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On Bing points in infinite-dimensional hereditarily indecomposable continua

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Abstract

Let $B_\infty(X)$ be the complement of the union of all non-trivial finite-dimensional continua in the infinite-dimensional hereditarily indecomposable continuum X, i.e., the set of Bing points in X. We construct examples showing that for countable-dimensional continua X, the variety of types of $B_\infty(X)$ is much greater than in the case of the set of Bing points in the finite-dimensional case investigated in [R. Pol, M. Reńska, Preprint]. A hereditarily indecomposable continuum X is constructed such that X is not strongly infinite-dimensional, but $B_\infty(X)$ has this property.

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1. Introduction

Our terminology follows [8,9,4]. All our spaces are metrizable separable. A continuum X is hereditarily indecomposable, abbreviated h.i., if for any two intersecting subcontinua K, L of X, either $K \subset L$ or $L \subset K$.

Bing [2] constructed h.i. continua separating the Hilbert cube. Therefore, there are h.i. continua of every finite dimension and there are strongly infinite-dimensional h.i. continua (see Section 2.4 for the definition).

Let X be a h.i. continuum of a finite dimension n or infinite dimension ∞. We say that x is a Bing point in X if for any non-trivial subcontinuum K of X containing x the

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dimension of K is equal to the dimension of X. Bing has shown that for any n-dimensional h.i. continuum X, the set $B_n(X)$ of Bing points is nonempty. Recently, it was demonstrated in [20] that for $n \geqslant 2$ the set $B_n(X)$ is 1-dimensional and not of type $G_{\delta\sigma}$ (being always a $G_{\delta\sigma\delta}$-set). A theorem of Henderson [6] shows also that the set of Bing points in a strongly infinite-dimensional h.i. continuum contains a strongly infinite-dimensional continuum (cf. Remark 6.2).

We shall discuss in this paper the sets $B_\infty(X)$ of Bing points in infinite-dimensional h.i. continua X.

An important class of infinite-dimensional spaces distinguished by Hurewicz is the class of countable-dimensional spaces (abbreviated c.d.), i.e., of those spaces that are countable unions of finite-dimensional subspaces. The c.d. compacta are the ones for which the transfinite extension ind of the inductive Menger–Urysohn dimension (or the extension Ind of the Brouwer dimension) is defined.

In Section 4 we describe, for each $\alpha < \omega_1$, h.i. continua with ind or Ind equal to α (the transfinite dimensions ind and Ind can differ for h.i. continua, cf. Remark 4.7).

We shall show in Section 5 that the sets $B_\infty(X)$ for X with infinite transfinite dimensions display a much greater variety of types than in the finite-dimensional case. However, we shall leave open the problem whether there are such continua with $\dim B_\infty(X) \geqslant 2$. An important role in our constructions is played by a method of condensation of singularities, described in Section 3.

Countable-dimensional spaces are not strongly infinite-dimensional. Of course, if X is a h.i. continuum that is c.d., $B_\infty(X)$ is also c.d.

We shall describe in Section 6 an example (obtained jointly with R. Pol) of a h.i. continuum X that is not strongly infinite-dimensional, but the set $B_\infty(X)$ is strongly infinite-dimensional.

This example also shows that there are uncountable-dimensional h.i. continua X all of whose composants are countable-dimensional (such X cannot be strongly infinite-dimensional).

2. Preliminaries

2.1. A continuum X is indecomposable if it is not the union of two proper subcontinua; otherwise X is called decomposable. A continuum X is hereditarily indecomposable (h.i.) if every subcontinuum of X is indecomposable.

A pseudo-arc $\boldsymbol{P}$ is a hereditarily indecomposable 1-dimensional chainable continuum (unique, up to a homeomorphism); see [9, §48, X].

2.2. Bing proved [2, Theorem 6] that for every disjoint pair of closed sets F_1 and F_2 of a continuum X there exists a closed partition L between F_1 and F_2 in X such that each component of L is hereditarily indecomposable. Partitions without decomposable subcontinua will be called Bing partitions.

2.3. Let us recall that $B_\infty(X)$ is the set of points in a continuum X that do not belong to any non-trivial finite-dimensional subcontinuum of X. Given $r \in N$, let $B_r(X)$ denote the set of all points in a continuum X that belong to some r-dimensional subcontinuum of X but avoid every non-trivial subcontinuum of dimension less than r. It was demonstrated in [20] that for any n-dimensional h.i. continuum X, $\dim B_r(X) = n - (r-1)$; in particular, $\dim B_n(X) = 1$.

2.4. We gave the definition of countable-dimensional spaces in the Introduction. The small (large) transfinite dimension ind (Ind) is the transfinite extension of the classical small (large) inductive dimension (see [4, Definitions 7.1.1 and 7.1.11]). A space X is strongly infinite-dimensional (shortly, s.i.d.), if there exists a sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X such that for every sequence $L_1, L_2, \ldots,$ where L_i is a partition between A_i and B_i, we have $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. Otherwise the space is called weakly infinite-dimensional (shortly, w.i.d.). A space X is a C-space if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of families of pairwise disjoint open subsets of X such that each $\mathcal{V}_i$ refines $\mathcal{U}_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ covers X. Every c.d. space is a C-space and every C-space is w.i.d. (see [4, Theorems 6.3.8 and 6.3.10]).

Lemma 2.5. *Let $\mathcal{K}$ be the class of weakly infinite-dimensional spaces or the class of C-spaces. Then $\mathcal{K}$ has the following properties*:

(i) *if $X \in \mathcal{K}$ and Y is homeomorphic to a closed subset of X, then $Y \in \mathcal{K}$,*
(ii) *a space which is a countable union of members of $\mathcal{K}$ is in $\mathcal{K}$,*
(iii) *if $f : X \to Y$ is a perfect mapping, Y is zero-dimensional, and all fibers $f^{-1}(y)$ are in $\mathcal{K}$, then $X \in \mathcal{K}$,*
(iv) *if $Y \subset X$, $Y \in \mathcal{K}$ and all closed in X sets disjoint from Y are in $\mathcal{K}$, then $X \in \mathcal{K}$.*

The class $\mathcal{K}$ of countable-dimensional spaces satisfies conditions (i) *and* (ii).

The proof can be found in [4, Chapters 5 and 6].

2.6. Recall that a subcontinuum K of a continuum X is terminal, if every subcontinuum of X which intersects both K and its complement must contain K. A continuous mapping from X onto Y is called atomic, if the inverse image of every point of Y is a terminal subcontinuum of X; equivalently, if f is monotone and for every subcontinuum K of X such that the set $f(K)$ is non-degenerate we have $K = f^{-1}f(K)$ (see [13, Proposition 4]).

2.7. If x is a point in a continuum X, the composant $C(x)$ of x in X is the union of all proper subcontinua of X containing x. The composants are connected and dense in X. If X is a non-trivial indecomposable continuum, then (see [9, §48, VI])

(i) every composant of X is an F_σ boundary subset of X,
(ii) different composants of X are disjoint, and
(iii) (Mazurkiewicz's theorem [15]) there exists a Cantor set in X which contains at most one point of each composant; in particular, X has continuum many composants.

