Advances in Mathematics **166**, 260–297 (2002)
doi:10.1006/aima.2001.2035, available online at http://www.idealibrary.com on IDEAL®

Constrained von Neumann Inequalities

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Received December 6, 2000; accepted August 21, 2001

An equivalent formulation of the von Neumann inequality states that the backward shift S^* on ℓ_2 is extremal, in the sense that if T is a Hilbert space contraction, then $\|p(T)\| \leqslant \|p(S^*)\|$ for each polynomial p. We discuss several results of the following type: if T is a Hilbert space contraction satisfying some constraints, then S^* restricted to a suitable invariant subspace is an extremal operator. Several operator radii are used instead of the operator norm. Applications to inequalities of coefficients of rational functions positive on the torus are given.

Key Words: Hilbert space operators; (constrained) von Neumann inequalities; operator radii; inequalities for positive trigonometric polynomials.

0. INTRODUCTION

Let T be a Hilbert space contraction, that is a bounded linear operator of norm at most one on a complex, separable Hilbert space H. A well-known inequality due to von Neumann [vN] asserts that

$$\|p(T)\| \leqslant \|p\|_\infty, \tag{0.1}$$

for every polynomial $p \in \mathbb{C}[X]$. Here

$$\|p\|_\infty = \sup\{|p(z)|: z \in \mathbb{C}, |z| \leqslant 1\}$$

0001-8708/02 $35.00

is the supremum norm of p, while

$$\|p(T)\| = \|p(T)\|_{\mathscr{B}(H)}$$

is the operator norm of $p(T)$ in $\mathscr{B}(H)$, the C^*-algebra of all bounded linear operators on H. The same inequality extends for functions in the disc algebra $A(\mathbb{D})$ and, if T is a completely non-unitary (c.n.u.) contraction, it extends to bounded analytic functions $f \in H^\infty(\mathbb{D})$ [NF]. Recall that a c.n.u. operator is one which has no unitary direct summand [NF].

Denote by S the forward unilateral shift on ℓ^2,

$$S(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots),$$

and by $S^* \in B(\ell^2)$,

$$S^*(x_0, x_1, \ldots) = (x_1, x_2, \ldots),$$

its adjoint (the backward shift).

An equivalent formulation of the von Neumann inequality (0.1) is the following: for every Hilbert space contraction T and every polynomial p we have

$$\|p(T)\|_{\mathscr{B}(H)} \leqslant \|p(S^*)\|_{\mathscr{B}(\ell^2)}. \tag{0.2}$$

We say that S^* is extremal. A proof of the inequality (0.2) will be sketched in Section 2.

We will discuss several results of the following type: if T is a Hilbert space contraction satisfying some constraints and ω is an operator radius, then there exists a suitable invariant subspace E of S^* such that

$$\omega(p(T)) \leqslant \omega(p(S^* \mid E));$$

that is S^* restricted to a suitable invariant subspace is an extremal operator.

Several results of this type are known in the literature. The following result was proved by Pták [P1, P2] in a particular case; the general case was proved by Pták and Young [PY]. Suppose that p and q are arbitrary analytic polynomials. Let T be a Hilbert space contraction of spectral radius smaller than one and suppose that $q(T) = 0$. Then

$$\|p(T)\| \leqslant \|p(S^* \mid \operatorname{Ker} q(S^*))\|.$$

The following extension was given by Sz.-Nagy [N]. Let f and g be two functions in $H^\infty(\mathbb{D})$. Let T be a Hilbert space c.n.u. contraction such that $g(T) = 0$. Then

$$\|f(T)\| \leqslant \|f(S^* \mid \operatorname{Ker} g(S^*))\|.$$

An equivalent form of Sz.-Nagy's result was stated by Williams [W]; Williams' proof is given in the survey paper [P3].

An apparently unrelated inequality due to Haagerup and de la Harpe [HH] asserts that each bounded linear nilpotent contraction T with $T^n = 0$, $n \geqslant 2$, satisfies the inequality

$$\omega_2(T) \leqslant \cos \frac{\pi}{n+1}. \tag{0.3}$$

Here $\omega_2(T)$ denotes the numerical radius of T defined by

$$\omega_2(T) = \sup\{|\langle Tx \mid x \rangle|: x \in H, \|x\| = 1\}.$$

To see how the Haagerup–de la Harpe inequality fits into the present framework, let S_n^* be the nilpotent Jordan cell

$$S_n^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

on the standard Euclidean space $\mathbb{C}^n$. Then [GR] $\cos(\pi/(n+1)) = \omega_2(S_n^*)$ and S_n^* is unitarily equivalent to $S^* \mid \mathbb{C}^n = S^* \mid \operatorname{Ker} u_n(S^*)$, where $u_n(z) = z^n$. Therefore the inequality of Haagerup and de la Harpe states that if $u_n(T) = 0$, then

$$\omega_2(T) \leqslant \omega_2(S^* \mid \operatorname{Ker} u_n(S^*)).$$

We refer to [Wu, Su, Po] for recent papers related to this inequality.

In [HH], inequality (0.3) is shown to be equivalent to an inequality, due to Fejer (1915), for the first coefficient c_1 of a positive trigonometric polynomial $\sum_{j=-n+1}^{n-1} c_j e^{ijt}$, namely

$$|c_1| \leqslant c_0 \cos\left(\frac{\pi}{n+1}\right).$$

We will prove other inequalities for coefficients of rational functions positive on the torus or for coefficients of positive trigonometric polynomials which are related to our constrained von Neumann inequalities. In particular, we obtain (Theorem 5.3) the following inequality for the sum of the absolute values of two coefficients of a positive trigonometric polynomial of degree n,

$$|c_k|+|c_l| \leqslant c_0 \left(1+\cos \frac{\pi}{\left[\frac{n-1}{k+l}\right]+2}\right)^{1/2} \left(1+\cos \frac{\pi}{\left[\frac{n-1}{|k-l|}\right]+2}\right)^{1/2},$$

for any distinct numbers k and l among $\{0, \ldots, n-1\}$.

Organization of the paper. We consider in the first section two classes of operator radii, called admissible and strongly admissible radii. The operator norm and the numerical radius belong to both classes as well as the more general radii ω_ρ for $\rho \leqslant 2$. We prove in Section 2 some constrained and unconstrained von Neumann inequalities for (strongly) admissible radii using the construction of analytic models of [AEM]. In Section 3 we prove some constrained von Neumann inequalities for radii which are associated to some bundles of operators; these radii are not necessarily admissible. The constraints in Section 2 are of algebraic type ($p(T)=0$ or $P(T^*, T)=0$) while in Section 3 they are of the type $u(T)=0$ for an inner function u. Several applications of the above general constrained von Neumann inequalities are given in Section 4. Applications to bounds of positive rational functions are presented in Section 5. In the last section we discuss constrained von Neumann inequalities with different types of constraints.

1. ADMISSIBLE AND STRONGLY ADMISSIBLE OPERATOR RADII

Admissible operator radii. In this paragraph w denotes a family of so-called operator radii $w=\{w_H\}$, one for each separable Hilbert space under consideration. An operator radius w_H is a map from $\mathscr{B}(H)$ to $[0, +\infty]$. For $T \in \mathscr{B}(H)$ we simply write $w(T)$ instead of the more correct $w_H(T)$ and say that w is an *operator radius*, or simply a radius.

DEFINITION 1.1. A radius w defined for all Hilbert space operators with values in $[0, +\infty]$ is called an *admissible radius* if it satisfies

(i) (unitary invariance) $w(U^*TU)=w(T)$ for each unitary $U\colon K \to H$ and each $T \in \mathscr{B}(H)$;

(ii) (isotonicity for restrictions) if $T \in \mathscr{B}(H)$ and $E \subset H$ is invariant for T, then $w(T \mid E) \leqslant w(T)$;

(iii) (ampliation) if $T^{(\infty)}$ denotes the countable orthogonal sum $T \oplus T \oplus \cdots$, then $w(T^{(\infty)})=w(T)$.

The order on the extended interval $[0, +\infty]$ uses the usual conventions. In most examples we are looking for radii with finite values.

Remark 1.1. Suppose condition (i) holds. Then the ampliation axiom (iii) is equivalent to

(iii') $w(T^{(n)}) = w(T)$, for n finite or $n = \infty$,

where $T^{(n)}$ denotes the orthogonal sum of n copies of T. Indeed, there is a unitary equivalence between $(T^{(n)})^{(\infty)}$ and $T^{(\infty)}$. By [FH, Lemma 15], the ampliation condition is also equivalent to $w(T \otimes I_E) = w(T)$, where $T \in \mathscr{B}(H)$ and $T \otimes I_E \in \mathscr{B}(H \otimes E)$. Here I_E is the identity on E. We refer to [FH] for other possible axioms of operator norms and several examples.

Note also that half of condition (iii), namely $w(T) \leqslant w(T^{(\infty)})$, is implied by conditions (i) and (ii).

Let $\mathscr{F} = \{\mathscr{F}_H\}$ be a collection of Hilbert space operators, that is for each considered separable Hilbert space H, $\mathscr{F}_H = \mathscr{F} \cap \mathscr{B}(H)$ is a given set.

Definition 1.2. Let $\mathscr{F}$ be a collection of Hilbert space operators. We say that $\mathscr{F}$ is *admissible* if it satisfies

(i) (unitary invariance) if $T \in \mathscr{F} \cap \mathscr{B}(H)$ and $U: K \to H$ is unitary, then $U^*TU \in \mathscr{F} \cap \mathscr{B}(K)$;

(ii) (stability for restrictions) if $T \in \mathscr{F} \cap \mathscr{B}(H)$ and $E \subset H$ is invariant for T, then $T \mid E \in \mathscr{F}$;

(iii) (ampliation) if $T \in \mathscr{F} \cap \mathscr{B}(H)$, then $T^{(\infty)} \in \mathscr{F}$.

Radius associated to a collection of operators. Let $\mathscr{F}$ be a collection of Hilbert space operators. Define the radius $w_{\mathscr{F}}$ associated to $\mathscr{F}$ by setting, for $T \in \mathscr{B}(H)$,

$$w_{\mathscr{F}}(T) := \inf\left\{r > 0 : \frac{1}{r} T \in \mathscr{F} \cap \mathscr{B}(H)\right\}.$$

Proposition 1.1. *The radius associated to an admissible collection is an admissible radius.*

Proof. Let $\mathscr{F}$ be an admissible collection. In order to show the unitary invariance of $w_{\mathscr{F}}$ let $T \in \mathscr{B}(H)$ and let $U: K \to H$ be a unitary operator. Fix $\varepsilon > 0$. There exists $r = r(\varepsilon)$ such that $0 < r < w_{\mathscr{F}}(T) + \varepsilon$ and $\frac{1}{r} T \in \mathscr{F} \cap \mathscr{B}(H)$. By the unitary invariance of $\mathscr{F}$, we have $\frac{1}{r} U^*TU \in \mathscr{F} \cap \mathscr{B}(K)$. This shows that $w_{\mathscr{F}}(U^*TU) \leqslant r < w_{\mathscr{F}}(T) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get

$$w_{\mathscr{F}}(U^*TU) \leqslant w_{\mathscr{F}}(T). \tag{1.1}$$

Replacing in this inequality T by UTU^* we obtain $w_{\mathcal{F}}(T) \leqslant w_{\mathcal{F}}(UTU^*)$; replacing now U by U^* we get

$$w_{\mathcal{F}}(T) \leqslant w_{\mathcal{F}}(U^*TU). \tag{1.2}$$

Using (1.1) and (1.2) we get that $w_{\mathcal{F}}$ is unitarily invariant.

