Some properties of the class of positive Dunford–Pettis operators

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ARTICLE INFO

Article history:
Received 31 March 2008
Available online 8 January 2009
Submitted by R. Curto

Keywords:
M-weakly compact operator
L-weakly compact operator
Dunford–Pettis operator
Order continuous norm

ABSTRACT

We characterize Banach lattices for which each positive Dunford–Pettis operator is M-weakly compact (resp. L-weakly compact) and we give some consequences.

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1. Introduction and notation

In [9], Meyer-Nieberg gave an interesting study of the class of M-weakly compact operators and the class of L-weakly compact operators. The introduction of these two classes of operators was justified by the difficulties meeting in the study of the class of weakly compact operators on Banach lattices. Recall that an operator \( T \) from a Banach lattice \( E \) into a Banach space \( F \) is said to be M-weakly compact if for each disjoint bounded sequence \( (x_n) \) of \( E \), we have \( \lim_n \| T(x_n) \| = 0 \). And an operator \( T \) from a Banach space \( E \) into a Banach lattice \( F \) is called L-weakly compact if for each disjoint bounded sequence \( (y_n) \), in the solid hull of \( T(B_E) \), we have \( \lim_n \| y_n \| = 0 \) where \( B_E \) is the closed unit ball of \( E \). Note that by Proposition 3.6.11 of [9], an operator between two Banach lattices is M-weakly compact (resp. L-weakly compact) if and only if its adjoint is M-weakly compact (resp. L-weakly compact).

On the other hand, an operator \( T \) from a Banach space \( E \) into another \( F \) is said to be Dunford–Pettis if it carries weakly compact subsets of \( E \) onto compact subsets of \( F \). Note that a M-weakly compact (resp. L-weakly compact) operator is not necessary Dunford–Pettis. The converse is not always true. More than, the class of Dunford–Pettis operators and the class of M-weakly compact (resp. L-weakly compact) operators do not coincide even when the Banach lattice \( E \) is reflexive. However, Meyer-Nieberg [9, Theorem 3.7.10] established that each Dunford–Pettis operator from a Banach lattice \( E \) into a Banach space \( F \) is M-weakly compact if and only if the norm of the topological dual of \( E \) is order continuous.

The objective of this paper is to establish necessary and sufficient conditions for which each positive Dunford–Pettis operator is M-weakly compact (resp. L-weakly compact). We will first prove that each Dunford–Pettis (resp. compact) operator \( T \), from a Banach lattice \( E \) into a Banach space \( F \), is M-weakly compact if and only if the norm of \( E' \) is order continuous or \( F = \{0\} \). As a consequence, we will give a generalization of Theorem 2.26 of [5] about the weak compactness of Dunford–Pettis operators. Next, we will establish an analogue to Riesz Theorem, by proving that the closed unit ball \( B_E \) of a Banach lattice \( E \), is L-weakly compact if and only if \( E \) is finite-dimensional. Finally, we will use the last result to show that each positive Dunford–Pettis operator \( T : E \to F \), between Banach lattices, is L-weakly compact if and only if \( E = \{0\} \) or \( F \) is finite-dimensional or the norms of \( E' \) and \( F \) are order continuous. Also, we will deduce some interesting consequences.

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To state our results, we need to fix some notations and recall some definitions. A vector lattice $E$ is an ordered vector space in which $\text{sup}(x, y)$ exists for every $x, y \in E$. A subspace $F$ of a vector lattice $E$ is said to be a sublattice if for every pair of elements $a, b$ of $F$ the supremum of $a$ and $b$ taken in $E$ belongs to $F$. A Banach lattice is a Banach space $(E, \| \cdot \|)$ such that $E$ is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq |y|$. If $E$ is a Banach lattice, its topological dual $E'$, endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\| \cdot \|$ of a Banach lattice $E$ is order continuous if for each generalized sequence $(x_n)$ such that $x_n \downarrow 0$ in $E$, the generalized sequence $(x_n)$ converges to 0 for the norm $\| \cdot \|$ where the notation $x_n \downarrow 0$ means that $(x_n)$ is decreasing, its infimum exists and $\inf(x_n) = 0$. A Banach lattice $E$ is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $|x + y| = \max\{|x|, |y|\}$. The Banach lattice $E$ is an AL-space if its topological dual $E'$ is an AM-space.

