M = \mathrm{Kdim}_{\hat{A}}\, \hat{M}$ *where ˆ is the completion.*

Proof. By faithful flatness of $\hat{A}$, $\operatorname{Kdim} \hat{M} \geq \operatorname{Kdim} M$. By 8.6, $\operatorname{lcd}(M) = \operatorname{lcd}(\hat{M})$. Hence by 6.7 and 8.3,

$$\operatorname{lcd}(M) \leq \operatorname{Kdim} M \leq \operatorname{Kdim} \hat{M} = \operatorname{lcd}(\hat{M}) = \operatorname{lcd}(M).$$

Therefore these numbers are all equal and (1) and (4) follow.

By [Va, Theorem B(iv)], a left artinian A-module is a left artinian $\hat{A}$-module and the category of left artinian A-module is equivalent to the category of left artinian $\hat{A}$-module. Therefore (2) and (3) follow from (4), 8.6, and 8.3(1b,c). ∎

9. EXAMPLES

EXAMPLE 9.1. Let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ be the algebra of upper triangular 2×2-matrices over the field k. Hence A is a noetherian, semilocal, complete PI ring. It is easy to see that A has two simple modules $S_1 = A/I$ and $S_2 = A/J$ where $I = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$ and $J = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$. Direct computations show that

(1) $\operatorname{depth} A = \operatorname{depth} S_1 = \operatorname{depth} S_2 = 0$.

(2) $\operatorname{gldim} A = 1$.

(3) $\operatorname{pd} S_1 = 0$ and $\operatorname{pd} S_2 = 1$.

Thus *A does not satisfy the Auslander–Buchsbaum formula*.

EXAMPLE 9.2. Let K be a field with a monomorphism f into itself such that $\dim K_{f(K)} = \infty$. This can be done for a field of infinite transcendence degree. Let $M = K$ be a K-bimodule with an ordinary left multiplication and a right multiplication via the map f. Then the trivial extension $A := K \oplus M$ with $M^2 = 0$ is local and left artinian, but not right noetherian. Every left artinian local algebra satisfies the weak χ condition. We want to show that A does not satisfies the χ condition. Let $E(K)$ be the injective hull of the simple module $K = A/M$. By [AF, Ex. 24.9(1)], $E(K)/K \cong \operatorname{Hom}_A({}_K M, {}_K K)$, which is not finite by the computation as in [Xue, 12.1] or [RZ]. Hence $\operatorname{Ext}^1_A(K, K) = \operatorname{Hom}_A({}_K M, {}_K K)$ is not of finite length. Therefore *A does not satisfy the χ condition*. Also [Xue, 12.1] shows that A does not have a Morita duality. Since $\operatorname{soc}(A_A) \neq 0$, A satisfies the weak χ and $\operatorname{lfPD}(A) = 0$ by 6.11.

EXAMPLE 9.3. This is taken from [Mc, Sect. IV]. Let $\mathfrak{g}$ be the 3-dimensional nilpotent Lie algebra over a field k of characteristic 2 with a basis x, y, z with $[x, y] = z$ and other commutators zero, and let A be the universal enveloping algebra of $\mathfrak{g}$. Then A is a noetherian PI domain of global dimension 3. By [Mc, Sect. IV] there is a *localizable* prime ideal $P \subset A$ such that the localization $B := A_P$ has the following properties.

(1) B is a noetherian local PI domain of global dimension 2.

(2) $B/\mathfrak{m}_B$ is a 2×2-matrix algebra over a division ring.

Hence the matrix units of $B/\mathfrak{m}_B$ cannot be lifted to the ring B. Therefore *B is not semiperfect*. The completion $\hat{B}$ is a 2×2-matrix algebra over a complete local ring. This example also shows that a completion of a domain at a maximal ideal may not be a domain.

EXAMPLE 9.4. Stafford [SZ2] gave an example of a noetherian graded ring not satisfying the χ condition (or even the weak χ condition). We modify his example into a local example. Ideas here are from [SZ2].

Let q be a nonzero element in the base field k with $q^n \neq 1$ for all $n \geq 1$. Let V be the scalar local complete algebra $k\langle\langle x, y\rangle\rangle/(yx - qxy - x^2)$ where $k\langle\langle x, y\rangle\rangle$ is the formal power series ring of two noncommuting variables. Replacing y by $y + (q-1)^{-1}x$ one sees that $V \cong k\langle\langle x, y\rangle\rangle/(yx - qxy)$. Therefore

(1) V is noetherian and scalar local.

(2) V is a domain of global dimension and Krull dimension 2.

(3) V is the completion of the connected graded algebra $k\langle x, y\rangle/(yx - qxy - x^2)$ with respect to the maximal graded ideal.