The inequalities $w_{\mathcal{F}}(T \mid E) \leqslant w_{\mathcal{F}}(T)$ and $w_{\mathcal{F}}(T^{(\infty)}) \leqslant w_{\mathcal{F}}(T)$ can be proved as (1.1) by using the stability for restrictions and the ampliation axiom for $\mathcal{F}$, respectively. Since T is unitarily equivalent to a restriction of $T^{(\infty)}$ to an invariant subspace, we also obtain $w_{\mathcal{F}}(T) \leqslant w_{\mathcal{F}}(T^{(\infty)})$. Thus $w_{\mathcal{F}}$ is admissible. ∎

In order to present some examples of admissible collections, we introduce the following notation. If z is the variable in the complex plane $\mathbb{C}$, we denote by $P(\mathbb{C})$ the algebra of all complex polynomial functions in $\bar{z}$ and z. If $T \in \mathcal{B}(H)$ and $P \in P(\mathbb{C})$, $P(\bar{z}, z) = \sum_{\alpha, \beta} c_{\alpha, \beta} \bar{z}^{\alpha} z^{\beta}$, we set

$$P(T^*, T) = \sum_{\alpha, \beta} c_{\alpha, \beta} T^{*\alpha} T^{\beta}.$$

This is part of the so-called hereditary functional calculus [A1] which is briefly described in the next section. We denote by $\sigma(T)$ the spectrum of an operator $T \in \mathcal{B}(H)$.

THEOREM 1.2. *Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of elements in $P(\mathbb{C})$.*

(a) *Let $\mathcal{F}$ be the collection of operators defined by the following positivity conditions*

$$T \in \mathcal{F} \cap \mathcal{B}(H) \qquad \textit{if and only if} \quad P_\lambda(T^*, T) \geqslant 0 \, (\lambda \in \Lambda).$$

Then $\mathcal{F}$ is admissible.

(b) *Define the collection $\mathcal{G}$ by*

$$T \in \mathcal{G} \cap \mathcal{B}(H) \qquad \textit{if and only if} \quad \sigma(T) \subseteq \bar{\mathbb{D}} \quad \textit{and} \quad P_\lambda(T^*, T) \geqslant 0 \, (\lambda \in \Lambda).$$

Then $\mathcal{G}$ is admissible.

Proof. Let $U\colon K \to H$ be a unitary operator and let $T \in \mathcal{B}(H)$. For each $\lambda \in \Lambda$ we have $P_\lambda(U^*TU) = U^* P_\lambda(T)\, U$. Therefore $\mathcal{F}$ is unitarily invariant. If E is an invariant subspace for T, then $(T \mid E)^{\beta} = (T^{\beta}) \mid E$ and $(T \mid E)^{*\alpha} = P_E T^{*\alpha} \mid E$, where P_E is the orthogonal projection onto E. This shows that for each $\lambda \in \Lambda$ we have $P_\lambda(T \mid E) = P_E P_\lambda(T) \mid E$, yielding the stability to restrictions property. The ampliation condition follows from the equality $P_\lambda(T^{(\infty)}) = P_\lambda(T)^{(\infty)}$.

For the second part, note that the spectrum satisfies $\sigma(U^*TU) = \sigma(T)$ and $\sigma(T^{(\infty)}) = \sigma(T)$. Let now $T \in \mathscr{B}(H)$ with $\sigma(T) \subseteq \bar{\mathbb{D}}$. Let $R = T \mid E \in \mathscr{B}(E)$, where E is an invariant subspace of T. Thus the matrix of T with respect to the decomposition $H = E \oplus E^{\perp}$ has the form

$$T = \begin{pmatrix} R & * \\ 0 & * \end{pmatrix}$$

and thus

$$T^n = \begin{pmatrix} R^n & * \\ 0 & * \end{pmatrix}.$$

This implies $\|R^n\| \leqslant \|T^n\|$, so the spectral radius of R is at most one. This completes the proof. ▮

Remark 1.2. Part (a) of the above Theorem also holds for more general positivity conditions, obtained by considering polynomials in $\bar{z}$ and z with matrix coefficients. We omit the details. Bounded collections satisfying such more general positivity conditions were characterized by Agler [A2] as bounded collections which are closed with respect to direct sums, with respect to unital C^*-algebraic representations and stable for restrictions. We refer to [A2] for the exact definition and for several examples of such collections.

Operators of class C_ρ. The main examples of operator radii we will use are the operator radii associated to the collection of operators of class C_ρ.

Operators in the class C_ρ are defined as operators having ρ-dilations: $T \in \mathscr{B}(H)$ is in C_ρ, $\rho > 0$, if there exist a larger Hilbert space $K \supset H$ and a unitary operator $U \in \mathscr{B}(K)$ such that

$$T^n h = \rho P_H U^n h, \qquad h \in H.$$

Contractions are operators of class C_1 and operators in C_2 coincides with numerical radius contractions, that is operators T such that $\omega_2(T) \leqslant 1$. We refer to [NF] and [R] for more information.

The operator radius ω_ρ associated to the class C_ρ is then defined by

$$\omega_\rho(T) = \inf\left\{r \colon r > 0, \frac{1}{r} T \in C_\rho\right\}.$$

It is determined by the conditions that it is homogeneous ($\omega_\rho(zT) = |z|\, \omega_\rho(T)$ for all complex z) and that $\omega_\rho(T) \leqslant 1$ if and only if $T \in C_\rho$. Then $\omega_1(T) = \|T\|$ and ω_2 is the numerical radius. It can also be proved that the limit of $\omega_\rho(T)$ as $\rho \to \infty$ is the spectral radius of T.

The radius ω_ρ is a (Banach space) norm if and only if $\rho \leqslant 2$. It is not an algebra norm; however, we always have [NF] $\omega_\rho(T^n) \leqslant \omega_\rho(T)^n$.

COROLLARY 1.3. *The radius ω_ρ is admissible for any $\rho > 0$.*

Proof. An operator T is in the class C_ρ [NF], [R] if and only if

$$\|x\|^2 - 2\left(1-\frac{1}{\rho}\right)\mathrm{Re}[z\langle Tx \mid x\rangle] + \left(1-\frac{2}{\rho}\right)|z|^2\,\|Tx\|^2 \geqslant 0$$

for every $x \in H$ and every $z \in \bar{\mathbb{D}}$. Therefore it suffices to set

$$P_\lambda(\bar{z}, z) = 1 - \left(1-\frac{1}{\rho}\right)\bar{\lambda}\bar{z} - \left(1-\frac{1}{\rho}\right)\lambda z + \left(1-\frac{2}{\rho}\right)|\lambda|^2\,\bar{z}z \qquad (\lambda \in \bar{\mathbb{D}})$$

in Theorem 1.2, Part (a). ∎

Strongly admissible operator radii. The following definition gives a smaller class of admissible radii.

DEFINITION 1.3. A radius v defined for all Hilbert space operators with values in $[0, +\infty]$ is called a *strongly admissible radius* if it satisfies

(ii') (isometry growth condition) For any isometry $V\colon K \to H$ and any $T \in B(H)$, we have $v(V^*TV) \leqslant v(T)$.

(iii) (ampliation) $v(T^{(\infty)}) = v(T)$ for every T.

PROPOSITION 1.4. *An operator radius v is strongly admissible if and only if it satisfies*

(i) *(unitary invariance) $v(U^*TU) = v(T)$ for each unitary $U\colon K \to H$ and each $T \in \mathscr{B}(H)$;*

(ii') *(isotonicity for compressions) If $T \in \mathscr{B}(H)$, if E is a closed subspace of H and $R = P_E T \mid E$, then $v(R) \leqslant v(T)$.*

(iii) *(ampliation) We have $v(T^{(\infty)}) = v(T)$.*

In particular, each strongly admissible radius is admissible.

Proof. Suppose that v is strongly admissible. Let $T \in \mathscr{B}(H)$ and let $U\colon K \to H$ be a unitary operator. Using the isometry growth condition for the isometry U we obtain $v(T_1) \leqslant v(T)$, where $T_1 = U^*TU$. The isometry growth condition for U^* yields $v(UT_1U^*) \leqslant v(T_1)$. Therefore

$$v(T) = v(UT_1U^*) \leqslant v(T_1) \leqslant v(T)$$

showing the unitary invariance. The isotonicity for compressions is obtained from $R = J^*TJ$, where $J: E \to H$ is the inclusion $Je = e$.

For the converse implication, note that every isometry $V: K \to H$ can be written as $V = JU$, where $U: K \to V(K)$, $Uk = Vk$, is unitary and $J: V(K) \to H$ is the inclusion map. Then

$$v(V^*TV) = v(U^*J^*TJU) = v(J^*TJ) = v(P_{V(K)}T \mid V(K)) \leqslant v(T).$$

The proof is now complete. ▮

A counterpart notion of *strongly admissible collection* of operators can be introduced as a collection which satisfies the unitary invariance, the stability for compressions and the ampliation properties. The radius associated to a strongly admissible collection is stongly admissible. We omit the details.

PROPOSITION 1.5. *Let $\rho > 0$. The radius w_ρ is stongly admissible if and only if $\rho \leqslant 2$, if and only if w_ρ is a norm.*

Proof. Suppose $\rho \leqslant 2$. Recall that $T \in C_\rho$ if and only if $P_\lambda(T^*, T) \geqslant 0$ for all $\lambda \in \bar{\mathbb{D}}$, where

$$P_\lambda(\bar{z}, z) = 1 - \left(1 - \frac{1}{\rho}\right)\bar{\lambda}\bar{z} - \left(1 - \frac{1}{\rho}\right)\lambda z + \left(1 - \frac{2}{\rho}\right)|\lambda|^2\, \bar{z}z.$$

The isometry growth condition for w_ρ follows from the fact that $V^*TV \in C_\rho$ whenever $\rho \leqslant 2$, $T \in C_\rho$ and $V: K \to H$ satisfies $V^*V = I_K$. Indeed, we have

$$\begin{aligned} P_\lambda(V^*T^*V, V^*TV) &= I - \left(1 - \frac{1}{\rho}\right)\bar{\lambda}V^*T^*V - \left(1 - \frac{1}{\rho}\right)\lambda V^*V \\ &\quad + \left(1 - \frac{2}{\rho}\right)|\lambda|^2\, V^*T^*VV^*TV \\ &= V^*P_\lambda(T^*, T)\, V + \left(\frac{2}{\rho} - 1\right)|\lambda|^2\, V^*[T^*(I - VV^*)\, T]\, V. \end{aligned}$$

Suppose now $\rho > 2$ and let $b > 0$ be a fixed, arbitrary positive number. Consider the following 2×2 matrix

$$T = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}.$$

We have [AN, Theorem 6]

$$w_\rho(T) = \frac{1}{\rho}\left[\sqrt{\frac{|b|^2}{4} + 1} + \sqrt{\frac{|b|^2}{4} + 1 + \rho(\rho - 2)}\right];$$

in particular $w_2(T) = [|b|^2/4+1]^{1/2}$. We can find a vector $e \in \mathbb{C}^2$ such that $\|e\| = 1$ and $w_2(T) = |\langle Te \mid e\rangle|$. Denote by V the isometry from $\mathbb{C}$ into $\mathbb{C}^2$ defined by $V(z) = ze$. We have $V^*TV = \langle Te \mid e\rangle\, e \otimes e$. Therefore

$$w_\rho(V^*TV) = |\langle Te \mid e\rangle| = \sqrt{|b|^2/4+1}.$$

We have $w_\rho(V^*TV) > w_\rho(T)$ for any $\rho > 2$. It follows that w_ρ is not a strongly admissible radius if $\rho > 2$. Recall [NF] also that w_ρ is a norm if and only if $\rho \leqslant 2$. ∎

Remark 1.3. There are other interesting examples of admisible and strongly admissible radii. For instance, if

$$W(T) = \{|\langle Tx \mid x\rangle| : x \in H, \|x\| = 1\}$$

denotes the numerical range of T, then the diameter of $W(T)$

$$\operatorname{diam} W(T) = \sup\{|\lambda - \mu| : \lambda, \mu \in W(T)\}$$

is a strongly admissible radius. Indeed (see for instance [GR] for properties of the numerical range), we have $W(U^*TU) = W(T)$, $W(P_E T \mid E) \subseteq W(T)$ and $W(T^{(\infty)}) = W(T)$. Note also that the sum, or even convex combinations, of (strongly) admissible radii are (strongly) admissible. For instance, $T \to \|T\| + \operatorname{diam} W(T)$ is strongly admissible.

