Dark energy and dust matter phases from an exact $f(R)$-cosmology model

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ABSTRACT

We show that dust matter–dark energy combined phases can be achieved by the exact solution derived from a power law $f(R)$ cosmological model. This example answers the query by which a dust-dominated decelerated phase, before dark-energy accelerated phase, is needed in order to form large scale structures.

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Very recently, alternative theories of gravity are playing an interesting role to describe the today observed Universe. Although being the best fit to a wide range of data [1], the ΛCDM model is affected by strong theoretical shortcomings [2] that have motivated the search for alternative models [3,4].

Dark energy models mainly rely on the implicit assumption that Einstein's GR is the correct theory of gravity indeed. Nevertheless, its validity on large astrophysical and cosmological scales has never been tested but only *assumed* [5], and it is therefore conceivable that both cosmic speed up and missing matter are nothing else but signals of a breakdown of GR. In this sense, GR could fail in giving self-consistent pictures both at ultraviolet scales (early universe) and at infrared scales (late universe).

Following this line of thinking, the "minimal" choice could be to take into account generic functions $f(R)$ of the Ricci scalar R. The task for this extended theories should be to match the data under the "economic" requirement that no exotic dark ingredients have to be added, unless these are going to be found by means of fundamental experiments [6,7]. This is the underlying philosophy of what are referred to as $f(R)$-gravity (see [4,8,9] and references therein).

Although higher order gravity theories have received much attention in cosmology, since they are naturally able to give rise to the accelerating expansion (both in the late and in the early universe [7]), it is possible to demonstrate that $f(R)$ theories can also play a major role at astrophysical scales. In fact, modifying the gravity Lagrangian affects the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the Solar System scale, this implication could represent an intriguing opportunity rather than a shortcoming for $f(R)$ theories. In fact, a corrected gravitational potential could offer the possibility to fit galaxy rotation curves without the need of huge amounts of dark matter [10–16]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted galaxy halo models which allow to reproduce dynamics and observations *without* dark matter [12].

However, extending the gravitational Lagrangian could give rise to several problems. These theories could have instabilities [17], ghost-like behaviors [18], and they have to be matched with the low energy limit experiments which quite fairly test GR.

In summary, it seems that the paradigm to adopt $f(R)$-gravity leads to interesting results at cosmological, galactic and Solar System scales but, up to now, no definite physical criterion has been found to *select* the final $f(R)$ theory (or class of theories) capable of matching the data at all scales. Interesting results have been achieved in this line of thinking [19–22] but the approaches are all

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phenomenological and are not based on some fundamental principle as the conservation or the invariance of some quantity or some intrinsic symmetry of the theory.

In some sense, the situation is similar to that of dark matter: we know very well its effect at large astrophysical scales but no final evidence of its existence has been found, up to now, at fundamental level. In the case of $f(R)$-gravity, we know that the paradigm is working: in principle, the missing matter and accelerated cosmic behavior can be addressed taking into account gravity (in some extended version), baryons and radiation but we do not know a specific criterion to select the final, comprehensive theory.

In this Letter, we want to show that a general exact solution, coming from the request of the existence of a Noether symmetry for $f(R)$ cosmological models, matches the two main important requirements that a cosmological solution should achieve to agree with data: a transient Friedmann dust-like phase, needed for structure formation, and an asymptotic accelerated behavior. Far to be the final model to explain the cosmic speed up, the presence of the Noether symmetry could be a physically motivated approach to select viable cosmological models.

The general features of the theory are the following. Let

$$\mathcal{A} = \int d^4x \sqrt{-g} f(R) + \mathcal{A}_m, \tag{1}$$

be the gravitational action where $f(R)$ is a generic function of the Ricci scalar R. GR is recovered in the particular case $f(R) = -R/16\pi G$, and $\mathcal{A}_m$ is the action for a perfect fluid minimally coupled with gravity

In the metric formalism, this action leads to 4th order differential equations

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R = -\frac{1}{2} T^m_{\mu\nu}, \tag{2}$$

where a subscript R denotes differentiation with respect to R and $T^m_{\mu\nu}$ is the matter fluid stress-energy tensor.