Lemma 2.8. *Let $f : X \to Y$ be an atomic mapping from a continuum X onto a (nondegenerate) continuum Y. Then the image $f(C(x))$ of the composant of x in X is the composant $C(f(x))$ of $f(x)$ in Y and $f^{-1}(C(f(x))) = C(x)$.*

Proof. Let L be a proper subcontinuum of Y containing $f(x)$. Since every atomic mapping is monotone, then $f^{-1}(L)$ is a proper subcontinuum of X containing x. Thus $f^{-1}(C(f(x))) \subset C(x)$ and $C(f(x)) \subset f(C(x))$. On the other hand, $f(C(x)) \subset C(f(x))$. Indeed, if $z \in C(x)$ then $f(z) = f(x)$ or there exists a proper subcontinuum L' of X containing x and z such that $f(L') \setminus \{f(x)\} \neq \emptyset$. Since f is atomic, then $f^{-1} f(L') = L'$, hence $f(L') \neq Y$. Thus $f(L')$ is a proper subcontinuum of Y containing $f(x)$ and $f(z)$ and thus $f(z) \in C(f(x))$. This shows that $f(C(x)) \subset C(f(x))$ and also $C(x) \subset f^{-1} f(C(x)) \subset f^{-1}(C(f(x)))$, which ends the proof. □

3. A method of condensation of singularities

In this section we generalize some techniques described in [3, Section 2], based on a classical idea of condensation of singularities (see [1]).

We shall use the following theorem of Maćkowiak.

Theorem 3.1 ([13, Theorem 15] and [14, 1.14]). *Let X be a continuum, A be a compact zero-dimensional set in X and let Z be a compactum that admits a continuous map onto A with connected fibers. Then there exists a continuum $\widetilde{X}$, containing Z as a boundary subset, which admits an atomic map r from $\widetilde{X}$ onto X such that $r|\widetilde{X} \setminus Z$ is a homeomorphism onto $X \setminus A$ and $r^{-1}(a)$ is a component of Z for every $a \in A$.*

We shall denote by $M(X, Z, A)$ and call a pseudosuspension of Z over X at A any space $\widetilde{X}$ satisfying conditions of Theorem 3.1. *The mapping $r : M(X, Z, A) \to X$ will be called a natural projection.*

Theorem 3.2. *Let X be a continuum, $\{Z_i\colon i \in N\}$ a sequence of compacta, $\{A_i\colon i \in N\}$ a sequence of* 0*-dimensional compact disjoint subsets of X and let each Z_i admit a continuous map onto A_i with connected fibers. Then there exist a continuum $L(X, Z_i, A_i)$ and a mapping $p : L(X, Z_i, A_i) \to X$ such that*:

- (i) *p is atomic,*
- (ii) *$p|p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) : p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) \to X \setminus \bigcup_{i=1}^{\infty} A_i$ is a homeomorphism,*
- (iii) *the set $p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i)$ is dense in $L(X, Z_i, A_i)$,*
- (iv) *$p^{-1}(A_i)$ is homeomorphic to Z_i for every $i \in N$ (hence $p^{-1}(a)$ is homeomorphic to a component of Z_i, if $a \in A_i$), and if $\bigcup_{i=1}^{\infty} A_i$ is dense in X then every non-empty open subset of $L(X, Z_i, A_i)$ contains $p^{-1}(a)$ for some $a \in \bigcup_{i=1}^{\infty} A_i$,*
- (v) *if n and α are ordinal numbers such that $\operatorname{Ind} X \leqslant n < \omega_0$ and $n \leqslant \operatorname{Ind} Z_i \leqslant \alpha < \omega_1$ for every $i \in N$ then $\operatorname{Ind} L(X, Z_i, A_i) \leqslant \alpha$,*
- (vi) *if n and α are ordinal numbers such that $\operatorname{ind} X \leqslant n < \omega_0$ and $n \leqslant \operatorname{ind} Z_i \leqslant \alpha < \omega_1$ for every $i \in N$ then $\operatorname{ind} L(X, Z_i, A_i) \leqslant \alpha$.*

Proof. We shall define an inverse sequence $\{L_i, p^i_j, N \cup \{0\}\}$ such that $L_0 = X$, $L_i = M(L_{i-1}, Z_i, (p_0^{i-1})^{-1}(A_i))$ is a pseudosuspension of Z_i over L_{i-1} at $(p_0^{i-1})^{-1}(A_i)$, p^i_{i-1} is a natural projection from L_i onto L_{i-1} and $p^i_j = p^{j+1}_j \circ \cdots \circ p^i_{i-1}$ for $i > j$. Let $L(X, Z_i, A_i)$ be the inverse limit of the inverse sequence, let $p_n : L(X, Z_i, A_i) \to L_n$ be the projection and let $p = p_0$ be the projection of the limit space onto $L_0 = X$. It is easy to see that the conditions (ii) and (iv) are satisfied. Since the projection $p^i_j : Z_i \to Z_j$ is a composition of finitely many atomic mappings then it is atomic (see [13, (1.4)]). Thus p is atomic (see [1, Theorem II]).

To prove (iii) it suffices to show that

(1) the set

$$G_n = p_n\left(p^{-1}\left(X \setminus \bigcup_{i=1}^{\infty} A_i\right)\right)$$

is dense in L_n for every $n \in N$.

Observe that $G_n = (p_0^n)^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i)$; in particular, $G_0 = X \setminus \bigcup_{i=1}^{\infty} A_i$ is dense in L_0 as the complement of a 0-dimensional set in the continuum L_0. Suppose that G_{n-1} is dense in L_{n-1} and put $V_{n-1} = L_{n-1} \setminus (p_0^{n-1})^{-1}(A_n)$. From the construction of L_n and p^n_{n-1} it follows that p^n_{n-1} restricted to $(p^n_{n-1})^{-1}(V_{n-1})$ is a homeomorphism and $(p^n_{n-1})^{-1}(V_{n-1})$ is dense in L_n. Since G_{n-1} is a dense subset of V_{n-1} and $G_n = (p^n_{n-1})^{-1}(G_{n-1})$, then G_n is dense in $(p^n_{n-1})^{-1}(V_{n-1})$ and thus in L_n. This ends the inductive proof of (1).

Let us check (v). First note that each L_i is the union of a closed subspace homeomorphic to the free union $\bigoplus_{i=1}^{n} Z_i$ and an open subspace of X. If α is a finite number then (v) follows from the countable sum theorem and the fact that the limit operation does not increase the dimension (see [4, Theorems 1.5.3 and 1.13.4]). Suppose now that $\alpha \geqslant \omega_0$ and let A, B be a pair of disjoint closed subsets of $L(X, Z_i, A_i)$. Then there exists $n \in N$ such that $p_n(A) \cap p_n(B) = \emptyset$. The set $D = \bigcup_{i=n+1}^{\infty} (p_0^n)^{-1}(A_i)$ is homeomorphic to the 0-dimensional subspace $\bigcup_{i=n+1}^{\infty} A_i$ of X and the set $F = \bigcup_{i=1}^{n} (p_0^n)^{-1}(A_i)$ is a closed subset of L_n homeomorphic to the free union $\bigoplus_{i=1}^{n} Z_i$. Therefore, $\operatorname{Ind} F \leqslant \alpha$. Let T be a partition in F between the sets $p_n(A) \cap F$ and $p_n(B) \cap F$ such that $\operatorname{Ind} T < \alpha$. Take a partition T' between the sets $p_n(A)$ and $p_n(B)$ in L_n such that $T' \cap F = T$ and $T' \cap D = \emptyset$. Then T' is the union of the closed subset T with $\operatorname{Ind} T < \alpha$ and an open finite-dimensional subset. Therefore, the inequality $\operatorname{Ind} T' < \alpha$ follows from the following fact, which can be readily justified by transfinite induction:

($\star$) *if a* (*metrizable, separable*) *space* Z' *is the union of an open* n*-dimensional set and a closed subset* Z *such that* $\operatorname{Ind} Z = \beta \geqslant n$ (*respectively,* $\operatorname{ind} Z = \beta \geqslant n$) *then* $\operatorname{Ind} Z' = \beta$ (*respectively,* $\operatorname{ind} Z' = \beta$).