2. (CONSTRAINED) VON NEUMANN INEQUALITIES USING ANALYTIC MODELS

The existence of a model for contractions is a key result in Sz.-Nagy and Foias dilation theory. In particular, a Hilbert space contraction with spectrum contained in the open unit disc is unitarily equivalent to a restriction of the backward shift of infinite multiplicity to an invariant subspace. This implies easily inequality (0.2) for strict contractions. If T is an arbitrary contraction, then, for any real $r < 1$, inequality (0.2) holds for the strict contraction rT. Making $r \to 1$ we obtain (0.2) for all contractions.

We show in this section how the existence of a model implies at once von Neumann and constrained von Neumann inequalities for different admissible radii. We use the recent construction of analytic models for n-tuples of operators due to Ambrozie, Engliš and Müller [AEM].

Hilbert spaces associated to a domain. We recall the context of [AEM], with some change of notation. We refer to [AEM] and the references cited therein for more information.

Let D be a nonempty open domain in $\mathbb{C}^n$. Set $D^* = \{\bar{z}: z \in D\}$. Let $\mathcal{H}$ be a *D-space*, that is $\mathcal{H}$ is a Hilbert space of functions analytic on D such that

(a) $\mathcal{H}$ is invariant under the operators Z_j, $j = 1, \ldots n$, of multiplication by the coordinate functions,

$$(Z_j f)(z) = z_j f(z); \qquad f \in \mathcal{H}, \quad z = (z_1, \ldots, z_n) \in D.$$

(b) For each $z \in D$, the evaluation functional $f \to f(z)$ is continuous on $\mathcal{H}$.

(c) $C(w, z) \neq 0$ for all $z \in D$ and $w \in D^*$.

Here $C(w, z)$ is the reproducing kernel of $\mathcal{H}$, that is $C(w, z) = C_{\bar{w}}(z)$, for $z \in D$ and $w \in D^*$, where C_ζ is a function in $\mathcal{H}$ such that $f(\zeta) = \langle f \mid C_\zeta \rangle$, $f \in \mathcal{H}$ (we use (b) and the Riesz representation theorem).

Let H be a Hilbert space. Denote by $\mathcal{H} \otimes H$ the completed Hilbertian tensor product. Consider the multiplication operators M_{z_j} on $\mathcal{H} \otimes H$ defined by

$$M_{z_j} = Z_j \otimes I_H; \qquad j = 1, \ldots, n.$$

Set

$$Z = (Z_1, \ldots, Z_n) \in \mathcal{B}(\mathcal{H})^n; \qquad M_z = (M_{z_1}, \ldots, M_{z_n}) \in \mathcal{B}(\mathcal{H} \otimes H)^n.$$

Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of operators. Denote by $\sigma(T)$ the Taylor spectrum of T, and let

$$M_T = (L_{T_1^*}, \ldots L_{T_n^*}, R_{T_1}, \ldots R_{T_n}).$$

Here $L_A(X) = AX$ and $R_A(X) = XA$ are the left and right multiplication operators by A on $\mathcal{B}(H)$. Let F be a analytic function on a neighborhood of $\sigma(M_T)$. Define $F(T^*, T) \in \mathcal{B}(H)$ by $F(T^*, T) = F(M_T)(I)$.

If $z = (z_1, \ldots, z_n)$ is the variable in complex Euclidean space $\mathbb{C}^n$, we denote by $P(\mathbb{C}^n)$ the algebra of all complex polynomial functions in $\bar{z}_1, \ldots, \bar{z}_n, z_1, \ldots, z_n$. If $F(w, z) = w^\alpha z^\beta$, then $F(T^*, T) = T^{*\alpha} T^\beta = P(T^*, T)$ for $P(\bar{z}, z) = F(\bar{z}, z) \in P(\mathbb{C}^n)$. We use the usual notation $T^\beta = T_1^{\beta_1} \cdots T_n^{\beta_n}$ for $\beta = (\beta_1, \ldots \beta_n) \in \mathbb{Z}_+^n$ and the like. Note that this differs slightly from [AEM] where T^* is written on the right.

Axiom (AEM). We will sometimes suppose that $\mathcal{H}$ satisfies *Axiom (AEM)*, that is $\mathcal{H}$ is a D-space such that the polynomials are dense in $\mathcal{H}$ and $\frac{1}{C}$ is a polynomial. Let (ψ_k) be a fixed orthonormal basis for $\mathcal{H}$

consisting of polynomials such that any finite polynomial is a finite linear combination of ψ_k. Set

$$f_m(w, z) = \sum_{k=m}^{+\infty} \overline{\psi_k(\bar{w})} \frac{1}{C}(w, z) \psi_k(z).$$

When $D = \mathbb{D}$ is the open unit disk and $\mathscr{H}$ is the Hardy space $H^2 = H^2(\mathbb{D})$ of the unit disk, then $C(w, z) = (1 - wz)^{-1}$ and M_z^* is the backward shift of multiplicity dim H. In this case $\mathscr{H}$ satisfies axiom (AEM) with $\psi_k(z) = z^k$ and $f_m(w, z) = w^m z^m$. We refer to [AEM] for other examples.

Unconstrained von Neumann inequalities for operator radii. We use notation as above.

THEOREM 2.1. *Let $T = (T_1, ..., T_n) \in \mathscr{B}(H)$ be an n-tuple of commuting operators. Suppose T and $\mathscr{H}$ satisfy one of the following two conditions*

(i) *$\mathscr{H}$ is a D-space, $\sigma(T) \subset D$ and $\frac{1}{C}(T^*, T) \geqslant 0$;*

(ii) *$\mathscr{H}$ is a D-space satisfying Axiom (AEM), $\frac{1}{C}(T^*, T) \geqslant 0$ and*

$$\lim_m f_m((T^*, T)) h = 0$$

for every $h \in H$.

Let $p(z) = \sum_{\beta \in \mathbb{Z}_+^n} c_\beta z^\beta$ be a fixed polynomial in the variable $z \in \mathbb{C}^n$ and let $P(w, z) = \sum_{\alpha, \beta \in \mathbb{Z}_+^n} c_{\alpha, \beta} w^\alpha z^\beta$ be a fixed polynomial in two variables. If ω is an admissible radius, then

$$\omega(p(T)) \leqslant \omega(p(Z^*));$$

if v is strongly admissible, then

$$v(P(T^*, T)) \leqslant v(P(Z, Z^*)).$$

Proof. Suppose T satisfies (i) or (ii). In either case, using [AEM, Corollary 7, Corollary 15], there is an isometry $V: H \to \mathscr{H} \otimes H$ such that $VT_j = M_{z_j}^* V$ for $j = 1, ..., n$. Note again that some care has to be taken when using the results of [AEM] because of the change of notation. This implies

$$V p(T) = p(M_z^*) V.$$

In particular, VH is invariant under M_z^* and T^β is unitarily equivalent to the restriction of $M_z^{*\beta}$ to the invariant subspace VH. Since ω is admissible, we have

$$\omega(p(T)) = \omega(p(M_z^*) \mid VH) \leqslant \omega(p(M_z^*)).$$

Using the ampliation axiom for ω and the fact that $M_z^* = Z^* \otimes I_H$, we obtain

$$\omega(p(T)) \leqslant \omega(p(Z^*)).$$

For the second part of the theorem, note that with respect to the decomposition $\mathscr{H} \otimes H = VH \oplus VH^\perp$, we can write

$$M_z^{*\beta} = \begin{pmatrix} T^\beta & * \\ 0 & * \end{pmatrix} \qquad \text{and} \qquad M_z^\alpha = \begin{pmatrix} T^{*\alpha} & 0 \\ * & * \end{pmatrix}.$$

This shows that

$$T^{*\alpha} T^\beta = P_{VH} M_z^\alpha M_z^{*\beta} \mid VH.$$

Since ν is strongly admissible, we have

$$\nu(P(T^*, T)) \leqslant \nu(P(M_z, M_z^*)) = \nu(P(Z, Z^*)).$$

The proof is complete. ∎

Constrained von Neumann inequalities. We start with a constrained von Neumann inequality for admissible radii.

THEOREM 2.2. *Let D be an open domain in $\mathbb{C}^n$. Suppose Hilbert space $\mathscr{H}$ of functions analytic on D and an n-tuple of operators T satisfy one of the two conditions* (i) *and* (ii) *in Theorem* 2.1. *Let p and q be one variable polynomials in n variables and suppose that $q(T) = 0$. If ω is an admissible radius, then*

$$\omega(p(T)) \leqslant \omega(p(Z^* \mid Ker\ q(Z^*))).$$

Proof. We use the notation of (the proof of) Theorem 2.1. Recall that T^β is unitarily equivalent to the restriction of $M_z^{*\beta}$ to the invariant subspace VH and $V p(T) = p(M_z^*) V$. Since $q(T) = 0$, we have

$$0 = V q(T) h = q(M_z^*) V h$$

for any $h \in H$. This shows that $VH \subseteq \operatorname{Ker} q(M_z^*)$. Therefore

$$\omega(p(T)) = \omega(p(M_z^*) \mid VH) \leqslant \omega(p(M_z^*) \mid \operatorname{Ker} q(M_z^*)).$$

By the ampliation axiom and the equality

$$M_z^* \mid \operatorname{Ker} q(M_z^*) = [Z^* \mid \operatorname{Ker} q(Z^*)] \otimes I_H,$$

we get

$$\omega(p(M_z^*) \mid \operatorname{Ker} q(M_z^*)) = \omega(p(Z^* \mid \operatorname{Ker} q(Z^*))).$$

This completes the proof. ∎

In some applications it is possible to avoid the hypothesis

$$\lim_m f_m((T^*, T))\, h = 0$$

in condition (ii) in Theorem 2.1. We refer to Corollary 4.1 and Corollary 4.5 for examples of results of this type.

The following result is a constrained von Neumann inequality for strongly admissible radii. Recall that $E \subseteq H$ is said to be *invariant* for the n-tuple $(T_1, \ldots, T_n) \in \mathscr{B}(H)$ if $T_j E \subseteq E$ for each j.

THEOREM 2.3. *Let D be an open domain in $\mathbb{C}^n$. Suppose Hilbert space $\mathscr{H}$ of functions analytic on D and an n-tuple of operators T satisfy one of the two conditions* (i) *and* (ii) *in Theorem* 2.1. *Suppose also that each operator $Z_j \in \mathscr{B}(\mathscr{H})$ is an isometry. Let P and Q be two elements of $P(\mathbb{C}^n)$ and suppose that $Q(T^*, T) = 0$. There exists an invariant subspace E for $Z^* \in \mathscr{B}(\mathscr{H})^n$ such that, for each strongly admissible radius v,*

$$v(P(T^*, T)) \leqslant v(P(Z_E, Z_E^*)),$$

where Z_E is defined by setting $Z_E^* := Z^* \mid E$. If $n = 1$, if Q is of degree less or equal than d and if $Q(e^{-it}, e^{it}) \neq 0$ for some $t \in \mathbb{R}$, then the dimension of E is less or equal than $2d$.