In order to derive the cosmological equations in a Friedmann–Robertson–Walker (FRW) metric, one can define a canonical Lagrangian $\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R})$, where $\mathcal{Q} = \{a, R\}$ is the configuration space and $\mathcal{TQ} = \{a, \dot{a}, R, \dot{R}\}$ is the related tangent bundle on which $\mathcal{L}$ is defined. The variables $a(t)$ and $R(t)$ are the scale factor and the Ricci scalar in the FRW metric, respectively. One can use the method of the Lagrange multipliers to set R as a constraint of the dynamics. Selecting the suitable Lagrange multiplier and integrating by parts, the Lagrangian $\mathcal{L}$ becomes canonical. In our case, we have

$$\mathcal{A} = 2\pi^2 \int dt\, a^3 \left\{ f(R) - \lambda \left[R + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}. \tag{3}$$

It is straightforward to show that, for $f(R) = -R/16\pi G$, one obtains the usual Friedman equations.

The variation with respect to R of the Lagrange multiplier gives $\lambda = f_R$. Therefore, integrating by parts, the point-like FRW Lagrangian is

$$\mathcal{L} = a^3(f - f_R R) + 6a^2 f_{RR}\dot{R}\dot{a} + 6f_R a\dot{a}^2 - 6kf_R a, \tag{4}$$

which is a canonical function of two coupled fields, R and a, both depending on time t. The total energy $E_\mathcal{L}$, corresponding to the $\{0, 0\}$-Einstein equation, is

$$E_\mathcal{L} = 6f_{RR}a^2\dot{a}\dot{R} + 6f_R a\dot{a}^2 - a^3(f - f_R R) + 6kf_R a = D, \tag{5}$$

where D represents the standard amount of dust fluid as, for example, measured today. The equations of motion for a and R are respectively

$$f_{RR}\left[R + 6H^2 + 6\frac{\ddot{a}}{a} + 6\frac{k}{a^2} \right] = 0, \tag{6}$$

$$6f_{RRR}\dot{R}^2 + 6f_{RR}\ddot{R} + 6f_R H^2 + 12f_R \frac{\ddot{a}}{a}$$
$$= 3(f - f_R R) - 12f_{RR}H\dot{R} - 6f_R \frac{k}{a^2}, \tag{7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Considering R and a as independent variables, we have, for consistency, that R coincides with the definition of the Ricci scalar in the FRW metric.

The form of the function $f(R)$ and the solution of the system (5), (6) and (7) can be achieved by asking for the existence of Noether symmetries. On the other hand, the existence of the Noether symmetries guarantees the reduction of dynamics and the eventual solvability of the system [23–25]. Here, we want to seek for viable $f(R)$ cosmological models.

We shall focus our attention on the fact that we need a cosmological solution of the field equations which exhibits not only an accelerated phase in recent universe, but also a decelerated period, which lasts for a long time, sufficient to allow the formation of structures. This issue has recently been argument of debate since the validity of $f(R)$ cosmology, which claims to avoid unknown ingredient as dark energy, strictly lies on this possibility [26]. Several works on $f(R)$-gravity have been devoted to the acceleration and the reconstruction of the models starting from data [7]. Numerical treatment is almost obliged and some educated, although arbitrary, guess on the functional form is often necessary. On the other hand, $f(R)$-cosmology should give rise to standard Friedmann dust-dominated phase, which is necessary for the structure formation mechanism, widely accepted and properly working. A first answer to this issue was given by means of a numerical reconstruction of the $f(R)$ function [19]. Here, we want to present a general exact solution of the equations, obtained by means of the so-called "Noether symmetry approach". A summary of the method can be found in [23–25].

We ask now for the existence of a vector field

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}}, \tag{8}$$

such that the Lie derivative of the Lagrangian is zero, i.e. $\mathcal{L}$ is conserved and X is a Noether symmetry. It is then possible to find

$$\alpha = 1/a; \qquad \beta = -2R/a^2; \qquad f(R) = -|R|^{3/2}. \tag{9}$$

The absolute value is needed, because (with our conventions) we have $R < 0$. Once the symmetry is found, we have an additional constant of the motion, and it is then easy to find a change of variables $\{a, R\} \to \{u, v\}$, such that one of the variables is cyclic. We have in fact

$$u = a^2 |R|; \qquad v = a^2/2 \tag{10}$$

and the new Lagrangian is

$$\mathcal{L}' = \frac{u^{3/2}}{2} + \frac{9}{2} \frac{\dot{u}\dot{v}}{\sqrt{u}} - 9k\sqrt{u}. \tag{11}$$