We will use the term operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. The operator $T$ is regular if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. It is well known that each positive linear mapping on a Banach lattice is continuous. For more informations on Banach lattice theory and positive operators, we refer the reader to [3].

2. M-weak compactness of positive Dunford–Pettis operators

A Banach space $E$ is said to have the Schur property if every weakly convergent sequence to 0 in $E$ is norm convergent to zero. For example, the Banach space $l^1$ has the Schur property.

The Banach lattice $E$ has the positive Schur property if each weakly null sequence with positive terms in $E$ converges to zero in norm. For example, the Banach lattice $l^1(\mathbb{N}, 1)$ has the positive Schur property but does not have the Schur property. For more informations about this notion see [12].

There exist operators which are not Dunford–Pettis nor M-weakly compact. In fact, the identity operator $Id_{l^1[0, 1]} : l^1(\{0, 1\}) \to l^1([0, 1])$ is not Dunford–Pettis nor M-weakly compact.

Also, since the Banach space $l^1$ admits the Schur property, its identity operator $Id_{l^1} : l^1 \to l^1$ is Dunford–Pettis but not M-weakly compact.

Recall that each M-weakly compact operator between two Banach lattices is weakly compact. But the converse is false in general. In fact, the identity operator $Id_{l^2} : l^2 \to l^2$ is weakly compact, however it is not M-weakly compact.

Now, we characterize Banach lattices $E$ and $F$ for which each positive Dunford–Pettis (resp. compact) operator $T : E \to F$ is M-weakly compact.

**Theorem 2.1.** Let $E$ and $F$ be two Banach lattices. Then the following assertions are equivalent:

1. Each positive Dunford–Pettis operator $T : E \to F$ is M-weakly compact.
2. Each positive compact operator $T : E \to F$ is M-weakly compact.
3. One of the following statements is valid:
   (i) the norm of $E'$ is order continuous;
   (ii) $F = \{0\}$.

**Proof.** (1) $\Rightarrow$ (2) Since each compact operator is Dunford–Pettis.

(2) $\Rightarrow$ (3) Assume that (3) is false i.e. the norm of $E'$ is not order continuous and $F \neq \{0\}$. It follows from Theorem 2.4.14 and Proposition 2.3.11 of Meyer-Nieberg [9] that $E$ contains a closed sublattice which is isomorphic to $l^1$ and there exists a positive projection $P : E \to l^1$.

On the other hand, as $F \neq \{0\}$, there exists a non-null element $y \in F^+$. If we consider the operator $S : l^1 \to F$ defined by

$$S((\lambda_n)) = \left(\sum_{n=1}^{\infty} \lambda_n\right) y \quad \text{for each } (\lambda_n) \in l^1.$$

It is clear that $S$ is well defined and positive. Also, $S$ is compact (because its rank is one). Hence the positive operator

$$T = S \circ P : E \to l^1 \to F$$

is compact. If we design by $(e_n)$ the canonical basis of $l^1 \subset E$, the sequence $(e_n)$ is disjoint and bounded in $E$, moreover we have $T((e_n)) = y$ for each $n \geq 1$. Then $\|T((e_n))\| \to 0$ (because $y \neq 0$). It follows that $T$ is not M-weakly compact and this proves the result.

(3)(i) $\Rightarrow$ (1) It is just the implication (i) $\Rightarrow$ (iii) of Theorem 3.7.10 of Meyer-Nieberg [9].

(3)(ii) $\Rightarrow$ (1) In this case we have $T = 0$, and hence $T$ is M-weakly compact. $\square$

Whenever $E = F$ in Theorem 2.1, we obtain the following characterization:

**Corollary 2.2.** Let $E$ be a Banach lattice. Then the following assertions are equivalent:

1. Each positive Dunford–Pettis operator $T : E \to E$ is M-weakly compact.
(2) Each positive compact operator $T : E \to E$ is $M$-weakly compact.