Proof. Let Q be a polynomial in $P(\mathbb{C}^n)$ of degree at most d, that is, the maximal power at which each w_j and z_j occurs is at most d. Recall from the proof of Theorem 2.1 that

$$T^{*\alpha} T^{\beta} = P_{VH} M_z^{\alpha} M_z^{*\beta} \mid VH$$

and thus $P(T^*, T) = J^* P(M_z, M_z^*)\, J$, where J denotes the inclusion $J \colon VH \to \mathscr{H} \otimes H$. The same equality, using the fact that $Q(T^*, T) = 0$, implies that the subspace $V(H)$ is contained in $\operatorname{Ker} Q(M_z, M_z^*)$. Since each Z_j and thus each M_{z_j} is an isometry, we get that $\operatorname{Ker} Q(M_z, M_z^*)$ is included in $E_0 = \operatorname{Ker}(M_z^{*d} Q(M_z, M_z^*))$ which is invariant by M_z^*. Denote

$$E = \operatorname{Ker}(Z^{*d} Q(Z, Z^*))$$

which is invariant under Z^*. Then, using the properties of the strongly admissible radius v, we obtain

$$\begin{aligned} v(P(T^*, T)) &= v(J^* P(M_z, M_z^*)\, J) = v(J^* P_{E_0} P(M_z, M_z^*)\, P_{E_0} J) \\ &\leqslant v(P_{E_0} P(M_z, M_z^*)\, P_{E_0}) = v(P_{E_0} [P(Z, Z^*) \otimes I]\, P_{E_0}) \\ &= v(P_E(p(Z, Z^*)\, P_E \otimes I) = v(P_E(p(Z, Z^*)\, P_E) \\ &= v(p(Z_E, Z_E^*)), \end{aligned}$$

where P_{E_0} and P_E are the orthogonal projections onto E_0, respectivelly E.

Finally, if $n = 1$ and if Q is of degree less or equal than d, then $Z^{*d} Q(Z, Z^*)$ is a polynomial in Z^* of degree less or equal to $2d$. Thus E, the kernel of $Z^{*d} Q(Z, Z^*)$, is a subspace of dimension no greater than $2d$, unless $Z^{*d} Q(Z, Z^*)$ is the null operator. This occurs if and only if $e^{ids} Q(e^{-is}, e^{is}) = 0$ for every $s \in \mathbb{R}$. The last equality is impossible if $Q(e^{-it}, e^{it}) \neq 0$ for some $t \in \mathbb{R}$. The proof is complete. ∎

3. INEQUALITIES FOR RADII ASSOCIATED TO BUNDLES OF OPERATORS

Constrained von Neumann inequalities for some operator radii which are not necessarily admissible are obtained in this section. The method also gives a different proof of constrained von Neumann inequalities for the radii ω_ρ.

Notation. We denote by $D(\alpha, r)$ the open disc of radius r and center α. Let $\mathbb{T}$ be the boundary of $\mathbb{D} = D(0, 1)$. The spaces $L^p = L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, are the usual Lebesgue function spaces relative to normalized Lebesgue measure on $\mathbb{T}$. The spaces $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, are the usual Hardy spaces. Denote

$$H_0^1 = \left\{ f \in L^1 : \int_0^{2\pi} f(e^{it})\, e^{int}\, dt = 0, n = 0, 1, \ldots \right\}.$$

For a given inner function u, denote $H(u) = H^2 \ominus uH^2$ and consider the operator $S(u) \in \mathscr{B}(H(u))$ defined by

$$S(u) = P_{H(u)} Z \mid H(u).$$

Recall that Z is the operator of multiplication by $z = e^{i\theta}$ on H^2. A proof that $S(u)$ and the extremal operator $S^* \mid \mathrm{Ker}(u(S)^*)$ are unitarily equivalent follows from the fact that they have the same characteristic function [NF]; a direct proof can be found in [P3].

If $T \in \mathcal{B}(H)$ is an absolutely continuous contraction, then, for any $x, y \in H$, there exists a function $x \overset{T}{\cdot} y \in L^1$ with the nth Fourier coefficient given by $\langle T^{*n}x \mid y\rangle$ if $n \geqslant 0$ and $\langle T^{-n}x \mid y\rangle$ if $n < 0$.

Let T be an operator whose spectrum is included in the closed unit disc. Consider the operator kernel $K_\alpha(T)$ defined by

$$K_\alpha(T) = (I - \bar{\alpha}T)^{-1} + (I - \alpha T^*)^{-1} - I; \qquad |\alpha| < 1.$$

For an absolutely continuous contraction T, $\langle K_{r\exp(it)}(T)\,x \mid y\rangle$ converges almost everywhere to $x \overset{T}{\cdot} y$ when r goes to 1.

Recall that a contraction $T \in \mathcal{B}(H)$ is said [NF] to be of class C_0 if T is c.n.u. and there is a nonzero function f in H^∞ such that $f(T) = 0$. Then there is a unique (up to a constant factor of modulus one) nonconstant inner function u, called the *minimal function* of T, such that $u(T) = 0$. The minimal function of $S(u)$ is u.

Bundles of selfadjoint operators and associated radii. Recall the following result. Let $\rho > 0$. An operator $T \in \mathcal{B}(H)$ whose spectrum is included in the closed unit disc is in C_ρ if and only if [CF] $K_\alpha(T) + \rho I \geqslant I$ for any $\alpha \in \mathbb{D}$.

DEFINITION 3.1. Suppose a collection $\mathcal{R}$ of *bundles* of self-adjoint operators is given, that is, for each separable Hilbert space H there is a map

$$\mathcal{R}_H\colon \mathbb{D} \times \mathcal{B}(H) \ni (\alpha, A) \to R_\alpha(A) \in \mathcal{B}(H)$$

with $R_\alpha(A) = R_\alpha(A)^*$. The collection $\mathcal{K} = \mathcal{K}_{\mathcal{R}}$ associated to $\mathcal{R}$ is defined by setting

$$A \in \mathcal{K} \cap \mathcal{B}(H) \qquad \text{if and only if}$$
$$\sigma(A) \subseteq \bar{\mathbb{D}} \quad \text{and} \quad K_\alpha(A) + R_\alpha(A) \geqslant I \qquad (\alpha \in \mathbb{D}).$$

The operator radius associated to $\mathcal{K} = \mathcal{K}_{\mathcal{R}}$ is then

$$\omega_{\mathcal{K}}(A) = \inf\left\{ r > 0 : \frac{1}{r} A \in \mathcal{K}_{\mathcal{R}} \right\}.$$

EXAMPLE 3.1. (a) Let $\rho > 0$. For the bundle $\mathcal{R}$ given by $R(\alpha) = \rho I$, the class $\mathcal{K}_{\mathcal{R}}$ coincides with the class C_ρ.

(b) Let A be an positive invertible operator and set $R(\alpha) = A$. Then the associated collection $\mathcal{K}_{\mathcal{R}}$ coincides with the class C_A introduced by Langer (cf. [NF, p. 54]).

(c) Let φ be a function in $H^\infty(\mathbb{D})$ and let $v\colon \mathscr{B}(H) \to [0, +\infty[$ a map which satisfies

$$v(T)\,\|T\| \leqslant 1 \qquad \text{for all} \quad T \in B(H).$$

Consider the bundle $\mathscr{R}$ by setting

$$R_\alpha(T) = \varphi(\alpha v(T)\,T) + \varphi(\alpha v(T)\,T)^*.$$

The associated radius to the collection $\mathscr{K}_{\mathscr{R}}$ is not necessarily admissible.

THEOREM 3.1. *Let $\mathscr{R}$ be a bundle such that*

$$\mathbb{D} \ni \alpha \to R_\alpha(A) \in \mathscr{B}(H)$$

is harmonic in $\mathbb{D}$ for each $A \in \mathscr{B}(H)$. Let $\mathscr{K} = \mathscr{K}_{\mathscr{R}}$ be the collection of operators associated to the bundle $\mathscr{R}$. Let T be a contraction of class C_0 with $u(T) = 0$, u an inner function, and let $f \in A(D)$. Assume that for any $\alpha \in D$ there exist a function g_α such that for any $r > 0$

$$R_\alpha(f(T)/r) = g_\alpha(f(T)/r) + g_\alpha(f(T)/r)^*$$

and

$$R_\alpha(f(S(u))/r) = g_\alpha(f(S(u))/r) + g_\alpha(f(S(u))/r)^*.$$

Then we have

$$\omega_{\mathscr{K}}(f(T)) \leqslant \omega_{\mathscr{K}}(f(S(u)).$$

Recall that $S(u)$ is unitarily equivalent to the extremal operator $S^* \mid \operatorname{Ker}(u(S)^*)$.

For the proof of Theorem 3.1, we need the following lemma which will be also used in Section 5.

LEMMA 3.2. *Let u be a inner function and let f be a positive function in the subspace $\bar{u}H_0^1$ of $L^1(\mathbb{T})$. Then there exists a function h in $H^2 \ominus uH^2$ such that $f = |h|^2$.*

Proof. Since $f \in \bar{u}H_0^1$ we have $f = \bar{u}f_1$, with $f_1 \in H_0^1$. Then $\log|f| = \log|f_1|$ is Lebesgue integrable. According to theorem of Hoffman [Ho] there exists an outer function g in H^2 such that $f = |g|^2$. Denote by $E = H(u)$ the orthogonal in H^2 of the subspace uH^2 and write $g = g_1 + ug_2$

with respect to the orthogonal decomposition $H^2 = E \oplus uH^2$. We have $g_1 \neq 0$ since g is an outer function. Using the fact that $g_1 \in E$, we obtain

$$\langle u\overline{g_1} \mid \bar{h} \rangle = \int_0^{2\pi} u(e^{it})\, \overline{g_1(e^{it})}\, h(e^{it})\, dm(t) = \overline{\langle g_1 \mid uh \rangle} = 0,$$

for all functions h in H^2. Using the theorem of F. and M. Riesz [Ho] we get

$$u\overline{g_1} \in H^2_0. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} uf = u\,|g|^2 &= u\,|g_1 + ug_2|^2 \\ &= u(g_1 + ug_2)(\overline{g_1 + ug_2}) \\ &= u\,|g_1|^2 + u\,|g_2|^2 + g_1\overline{g_2} + u^2\overline{g_1}g_2. \end{aligned}$$

Therefore

$$g\overline{g_2} = (g_1 + ug_2)\,\overline{g_2} = u\,|g_2|^2 + g_1\overline{g_2} = uf - u\,|g_1|^2 - u^2\overline{g_1}g_2.$$

Since $f \in \bar{u}H^1_0$ and using (3.1), we see that the three last terms belong to H^1_0. Hence $g\overline{g_2} \in H^1_0$ and for any polynomial p we have

$$\langle pg \mid g_2 \rangle = \int_0^{2\pi} p(e^{it})\, g(e^{it})\, \overline{g_2(e^{it})}\, dm(t) = 0.$$

Since g is an outer function, it follows that $g_2 = 0$. The proof of the lemma is now complete. ∎

Proof (of Theorem 3.1). By the canonical factorization theorem, u can be decomposed as

$$u(z) = B(z) \exp\left[-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \right],$$

where B is a Blaschke product and μ is a positive measure on ∂D which is singular with respect to the Lebesgue measure. Using the spectral mapping theorem of a C_0 operator, we have

$$\sigma(T) \subseteq \overline{B^{-1}\{0\}} \cup Supp(\mu) = \sigma(S(u)),$$

where $Supp(\mu)$ is the support of μ.