The Noether charge is then $\Sigma_1 = \dot{u}/\sqrt{u}$, leading to immediate integration for u. Introducing the solution into $E_\mathcal{L} = D$, and solving for v we obtain

$$u = \frac{1}{4}(\Sigma_1 t + \Sigma_0)^2, \tag{12}$$

$$v = \frac{\Sigma_1^2}{288} t^4 + \frac{\Sigma_1 \Sigma_0}{72} t^3 + \left(\frac{\Sigma_0^2}{48} - \frac{k}{2} \right) t^2$$
$$+ \left(\frac{\Sigma_0^3}{72\Sigma_1} - k\frac{\Sigma_0}{\Sigma_1} + \frac{2D}{9\Sigma_1} \right) t + v_0. \tag{13}$$

The parameters Σ_0, Σ_1, D, and v_0 are the integration constants of the equations. They are four since this is a general solution of a fourth order problem.
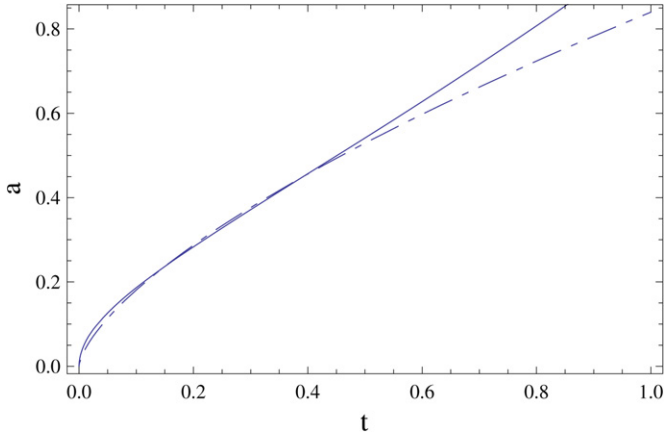
Fig. 1. Scale factor versus time in Standard Model (dashed) and our model (continuous).

Fig. 2. Percentage difference δa of the two scale factors, for a range in time corresponding to $z = 2$–4. It is less than 3%.

Coming back to $a(t)$, and setting, for the sake of simplicity $a(0) = 0$, i.e. $v_0 = 0$, we get

$$a = \sqrt{a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t}, \tag{14}$$

with

$$a_4 = \frac{\Sigma_1^2}{144}; \qquad a_3 = \frac{\Sigma_1 \Sigma_0}{36}; \qquad a_2 = \frac{\Sigma_0^2}{24} - k;$$

$$a_1 = \frac{\Sigma_0^3}{36 \Sigma_1} - 2k \frac{\Sigma_0}{\Sigma_1} + \frac{4D}{9 \Sigma_1}.$$

We see that this solution is $a \propto t^2$ for large t, and $a \propto t^{1/2}$, for small t. There is thus room for a smooth transition, passing through a period during which the solution approximates reasonably well a Friedmann dust-transient like $a_f \propto t^{2/3}$. In order to see this, we have to consider suitable values of the integration constants a_i. All computations and results are simplified if we fix the time unit, by setting the current time $t_0 = 1$. This will not affect the results but the value of H_0 has to be recast with respect to physical units. We assume also $H_0 = 1$ for simplicity. A unitary value for a_0 can be also set, if no restriction on the value of k is imposed. Finally, we consider a value of the deceleration parameter $q_0 = -0.4$, which could describe a reasonable current acceleration. These considerations yield a model depending only on one parameter. Taking $a_4 = 0.106$, the scale factor turns out to be expressed as

$$a = \sqrt{\frac{t}{5} \left[2 + 0.53(t-1)^3 + t + 2t^2 \right]}. \tag{15}$$

Comparing this solution with $a_f = a_{0f} t^{2/3}$ and noting that a_{0f} must be less than a_0, we obtain the very good coincidence of Fig. 1. The difference is close to 3% in the interval $2 \leqslant z \leqslant 4$, enough for a phase dominated by galaxies (see Fig. 2).

