Applied Mathematics Letters 25 (2012) 257–261

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Conservation laws for some compacton equations using the multiplier approach

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ARTICLE INFO

Article history:
Received 14 February 2011
Received in revised form 10 August 2011
Accepted 17 August 2011

Keywords:
Multipliers
Compacton $k(2, 2)$ equation
Compacton $k(3, 3)$ equation
Compacton $Zk(2, 2)$ equation

ABSTRACT

This paper is an application of the variational derivative method to the derivation of the conservation laws for partial differential equations. The conservation laws for (1+1) dimensional compacton $k(2, 2)$ and compacton $k(3, 3)$ equations are studied via multiplier approach. Also the conservation laws for (2+1) dimensional compacton $Zk(2, 2)$ equation are established by first computing the multipliers.

1. Introduction

The conservation laws for partial differential equations are important in solution and reduction point of view. There are many different approaches to the construction of conservation laws. For variational problems, the Noether theorem [1] can be used for the derivation of conservation laws. The Laplace Direct method [2], Kara and Mahomed symmetry condition [3] and partial Noether approach [4] are useful for the construction of conservation laws for both variational and non-variational problems. There are some other approaches in which conservation laws are expressed in the form of characters [5–7]. Wolf [8,9], Göktas and Hereman [10], Hereman et al. [11–13] and Cheviakov [14] developed powerful software packages to compute conservation laws for partial differential equations.

The multiplier approach (also known as the variational derivative method) is adopted in this paper. It was successfully applied to the construction of conservation laws for nonlinear partial differential equations [15,16]. In [17], conservation laws for Zk equation were derived. The compacton $k(2, 2)$ equation, the compacton $k(3, 3)$ equation and the compacton $Zk(2, 2)$ equation [18–20] are considered in this work. To the best of our knowledge, the conservation laws for these equations are not computed and are the subject of this paper.

The detailed outline of the paper is as follows. In Section 2, some definitions related with the multiplier approach are given. In Section 3, conservation laws for the $k(2, 2)$ equation are derived by first computing the multipliers. The conservation laws for the compacton $k(3, 3)$ equation and the compacton $Zk(2, 2)$ equation are established in Sections 4 and 5, respectively. Finally, conclusions are summarized in Section 6.

2. Preliminaries

Let $x^i,\ i = 1, 2, \dots, n$ be n independent variables and u be the dependent variable.

1. The total derivative operator with respect to x^i is

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \dots, n, \tag{1}$$

where u_i denotes the derivative of u with respect to x^i. Similarly u_{ij} denotes the derivative of u with respect to x^i and x^j.

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doi:10.1016/j.aml.2011.08.019

2. The Euler operator is defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_{ij} \frac{\partial}{\partial u_{ij}} - D_{ijk} \frac{\partial}{\partial u_{ijk}} + \cdots. \tag{2}$$

Consider a kth-order partial differential equation of n independent and one dependent variable

$$E(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0. \tag{3}$$

3. An n-tuple $T = (T^1, T^2, \ldots, T^n),\ i = 1, 2 \cdots n$, such that

$$D_i T^i = 0 \tag{4}$$

holds for all solutions of (3) is known as the conserved vector of (3).

4. The multiplier Λ of system (3) has the property

$$D_i T^i = \Lambda E, \tag{5}$$

for arbitrary function $u(x^1, x^2, \ldots, x^n)$ [5,6].

5. The determining equations for multipliers are obtained by taking the variational derivative of (5) (see [6]):

$$\frac{\delta}{\delta u}(\Lambda E) = 0. \tag{6}$$

Eq. (6) holds for arbitrary function $u(x^1, x^2, \ldots, x^n)$ not only for solutions of system (3).

Once the multipliers are computed from (6), the conserved vectors can be derived systematically using (5) as the determining equation. But in some problems it is not difficult to construct the conserved vectors by elementary manipulations after the determination of the multipliers.

The summation convention is adopted in which there is summation over repeated upper and lower indices.