The set $p^{-1}(T')$, which is homeomorphic with T', is a partition between A and B in $L(X, Z_i, A_i)$ with $\operatorname{Ind} p^{-1}(T') < \alpha$, which ends the proof.

The proof of (vi) is similar to the proof of (v). □

Remark 3.3. If, for some continuum K, every Z_i is homeomorphic to K, every A_i is a singleton $\{a_i\}$ and $A = \{a_1, a_2, \ldots\}$, then $L(X, Z_i, A_i)$ is the space $S(X, K, A)$ described in Section 2 of [3].

Proposition 3.4. *Let X, $\{Z_i\colon i \in N\}$ and $\{A_i\colon i \in N\}$ be as in Theorem* 3.2.

(a) *If X is hereditarily indecomposable and no Z_i contains any decomposable subcontinuum, then the continuum $L(X, Z_i, A_i)$ is hereditarily indecomposable.*

(b) *If X and all Z_i are countable-dimensional then so is $L(X, Z_i, A_i)$.*

Proof. The proof of (a) follows from Proposition 11(ii) in [13]. The proof of (b) follows instantly from the fact that $L(X, Z_i, A_i)$ is the union of topological copies of Z_i for $i = 1, 2, \ldots$ and a subspace homeomorphic to $X \setminus \bigcup_{i=1}^{\infty} A_i$. □

Remark 3.5. If X is a continuum, A is a 0-dimensional closed subset of X and Z is a compactum then

(a) if X is hereditarily indecomposable and Z does not contain any decomposable subcontinuum, then the continuum $M(X, Z, A)$ is hereditarily indecomposable,

(b) if X and Z are countable-dimensional then so is $M(X, Z, A)$,

(c) if $\operatorname{Ind} X < \omega_0$ and $\operatorname{Ind} Z \geqslant \operatorname{Ind} X$, then $\operatorname{Ind} M(X, Z, A) = \operatorname{Ind} Z$,

(d) in (c), Ind can be replaced by ind.

Indeed, (a) follows from Proposition 11(ii) in [13]. Note that $M(X, Z, A)$ is the union of a closed subspace homeomorphic to Z and an open subspace homeomorphic to $X \setminus A$. This implies (b), while (c) and (d) follow from the fact ($\star$) given at the end of the proof of Theorem 3.2.

4. Bing partitions in Smirnov and Henderson compacta

In this section, for every $\alpha < \omega_1$, we shall describe hereditarily indecomposable continua X, Y with $\operatorname{Ind} X = \operatorname{ind} Y = \alpha$. We will construct such continua as Bing partitions between "opposite faces" of some of Henderson's compacta. Other constructions of such spaces, with some additional properties, are given in the next section.

Let us recall that Smirnov's compacta S_α, for $\alpha < \omega_1$, are defined by transfinite induction in the following way: $S_n = I^n$ is the Euclidean n-cube, $S_{\alpha+1} = S_\alpha \times I$ and, for a limit ordinal α, S_α is the one-point compactification of the free union $\bigoplus\{S_\beta\colon \beta < \alpha\}$.

For every countable ordinal α, $\operatorname{Ind} S_\alpha = \alpha$ and S_α has only countably many components, each being a finite-dimensional cube (see [4, Example 7.1.33]).

Henderson's compactum H_α, for $\alpha < \omega_1$, is an absolute retract topologically containing S_α with $\operatorname{Ind} H_\alpha = \alpha$ and such that $H_{\alpha+1} = H_\alpha \times I$ (see [5]).

Let us notice that not every countable-dimensional continuum contains a h.i. infinite-dimensional subcontinuum. Indeed, if K is a continuum containing a copy of S_α with $K \setminus S_\alpha$ being the countable union of open arcs, then any hereditarily indecomposable continuum in K is finite-dimensional.

Theorem 4.1. *If L is a Bing partition between $X \times \{0\}$ and $X \times \{1\}$ in $X \times I$, where X is either a Smirnov compactum S_α or a Henderson compactum H_α, then* $\operatorname{Ind} L = \operatorname{Ind} X$.

In addition, if $\operatorname{ind}(X \times I) = \operatorname{ind} X + 1$, *then* $\operatorname{ind} L = \operatorname{ind} X$.

In the proof the following lemma will be used (recall that a mapping f is light if every fiber of f is 0-dimensional).

Lemma 4.2. *Let $f: Z \to X$ be a light mapping, where X is either a Smirnov compactum S_α or a Henderson compactum H_α and Z is a compact space. Then* $\operatorname{Ind} Z \leqslant \operatorname{Ind} X = \alpha$.

Proof. Let D be the D-dimension as defined in [4, Section 7.3]. From the definition of D-dimension and from the theorem on dimension lowering mappings (see [4, Theorem 1.12.4]) it easily follows that $D(Z) \leqslant D(X)$. Thus, by Theorems 7.3.17, 7.3.18 and Problem 7.3.D of [4] we have $\operatorname{Ind} Z \leqslant D(Z) \leqslant D(X) = \alpha$. □

We shall need also the following fact, whose standard proof is included for the reader's convenience.

Lemma 4.3. *Let $f: Z \to X$ be a light mapping from a compactum Z onto a compactum X with* $\operatorname{ind} X \leqslant \alpha$. *Then* $\operatorname{ind} Z \leqslant \alpha$.

Proof. We proceed by induction on α. For $\alpha < \omega_0$ this is a classical result, cf. [4, Theorem 1.12.4]. Suppose that it is true for every $\beta < \alpha$, where $\alpha \geqslant \omega_0$. Let $\widehat{Z} = \{(f(z), z): z \in Z\} \subset X \times Z$ and let $\hat{f} = p|\widehat{Z}$, where $p: X \times Z \to X$ is the projection. Then $\widehat{Z}$ is homeomorphic to Z and $\hat{f}$ is light. Let z be a point of $\widehat{Z}$ and U be an arbitrary open neighbourhood of z in $X \times Z$. Since $\operatorname{ind} \hat{f}^{-1}(\hat{f}(z)) = 0$ and $\hat{f}^{-1}(\hat{f}(z)) = (\{\hat{f}(z)\} \times Z) \cap \widehat{Z}$, there exists an open subset $V = V_1 \times V_2$ of $X \times Z$ containing z, where $V_1 \subset X$, $V_2 \subset Z$, such that $\overline{V} \subset U$ and $(X \times \operatorname{Fr} V_2) \cap \hat{f}^{-1}(\hat{f}(z)) = \emptyset$. We have $\hat{f}(z) \notin p((X \times \operatorname{Fr} V_2) \cap \widehat{Z})$ and hence there exists an open set V_3 of X containing $\hat{f}(z)$ with $\operatorname{ind}(\operatorname{Fr} V_3) < \alpha$ and such that $\overline{V_3} \subset V_1 \setminus p((X \times \operatorname{Fr} V_2) \cap \widehat{Z})$. Therefore $\operatorname{Fr}(V_3 \times V_2) \cap \widehat{Z} = (\operatorname{Fr} V_3 \times \overline{V_2}) \cap \widehat{Z}$. Applying the inductive assumption to the restriction of $\hat{f}$ to $\operatorname{Fr}(V_3 \times V_2) \cap \widehat{Z}$, we conclude that $\operatorname{ind}(\operatorname{Fr}(V_3 \times V_2) \cap \widehat{Z}) < \alpha$. □