It is interesting to come back to the original parameters, in particular for what is concerning the spatial curvature. We have $k \simeq -0.49$, which yields $\Omega_{k,0} = kG_{\text{eff}}/(3H_0^2 a_0^2) \simeq -0.02$, with $G_{\text{eff}} = 1/[2f'(R)]$. Therefore, this model describes a spatially open universe instead of a spatially flat $k = 0$. Indeed, what is physically relevant is not the value of k, which is connected with the normalization of a, but the dimensionless parameter Ω_k. Moreover, the alleged statement $\Omega_k \simeq 0$, is obtained from the spectrum of the CMBR radiation and strongly depends on the standard ΛCDM model. Another relevant parameter is the matter content. With our choice of the parameters we get $D \simeq 0.88$, this value implies $\Omega_{m,0} \simeq 0.042$, which is very close to the expected content of baryonic matter in the Universe. One could consider an observer living within a universe described by our model. If this observer
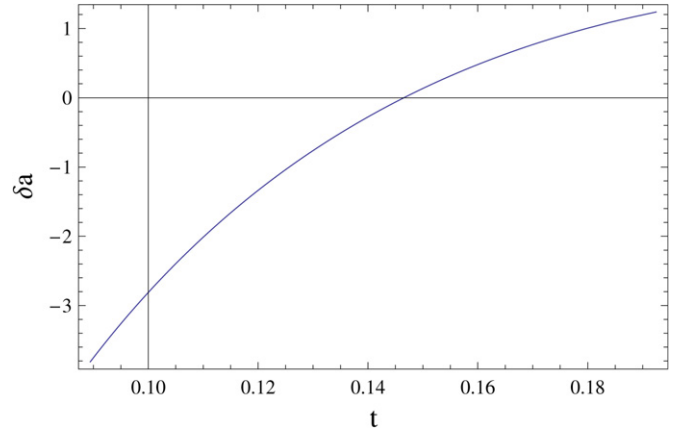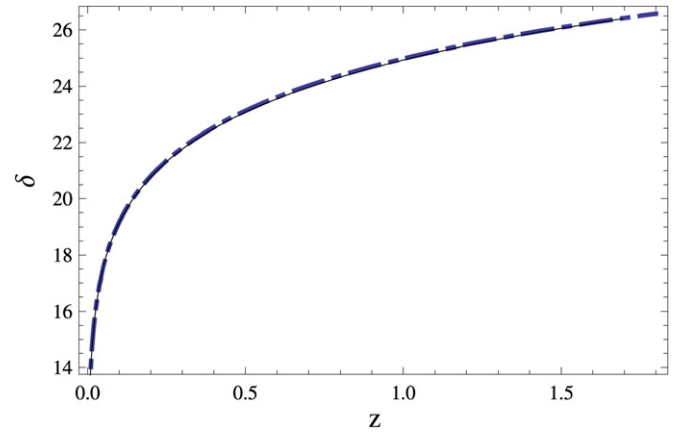

Fig. 3. Comparison of the distance modulus δ. Our model (continuous) and ΛCDM (dashed). The agreement is almost perfect.

is unaware of the fact that the function $f(R)$ in the Lagrangian is $f(R) = -|R|^{3/2}$ and not $f(R) = R$, he would perform all calculations taking into account G_N (and not with G_{eff}), obtaining $\Omega'_{m,0} \sim 0.29$. This value is the expected one for all the matter content in the Universe, included the dark matter. Therefore, in this framework, it seems that taking into account dark matter could be nothing else but an assumption due to the ignorance of the physical theory behind the cosmological model.

It can also be noted that $\Omega_{m,0}$ has nearly the same value of $-\Omega_{k,0}$. Since we have $\Omega_{m,0} + \Omega_{k,0} + \Omega_{R,0} = 1$, the current dynamic of this universe results almost totally driven by the curvature, being $\Omega_{R,0} \simeq 0.98$.

In order to check our model in another way, we consider the distance modulus given by the SNIa and we compare our solution with the standard ΛCDM model, as we know that it fits data very well. Taking as reference the standard solution for ΛCDM model, with $\Omega_m \simeq 0.27$, we get Fig. 3. The coincidence is very good and it is difficult to distinguish between the two models.