(3) The norm of $E'$ is order continuous.

As a consequence of Theorem 2.1, we obtain Theorem 2.2 of [4].

**Corollary 2.3.** Let $E$ and $F$ be two Banach lattices. If the norm of $E'$ is order continuous, then each positive Dunford–Pettis operator $T : E \to F$ is weakly compact.

To give another consequence of Theorem 2.1, recall that an operator $T$ from a Banach space $E$ into a Banach lattice $F$ is said to be semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where $B_H$ is the closed unit ball of $H = E$ or $F$ and $F^+ = \{x \in F : 0 \leq x\}$.

As we asked in [6], a semi-compact operator is not necessary Dunford–Pettis, and conversely a Dunford–Pettis operator is not necessary semi-compact. For example, the identity operator $Id_{l^1} : l^1 \to l^1$ is Dunford–Pettis but it is not semi-compact and conversely, the identity operator $Id_c : c \to c$ is semi-compact but it is not Dunford–Pettis where $c$ is the Banach lattice of all convergent sequences. And in [6], the semi-compactness of Dunford–Pettis operators was studied.

As a consequence of Theorems 2.1, 2.2 of [4] and Theorem 2.2 of [6] we obtain the following characterization:

**Corollary 2.4.** Let $E$ be a Banach lattice. Then the following assertions are equivalent:

(1) Each positive Dunford–Pettis operator $T : E \to E$ is $M$-weakly compact.

(2) Each positive Dunford–Pettis operator $T : E \to E$ is weakly compact.

(3) Each positive compact operator $T : E \to E$ is $M$-weakly compact.

(4) Each positive Dunford–Pettis operator from $E$ into $E$ is semi-compact.

(5) For all operators $S$ and $T$ from $E$ into $E$ such that $0 \leq S \leq T$ and $T$ is Dunford–Pettis, the operator $S$ is weakly compact.

(6) The norm of $E'$ is order continuous.

By an analogue proof to Theorem 2.1, we obtain the following result, which is a generalization of Theorem 3.7.10 of Meyer-Nieberg [9]:

**Theorem 2.5.** Let $E$ be a Banach lattice and $F$ a Banach space. Then the following assertions are equivalent:

(1) Each Dunford–Pettis operator $T : E \to F$ is $M$-weakly compact.

(2) Each compact operator $T : E \to F$ is $M$-weakly compact.

(3) One of the following statements is valid:
   (i) the norm of $E'$ is order continuous;
   (ii) $F = \{0\}$.

As a consequence of Theorem 2.5, we obtain a generalization of Theorem 2.2 of [4].

**Corollary 2.6.** Let $E$ be a Banach lattice and $F$ a Banach space. If the norm of $E'$ is order continuous, then each Dunford–Pettis operator $T : E \to F$ is weakly compact.

Finally, we establish a generalization of Theorem 2.26 of [5].

**Theorem 2.7.** Let $E$ be a Banach lattice and $F$ a Banach space. Then the following assertions are equivalent:

(1) Each Dunford–Pettis operator $T : E \to F$ is weakly compact.

(2) One of the following statements is valid:
   (i) the norm of $E'$ is order continuous;
   (ii) $F$ is reflexive.

**Proof.** (1) $\Rightarrow$ (2) It suffices to establish that if the norm of $E'$ is not order continuous, then $F$ is reflexive. Indeed, assume that $E'$ does not have an order continuous norm. By Theorem 2.4.14 of Meyer-Nieberg [9] we may assume that $l^1$ is a closed sublattice of $E$, and it follows from Proposition 2.3.11 of Meyer-Nieberg [9] the existence of a positive projection $P : E \to l^1$. It is clear that $P$ is a Dunford–Pettis operator.