Proof of Theorem 4.1. The inequalities "$\geqslant$" follow from [16, Theorem 2.1] and [17, Theorem 2.1]. To prove the inequalities "$\leqslant$" observe that the restriction $f: L \to X$ of the projection of $X \times I$ onto X to the subspace L is light. Indeed, for every $x \in X$ we have $f^{-1}(x) \subset \{x\} \times I$ and $f^{-1}(x)$ does not contain any subset homeomorphic to I, since L is a Bing partition. Therefore $\operatorname{Ind} f^{-1}(x) \leqslant 0$. Applying Lemma 4.2. (Lemma 4.3) to the mapping $f: L \to X$ we get the inequality "$\leqslant$" for Ind (respectively, for ind). □

Remark 4.4. For every ordinal number α there exist ordinal numbers β, γ such that $\operatorname{ind} S_\beta = \operatorname{ind} H\gamma = \alpha$ and $\operatorname{ind} S_{\beta+1} = \operatorname{ind} H_{\gamma+1} = \alpha + 1$ (see [4, Example 7.2.12]; the proof for Henderson's compacta can be found, for example, in [16,17]).

Lemma 4.5. *If X is a locally connected continuum then there exists a connected Bing partition K between $X \times \{0\}$ and $X \times \{1\}$ in $X \times I$.*

Proof. Let L be a Bing partition between $X \times \{0\}$ and $X \times \{1\}$ in $X \times I$. Since the cone over X is unicoherent (see [9, §57, I, Theorem 9 and III, Theorem 3]) and locally connected, there exists a connected partition $K \subset L$ between $X \times \{0\}$ and $X \times \{1\}$ in $X \times I$ (as K one can take any irreducible partition between $X \times \{0\}$ and $X \times \{1\}$ in $X \times I$ contained in L), see [9, §57, III, Theorem 1]. This completes the proof of the lemma. □

Theorem 4.6. *For every countable infinite ordinal α there exist hereditarily indecomposable continua K, L with* $\operatorname{Ind} K = \alpha = \operatorname{ind} L$.

Proof. Let β be an ordinal such that $\operatorname{ind} H_\beta + 1 = \operatorname{ind} H_{\beta+1} = \alpha + 1$ (see Remark 4.4). The Henderson continua are locally connected and hence Lemma 4.5 provides a connected Bing partition K (respectively, L) between the top and the bottom of the cylinder $H_\alpha \times I$ (respectively, $H_\beta \times I$). By Theorem 4.1, we have $\operatorname{Ind} K = \operatorname{ind} L = \alpha$, which ends the proof. □

Remark 4.7. Notice that the transfinite dimensions ind and Ind may differ for h.i. continua. Indeed, there exist ordinals β and α such that $\operatorname{ind} H_\beta + 1 = \operatorname{ind} H_{\beta+1} = \alpha + 1 < \beta + 1$ (see [12]; cf. [17]). As we showed in the proof of Theorem 4.6, there exists a connected h.i. partition L between $H_\beta \times \{0\}$ and $H_\beta \times \{1\}$ in $H_\beta \times I$ and $\operatorname{Ind} L = \beta > \operatorname{ind} L = \alpha$.

5. The set of Bing points in countable-dimensional spaces

For any n-dimensional hereditarily indecomposable continuum X, $n \geqslant 2$, the set $B_n(X)$ of Bing points is 1-dimensional and not of type $G_{\delta\sigma}$ in X (being always a $G_{\delta\sigma\delta}$-set, see [20]). For infinite-dimensional hereditarily indecomposable continua X with defined transfinite dimensions, the corresponding sets $B_\infty(X)$ of Bing points display a greater variety of types. The aim of this section is to illustrate with examples that: $B_\infty(X)$ may be a copy of any closed subset of the Cantor set (including the empty one), or it may be a copy of the irrationals, or else it may be a 1-dimensional G_δ-set in X. We did not succeed, however, in constructing continua X with $B_\infty(X)$ of dimension greater than 1. Notice that for any countable-dimensional continuum X the set $B_\infty(X)$ contains no non-trivial continuum (otherwise it would contain a finite-dimensional subcontinuum, contrary to the definition of $B_\infty(X)$; cf. [4, 5.2.5]). Moreover, if a continuum X is the union of countably many closed finite-dimensional subsets, then the set $B_\infty(X)$ is at most 0-dimensional (see Proposition 5.13).

Concerning the hereditarily indecomposable continua X with finite $\dim X = n \geqslant 2$, we shall show that the Baire category of $B_n(X)$ depends on X.

We start from the following example.

Example 5.1. For every ordinal number $\alpha < \omega_1$ there exist hereditarily indecomposable continua X_α and Y_α such that $\operatorname{Ind} X_\alpha = \alpha = \operatorname{ind} Y_\alpha$ and every composant of X_α or Y_α is finite-dimensional; in particular, $B_\infty(X_\alpha) = \emptyset = B_\infty(Y_\alpha)$.

First we will construct X_α. Let Z be a Bing partition between $S_\alpha \times \{0\}$ and $S_\alpha \times \{1\}$ in the product $S_\alpha \times I$ of the Smirnov compactum S_α and the interval I. By Theorem 4.1, $\operatorname{Ind} Z = \alpha$. Since every component of S_α is finite-dimensional, Z has the same property.

Let A be a 0-dimensional compact subspace of the pseudoarc $\boldsymbol{P}$ homeomorphic with the decomposition space of Z into components. By 2.7(iii) we can assume that A contains at most one point from each composant of $\boldsymbol{P}$. Let

$$X_\alpha = M(\boldsymbol{P}, Z, A)$$

be a pseudosuspension of Z over $\boldsymbol{P}$ at A and

$$p : X_\alpha \to \boldsymbol{P}$$

be the natural projection. Then X_α is a h.i. continuum satisfying $\operatorname{Ind} X_\alpha = \alpha$, by Remark 3.5 (a) and (c). If $C(x)$ is the composant of x in X_α, then, by Lemma 2.8, $C(x) = p^{-1}(C(p(x)))$, where $C(p(x))$ is the composant of $p(x)$ in $\boldsymbol{P}$. Hence $C(x)$ is homeomorphic to the composant $C(p(x))$ of $p(x)$ or else it is the union of some component of Z and a set homeomorphic with a subset of $C(p(x))$. Since every component of Z is finite-dimensional, every composant of X_α is finite-dimensional, which implies $B_\infty(X_\alpha) = \emptyset$.