3. Conservation laws for the compacton $k(2, 2)$ equation

The compacton $k(2, 2)$ equation [18,19], which takes the form

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \tag{7}$$

or alternatively

$$u_t + 2uu_x + 6u_x u_{xx} + 2uu_{xxx} = 0. \tag{8}$$

We will derive the conservation laws for (8) by the multiplier approach. The determining equation for multiplier $\Lambda(t, x, u)$, from (6), is

$$\frac{\delta}{\delta u}[\Lambda(u_t + 2uu_x + 6u_x u_{xx} + 2uu_{xxx})] = 0. \tag{9}$$

The standard Euler operator $\delta/\delta u$ from (2) can be defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}} - \cdots, \tag{10}$$

and total derivative operators D_t and D_x using (1) are

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots, \tag{11}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots. \tag{12}$$

Eq. (9) after expansion and simplification takes the following form;

$$u_{xx}[6u\Lambda_{xu} + 6uu_x\Lambda_{uu} - 6u_x\Lambda_u] + 2uu_x^3\Lambda_{uuu} + 6uu_x^2\Lambda_{uux} + 6uu_x\Lambda_{xxu} + 2u\Lambda_{xxx} + 2u\Lambda_x + \Lambda_t = 0, \tag{13}$$

which yields

$$\Lambda = c_1 + \frac{c_2}{2}u^2 + c_3 \sin x + c_4 \cos x. \tag{14}$$

From (5) and (14), we have

$$\left(c_1 + \frac{c_2}{2}u^2 + c_3 \sin x + c_4 \cos x\right)(u_t + 2uu_x + 6u_x u_{xx} + 2uu_{xxx})$$
$$= D_t\left[c_1 u + \frac{c_2}{6}u^3 + c_3 u \sin x + c_4 u \cos x\right] + D_x\Bigg[c_1(u^2 + 2uu_{xx} + 2u_x^2) + c_2\left(\frac{u^4}{4} + u^3 u_{xx}\right)$$
$$+ c_3(2uu_{xx}\sin x + 2u_x^2 \sin x - 2uu_x \cos x) + c_4(2uu_{xx}\cos x + 2u_x^2 \cos x + 2uu_x \sin x)\Bigg], \tag{15}$$

for arbitrary functions $u(t, x)$ and $v(t, x)$. When $u(t, x)$ and $v(t, x)$ are solutions of Eq. (8) then left hand side of (15) vanishes and we obtain

$$D_t\left[c_1 u + \frac{c_2}{6}u^3 + c_3 u \sin x + c_4 u \cos x\right] + D_x\Bigg[c_1(u^2 + 2uu_{xx} + 2u_x^2) + c_2\left(\frac{u^4}{4} + u^3 u_{xx}\right)$$
$$+ c_3(2uu_{xx}\sin x + 2u_x^2 \sin x - 2uu_x \cos x) + c_4(2uu_{xx}\cos x + 2u_x^2 \cos x + 2uu_x \sin x)\Bigg] = 0. \tag{16}$$

Therefore the conserved vectors for $k(2, 2)$ equation (8) are

$$T_1^1 = u, \qquad T_1^2 = u^2 + 2uu_{xx} + 2u_x^2, \tag{17}$$

$$T_2^1 = \frac{u^3}{6}, \qquad T_2^2 = \frac{u^4}{4} + u^3 u_{xx}, \tag{18}$$

$$T_3^1 = u \sin x, \qquad T_3^2 = 2uu_{xx}\sin x + 2u_x^2 \sin x - 2uu_x \cos x, \tag{19}$$

$$T_4^1 = u \cos x, \qquad T_4^2 = 2uu_{xx}\cos x + 2u_x^2 \cos x + 2uu_x \sin x. \tag{20}$$

The variational derivative approach for the compacton $k(2, 2)$ equation gives four multipliers of the form $\Lambda(t, x, u)$ and hence four conserved vectors are obtained. The first-order multipliers $\Lambda(t, x, u, u_t, u_x)$ and the higher order multipliers determining equations are too complicated and cannot be separated manually. The computer program in [8,9] can be used to search for the higher order multipliers.

4. Conservation laws for the compacton $k(3, 3)$ equation

Consider the compacton $k(3, 3)$ equation [18,19] which have the form

$$u_t + (u^3)_x + (u^3)_{xxx} = 0, \tag{21}$$

or

$$u_t + 3u^2 u_x + 6u_x^3 + 18uu_x u_{xx} + 3u^2 u_{xxx} = 0. \tag{22}$$

The determining equation for multiplier $\Lambda(t, x, u)$ after expansion takes the following form:

$$\Lambda_u(u_t + 3u^2 u_x + 6u_x^3 + 18uu_x u_{xx} + 3u^2 u_{xxx}) + 6uu_x\Lambda + 18u_x u_{xx}\Lambda + 6uu_{xxx}\Lambda$$
$$- D_t(\Lambda) - D_x[\Lambda(3u^2 + 18u_x^2 + 18uu_{xx})] + D_x^2(18uu_x\Lambda) - D_x^3(3u^2\Lambda) = 0. \tag{23}$$