Let f be a non-identically zero function in $A(\mathbb{D})$. Using the spectral mapping theorem, we get

$$\sigma(f(T)) = f(\sigma(T)) \subseteq f(\sigma(S(u))) = \sigma(f(S(u))). \tag{3.2}$$

Fix $r > \omega_{\mathscr{K}}(f(S(u)))$ and let $\alpha \in D(0, 1/(r\,\|f\|_\infty))$, $\alpha \neq 0$. We deduce from (3.2) that r/α belongs to the resolvent of T. Therefore, for any $x \in H$ and every α in $D(0, 1/(r\,\|f\|_\infty))$, we can write

$$\begin{aligned}
&\left\langle \left[K_\alpha\left(\frac{f(T)}{r}\right) + R_\alpha\left(\frac{f(T)}{r}\right) - I\right] x \mid x\right\rangle \\
&= \left\langle \left[\left(I - \alpha\frac{f(T)^*}{r}\right)^{-1} + \left(I - \bar{\alpha}\frac{f(T)}{r}\right)^{-1} - 2I + g_\alpha\left(\frac{f(T)}{r}\right)\right.\right. \\
&\qquad \left.\left. + g_\alpha\left(\frac{f(T)}{r}\right)^*\right] x \mid x\right\rangle.
\end{aligned}$$

Recall that for any absolutely continuous contraction T and for any $x, y \in H$, the function $\langle K_{r,t}(T)\, x \mid y\rangle$ converge almost everywhere to a function $x \overset{T}{\cdot} y \in L^1(\partial D)$ when r goes to 1. Since T is a C_0 contraction such that $u(T) = 0$, it follows [CCC, Lemma 5.2] that $x \overset{T}{\cdot} x \in \bar{u} H_0^1$. From Lemma 3.2 we get the existence of a function h in $H^2 \ominus uH^2 = E$ such that $x \overset{T}{\cdot} x(e^{it}) = |h(e^{it})|^2$. We obtain

$$\begin{aligned}
&\left\langle \left[K_\alpha\left(\frac{f(T)}{r}\right) + R_\alpha\left(\frac{f(T)}{r}\right) - I\right] x \mid x\right\rangle \\
&= \int_0^{2\pi} \left[\frac{1}{1 - \bar{\alpha}\dfrac{f(e^{it})}{r}} + \frac{1}{1 - \alpha\dfrac{\overline{f(e^{it})}}{r}} - 2\right. \\
&\qquad \left. + g_\alpha\left(\frac{f(e^{it})}{r}\right) + \overline{g_\alpha\left(\frac{f(e^{it})}{r}\right)}\right] x \overset{T}{\cdot} x(e^{it})\, dm(t) \\
&= \int_0^{2\pi} \left[\frac{1}{1 - \bar{\alpha}\dfrac{f(e^{it})}{r}} + \frac{1}{1 - \alpha\dfrac{\overline{f(e^{it})}}{r}} - 2\right. \\
&\qquad \left. + g_\alpha\left(\frac{f(e^{it})}{r}\right) + \overline{g_\alpha\left(\frac{f(e^{it})}{r}\right)}\right] |h(e^{it})|^2\, dm(t)
\end{aligned}$$

$$= \left\langle \left[\left(I - \alpha \frac{f(S_u)^*}{r} \right)^{-1} + \left(I - \bar{\alpha} \frac{f(S_u)}{r} \right)^{-1} - 2I \right.\right.$$

$$\left.\left. + g_\alpha \left(\frac{f(S_u)}{r} \right) + g_\alpha \left(\frac{f(S_u)}{r} \right)^* \right] h \mid h \right\rangle$$

$$= \left\langle \left[K_\alpha \left(\frac{f(S_u)}{r} \right) + R_\alpha \left(\frac{f(S_u)}{r} \right) - I \right] h \mid h \right\rangle.$$

Since both sides of the previous equalities are harmonic inside the unit disc (with respect to the variable α) and coincide inside the disc $D(0, 1/(r \|f\|_\infty))$, we get that for any α in the unit disc

$$\left\langle \left[K_\alpha \left(\frac{f(T)}{r} \right) + R_\alpha \left(\frac{f(T)}{r} \right) - I \right] x \mid x \right\rangle \tag{3.3}$$

and

$$\left\langle \left[K_\alpha \left(\frac{f(S_u)}{r} \right) + R_\alpha \left(\frac{f(S_u)}{r} \right) - I \right] h \mid h \right\rangle \tag{3.4}$$

coincide. As $r > \omega_{\mathscr{K}}(f(S_u))$, we get the positivity of (3.3). We obtain $r > \omega_{\mathscr{K}}(f(T))$ and the proof is now complete. ▮

4. APPLICATIONS OF THE PREVIOUS RESULTS

We show in this section how the above constrained von Neumann inequalities can be applied in a variety of situations. We are not always looking for the most possible general inequalities.

Applications of Theorem 2.2. We denote by $[x]$ the integer part of x, that is the least integer no greater than x.

COROLLARY 4.1. *Let $n \geqslant 2$. Let $T \in \mathscr{B}(H)$ be a contraction such that $T^n = 0$. Then, for each $\rho > 0$ and each analytic polynomial p, we have*

$$\omega_\rho(p(T)) \leqslant \omega_\rho(p(S_n^*)).$$

In particular, for any m we have

$$\omega_2(T^m) \leqslant \cos \frac{\pi}{k(m, n)+2}, \qquad k(m, n) := \left[\frac{n-1}{m} \right].$$

Proof. Let $r<1$ be a positive number. Let $\mathscr{H}$ be H^2 in Theorem 2.2. Then Theorem 2.2, (ii), applied with rT instead of T, $q(z)=z^n$ and $\omega=\omega_\rho$, gives

$$\omega_\rho(p(rT)) \leqslant \omega_\rho(p(S_n^*)).$$

Make now r tends to 1.

For the proof of the last part note that a majorant of the left-hand side will be $\omega_2(S_n^{*m})$. But S_n^{*m} is unitarily equivalent to an orthogonal sum of shifts of smaller dimension, the largest dimension being $k(m,n)+1$. Therefore $\omega_2(S_n^{*m})=\omega_2(S_{k(m,n)+1}^*)$ is equal to $\cos\frac{\pi}{k(m,n)+2}$. The same computation follows from [GR, page 120]. ∎

Remark 4.1. The inequality $\omega_2(T^m)\leqslant\cos(\pi/(k(m,n)+2))$ can be deduced from the inequality (0.3) of Haagerup and de la Harpe. Indeed, $k=k(m,n)=[\frac{n-1}{m}]$ implies that $mk+m>n-1$ and thus $(T^m)^{k+1}=0$. Apply the Haagerup–de la Harpe inequality for T^m.

In the general case, if $p(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}$ is a polynomial of degree less or equal than $n-1$, then $p(S_n^*)$ is the following triangular Toeplitz matrix

$$p(S_n^*)=\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ & a_0 & a_1 & \cdots & a_{n-2} \\ & & a_0 & \cdots & a_{n-3} \\ & & & \ddots & \vdots \\ & & & & a_0 \end{pmatrix}.$$

Recall that we have the following reciprocity law of Ando and Nishio:

$$\omega_\rho(T)=\left(\frac{2}{\rho}-1\right)\omega_{2-\rho}(T).$$

This shows that computations of ω_ρ for $0<\rho<1$ follows from computations for $1<\rho<2$. Using interpolation properties of ω_ρ (see [FH, p. 296]), the law of Ando and Nishio, Corollary 4.1 and a result from [E, Lemma 5] concerning the numerical range of Toeplitz matrices we get the next result.

COROLLARY 4.2. *Let $n\geqslant 2$. Let $T\in\mathscr{B}(H)$ be a contraction such that $T^n=0$. Let $p(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}$ be an analytic polynomial of degree at most $n-1$. We have*

$$\omega_\rho(p(T))\leqslant\left(\frac{2}{\rho}-1\right)\|p\|_\infty^\rho\,[\inf_{\theta\in\mathbb{R}}\sup\{|p(\zeta)|:\zeta\in\mathbb{C},\zeta^{2n-1}=e^{i\theta}\}]^{1-\rho}$$

if $\rho \in]0, 1]$, *and*

$$\omega_\rho(p(T)) \leqslant \|p\|_\infty^{2-\rho} \,[\inf_{\theta \in \mathbb{R}} \sup \{|p(\zeta)| : \zeta \in \mathbb{C}, \zeta^{2n-1} = e^{i\theta}\}]^{\rho-1}$$

if $\rho \in [1, 2]$.

We refer to the proof of Theorem 5.3 for a better estimate of $\omega_2(p(S_n^*))$ for polynomials of the form $p(z) = z^k + e^{i\gamma} z^l$; this yields (Theorem 6.2) an estimate for $\omega_2(T^k + T^l)$.

If the polynomial q of degree d is given by

$$q(z) = b_0 + b_1 z + \cdots + b_d z^d,$$

then $\operatorname{Ker}(q(S^*))$ consists of all sequences $(y_r) \in \ell_2$ satisfying

$$b_d\, y_{r+d} + b_{d-1}\, y_{r+d-1} + \cdots b_0\, y_r = 0$$

for $r = 0, 1, 2, \ldots$. This linear recurrence has a d-dimensional solution space and if all the zeros of q have modulus less than one then all solutions lie in ℓ_2. In this case $\operatorname{Ker}(q(S^*))$ has dimension d.

We refer to [PY] for the matrix of $S^* \,|\, \operatorname{Ker}(q(S^*))$ with respect to some orthonormal basis of $\operatorname{Ker}(q(S^*))$ and, for instance, to [GR] and the references therein for a discussion on how the numerical radius of a matrix can be estimated/computed.

Recall [A1] that $T \in \mathscr{B}(H)$ is called a 2-hypercontraction if

$$I - T^*T \geqslant 0 \qquad \text{and} \qquad I - 2T^*T + T^{*2}T^2 \geqslant 0.$$

COROLLARY 4.3. *Let* $T \in \mathscr{B}(H)$ *be a nilpotent* 2-*hypercontraction with* $T^n = 0$, $n \geqslant 2$. *Then*

$$\omega_\rho(p(T)) \leqslant \omega_\rho(p(B_n^*))$$

for all $\rho > 0$ *and all polynomials* p. *Here* $B_n^* \in \mathscr{B}(\mathbb{C}^n)$ *is given by the matrix*

$$B_n^* = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{n}{n+1}} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Proof. Let $r<1$ be a fixed positive real number. Consider $\mathscr{H}=L_a^2(\mathbb{D})$ the Bergman space of all analytic functions on $\mathbb{D}$ satisfying

$$\|f\|^2=\frac{1}{\pi}\int_{\mathbb{D}}|f(re^{it})|^2\,dA<\infty,$$

where dA is the area Lebesgue measure. In this case $C(w,z)=(1-2wz+w^2z^2)^{-1}$ and $\mathscr{H}$ is a $\mathbb{D}$-space satisfying axiom (AEM) with $\psi_j(z)=\sqrt{j+1}\,z^j$ and $f_m(w,z)=(m+1)\,w^m z^m-mw^{m+1}z^{m+1}$. Then Z^* is unitarily equivalent to the Bergman shift B^*, where B is given by $Be_p=\sqrt{\frac{p+1}{p+2}}\,e_{p+1}$ for a suitable orthonormal basis (e_p).

We have $\|rT\|\leqslant 1$, $(rT)^m\to 0$ strongly and also [A1]

$$I-2(rT)^*(rT)+(rT)^{*2}(rT)^2\geqslant 0.$$

It follows from [AEM, Example 2] that rT satisfies condition (ii) of Theorem 2.1. It follows from Theorem 2.2 that

$$\omega_\rho(p(rT))\leqslant\omega_\rho(p(B_n^*)),$$

since B_n^* is unitarily equivalent to $B^*\,|\,\mathrm{Ker}(B^{*n})$. This holds for all $r<1$; it also holds for $r=1$. ∎

The numerical radius of B_n^* can be expressed [S] in terms of the smallest positive root of a polynomial involving circularly symmetric functions. To give a flavor of what can be done, we prove here the following inequalities.

Corollary 4.4. *Suppose $T\in\mathscr{B}(H)$ satisfies $\|T\|\leqslant 1$, $T^3=0$ and $I-2T^*T+T^{*2}T^2\geqslant 0$. Then*

$$\omega_2(T)\leqslant\sqrt{\tfrac{7}{24}}\qquad\textit{and}\qquad\omega_2(T^2)\leqslant\sqrt{\tfrac{1}{12}}$$

and these constants are the best possible ones.