We have to prove that $F$ is reflexive i.e. the closed unit ball of $F$ is compact for $\sigma(F, F')$. By applying Eberlein–Smulian Theorem (see for example Theorem 19.4 of Aliprantis and Burkinshaw [1]) it suffices to show that every sequence $(y_n)$ in
the closed unit ball of $F$ has a subsequence that is convergent to an element of $F$ for $\sigma(F, F')$. Given such a sequence $(y_n)$. Again, define an operator $S : l^1 \to F$ by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for each } (\lambda_n) \in l^1.$$ 

Then the composed operator $T = S \circ P : E \to l^1 \to F$ is also Dunford–Pettis. From the assumption we have $T$ is weakly compact. As $T(e_n) = y_n$ for all $n \geq 1$, where $e_n$ is the sequence with the $n$th entry equals to 1 and all others are zero, we conclude that $(y_n)$ has a subsequence that is convergent to an element of $F$ for $\sigma(F, F')$.

(2)(i) $\Rightarrow$ (1) follows from Corollary 2.3.

(2)(ii) $\Rightarrow$ (1) In this case, each operator from $E$ into $F$ is weakly compact. □

3. L-weak compactness of positive Dunford–Pettis operators

To study the L-weak compactness of positive Dunford–Pettis operators, we need to recall some definitions. A subset $A$ of a Banach lattice $E$ is said to be almost order bounded or order quasi-precompact if for each $\epsilon > 0$ there exists some $x \in E^+$ such that $A \subset [-x, x] + \epsilon B_E$. A non-empty bounded subset $A$ of $E$ is said to be L-weakly compact if for each disjoint sequence $(x_n)$ in the solid hull of $A$, we have $\|x_n\| \to 0$.

Each L-weakly compact subset of $E$ is relatively weakly compact. But the converse is false in general. In fact, the closed unit ball of $l^2$ is weakly compact but it is not a L-weakly compact subset of $l^2$. However, if $E$ has the positive Schur property, then a subset $A$ of $E$ is L-weakly compact if and only if $A$ is relatively weakly compact (Theorem 3.1 of Chen and Wickstead [7]).

Also, if the norm of $E$ is order continuous, then each relatively compact (resp. order quasi-precompact) subset of $E$ is L-weakly compact.

Recall that a Banach space $E$ has the Dunford–Pettis property if each weakly compact operator on $E$, into another Banach space $F$, is Dunford–Pettis.

To prove our next result, we need to establish the following theorem, which is an analogue result to the famous Riesz Theorem on the compactness of the closed unit ball of a finite-dimensional Banach space.

**Theorem 3.1.** Let $(E, \| \cdot \|)$ be a Banach lattice. The closed unit ball $B_E$ of $E$ is L-weakly compact if and only if $E$ is finite-dimensional.

**Proof.** First, we observe that the closed unit ball $B_E$ of $E$ is L-weakly compact if and only if, for each disjoint sequence $(x_n)$ in $B_E$ we have $\|x_n\| \to 0$, if and only if, for each norm bounded disjoint sequence $(x_n)$ in $E$ we have $\|x_n\| \to 0$.

(1) Assume that $B_E$ is L-weakly compact, then it is almost order bounded. In fact, note that for every disjoint sequence $(x_n)$ in the solid hull of $B_E$ we have $\|x_n\| \to 0$. It follows from Corollary 2.10 of Dodds and Fremlin [8] that for each $\epsilon > 0$, there exists $u \in E^+$ such that $\|u(x - u)\| \leq \epsilon$ for every $x \in B_E$. Hence, Theorem 122.1 of Zaanen [13] implies the result.

Now, by Exercise 122.8 of Zaanen [13], there exists some $e \in E^+$ such that $B_E \subset [-2e, 2e]$. Hence, $e$ is a strong unit in $E$; in other words, the order ideal $A_e$ generated by $e$ satisfies $A_e = E$. But then $E$ is a Banach lattice with respect to the $\epsilon$-uniform norm defined by $\|x\|_e = \inf \{\lambda > 0 : \|x\| \leq \lambda e\}$ for all $x \in E$, i.e., $(E, \| \cdot \|_e)$ is an AM-space with unit $e$ (see Corollary of Proposition 7.2 of Schaefer [10, p. 102]). Hence $(E, \| \cdot \|_e)$ has the Dunford–Pettis property.