The space Y_α can be constructed similarly, starting from a Bing partition Z between $S_\beta \times \{0\}$ and $S_\beta \times \{1\}$, where β is an ordinal such that $\operatorname{ind} S_\beta + 1 = \operatorname{ind} S_{\beta+1} = \alpha + 1$.

Example 5.2. Let D be any 0-dimensional non-empty compactum and α an infinite countable ordinal. Then there exist hereditarily indecomposable continua X and Y with $\operatorname{Ind} X = \alpha = \operatorname{ind} Y = \alpha$ such that $B_\infty(X)$ and $B_\infty(Y)$ are homeomorphic to D.

We shall assume that D is embedded into the pseudoarc $\boldsymbol{P}$. Take a sequence $U_1 \supset U_2 \supset \cdots$ of neighborhoods of the set D in $\boldsymbol{P}$ such that $\overline{U_{i+1}} \subset U_i$, $\operatorname{ind} \operatorname{Fr} U_i = 0$ and U_i is contained in the $1/i$-neighbourhood of D. For every $i \in N$, let $A_i = \operatorname{Fr} U_i$ and let X_α be the continuum such that $\operatorname{Ind} X_\alpha = \alpha$ and $B_\infty(X_\alpha) = \emptyset$, constructed in Example 5.1. Finally, let $X = L(\boldsymbol{P}, X_\alpha \times A_i, A_i)$ and $p : X \to \boldsymbol{P}$ be a continuum and a mapping satisfying conditions (i)–(vi) of Theorem 3.2. By Proposition 3.4(a), X is h.i.

We will show that $B_\infty(X) = p^{-1}(D)$ and hence is homeomorphic to D. Since the set $D \cup \bigcup_{i=1}^\infty A_j$ is closed in $\boldsymbol{P}$, every point x not belonging to $p^{-1}(D \cup \bigcup_{i=1}^\infty A_j)$ has a 1-dimensional neighbourhood in X and thus $x \notin B_\infty(X)$. If $x \in p^{-1}(a)$ for some $a \in A_i$, then $x \notin B_\infty(X)$, since $p^{-1}(a)$ is homeomorphic to X_α and every composant of X_α is finite-dimensional. Suppose now that $p(x) \in D$ and let L be an arbitrary non-trivial continuum in X containing x. Then $p(L)$ is a non-degenerate continuum in $\boldsymbol{P}$ containing $p(x)$, and hence $p(L) \cap A_n \neq \emptyset$ for some n. Let $a \in p(L) \cap A_n$. Since $p^{-1}(a)$ is a terminal continuum (see 3.2(i)) in X homeomorphic to X_α, L topologically contains X_α. This shows that $x \in B_\infty(X)$.

The proof that $\operatorname{Ind} X \leqslant \alpha$ follows from condition (v) of Theorem 3.2. Since X contains a closed copy of X_α, then $\operatorname{Ind} X \geqslant \operatorname{Ind} X_\alpha = \alpha$.

The space Y can be constructed in a similar way using the space Y_α constructed in Example 5.1.

Example 5.3. Let α be an infinite countable ordinal. Then there exists a hereditarily indecomposable continuum Z with $\operatorname{Ind} Z = \alpha$ ($\operatorname{ind} Z = \alpha$) such that $B_\infty(Z)$ is a 0-dimensional dense G_δ-subset of Z, homeomorphic with the irrationals.

Let X_α (Y_α) be the continuum constructed in Example 5.1. Let us split the pseudoarc $\boldsymbol{P}$ into a 0-dimensional G_δ-set G and a 0-dimensional F_σ-set F disjoint with G. Decompose F into a sequence $A_1, A_2, \ldots$ of disjoint compact 0-dimensional subsets.

Let $Z = L(\boldsymbol{P}, X_\alpha \times A_i, A_i)$ ($Z = L(\boldsymbol{P}, Y_\alpha \times A_i, A_i)$) and $p : Z \to \boldsymbol{P}$ be as in Theorem 3.2. By Proposition 3.4(a), Z is h.i.

We will show that $B_\infty(Z) = p^{-1}(G)$. Indeed, since G does not contain any non-trivial continuum and every $p^{-1}(a)$, for $a \in F$, is terminal (see 3.2(i)), $p^{-1}(G) \subset B_\infty(Z)$. On the other hand, every composant of X_α (Y_α) is finite-dimensional, hence every point $x \in p^{-1}(F)$ belongs to a finite-dimensional continuum and therefore $B_\infty(Z) \subset p^{-1}(G)$.

By conditions (ii) and (iii) of Theorem 3.2, $B_\infty(Z)$ is a 0-dimensional dense G_δ-subset of Z, homeomorphic to G. It follows that $B_\infty(Z)$ is homeomorphic with the irrationals (cf. [4, Problem 1.3.E(b)]).

We have $\operatorname{Ind} Z \leqslant \alpha$ ($\operatorname{ind} Z \leqslant \alpha$) by conditions (v) and (vi) of Theorem 3.2, while $\operatorname{Ind} Z \geqslant \operatorname{Ind} X_\alpha = \alpha$ ($\operatorname{ind} Z \geqslant \operatorname{ind} Y_\alpha = \alpha$).

Recall that a space is punctiform if it does not contain any non-trivial subcontinuum.

Lemma 5.4. *There exists a* 1*-dimensional punctiform G_δ-set G in the pseudoarc $\boldsymbol{P}$ such that the set $\boldsymbol{P} \setminus G$ is* 0*-dimensional.*

Proof. We shall begin with the construction of a 1-dimensional punctiform G_δ-subset H of $\boldsymbol{C} \times \boldsymbol{P}$, following [4, Example 6.2.4].

Let $p_1 : \boldsymbol{I} \times \boldsymbol{P} \to \boldsymbol{I}$ be the projection and

$$\mathcal{S} = \{K \in 2^{(\boldsymbol{I} \times \boldsymbol{P})}\colon\ K \text{ is a continuum joining } \{0\} \times \boldsymbol{P} \text{ and } \{1\} \times \boldsymbol{P}\},$$

a subspace of the hyperspace $2^{(\boldsymbol{I} \times \boldsymbol{P})}$. Since $\mathcal{S}$ is closed in $2^{(\boldsymbol{I} \times \boldsymbol{P})}$, there exists a continuous mapping f of $\boldsymbol{C} \subset \boldsymbol{I}$ onto $\mathcal{S}$. Note that every set $p_1^{-1}(x) \cap f(x)$ is non-empty and the set $Y = \bigcup\{p_1^{-1}(x) \cap f(x)\colon\ x \in \boldsymbol{C}\}$ is closed in $\boldsymbol{C} \times \boldsymbol{P}$; hence there exists a G_δ-set $H \subset Y$ which intersects each fibre $p_1^{-1}(x)$ in exactly one point (see [4, Theorem 6.2.3]). Since $p_1|H : H \to \boldsymbol{C}$ is one-to-one, the set H is punctiform. The equality $\operatorname{ind} H = 1$ follows from a version of the classical Mazurkiewicz theorem stating that if a set $H \subset \boldsymbol{I} \times X$ intersects every continuum in $I \times X$ joining $\{0\} \times X$ and $\{1\} \times X$, where X is an n-dimensional compactum, then $\operatorname{ind} M \geqslant n$ (see, for example, [20, Theorem 2.2]).