Let $A = k + Vy$. Then A is a domain. For every element $a \in Vy$, $1 + a$ has an inverse in A. Thus Vy is the Jacobson radical of A and A is scalar local. Let $\mathfrak{n}_n = \mathfrak{m}_V^n \cap A = \mathfrak{m}_V^n \cap Vy$. Then $A = \varprojlim A/\mathfrak{n}_n$; namely, A is complete with respect to $\{\mathfrak{n}_n \mid n \geq 0\}$. The associated graded ring $\bigoplus_{n\geq 0} \mathfrak{n}_n/\mathfrak{n}_{n+1}$ is the graded ring R defined in [SZ2, 2.1]. Recall that $U = k\langle x, y\rangle/(yx - qxy - x^2)$ and $R = k + Uy$. By [SZ2, 2.3] R is noetherian. It follows by [NV, IV.5] that A is noetherian. Let $\operatorname{gr} R$ be the category of finite graded left R-modules with morphism being graded homomorphisms of degree zero. The completion functor $F: M \to \varprojlim M_{\leq n}$ with respect to the grading of M is an exact functor from $\operatorname{gr} R$ to $\operatorname{Mod} A$. Similar to 2.4(1) one sees that this completion functor is equivalent to the tensor product $A \otimes_R -$ on $\operatorname{gr} R$, whence A_R is flat on the category $\operatorname{gr} R$. Similarly ${}_RA$ is flat on the category of finite graded right R-modules. Similar to the proof of 4.1, one can show the following.

LEMMA 9.5. *Let R and A be as above. Suppose N is a noetherian graded R-bimodule such that $N \otimes_R A = A \otimes_R N$. Then for all $M \in \operatorname{gr} R$ and all $i \geq 0$,*

$$\operatorname{Ext}^i_R(M, N) \otimes_R A \cong \operatorname{Ext}^i_A(A \otimes_R M, N \otimes_R A)$$

as right A-modules.

We list below some other properties analogous to the results in [SZ2].

PROPOSITION 9.6. *Let A be defined as above. Then*

(1) *A is a noetherian, complete, scalar local domain of infinite global dimension.*

(2) *A and A^{op} do not satisfy the χ condition or the weak χ condition.*

(3) *The Auslander–Buchsbaum formula does not hold for A and A^{op}.*

Proof. (1) We have already seen that A is a noetherian scalar local domain. It follows by [SZ2, 2.2(ii)] that $\{\mathfrak{n}_n \mid n \geq 0\}$ is cofinal with $\{\mathfrak{m}^n \mid n \geq 0\}$ where $\mathfrak{m}$ is the Jacobson radical of A. Hence A is complete with respect to its Jacobson radical. Since $A_0 = R_0 = k$, k denotes the trivial A-module and the trivial graded R-module. Since R has infinite global dimension, $\mathrm{Ext}^i_R(k, k) \neq 0$ for all i. By 9.5, $\mathrm{Ext}^i_A(k, k) \neq 0$ for all i. Hence A has infinite global dimension.

We prove (2) and (3) for A. Because R, and hence A, has a kind of symmetry described in the proof of [SZ2, 2.3], the same proofs work for A^{op}. (2) follows by (3) and 3.3. It remains to show (3). Since A is a domain, $\mathrm{Hom}_A(k, A) = 0$. By the left-module version of [SZ2, 2.3], $\mathrm{Ext}^1_R(k, R)$ is an infinite-dimensional, noetherian, graded right R-module. Hence, by 9.5, $\mathrm{Ext}^1_A(k, A) = \mathrm{Ext}^1_R(k, R) \otimes_R A$ is an infinite-dimensional, noetherian, right A-module. This implies that depth $A = 1$. Also follows by [SZ2, 2.3] that y is a non-zero-divisor on $\mathrm{Ext}^1_R(k, R)$. Let $L = \mathrm{Ext}^1_R(k, R)$. Then the right multiplication by y induces a map $r_y\colon L_{\leq(n-1)}[-1] \to L_{\leq n}$ with the kernel in degree n. By taking $\varprojlim$, we see that r_y is an injective map from $\hat{L}$ to $\hat{L}$. Since L is finite, $\hat{L} = L \otimes_R A = \mathrm{Ext}^1_A(k, A)$. This proves that y is a non-zero-divisor on $\mathrm{Ext}^1_A(k, A)$. By the long exact sequence $\mathrm{Hom}_A(k, A/Ay)$ is the kernel of the map $r_y\colon \mathrm{Ext}^1_A(k, A) \to \mathrm{Ext}^1_A(k, A)$. Hence $\mathrm{Hom}_A(k, A/Ay) = 0$. Using the above argument we also have that the cokernel of the map $r_y\colon \mathrm{Ext}^1_A(k, A) \to \mathrm{Ext}^1_A(k, A)$ is 1-dimensional. By the long exact sequence, the cokernel of r_y is a submodule of $\mathrm{Ext}^1(k, A/Ay)$. This implies that depth $A/Ay = 1$. It is clear that pd $A/Ay = 1$. Thus

$$\mathrm{pd}\, A/Ay + \mathrm{depth}\, A/Ay = 2 > 1 = \mathrm{depth}\, A$$

and the Auslander–Buchsbaum formula fails. ∎

A similar statement of 9.6 holds in the connected graded case.

ACKNOWLEDGMENTS

The paper was finished during the first author's visit to the Department of Mathematics at the University of Washington and was supported by a research fellowship from the China

Scholarship Council. He thanks these two institutions for their hospitality and support. The second author was supported in part by the NSF and a Sloan Research Fellowship. The authors thank the referee for her/his careful reading of the manuscript and valuable comments.