Eq. (23) is separated according to different combinations of derivatives of u and an overdetermined system of equations for multiplier Λ is obtained which gives

$$\Lambda = c_1 + \frac{c_2}{3}u^3 + c_3 \sin x + c_4 \cos x. \tag{24}$$

From (5) and (24), we obtain following conserved vectors:

$$T_1^1 = u, \qquad T_1^2 = u^3 + 3u^2 u_{xx} + 6uu_x^2, \tag{25}$$

$$T_2^1 = \frac{u^4}{12}, \qquad T_2^2 = \frac{u^6}{6} + u^5 u_{xx} + \frac{1}{2}u^4 u_x^2, \tag{26}$$

$$T_3^1 = u \sin x, \qquad T_3^2 = 3u^2 u_{xx}\sin x + 6uu_x^2 \sin x - 3u^2 u_x \cos x, \tag{27}$$

$$T_4^1 = u \cos x, \qquad T_4^2 = 3u^2 u_{xx}\cos x + 6uu_x^2 \cos x + 3u^2 u_x \sin x. \tag{28}$$

The variational derivative approach for the compacton $k(3, 3)$ equation gives four conserved vectors corresponding to four multipliers of the form $\Lambda(t, x, u)$. The computer program in [8,9] can be used to search for the higher order multipliers.

5. Conservation laws for the compacton $Zk(2, 2)$ equation

The compacton $Zk(2, 2)$ equation [18,20], which takes the form

$$u_t + (u^2)_x + (u^2)_{xxx} + +(u^2)_{xyy} = 0, \tag{29}$$

or alternatively

$$u_t + 2uu_x + 6u_xu_{xx} + 2uu_{xxx} + 2u_xu_{yy} + 4u_yu_{xy} + 2uu_{xyy} = 0. \tag{30}$$

The determining equation for multiplier $\Lambda(t, x, u)$, from (6), is

$$\frac{\delta}{\delta u}[\Lambda(u_t + 2uu_x + 6u_xu_{xx} + 2uu_{xxx} + 2u_xu_{yy} + 4u_yu_{xy} + 2uu_{xyy})] = 0, \tag{31}$$

where $\delta/\delta u$ is the standard Euler operator which from (2) is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_xD_t\frac{\partial}{\partial u_{tx}} + D_yD_t\frac{\partial}{\partial u_{ty}} + D_xD_y\frac{\partial}{\partial u_{xy}} - \cdots. \tag{32}$$

Eq. (31), after expansion and simplification, becomes

$$u_{xx}[6u\Lambda_{xu} + 6uu_x\Lambda_{uu} - 6u_x\Lambda_u] + u_{xy}[4u\Lambda_{yu} + 4uu_y\Lambda_{uu} - 4u_y\Lambda_u] + u_{yy}[2u\Lambda_{xu} + 2uu_x\Lambda_{uu} - 2u_x\Lambda_u] + 2uu_x^3\Lambda_{uuu} + 2uu_xu_y^2\Lambda_{uuu} + 6uu_x^2\Lambda_{uux} + 2uu_y^2\Lambda_{uux} + 4uu_xu_y\Lambda_{uuy} + 6uu_x\Lambda_{xxu} + 2uu_x\Lambda_{yyu} + 4uu_y\Lambda_{uxy} + 2u\Lambda_{xxx} + 2u\Lambda_{xyy} + 2u\Lambda_x + \Lambda_t = 0. \tag{33}$$

Eq. (33) gives

$$\Lambda = \frac{c_1}{2}u^2 + A(x, y), \tag{34}$$

where $A(x, y)$ is an arbitrary function which satisfies

$$A_x + A_{xxx} + A_{xyy} = 0. \tag{35}$$

Thus there is an infinite number of multipliers $\Lambda = A(x, y)$ satisfying (35).