Now, let D be any 0-dimensional G_δ-set in $\boldsymbol{P}$ such that $\boldsymbol{P} \setminus D$ is 0-dimensional and let $H' = H \cup (\boldsymbol{C} \times D)$. Then H' is punctiform, since $p_1^{-1}(x) \cap H'$ is 0-dimensional for every $x \in \boldsymbol{C}$.

Now, going back to the pseudoarc, we shall use a theorem of Lelek [10] providing an embedding $i : (\boldsymbol{C} \times \boldsymbol{P}) \to \boldsymbol{P}$ (see Remark 5.5 below). Let B be any 0-dimensional G_δ-

subset of $\boldsymbol{P} \setminus i(\boldsymbol{C} \times \boldsymbol{P})$ such that $(\boldsymbol{P} \setminus i(\boldsymbol{C} \times \boldsymbol{P})) \setminus B$ is 0-dimensional. Then the set $G = i(H') \cup B$ satisfies the required conditions. □

Example 5.5. For every countable ordinal $\alpha \geqslant \omega_0$ there exists a hereditarily indecomposable continuum Z such that $B_\infty(Z)$ is a 1-dimensional dense G_δ-subset of Z and $\operatorname{Ind} Z = \alpha$ (respectively, $\operatorname{ind} Z = \alpha$).

By Lemma 5.4, there exists a 1-dimensional punctiform G_δ-subset G of $\boldsymbol{P}$ such that the set $\boldsymbol{P} \setminus G$ is 0-dimensional. Let us decompose $\boldsymbol{P} \setminus G$ into a sequence of disjoint compact 0-dimensional sets $A_1, A_2, \ldots$. Let X_α (Y_α) be the space constructed in Example 5.1. Let $Z = L(\boldsymbol{P}, X_\alpha \times A_i, A_i)$ ($Z = L(\boldsymbol{P}, Y_\alpha \times A_i, A_i)$) and $p : Z \to \boldsymbol{P}$ be as in Theorem 3.2. Similarly as in Example 5.3 one shows that Z is a h.i. continuum such that $B_\infty(Z) = p^{-1}(G)$ is a dense G_δ-set in Z homeomorphic to G and $\operatorname{Ind} Z = \alpha$ ($\operatorname{ind} Z = \alpha$).

Remark 5.6. Let us notice that in Example 5.5 we can replace the pseudoarc $\boldsymbol{P}$ and $G \subset \boldsymbol{P}$ by a continuum X and $G' \subset X$ constructed in the following way. Let A be a subset of $\boldsymbol{P}$ homeomorphic to the Cantor set and let $X = M(\boldsymbol{P}, A \times \boldsymbol{P}, A)$. Then X is a 1-dimensional h.i. continuum containing a copy of $\boldsymbol{C} \times \boldsymbol{P}$. Thus X contains a 1-dimensional punctiform G_δ-set G' such that $X \setminus G'$ is 0-dimensional. The proof of this fact is analogous to the proof of Lemma 5.4 but does not require the use of the theorem of Lelek cited above.

The next two examples concern the Baire category of the set of Bing points in finite-dimensional h.i. continua (see Section 2.3).

Example 5.7. For every $n = 1, 2, \ldots$ there exists a hereditarily indecomposable n-dimensional continuum Z_n such that the set $B_n(Z_n)$ of Bing points of Z_n is residual, i.e., it contains a dense G_δ-subset of Z_n.

Indeed, let $G \subset \boldsymbol{P}$ and $A_1, A_2, \ldots$ be as in Example 5.3 (or 5.5) and let K be any n-dimensional h.i. continuum. Let $Z_n = L(\boldsymbol{P}, K \times A_i, A_i)$ and $p : Z_n \to \boldsymbol{P}$ be as in Theorem 3.2. By condition (v) of Theorem 3.2 and Proposition 3.4(a), the space Z_n is an n-dimensional h.i. continuum. Since G does not contain any non-trivial continuum and every $p^{-1}(a)$, where $a \in \bigcup_{i=1}^\infty A_i$, is terminal, then $p^{-1}(G) \subset B_n(Z_n)$. By condition (iv) of Theorem 3.2, the set $p^{-1}(G) = Z_n \setminus p^{-1}(\bigcup_{i=1}^\infty A_i)$ is a dense G_δ-subset of Z_n and thus $B_n(Z_n)$ is residual in Z_n.

Example 5.8. For every $n = 2, 3, \ldots$ there exists a h.i. n-dimensional continuum T_n such that the set $T_n \setminus B_n(T_n)$ is residual.

Indeed, let a be any point of the pseudoarc $\boldsymbol{P}$ and let K be any n-dimensional h.i. continuum. Then the pseudosuspension $T_n = M(\boldsymbol{P}, K, \{a\})$ is an n-dimensional h.i.

continuum such that $p^{-1}(\boldsymbol{P} \setminus \{a\})$ is a dense G_δ-subset of T_n disjoint with $B_n(T_n)$ (where $p : T_n \to \boldsymbol{P}$ is the natural projection).

Remark 5.9. Obviously, if $B_\infty(X)$ is nonempty, then X has an infinite-dimensional composant. The converse is not true. Indeed, if we take a point a in $\boldsymbol{P}$ and the continuum X_α constructed in Example 5.1 for any $\alpha \geqslant \omega_0$, then the pseudosuspension $Z = M(\boldsymbol{P}, X_\alpha, \{a\})$ has an infinite-dimensional composant but $B_\infty(Z) = \emptyset$.

Remark 5.10. Suppose that d is the small transfinite dimension ind or the large transfinite dimension Ind. Then every countable dimensional h.i. continuum X with $\mathrm{d}X = \alpha$ can be decomposed into the layers $\{B_\beta(X)\colon 1 \leqslant \beta \leqslant \alpha\}$, where $B_\beta(X)$ is the set of points in X which belong to a subcontinuum K of X with $\mathrm{d}K = \beta$ but avoid every non-trivial subcontinuum L of X with $\mathrm{d}L < \beta$. If $\omega_0 \leqslant \mathrm{d}X < \omega_1$ then X contains subcontinua of all finite dimensions; in particular, $B_n(X)$ is infinite-dimensional for every $n \in N$ by [20, Remark 4.1]. Example 5.1 demonstrates that all layers $B_\beta(X)$ may be empty for $\omega_0 \leqslant \beta \leqslant \mathrm{d}X$. Spaces constructed in Examples 5.2, 5.3 and 5.5 show that for every countable infinite ordinal α there exist spaces X such that $\mathrm{d}X = \alpha$ and $B_\infty(X) = B_\alpha(X)$, so $B_\beta(X) = \emptyset$ for all $\omega_0 \leqslant \beta < \alpha$. The method of condensation of singularities yields also easily examples of h.i. continua X with $\mathrm{d}X = \alpha$ for which all the layers (or some given layers) B_β, for $\beta \leqslant \alpha$, are nonempty.

As we have already mentioned, we have no examples of c.d. spaces X with $\operatorname{ind} B_\infty(X) \geqslant 2$. However, one can prove that $\operatorname{ind} B_\alpha(X) \leqslant 1$ for any h.i. continuum X with $\operatorname{ind} X = \alpha$. We shall precede a justification of this fact by a simple observation.