REFERENCES

[AF] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," 2nd ed., Graduate Texts in Mathematics **13**, Springer-Verlag, New York, 1992.

[AZ] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109** (1994), 228–287.

[Br] A. Braun, Completions of Noetherian PI rings, *J. Algebra* **133** (1990), 340–350.

[BH1] K. A. Brown and C. R. Hajarnavis, Homologically homogeneous rings, *Trans. Amer. Math. Soc.* **281** No. 1 (1984), 197–208.

[BH2] K. A. Brown and C. R. Hajarnavis, Injectively homogeneous rings, *J. Pure Appl. Algebra* **51**, No. 1–2 (1988), 65–77.

[BL] K. A. Brown and T. Levasseur, Cohomology of bimodules over enveloping algebras, *Math. Z.* **189** (1985), 393–413.

[CH] A. W. Chatters and C. R. Hajarnavis, "Rings with Chain Conditions," Research Notes in Mathematics, Vol. 44, Pitman, Boston/London, 1980.

[GW] K. R. Goodearl and R. B. Warfield, Jr., "An Introduction to Noncommutative Noetherian Rings," London Math. Soc. Student Texts, Vol. **16**, Cambridge Univ. Press, Cambridge, UK, 1989.

[GN] S. Goto and K. Nishida, On Gorenstein R-algebras, manuscript (1997).

[Gr] A. Grothendieck, "Local Cohomology," Lecture Notes in Math., Vol. **41**, Springer-Verlag, Berlin, 1967.

[Ha1] R. Hartshorne, "Algebraic Geometry," Graduate Texts in Mathematics, Vol. **52**, Springer-Verlag, New York/Heidelberg, 1977.

[Ha2] R. Hartshorne, "Residues and Duality," Lecture Notes in Math., Vol. **20**, Springer-Verlag, Berlin, 1966.

[Jo1] P. Jørgensen, Local cohomology for non-commutative graded algebras, *Comm. Algebra* **25** (1997), 575–591.

[Jo2] P. Jørgensen, Non-commutative graded homological identities, *J. London Math. Soc.* **57**, No. 2 (1998), 336–350.

[KKS] E. Kirkman, J. Kuzmanovich, and L. Small, Finitistic dimensions of Noetherian rings, *J. Algebra* **147**, No. 2 (1992), 350–364.

[MS] I. G. Macdonald and R. Y. Sharp, An elementary proof of the non-vanishing of certain local cohomology modules, *Quart. J. Math. Oxford (2)* **23** (1972), 197–204

[Mc] J. C. McConnell, Localisation in enveloping rings, *J. London Math. Soc.* **43** (1968), 421–428.

[MR] J. C. McConnell and J. C. Robson, "Noncommutative Noetherian Rings," Wiley, Chichester, 1987.

[NV] C. Năstăsescu and F. van Oystaeyen, "Graded Ring Theory," North-Holland Mathematical Library, Vol. 28, North-Holland, Amsterdam/New York, 1982.

[Ni] K. Nishida, A characterization of Gorenstein orders. *Tsukuba J. Math.* **12**, No. 2 (1988), 459–468.

[Ra] J. Rainwater, Global dimension of fully bounded Noetherian rings, *Comm. Algebra* **15**, No. 10 (1987), 2143–2156.

[RZ] A. Rosenberg and D. Zelinsky, Finiteness of the injective hull, *Math. Z.* **70** (1959), 372–380.

[Rot] J. J. Rotman, “An Introduction to Homological Algebra,” Pure and Applied Mathematics, Vol. **85**, Academic, New York/London, 1979.

[Row] L. H. Rowen, “Ring Theory, Student Edition,” Academic Press, San Diego, 1991.

[Sc] A. H. Schofield, Stratiform simple Artinian rings, *Proc. London Math. Soc. (3)* **53**, No. 2 (1986), 267–287.

[SZ1] J. T. Stafford and J. J. Zhang, Homological properties of (graded) noetherian PI rings, *J. Algebra* **168** (1994), 988–1026.

[SZ2] J. T. Stafford and J. J. Zhang, Examples in non-commutative projective geometry, *Math. Proc. Cambridge Philos. Soc.* **116** (1994), 415–433.

[Va] P. Vámos, Semi-local Noetherian *PI*-rings, *Bull. London Math. Soc.* **9**, No. 3 (1977), 251–256.

[We] C. A. Weibel, “An Introduction to Homological Algebra,” Cambridge Studies in Advanced Mathematics, Vol. 38, Cambridge Univ. Press, Cambridge, UK, 1994.

[WZ] Q. S. Wu and J. J. Zhang, Dualizing complexes over noetherian local algebras, in preparation.

[Xue] W. Xue, “Rings with Morita Duality,” Lecture Notes in Mathematics, Vol. 1523, Springer-Verlag, Berlin, 1992.

[Ye] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, *J. Algebra* **153** (1992), 41–84.

[YZ] A. Yekutieli and J. J. Zhang, Rings with Auslander dualizing complexes, *J. Algebra* **213**, No. 1 (1999), 1–51.