A conserved vector for (30) has three components T^1, T^2, T^3. The conserved vector corresponding to multiplier $u^2/2$ is

$$T_1^1 = \frac{u^3}{6}, \qquad T_1^2 = \frac{u^4}{4} + u^3u_{xx} + \frac{1}{3}u^3u_{yy}, \qquad T_1^3 = u^3u_{xy} - \frac{1}{3}u^3u_{xy}. \tag{36}$$

The conserved vector corresponding to multiplier $\Lambda = A(x, y)$ is

$$T_A^1 = uA(x, y), \qquad T_A^2 = A(x, y)[u^2 + 2uu_{xx} + 2u_x^2] + u^2A_{xx} - 2uu_yA_y - 2uu_xA_x$$
$$T_A^3 = 2A(x, y)[u_xu_y + uu_{xy}] + u^2A_{xy}, \tag{37}$$

where $A(x, y)$ satisfies (35). There is an infinite number of conserved vectors (37) for multipliers $\Lambda = A(x, y)$ satisfying (35), some of these are given in Table 1.

6. Conclusions

The conservation laws for the compacton $k(2, 2)$ equation, the compacton $k(3, 3)$ equation and the compacton $Zk(2, 2)$ equation were constructed by utilizing the multiplier approach. The multiplier approach on compacton $k(2, 2)$ and $k(3, 3)$ equations yielded four multipliers and thus four local conserved vectors were obtained in each case. The conserved vectors obtained here can be used in reductions and solutions of these partial differential equations. For the compacton $Zk(2, 2)$ equation, an infinite number of multipliers were obtained for arbitrary function $A(x, y)$ satisfying relation (35), some of which were given in Table 1.

The multiplier approach only yields the multipliers for the local conserved vectors and there exists a conserved vector corresponding to each multiplier. In this paper, only multipliers of the form $\Lambda(t, x, u)$ were considered since the higher order multipliers determining equations are too complicated and cannot be separated manually. The higher order multipliers can be found using the computer packages [8,9].

Acknowledgment

Rehana Naz is grateful to the Lahore school of Economics for providing nice environment and funding to complete this research work.

Table 1
Multipliers and conserved vectors for the compacton $Zk(2, 2)$ equation.

Multiplier	Conserved vector
$\Lambda = 1$	$T_2^1 = u,\ T_2^2 = u^2 + 2uu_{xx} + 2u_x^2,\ T_2^3 = 2u_x u_y + 2uu_{xy}$
$\Lambda = f(y)$	$T_3^1 = uf(y),\ T_3^2 = f(y)[u^2 + 2uu_{xx} + 2u_x^2] - 2uu_y f'(y)$ $T_3^3 = 2f(y)[u_x u_y + uu_{xy}]$
$\Lambda = \sin x$	$T_4^1 = u\sin x,\ T_4^2 = \sin x[2uu_{xx} + 2u_x^2] - 2uu_x \cos x$ $T_4^3 = 2\sin x[u_x u_y + uu_{xy}]$
$\Lambda = \cos x$	$T_5^1 = u\cos x,\ T_5^2 = \cos x[2uu_{xx} + 2u_x^2] + 2uu_x \sin x$ $T_5^3 = 2\cos x[u_x u_y + uu_{xy}]$
$\Lambda = x\cos y$	$T_6^1 = ux\cos y,\ T_6^2 = x\cos y[u^2 + 2uu_{xx} + 2u_x^2] - 2uu_x \cos y + 2xuu_y \sin y$ $T_6^3 = 2x\cos y[u_x u_y + uu_{xy}] - u^2 \sin y$
$\Lambda = x\sin y$	$T_7^1 = ux\sin y,\ T_7^2 = x\sin y[u^2 + 2uu_{xx} + 2u_x^2] - 2uu_x \sin y - 2xuu_y \cos y$ $T_7^3 = 2x\sin y[u_x u_y + uu_{xy}] + u^2 \cos y$
$\Lambda = x^2\cos y$	$T_8^1 = x^2 u\cos y$ $T_8^2 = x^2\cos y[u^2 + 2uu_{xx} + 2u_x^2] - 4xuu_x \cos y + 2x^2 uu_y \sin y + 2u^2 \cos y$ $T_8^3 = 2x^2\cos y[u_x u_y + uu_{xy}] - 2xu^2 \sin y$
$\Lambda = x^2\sin y$	$T_9^1 = x^2 u\sin y$ $T_9^2 = x^2\sin y[u^2 + 2uu_{xx} + 2u_x^2] - 4xuu_x \sin y - 2x^2 uu_y \cos y + 2u^2 \sin y$ $T_9^3 = 2x^2\sin y[u_x u_y + uu_{xy}] + 2xu^2 \cos y$

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