Lemma 5.11. *The set $Z(X)$ of all points in a compactum X which belong to a trivial component of X is at most 0-dimensional.*

Proof. Let $q : X \to q(X)$ be the quotient map of X onto the space of components of X (see [9, §46, Va]). Then $q|Z(X)$ is a homeomorphism, so $\operatorname{ind} Z(X) \leqslant 0$. □

Proposition 5.12. *Given a compactum Y, let $Z(Y)$ be as in Lemma 5.11 and $B'_\alpha(Y)$ be the set of all points in Y which do not belong to any non-trivial subcontinuum of Y with $\operatorname{ind} < \alpha$.*

If X is a compact space which does not contain any decomposable continuum and $\operatorname{ind} X \leqslant \alpha + \beta$, then $\operatorname{ind}(Z(X) \cup B'_\alpha(X)) \leqslant 1 + \beta$. In particular, if $\operatorname{ind} X = \alpha$ then $\operatorname{ind} B'_\alpha(X) \leqslant 1$.

Proof. We use induction with respect to β. Assume first that $\beta = 0$ and let $\mathcal{V}$ be a base of X such that $\operatorname{ind} \operatorname{Fr} V < \alpha$ for every $V \in \mathcal{V}$. For every $V \in \mathcal{V}$ we have $\operatorname{Fr} V \cap (Z(X) \cup B'_\alpha(X)) \subset Z(\operatorname{Fr} V)$; hence the space $Z(X) \cup B'_\alpha(X)$ has a base consisting of sets with 0-dimensional boundaries. This ends the proof for $\beta = 0$. Suppose now that the theorem is true for every ordinal $< \beta$ and let $\operatorname{ind} X \leqslant \alpha + \beta$. Then X has a base $\mathcal{V}$ such that $\operatorname{ind} \operatorname{Fr} V < \alpha + \beta$ for every $V \in \mathcal{V}$. Observe that

$$\operatorname{Fr} V \cap \big(Z(X) \cup B'_\alpha(X)\big) \subset B'_\alpha(\operatorname{Fr} V) \cup Z(\operatorname{Fr} V)$$

for every $V \in \mathcal{V}$. From this and the inductive assumption it follows that the space $Z(X) \cup B'_\alpha(X)$ has a base consisting of sets with boundaries having $\operatorname{ind} < 1 + \beta$. This completes the proof. □

Proposition 5.13. *If a continuum X is the union of countably many closed finite-dimensional subsets, then* $\operatorname{ind} B_\infty(X) \leqslant 0$.

Proof. Suppose that $X = \bigcup_{j=1}^\infty F_j$, where F_j is closed in X and $\operatorname{ind} F_j < \infty$ for $j = 1, 2, \ldots$. Then $F_j \cap B_\infty(X) \subset Z(F_j)$; hence $\operatorname{ind}(F_j \cap B_\infty(X)) \leqslant 0$ by Lemma 5.11. Thus $\operatorname{ind} B_\infty(X) \leqslant 0$ by the countable sum theorem. □

6. On Bing points and composants in uncountable-dimensional spaces

Let X be a h.i. continuum. Then, by a theorem of Levin [11, Theorem 3],

(1) the union of all non-trivial finite-dimensional subcontinua, i.e., the set $X \setminus B_\infty(X)$, is countable-dimensional.
Thus, by the sum theorem, see Lemma 2.5(ii), it follows that

(2) the set $B_\infty(X)$ is uncountable-dimensional (respectively, s.i.d., respectively, not a C-space) for any uncountable-dimensional (respectively, s.i.d., respectively, not a C-space) h.i. continuum X.
It was also observed by Levin (see [11, Theorem 8]) that

(3) the union of all non-trivial w.i.d. subcontinua of a h.i. continuum is w.i.d.
This fact can be generalized as follows.

Proposition 6.1. *Let $\mathcal{K}$ be a class of spaces satisfying conditions* (i)–(iii) *of Lemma* 2.5. *Let $\mathcal{B}_{\mathcal{K}}(X)$ denote the set of points x in a h.i. continuum X such that no non-trivial subcontinuum of X containing x belongs to $\mathcal{K}$. Then the set $X \setminus B_{\mathcal{K}}(X)$ belongs to $\mathcal{K}$.*

Proof. We use the idea of Levin [11]. Let $U_1, U_2, \ldots$ be a countable base of X. For every $n \in N$ let W_n be the union of all non-trivial components of $\overline{U_n}$ which belong to $\mathcal{K}$. Considering the restriction $q|W_n$ of the quotient mapping q of $\overline{U_n}$ onto the space of components of $\overline{U_n}$, we deduce from (iii) that W_n belongs to $\mathcal{K}$. By (ii), the set $\bigcup_{n=1}^\infty W_n$ is in $\mathcal{K}$. Thus to end the proof it suffices to show that $X \setminus B_{\mathcal{K}}(X) = \bigcup_{n=1}^\infty W_n$. If $x \in W_n$ for some n, then x lies in a non-trivial continuum belonging to $\mathcal{K}$ and so $x \in X \setminus B_{\mathcal{K}}(X)$. Suppose now that $x \notin \bigcup_{n=1}^\infty W_n$ and let K be any non-trivial continuum in X containing x. Take U_n containing x with $\operatorname{diam} U_n < \operatorname{diam} K$. Since the component S of x in $\overline{U_n}$ is non-trivial and $x \notin W_n$, S does not belong to $\mathcal{K}$. Since $x \in S \cap K$ and $\operatorname{diam} S < \operatorname{diam} K$, $S \subset K$, X being hereditarily indecomposable. Thus, by (i), K does not belong to $\mathcal{K}$. This shows that $x \in B_{\mathcal{K}}(X)$ and the proof is completed. □

Remark 6.2. A theorem of Henderson [6] stating that every s.i.d. continuum contains an s.i.d. continuum without any non-trivial finite-dimensional subcontinua implies that for every h.i. continuum X that is s.i.d., the set $B_\infty(X)$ contains a s.i.d. continuum. This fact is also an easy consequence of Levin's result (3) (see [11]). In fact, Proposition 6.1 immediately implies that if $\mathcal{K}$ is a class of spaces satisfying conditions (i)–(iv) of Lemma 2.5, then for every h.i. continuum X not belonging to $\mathcal{K}$ the set $B_\infty(X)$ contains a continuum which is not in $\mathcal{K}$.

Corollary 6.3. *Let X be a hereditarily indecomposable continuum. Then*

(a) *if every composant of X is finite-dimensional, then X is countable-dimensional,*

(b) *if every composant of X is weakly infinite-dimensional* (*respectively, is a C-space*), *then X is weakly infinite-dimensional* (*respectively, is a C-space*).

Proof. (a) If X is not c.d. then, by (2), $B_\infty(X)$ is not c.d.; hence it contains a point x_0. Then the composant $C(x_0)$ of x_0 is infinite-dimensional.

(b) The case of w.i.d. spaces follows similarly from (3). Suppose that X is not a C-space. If $\mathcal{K}$ is the class of C-spaces then, by Proposition 6.1, the set $B_{\mathcal{K}}(X)$ is not a C-space, in particular, there exists a point $x_0 \in B_{\mathcal{K}}(X)$. It follows that the composant of x_0 is not a C-space. □

We cannot claim, however, that every uncountable-dimensional space has a composant which is uncountable-dimensional. This is illustrated by the following example, obtained jointly with R. Pol.