Proof. We have to compute $\omega_2(B_n^*)$ and $\omega_2(B_n^{*2})$ for $n=3$. This can be done using [S] or in the following (equivalent) way. Consider the symmetric $n\times n$ matrix

$$A_n = B_n^* + B_n = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{3}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{n}{n+1}} \\ 0 & 0 & 0 & \cdots & \sqrt{\frac{n}{n+1}} & 0 \end{pmatrix}.$$

Let θ be a real number. If $D(\theta)$ denotes the diagonal matrix with $e^{ij\theta}$, $j = 1, ..., n$, on the main diagonal, then we have $D(\theta)^* \, (e^{i\theta} B_n + e^{-i\theta} B_n^*) \, D(\theta) = A_n$. Recall that

$$\omega_2(T) = \tfrac{1}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta} T + e^{-i\theta} T^*\|.$$

Therefore

$$\begin{aligned} \omega_2(B_n^*) &= \tfrac{1}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta} B_n + e^{-i\theta} B_n^*\| \\ &= \tfrac{1}{2} \sup_{\theta \in \mathbb{R}} \|D(\theta)^* \, (e^{i\theta} B_n + e^{-i\theta} B_n^*) \, D(\theta)\| \\ &= \tfrac{1}{2} \|A_n\|. \end{aligned}$$

Since $\frac{1}{2} A_n$ is hermitian, its norm coincides with its largest eigenvalue. For $n = 3$ it is equal to $\sqrt{7/24}$. In a similar way, the numerical radius of B_3^{*2} is the spectral radius of $\frac{1}{2}(B_3^{*2} + B_3^2)$, that is $\sqrt{1/12}$. ∎

Note that the inequality

$$\omega_2(T^2) \leqslant \sqrt{\tfrac{1}{12}} = 0.2886...$$

is an improvement of the inequality

$$\omega_2(T^2) \leqslant \omega_2(T)^2 \leqslant \tfrac{7}{24} = 0.2916....$$

Inequalities for n-tuples of operators. Theorem 2.2 can be applied also for n-tuples of commuting operators $T = (T_1, ..., T_n) \in \mathcal{B}(H)^n$, $n \geqslant 1$. In fact, anytime we dispose of a model operator, the techniques of Section 2 can be used to obtain constrained von Neumann inequalities. We give only one example using the model of Vasilescu [V]. It corresponds, using the notation of Section 2, to the domain

$$D = \left\{ z \in \mathbb{C}^n : \sum_j c_{ij} |z_j|^2 < 1, 1 \leqslant i \leqslant m \right\}.$$

This generalizes previous models for the unit ball in $\mathbb{C}^n$ and for the unit polydisc (cf . the references in [V]).

Let $m \geqslant 1$ be a fixed integer. Let $p = (p_1, ..., p_m)$ be a family of complex polynomials

$$p_j(z) = 1 - c_{j1} z_1 - \cdots - c_{jn} z_n,$$

for $j = 1, ..., m$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$ such that

- $c_{jk} \geqslant 0$ for all indices j and k;
- for every $k \in \{1, ..., n\}$ there is $j \in \{1, ..., m\}$ such that $c_{jk} \neq 0$;
- p_j is identical 1 for no indice j.

The case

$$p_j(z) = 1 - z_j \qquad 1 \leqslant j \leqslant n,$$

corresponds to the unit polydisc in $\mathbb{C}^n$, while

$$p_1(z) = 1 - z_1 - \cdots - z_n$$

corresponds to the unit ball.

If $\gamma = (\gamma_1, ..., \gamma_m) \in \mathbb{Z}_+^m$, we set

$$p^\gamma(z) = p_1(z)^{\gamma_1} \cdots p_m(z)^{\gamma_m} \qquad (z \in \mathbb{C}^n).$$

Define

$$V_{T,j} = \sum_{k=1}^n c_{jk} M_{T_k}, \qquad j = 1, ..., m,$$

and $V_T = (V_{T_1}, ..., V_{T_n})$. Define

$$\Delta_T^\gamma = (I - V_{T,1})^{\gamma_1} \cdots (I - V_{T,m})^{\gamma_m} (I_H),$$

where I_H is the identity on H and $I = I_{\mathscr{B}(H)}$ is the identity on $\mathscr{B}(H)$. Let $\gamma \geqslant (1, ..., 1)$. We say [V] that $T \in \mathscr{B}(H)^n$ satisfies the positivity condition (p, γ) if

$$\Delta_T^\beta \geqslant 0, \qquad \text{for all } \beta, \quad 0 \leqslant \beta \leqslant \gamma.$$

We denote by $S^{(p,\gamma)} \in \mathscr{B}(\ell_2(\mathbb{Z}_+^n, \mathbb{C}))$ the *backwards multishift* of type (p, γ) as defined in [V] (in fact, $S^{(p,\gamma)} \otimes I_H$ is the model there).

COROLLARY 4.5. *Suppose, with notation as above, that $\gamma \geqslant (1, ..., 1)$. Let $T \in \mathscr{B}(H)^n$ be a n-tuple of commuting operators satisfying the positivity condition (p, γ) and the constraint $q(T) = 0$ for a fixed polynomial q in n variables. Then, for any admissible radius ω and any polynomial f in n variables, we have*

$$\omega(f(T)) \leqslant \omega(f(S^{(p,\gamma)}) \mid Ker\; q(S^{(p,\gamma)}))).$$

Proof. Let $r \in \,]0, 1[$. It was proved in [V, Proposition 3.15] that rT is unitarily equivalent to the restriction of $S^{(p,\gamma)} \otimes I_H$ to an invariant subspace. Using the admissibility of ω, the fact that $q(T) = 0$, and making $r \to 1$ at the end, we obtain the desired inequality. ∎

A proof of the above corollary can be given using directly Theorem 2.2 (cf. Example (5) in [AEM]). The unconstrained von Neumann inequality in this case, for the operator norm, is [V, Proposition 3.15].

Applications of Theorem 2.3. The following result is obtained from Theorem 2.3 in the classical case $\mathscr{H} = H^2$.

COROLLARY 4.6. *Let $T \in \mathscr{B}(H)$ be a Hilbert space contraction such that $Q(T^*, T) = 0$ for a given $Q \in P(\mathbb{C})$ of degree d with $Q(e^{-it}, e^{it}) \neq 0$ for some $t \in \mathbb{R}$. Then there exists an invariant subspace E for the backward shift S^* on H^2 such that*

$$\omega_\rho(P(T^*, T)) \leqslant \omega_\rho(P(S_E, S_E^*))$$

for all $\rho \in \,]0, 2]$ and all $P \in P(\mathbb{C})$. Here $S_E \in \mathscr{B}(E)$ is the adjoint of $S_E^* = S^* \mid E$.

It follows from the proof of Theorem 2.3 that the space E in the above corollary is given by $E = \operatorname{Ker} S^{*d} Q(S, S^*)$. The following is a possible application.

COROLLARY 4.7. *Let $m \geqslant n \geqslant 1$ be two positive integers. Let $T \in \mathscr{B}(H)$ be a contraction and suppose that $T^{*m} = T^n$. Let $\rho \in \,]0, 2]$ and let $P \in P(\mathbb{C})$. Then*

$$\omega_\rho(P(T^*, T)) \leqslant \omega_\rho(P(S_{m+n}, S_{m+n}^*));$$

in particular,

$$\omega_2(T^l)) \leqslant \cos \frac{\pi}{[(m+n-1)/l]+2}$$

for all l with $1 \leqslant l \leqslant m+n-1$.

Proof. Set $Q(w, z) = w^m - z^n$. We have $Q(e^{-it}, e^{it}) \neq 0$ for some $t \in \mathbb{R}$. Note that S on H^2 is unitarily equivalent to the forward shift on S on ℓ_2. We have $S^{*m}Q(S, S^*) = I - S^{*(m+n)}$. Then $E = \operatorname{Ker} S^{*m}Q(S, S^*)$ is given by

$$E = \{(h_0, h_1, \ldots, h_{p-1}, h_0, h_1, \ldots, h_{p-1}, h_0, \ldots) \colon h_k \in \mathbb{C} \text{ for } 0 \leqslant k \leqslant p\},$$

where $p = m+n$. Thus $S_E^* = S^* \mid E$ is unitarily equivalent to $S_p^{*(\infty)}$. Since ω_ρ is strongly admissible for $\rho \leqslant 2$, we obtain

$$\omega_\rho(P(T^*, T)) \leqslant \omega_\rho(P(S_{m+n}, S_{m+n}^*)).$$

The second inequality is obtained for $P(w, z) = z^l$. ∎

Applications of Theorem 3.1. Theorem 3.1 can be applied for instance to bundles of the following type. Let $(p_n)_{n \geqslant 0}$ be a sequence of polynomials which is uniformly bounded on the closed unit disc. Suppose $v \colon \mathscr{B}(H) \to [0, \infty[$ is such that $v(T)\,\|T\| \leqslant 1$. Let $\mathscr{U}$ be a non trivial ultrafilter on $\mathbb{Z}_+$. For any $\alpha \in \mathbb{D}$ and any $T \in \mathscr{B}(H)$, set

$$R_\alpha(T) = \lim_{\mathscr{U}} \, [p_n(v(T)\,T) + p_n(v(T)\,T)^*].$$

Denote $\mathscr{K} = \mathscr{K}_R$ the collection of operators associated to the previous defined bundle R. Let $\omega_{\mathscr{K}}$ be the associated operator radius. With these notations and using Theorem 3.1, we obtain the following result.

Corollary 4.8. *Assume that u is a finite Blaschke product. Suppose that T is a C_0 contraction such that $u(T) = 0$ and $v(T) = v(S_u)$. Then, with notation as above,*

$$\omega_{\mathscr{K}}(f(T)) \leqslant \omega_{\mathscr{K}}(f(S(u)))$$

for each $f \in A(\mathbb{D})$.

5. BOUNDS OF COEFFICIENTS OF POSITIVE RATIONAL FUNCTIONS

There are many classical inequalities for coefficients of (positive) trigonometric polynomials. The next result shows the links between the numerical radius of the extremal operator in the constrained von Neumann inequalities and the Taylor coefficients of rational functions positive on $\mathbb{T}$.

THEOREM 5.1. *Let $F = P/Q$ be a rational function with no principal part and which is positive on the torus. Then the Taylor coefficient c_k of order k satisfies the following inequality*

$$|c_k| \leqslant c_0 \omega_2(R^k),$$

where $R = S^* \mid Ker(Q(S^*))$.