Despite these good results, some comments are in order. As we have seen, in our model, the dynamical history of the universe is described by the scale factor $a(t) \sim t^{1/2}$ at early epochs and $a(t) \sim t^2$ at late times giving rise to a matter-dust-like stage at intermediate times. This behavior addresses, in principle, the two main issues of dark energy models: (i) producing a Friedmann-like epoch suitable for LSS formation and (ii) an accelerated present
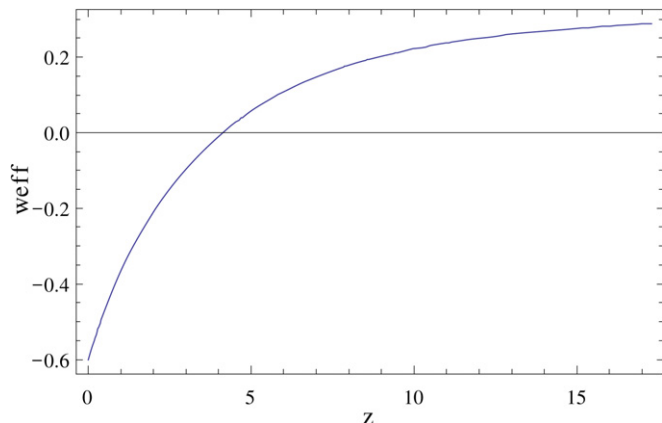
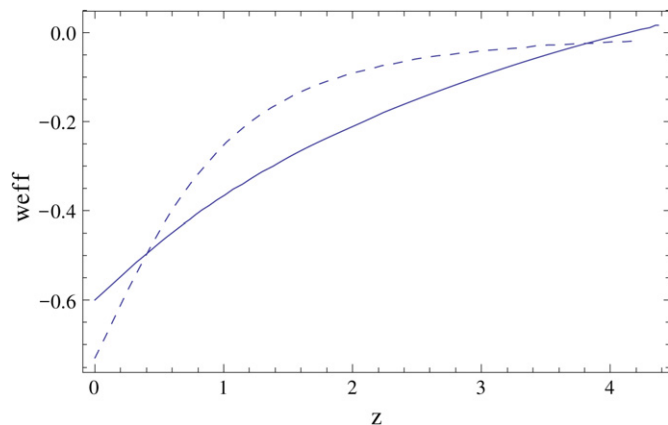**Fig. 4.** w_{eff} versus redshift derived from the model.

Fig. 5. Comparison of the effective equation of state parameter w_{eff} for our model (continuous) and ΛCDM (dashed). For z larger than 4, the radiation epoch should be carefully considered.

epoch stage. In Figs. 4 and 5, we have plotted the behavior of the effective equation of state parameter

$$w_{\text{eff}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}, \tag{16}$$

for our model and compared it with the ΛCDM model. Clearly, also if the model is accelerating at present epoch ($z \sim 0$), the power is not enough to completely fit the prescription $w_{\text{eff}} \simeq -1$ for the cosmological constant (see Fig. 5). However also the ΛCDM model does not produce exactly $w_{\text{eff}} = 1$ (since there is also the matter component); in fact, if we consider $\Omega_m \simeq 0.27$, we have $w_{\text{eff}} \simeq -0.73$ (in the case $\Omega_m = 0.3$, $w_{\text{eff}} = -0.7$). Therefore, the value of our w_{eff} (absolute value) is smaller than the desiderated value, but if we compare this with w_{eff} of the ΛCDM, it is not so far as if we compare it with the pure Λ-case $w_\Lambda = -1$.

Furthermore, radiation should be included into dynamics. This fact could destroy the nice feature achieved here, i.e., the smooth transition between an unstable dust epoch to a stable, asymptotic accelerated phase. In this perspective, more accurate models, including e.g., non-local gravitational corrections, should be taken into account as done in [27].

Finally, our discussion takes into account only the background while fluctuations are not considered. In fact, at the background level, we are able to obtain matter-like regime but things could not

work when fluctuations are included so one should try to mimic matter-like behavior by modifying gravity or including a dynamical equation of state similar to the Chaplygin gas model which well address this goal. This will be the argument of future investigations.

In summary, we have shown that suitable values of the parameters in the presented general solution (14) allow to reproduce the requested behavior of a Friedmann dust-like solution evolving into an accelerated behavior as prescribed by observations. This model, physically consistent, has been derived by asking for a Noether symmetry in the $f(R)$ function. The existence of such a symmetry fixes the form of $f(R)$ and allows physically viable models. However, starting from this approach, more accurate models should be considered in order to address all the issues related to the theory of perturbations and the observational data sets.

Acknowledgements

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