Example 6.4. There exists a weakly infinite-dimensional hereditarily indecomposable continuum M each composant of which is countable-dimensional (in fact, $\sup\{\operatorname{ind} C\colon C$ is a composant of $M\} < \omega_1$) such that the set $B_\infty(M)$ is strongly infinite-dimensional. Moreover, M is a C-space.

We will need the following variant of the example of a w.i.d. compactum which is not countable-dimensional described in [18].

There exists a w.i.d. compact space S and a G_δ-set $H \subset S$ such that:

(4) $\operatorname{ind} H = \infty$ and each subset of H of positive dimension is strongly infinite-dimensional,

(5) $S \setminus H = \bigcup_{j=1}^{\infty} A_j$, where A_j are compact and finite-dimensional,

(6) all components of S are countable-dimensional (in fact, $\sup\{\operatorname{ind} D\colon D$ is a component of $S\} \leqslant \alpha_0$ for some infinite ordinal $\alpha_0 < \omega_1$).

To be more specific, let us consider the projection $p : \boldsymbol{C} \times I^\infty \to \boldsymbol{C}$. As shown in [18], there exists an s.i.d. G_δ-set $X \subset \boldsymbol{C} \times I^\infty$ which intersects each fiber $p^{-1}(t)$ in exactly one point. Let Z be the closure of X in $\boldsymbol{C} \times I^\infty$. Following the proof of Lemma 5.3.1 in [4] one can find a continuous map $F : Z \to I^\infty$ such that if k is the mapping of Z into $\boldsymbol{C} \times I^\infty$ defined by $k(x) = (p(x), F(x))$ then:

(7) k restricted to X is an embedding, and

(8) $Y \setminus k(X)$ is the union of countably many compact finite-dimensional subspaces, where Y is the closure of $k(X)$ in $\boldsymbol{C} \times I^\infty$.

Let q be the restriction to Y of the projection of $\boldsymbol{C} \times I^\infty$ to $\boldsymbol{C}$. Each fiber $q^{-1}(t)$ is the union of the c.d. set $(Y \setminus k(X)) \cap q^{-1}(t)$ and the one-point set $\{k(x)\}$, where $\{x\} = X \cap p^{-1}(t)$. Thus $q^{-1}(t)$ is c.d. for every $t \in \boldsymbol{C}$. Moreover, by Theorem 3.6 of [19], there exists a c.d. compactum which contains topologically every fiber $q^{-1}(t)$; in particular, $\sup\{\operatorname{ind} q^{-1}(t)\colon\ t \in \boldsymbol{C}\} \leqslant \alpha_0$ for some infinite ordinal $\alpha_0 < \omega_1$. Since Y admits a perfect mapping q with countable-dimensional fibers onto $\boldsymbol{C}$, then Y is a C-space (see Lemma 2.5(iii)). In particular, Y is w.i.d. Since the space $k(X)$ is s.i.d. by (7), then by a theorem of Rubin [21] it contains a closed s.i.d. subspace H without any w.i.d. subsets of positive dimension. Let S be the closure of H in Y. Then S is a w.i.d. space (and a C-space) as a closed subspace of the C-space Y (see Lemma 2.5(i)). Since $H = S \cap k(X)$, then $S \setminus H = S \cap (Y \setminus k(X))$ is a closed subset of $Y \setminus k(X)$. Therefore, by (8), the complement $S \setminus H$ is the union of countably many compact finite-dimensional sets $A_1, A_2, \ldots$. Moreover, every component D of S is a subset of $q^{-1}(t) \cap S$ for some $t \in \boldsymbol{C}$; hence $\operatorname{ind} D \leqslant \operatorname{ind} q^{-1}(t) \leqslant \alpha_0$. It follows that $\sup\{\operatorname{ind} D\colon\ D$ is a component of $S\} \leqslant \alpha_0 < \omega_1$.

Having the space S constructed, one can find two disjoint closed subsets F_1 and F_2 of S such that each partition in S between F_1 and F_2 intersects H in an infinite-dimensional set. To show that such two sets exist suppose that this is not true. Then every point of H has an arbitrary small neighbourhood U in S such that $\operatorname{Fr} U \cap H$ is finite-dimensional. It follows that the space H has a countable base of open sets with finite-dimensional boundaries, which implies that H is c.d., contrary to the fact that it is s.i.d.

Now, let K be a partition in S between F_1 and F_2 such that each component of K is hereditarily indecomposable (see 2.2). Let A be a 0-dimensional compact subset of the pseudo-arc $\boldsymbol{P}$ homeomorphic with the decomposition space of K into components such that each composant of $\boldsymbol{P}$ intersects A in at most one point (see 2.7(iii)). Finally, let $M = M(\boldsymbol{P}, K, A)$ be a continuum containing K which admits an atomic map r onto $\boldsymbol{P}$ such that $r|r^{-1}(\boldsymbol{P} \setminus A)$ is a homeomorphism and $r^{-1}(a)$ is a component of K if $a \in A$ (see Theorem 3.1). Then M is a h.i. continuum by Remark 3.5.

Let C be a composant of M. By Lemma 2.8, C is the preimage under r of some composant C' of $\boldsymbol{P}$. If $A \cap C' = \emptyset$, then C is homeomorphic to C' and hence $\operatorname{ind} C = 1$. If $A \cap C' = \{a\}$, then C is the union of the 1-dimensional set $r^{-1}(C' \setminus \{a\})$ and the set $r^{-1}(a)$, which is homeomorphic to some component D of the space K. Since K is a subspace of S, then $\operatorname{ind} D \leqslant \alpha_0$ by (6), and thus $\operatorname{ind} C \leqslant \alpha_0$ (see the fact $(\star)$ at the end of the proof of Theorem 3.2). In particular, all composants of M are countable-dimensional.

The space M is w.i.d. (even a C-space) as the union of a w.i.d. compactum K (which is a C-space) and an open subset homeomorphic with $\boldsymbol{P} \setminus A$ (see Lemma 2.5(ii)).

It remains to show that $B_\infty(M)$ is s.i.d. Consider $G = H \cap K$ and $F_j = A_j \cap K$ as subspaces of M. By the choice of K the set G is infinite-dimensional; hence it is s.i.d. by (4). It follows that M is not c.d. Since $\operatorname{ind} F_j < \infty$, $F_j \cap B_\infty(M) \subset Z(F_j)$ and so $\operatorname{ind}(F_j \cap B_\infty(M)) \leqslant 0$ (see Lemma 5.11). Thus the set $E = \bigcup_{j=1}^{\infty}(F_j \cap B_\infty(M))$ is at most 0-dimensional. Let E^* be a zero-dimensional G_δ-set in M containing E. Since, by

(1), the set $B_\infty(M)$ is not c.d., then $B_\infty(M) \setminus E^*$ is an infinite-dimensional subset of $G \subset H$. By (4), $B_\infty(M) \setminus E^*$ must be strongly infinite-dimensional. But $B_\infty(M) \setminus E^*$ is an F_σ-subset of the space $B_\infty(M)$, so $B_\infty(M)$ is also s.i.d. (see Lemma 2.5(i) and (ii)).

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