Proof. First, observe that by continuity we may assume that F is strictly positive on the torus. Let $F = P/Q$ be a rational function without principal part, that is we have $d^\circ(P) < d^\circ(Q)$ for the degrees. Assume that $F(z) > 0$ for every $z \in \mathbb{T}$. Denote by $\beta_1, ..., \beta_q$ the zeros of Q which are contained in the open unit disc $\mathbb{D}$ and write $Q(z) = (z-\beta_1)^{d_1} ... (z-\beta_q)^{d_q} Q_2(z)$, where Q_2 has no zero in $\mathbb{D}$. Consider the function $G(z) = \overline{F(1/\bar{z})}$ which is analytic, except at a finite set of complex numbers. Since F is real on the torus, we have $G(e^{it}) = \overline{F(e^{it})} = F(e^{it})$ for every $t \in \mathbb{R}$. The analytic extension principle implies that $F(z) = G(z)$ except for a finite set in $\mathbb{C}$. Thus $F(z)$ can be written in the following way

$$F(z) = \frac{P(z)}{Q_1(z)\, Q_2(z)},$$

where $Q_1(z) = (z-\beta_1)^{d_1} ... (z-\beta_q)^{d_q}$ and $Q_2(z) = (1-\overline{\beta_1} z)^{d_1} ... (1-\overline{\beta_q} z)^{d_q}$. Because of the condition $F(z) = \overline{F(1/\bar{z})}$, we have $P(z) = z^{2d}\overline{P(1/\bar{z})}$ where $d = d_1 + \cdots + d_q = d^\circ(Q)/2$. If $P(\alpha) = 0$, with $\alpha \neq 0$, then necessarily $P(1/\bar{\alpha}) = 0$. Therefore P can be written as

$$P(z) = cz^{m_0}(z-\alpha_1)^{m_1} ... (z-\alpha_p)^{m_p}(1-\overline{\alpha_1} z)^{m_1} ... (1-\overline{\alpha_p} z)^{m_p}$$

with a suitable constant c. We have $d = m_1 + \cdots + m_p$. Finally, we get

$$F(e^{it}) = c \left| \frac{P_1(e^{it})}{Q_2(e^{it})} \right|^2$$

with $d^\circ(P_1) < d^\circ(Q_2)$ and $c > 0$. Note that

$$\frac{P_1(z)}{Q_1(z)} = \sum_{k=1}^{q} \sum_{i=1}^{m_k} \frac{a_{k,i}}{(1-\overline{\alpha_k} z)^i}$$

for some $a_{k,i} \in \mathbb{C}$. It follows that $P_1(z)/Q_1(z) \in E := H^2 \ominus bH^2$, where b is the associated Blaschke product defined by

$$b(z) = \prod_{k=1}^{q} \left(\frac{z-\alpha_k}{1-\overline{\alpha_k} z}\right)^{m_k}.$$

It follows from Lemma 3.2 that we have $F = |f|^2$ with a suitable $f \in E$.

Denote by R the restriction of the backward shift S^* to the invariant subspace Ker $Q(S)^*$. Then, for any integer k, we get

$$|c_k| = |\langle R^k f \mid f \rangle| \leqslant \omega_2(R^k) \, \|f\|_2^2 = \omega_2(R^k) \, \|F\|_1 = \omega_2(R^k) \, c_0.$$

This ends the proof. ∎

Setting $Q(z) = z^{n-1}$ in the previous theorem, and using previous computations of the numerical radii, we obtain the following classical inequality due to Egerváry and Százs (1927). The bound for c_1 is due to Fejer (1915).

Corollary 5.2 (Egerváry–Százs). *Let $P(e^{it}) = \sum_{j=-n+1}^{n-1} c_j e^{ijt}$ be a positive trigonometric polynomial ($n \geqslant 2$). Then*

$$|c_k| \leqslant c_0 \cos\left(\frac{\pi}{\left[\frac{n-1}{k}\right]+2}\right) \qquad \textit{for} \quad 1 \leqslant k \leqslant n-1.$$

Remark 5.1. We note the amuzing consequence that Fejer's inequality for $|c_1|$ implies, via operator inequalities, the Egerváry–Százs inequality. Indeed, by [HH], Fejer's inequality implies the Haagerup–de la Harpe inequality (0.3). By Remark 4.1 this implies a bound for $\omega_2(T^m)$, which in turn implies, as in [HH], the Egerváry–Százs inequality.

The next result gives estimates involving two coefficients of a positive trigonometric polynomial.

Theorem 5.3. *Let $P(e^{it}) = \sum_{j=-n+1}^{n-1} c_j e^{ijt}$ be a positive trigonometric polynomial ($n \geqslant 2$). Then, for every distinct numbers k and l among $\{0; ..., n-1\}$, there exists $\gamma \in \mathbb{R}$ such that*

$$|c_k| + |c_l| \leqslant c_0 \omega_2(S_n^k + e^{i\gamma} S_n^l).$$

In particular, we have

$$|c_k|+|c_l|\leqslant c_0\left(1+\cos\frac{\pi}{\left[\frac{n-1}{k+l}\right]+2}\right)^{1/2}\left(1+\cos\frac{\pi}{\left[\frac{n-1}{|k-l|}\right]+2}\right)^{1/2}.$$

Proof. We can assume that $c_0=1$. Since P is positive, we have $P=|Q|^2$ for some $Q\in\mathbb{C}_{n-1}[X]$, the space of all polynomials of degree less or equal to $n-1$. For any k, l, there exists γ such that

$$|c_k|+|c_l|=|c_k+e^{i\gamma}c_l|=\left|\int_0^{2\pi}(e^{ik\theta}+e^{i(l\theta+\gamma)})\,|Q(e^{il\theta})|^2\,dm(\theta)\right|.$$

We deduce from the equality $\|Q\|_2=c_0=1$ that

$$|c_k|+|c_l|\leqslant\omega_2(S_n^k+e^{i\gamma}S_n^l).$$

Denote $M=\omega_2(S_n^k+e^{i\gamma}S_n^l)$. We have

$$\begin{aligned}
M&=\sup_{\|R\|_2\leqslant1}\sup_{\alpha\in\mathbb{R}}\operatorname{Re}e^{i\omega}\int_0^{2\pi}(e^{ik\theta}+e^{i(l\theta+\gamma)})\,|R(e^{il\theta})|^2\,dm(\theta)\\
&=2\sup_{\|R\|_2\leqslant1}\sup_{\alpha\in\mathbb{R}}\int_0^{2\pi}\cos(\tfrac12[(k+l)\,\theta+\gamma+2\alpha])\cos(\tfrac12[(k-l)\,\theta-\gamma])\\
&\quad\times|R(e^{il\theta})|^2\,dm(\theta)\\
&\leqslant2\left(\sup_{\|R\|_2\leqslant1}\sup_{\alpha\in\mathbb{R}}\int_0^{2\pi}\cos^2(\tfrac12[(k+l)\,\theta+\gamma+2\alpha])\,|R(e^{il\theta})|^2\,dm(\theta)\right)^{1/2}\\
&\quad\times\left(\sup_{\|R\|_2\leqslant1}\int_0^{2\pi}\cos^2(\tfrac12[(k-l)\,\theta-\gamma])\,|R(e^{il\theta})|^2\,dm(\theta)\right)^{1/2}.
\end{aligned}$$

Let R be in $\mathbb{C}_{n-1}[X]$ with $\|R\|_2\leqslant1$. Since $L(e^{it})=R(e^{i\left(t-\frac{\gamma+2\omega}{k+l}\right)})$ is also in $\mathbb{C}_{n-1}[X]$ and of norm less or equal to one, we obtain, using the rotation invariance of the Haar measure, that

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{R}}\sup_{\|R\|_2\leqslant1}\int_0^{2\pi}\cos^2\left(\frac12[(k+l)\,\theta+\gamma+2\alpha]\right)|R(e^{il\theta})|^2\,dm(\theta)\\
&\quad=\sup_{\|L\|_2\leqslant1}\int_0^{2\pi}\cos^2\left(\left(\frac{k+l}{2}\right)t\right)|L(e^{ilt})|^2\,dm(\theta)\\
&\quad=\frac12+\frac12\sup_{\|L\|_2\leqslant1}\int_0^{2\pi}\cos((k+l)\,t)\,|L(e^{ilt})|^2\,dm(\theta)=\frac12(1+\omega_2(S_n^{k+l})).
\end{aligned}$$

In a similar way

$$\sup_{\|R\|_2 \leqslant 1} \int_0^{2\pi} \cos^2(\tfrac{1}{2}[(k+l)\,\theta-\gamma])\,|R(e^{il\theta})|^2\,dm(\theta)=\tfrac{1}{2}\,(1+\omega_2(S_n^{k-l})).$$

Finally, we obtain

$$|c_k|+|c_l| \leqslant \sqrt{1+\omega_2(S_n^{k+l})}\,\sqrt{1+\omega_2(S_n^{k-l})}.$$

Since $\omega_2(S_n^p)=\cos(\frac{\pi}{[n-1/p]+2})$, we get the desired result. ▮

Remark 5.2. (a) For $l=0$ we reobtain the Egerváry–Százs inequality.

(b) When $k+l>n-1$, we get from Theorem 5.3 that

$$|c_k|+|c_l| \leqslant c_0 \left(1+\cos\frac{\pi}{\left[\dfrac{n-1}{|k-l|}\right]+2}\right)^{1/2}.$$

In particular, if $n \geqslant 4$, we obtain

$$|c_1|+|c_{n-1}| \leqslant c_0\,\sqrt{3/2}.$$

This estimate is better than that one obtained by applying twice the Egerváry–Százs inequality.

(c) In some particular cases, it is possible to compute exactly the numerical radius $M=\omega_2(S_n^k+e^{i\gamma}S_n^l)$. Suppose $n=9$, $k=3$, $l=7$. It follows from [DH] that $M=\cos(\pi/10)$ if $\gamma=0$. The method from [DH] does not seem to apply for an arbitrary γ.

6. OTHER TYPE OF CONSTRAINTS

The constraints until now were of algebraic type ($q(T)=0$ or $Q(T^*,T)=0$). We discuss briefly constraints of different nature.

Some positivity conditions. We discuss constrained von Neumann inequalities for the numerical radius ω_2 of an operator satisfying some positivity conditions $R_\lambda(T^*,T) \geqslant 0$ for $\lambda \in \mathbb{T}$.

PROPOSITION 6.1. *Let $n \geqslant 2$ be a positive integer and let ρ_k, $0 \leqslant k \leqslant n-1$, be n positive reals with $\rho_0=1$. Let $T \in \mathscr{B}(H)$ be an operator such that $R_\lambda(T^*,T) \geqslant 0$ for $\lambda \in \mathbb{T}$, where*

$$R_\lambda(w,z)=1+\sum_{k=1}^{n-1}\frac{\lambda^k}{\rho_k}\,w^k+\sum_{k=1}^{n-1}\frac{\bar{\lambda}^k}{\rho_k}\,z^k(\lambda \in \mathbb{T}). \tag{6.1}$$

Then

$$\omega_2(T^m) \leqslant \rho_m \cos \frac{\pi}{\left[\dfrac{n-1}{m}\right]+2}$$

for each $m \in \{1, 2, \ldots, n-1\}$.

Proof. Let $h \in H$ be a norm-one vector and let $\theta \in \mathbb{R}$. Set

$$c_k = \begin{cases} 1 & : \text{ if } \quad k = 0 \\ \dfrac{1}{\rho_k} \langle T^k h \mid h \rangle & : \text{ if } \quad k > 0 \\ \dfrac{1}{\rho_{|k|}} \langle h \mid T^{|k|} h \rangle & : \text{ if } \quad k < 0 \end{cases}$$

and

$$t_n(\theta) = \sum_{k=-n+1}^{n-1} c_k e^{ik\theta}.$$

Then t_n is a positive trigonometric polynomial since

$$t_n(\theta) = \langle R_{\exp(it)}(T^*, T)\, h \mid h \rangle.$$

According to the Egerváry–Százs inequality, we have

$$\frac{1}{\rho_m} |\langle T^m h \mid h \rangle| = |c_m| \leqslant \cos\left(\frac{\pi}{\left[\dfrac{n-1}{m}\right]+2}\right)$$

which gives the desired inequality. ∎

If $\rho_k = 1$, for each $k \leqslant n-1$, then $R_\lambda(T^*, T)$ in Equation (6.1) are the nth sections of the operator kernel $K_\lambda(T)$.

In fact, the following more general result holds.

THEOREM 6.2. *Let* $n \geqslant 2$ *be a positive integer and let* ρ_k, $0 \leqslant k \leqslant n-1$, *be* n *positive reals with* $\rho_0 = 1$. *Let* $T \in \mathcal{B}(H)$ *be an operator such that* $R_\lambda(T^*, T) \geqslant 0$ *for* $\lambda \in \mathbb{T}$, *where* $R_\lambda(w, z)$ *are given by* (6.1). *Then, for any strongly admissible radius* v *and any* $m \in \{0, 2, \ldots, n-1\}$, *we have*

$$v(T^m) \leqslant \rho_m v(S_n^{*m}).$$

Moreover, if m and l are distinct numbers among $\{0; ..., n-1\}$ and if additionally $\rho_m = \rho_l$, then

$$v(T^m + T^l) \leqslant \rho_m v(S_n^{*m} + S_n^{*l}).$$

In particular, we have

$$\omega_\rho(T^m+T^l) \leqslant \left(\frac{2}{\rho}-1\right) 2^\rho \rho_m \left[1+\cos\frac{\pi}{\left[\frac{n-1}{m+l}\right]+2}\right]^{\frac{1-\rho}{2}} \left[1+\cos\frac{\pi}{\left[\frac{n-1}{|m-l|}\right]+2}\right]^{\frac{1-\rho}{2}}$$

if $\rho \in]0, 1]$, and

$$\omega_\rho(T^m+T^l) \leqslant 2^{2-\rho}\rho_m^{\rho-1} \left[1+\cos\frac{\pi}{\left[\frac{n-1}{m+l}\right]+2}\right]^{\frac{\rho-1}{2}} \left[1+\cos\frac{\pi}{\left[\frac{n-1}{|m-l|}\right]+2}\right]^{\frac{\rho-1}{2}}$$

if $\rho \in]1, 2]$.

Its proof follows from Theorem 5.3, interpolation properties of ω_ρ (see [FH, p. 296]) and the following generalization of a result of Arveson (obtained in [Ar] for $\rho_k = 1$, $k \geqslant 1$).

THEOREM 6.3. *Let $T \in \mathscr{B}(H)$ be a contraction and let $n \geqslant 2$. Suppose T satisfies $R_\lambda(T^*, T) \geqslant 0$ for all $\lambda \in \mathbb{T}$, where $R_\lambda(w, z)$ are given by* (6.1). *Then there is a Hilbert space $K \supset H$ and a nilpotent contraction $N \in \mathscr{B}(K)$ such that $N^n = 0$, N is unitarily equivalent to $S_n^{*(d)}$, d finite or ∞, and $T^j = \rho_j P_H N^j \mid H$ for $j = 0, 1, ..., n-1$.*

Proof. The idea of the proof is that of [Ar] and some details will be omitted below. Define a linear map φ from *span*$\{S_n^{*j}: 0 \leqslant j \leqslant n-1\}$ onto *span*$\{\frac{1}{\rho_j} T^j: 0 \leqslant j \leqslant n-1\}$ by $\varphi(S_n^{*j}) = \frac{1}{\rho_j} T^j$ and by linearity. Define the map $\psi: C(\mathbb{T}) \to \mathscr{B}(H)$ by

$$\psi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, R_{\exp(i\theta)}(T^*, T)\, d\theta.$$

It is a positive linear map. Note that $\psi(z^j) = \frac{1}{\rho_j} T^j$ for $j = 0, 1, ..., n-1$ and $\psi(z^j) = 0$ for $j \geqslant n$, where $z(\theta) = \theta$. It is known that a positive linear map on a commutative C*-algebra is completely positive and a completely positive map which preserves the identity is completely contractive [Pa]. The restriction ψ_0 of ψ on the disc algebra (the closed linear span in $C(\mathbb{T})$ of

$1, z, z^2, \ldots)$ is a completely contractive linear map such that $\psi_0(z^j) = \frac{1}{\rho_j} T^j$ for $j = 0, 1, \ldots, n-1$ and $\psi_0(z^j) = 0$ for $j \geqslant n$. It vanishes on the ideal $z^n A$ and thus it induces a completely contractive linear map ψ_{00} of the quotient $A/z^n A$ into $\mathscr{B}(H)$. It was proved in [Ar] that $\mu(S_n^{*j}) = z^j + z^n A$ defines a completely isometric linear map of $span\{I, S_n^*, \ldots, S_n^{*n-1}\}$ onto $A/z^n A$. The original map $\varphi = \psi_{00}\mu$ is thus completely contractive. Since $\varphi(I) = I$, φ has [Pa] a completely positive extension to $C^*(S_n^*) = \mathscr{B}(\mathbb{C}^n)$. Stinespring's theorem [Pa] furnishes then a unital C*-representation π. Then $N = \pi(S_n^*)$ gives, as in [Ar], the desired representation. ∎

In the case $\rho_k = 1$ for all k, studied in [Ar], the converse of Theorem 6.3 also holds. Also, an operator T satisfies

$$I + 2\,\mathrm{Re} \sum_{k=1}^{n-1} z^k T^k \geqslant 0, \qquad \text{for each} \quad z \in \mathbb{T},$$

if and only if [Ar]

$$2\,\mathrm{Re}(I - zT)^*\, z^n T^n \leqslant I - T^*T, \qquad \text{for each} \quad z \in \mathbb{T}.$$

In particular this holds if T satisfies $T^n = 0$ and $I - T^*T \geqslant 0$.

Stability of the algebraic constraints. In what follows $\varepsilon > 0$ is supposed to be a (fixed) small positive number. We study what happens if the constraint $q(T) = 0$ is replaced by $\|q(T)\| \leqslant \varepsilon$.

PROPOSITION 6.4. *Let q be a polynomial. For each $\varepsilon > 0$ there exists $\delta > 0$ such that every contraction $T \in \mathscr{B}(H)$ with $\|q(T)\| \leqslant \delta$ satisfies*

$$\omega_\rho(T) \leqslant \varepsilon + \omega_\rho(S^* \mid Ker\; q(S^*))$$

for every $\rho \in\ ]0, 2]$.

Proof. By [He, Corollary 2.22], for every $\varepsilon > 0$ there is $\delta > 0$ such that, if $\|T\| \leqslant 1$ and $\|q(T)\| \leqslant \delta$, then there exists $T' \in \mathscr{B}(H)$ such that $q(T') = 0$ and $\|T - T'\| \leqslant \varepsilon$. Note also that ω_ρ is a norm for $\rho \leqslant 2$. We thus have

$$\begin{aligned} \omega_\rho(T) &\leqslant \omega_\rho(T - T') + \omega_\rho(T') \\ &\leqslant \|T - T'\| + \omega_\rho(T') \\ &\leqslant \varepsilon + \omega_\rho(S^* \mid \mathrm{Ker}\; q(S^*)). \end{aligned}$$

The proof is complete. ∎

It was proved in [He, Theorem 2.21] that if $\|T\| \leqslant 1$ and $\|T^n\| \leqslant \varepsilon$, then there exists $T' \in \mathscr{B}(H)$ such that $T'^n = 0$ and $\|T - T'\| \leqslant \delta_n(\varepsilon)$, where $\delta_n(\varepsilon)$ is defined inductively by

$$\delta_2(\varepsilon) = (2\varepsilon)^{1/2} \qquad \text{and} \qquad \delta_k(\varepsilon) = \{\varepsilon + [\delta_{k-1}((k-1)\sqrt{\varepsilon})]^2\}^{1/2}.$$

This implies that if $\|T\| \leqslant 1$ and $\|T^n\| \leqslant \varepsilon$ then

$$\omega_2(T) \leqslant \cos\left(\frac{\pi}{n+1}\right) + \delta_n(\varepsilon).$$

Note that $\lim_{\varepsilon \to 0} \delta_n(\varepsilon) = 0$.

The following result gives a better bound for small ε; we obtain the Haagerup–de la Harpe inequality for $\varepsilon \to 0$.

THEOREM 6.5. *Let $n \geqslant 2$ be a positive integer. Suppose $T \in \mathscr{B}(H)$ is a contraction satisfying $\|T^n\| \leqslant \varepsilon$ and*

$$\sum_{k > n+1} \|T^k\| < +\infty.$$

Then

$$\omega_2(T) \leqslant \cos\left(\frac{\pi}{n+1}\right) + 3\left[\pi \cos^4 \frac{\pi}{2(n+1)}\right]^{1/3} \left(\frac{\varepsilon}{n+1}\right)^{2/3}$$
$$\leqslant \cos\left(\frac{\pi}{n+1}\right) + 3\sqrt[3]{\pi}\left(\frac{\varepsilon}{n+1}\right)^{2/3}.$$

The proof uses the following epsilonized Fejer inequality. Note that an epsilonized version of the Egerváry–Százs inequality can be proved along the same lines.

LEMMA 6.6 (The epsilonized Fejer inequality). *Let h be a positive function,*

$$h(\theta) = \sum_{m \in \mathbb{Z}} c_m e^{im\theta},$$

such that $\sum_{m \in \mathbb{Z}} |c_m| < \infty$ with $c_0 = 1$ and $|c_k| \leqslant \varepsilon$ for $k \geqslant n$. Then

$$|c_1| \leqslant \cos\left(\frac{\pi}{n+1}\right) + 3\left[\pi \cos^4 \frac{\pi}{2(n+1)}\right]^{1/3} \left(\frac{\varepsilon}{n+1}\right)^{2/3}.$$

Proof. The following result has been proved in [J, Example 4(a)]: Let f be the Fourier transform of a non-negative integrable function φ:

$$f(x) = \int_{-\infty}^{\infty} e^{ixt} \varphi(t)\, dt.$$

Let $u > 0$ and suppose that $f(0) = 1$ and $|f(ku)| \leqslant \varepsilon$ for $k \geqslant n$. Then

$$|f(u)| \leqslant \cos\left(\frac{\pi}{n+1}\right) + 3\left[\pi \cos^4 \frac{\pi}{2(n+1)}\right]^{1/3} \left(\frac{\varepsilon}{n+1}\right)^{2/3}.$$

This is a generalization of a result due to Boas and Kac [BK] for band-limited functions.

Set now $\varphi(t) = h(-t)$, $t \in [-\pi, \pi]$. Consider f the Fourier transform of φ. Then $f(0) = c_0 = 1$, $f(k) = c_k$ and thus $|f(k)| \leqslant \varepsilon$ for $k \geqslant n+1$. We can now apply [J] with $u = 1$. ∎

Proof (of Theorem 6.5). The proof is similar to the proof of Proposition 6.1. By replacing eventually T by rT, $0 < r < 1$, it is possible to assume that the spectrum of T is contained in $\mathbb{D}$. For each norm-one vector $h \in H$ and $\theta \in \mathbb{R}$, set

$$c_k (= c_k(h)) = \begin{cases} 1 & : \text{ if } \; k = 0 \\ \langle T^k h \,|\, h \rangle & : \text{ if } \; k > 0 \\ \langle h \,|\, T^{|k|} h \rangle & : \text{ if } \; k < 0 \end{cases}$$

and

$$h(\theta) = \sum_{k \in \mathbb{Z}} c_k e^{ik\theta}.$$

Then $\sum_{m \in \mathbb{Z}} |c_m| < \infty$. Note also that

$$h(\theta) = \langle K_{\exp(it)}(T)\, h \,|\, h \rangle$$

and the operator kernel

$$K_{\exp(it)}(T) = (I - e^{it} T^*)^{-1} (I - T^*T)(I - e^{-it} T)^{-1}$$

is positive since T is a contraction. We use now the epsilonized Fejer inequality. ∎

COROLLARY 6.7. *Let n and m be two positive integers such that $m \geqslant n \geqslant 2$. Suppose $T \in \mathcal{B}(H)$ is a contraction satisfying $\|T^n\| \leqslant \varepsilon$ and $T^m = 0$. Then*

$$\omega_2(T) \leqslant \min\left[\cos\left(\frac{\pi}{m+1}\right); \cos\left(\frac{\pi}{n+1}\right) + 3\sqrt[3]{\pi}\left(\frac{\varepsilon}{n+1}\right)^{2/3}\right].$$

ACKNOWLEDGMENT

This paper was written during several visits of the authors to the Mathematical Departements of Universities of Lille and Lyon. We wish to thank both institutions for their help.

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