Since $B_E \subset [-2e, 2e]$ and $[-e, e] \subset \| \cdot \|_e$, the two norms $\| \cdot \|_e$ and $\| \cdot \|_e$ are equivalent (because we have $\frac{1}{2}\|x\|_e \leq \|x\| \leq \|x\|_e$ for all $x \in E$). And then $(E, \| \cdot \|_e)$ and $(E, \| \cdot \|_e)$ have the same norm bounded subsets. Hence, for every sequence $(x_n)$ of $E$, we have

$$\|x_n\| \to 0 \quad \text{if and only if} \quad \|x_n\|_e \to 0. \quad (*)$$

On the other hand, as the closed unit ball $B_E$ of $(E, \| \cdot \|_e)$ is L-weakly compact, then for every disjoint sequence $(x_n)$ of $E$, bounded for the norm $\| \cdot \|_e$, we have $\|x_n\| \to 0$. It follows from ($*$) that for every disjoint sequence $(x_n)$ of the closed unit ball of $(E, \| \cdot \|_e)$, we have $\|x_n\|_e \to 0$. Hence, the identity operator $I_E : (E, \| \cdot \|_e) \to (E, \| \cdot \|_e)$ is M-weakly compact, and so it is weakly compact. This proves that $(E, \| \cdot \|_e)$ is reflexive. Finally, Corollary 2 of Theorem 9.9 of Schaefer [10] implies that $E$ is finite-dimensional.

(2) Conversely, assume that $E$ is finite-dimensional. It follows from Corollary 2 of Theorem 1.5 of Schaefer [10] that each family of non-null disjoint elements of $E$ is free, and then each disjoint sequence $(x_n)$ of $E$ contains only a finite number of non-null terms i.e. there exists some $n_0 \in N$ such that $x_n = 0$ for each $n \geq n_0$. And then $\|x_n\| \to 0$. This implies that the closed unit ball $B_E$ of $E$ is L-weakly compact. □

As a consequence, we obtain

**Corollary 3.2.** Let $E$ and $F$ be two Banach lattices.

(1) If $E$ is finite-dimensional, then each operator $T : E \to F$ is $M$-weakly compact.
(2) If $F$ is finite-dimensional, then each operator $T : E \to F$ is $L$-weakly compact.

Now, we characterize Banach lattices for which each positive Dunford–Pettis operator is $L$-weakly compact.

**Theorem 3.3.** Let $E$ and $F$ be two Banach lattices. Then the following assertions are equivalent:

1. Each positive Dunford–Pettis operator $T : E \to F$ is $L$-weakly compact.
2. One of the following statements is valid:
   (i) $E = \{0\}$;
   (ii) $F$ is finite-dimensional;
   (iii) the norms of $E'$ and $F$ are order continuous.

**Proof.** (1) $\Rightarrow$ (2) Assume that (1) holds. The proof follows along the lines of the proof of Theorem 1 of Wickstead [11] if we prove separately the two following assertions:

(a) If the norm of $E'$ is not order continuous, then $F$ is finite-dimensional.

(b) If the norm of $F$ is not order continuous, then $E = \{0\}$.

Assume that (a) is false, i.e. the norm of $E'$ is not order continuous and $F$ is infinite-dimensional. Then $E$ contains a sublattice isomorphic to $l^1$ and there exists a positive projection $P : E \to l^1$. It is clear that $P$ is a Dunford–Pettis operator.

Now, since $F$ is infinite-dimensional, Theorem 3.1 implies the existence of a disjoint norm bounded sequence $(y_n)$ of $F^+$ which does not converge to zero in norm. We consider the operator $S : l^1 \to F$ defined by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n$$

for each $(\lambda_n) \in l^1$.

It is clear that $S$ is well defined and positive.

On the other hand, the operator

$$T = S \circ P : E \to l^1 \to F$$

is positive and Dunford–Pettis. But it is not $L$-weakly compact. In fact, we have $T(e_n) = y_n$ for all $n \geq 1$, where $e_n$ is the sequence with the $n$th entry equals to 1 and all others are zero. If $T$ is $L$-weakly compact, since $(y_n)$ is disjoint, it follows that $\|y_n\| \to 0$ as $n \to \infty$, which is a contradiction.

Assume now that (b) is false i.e. the norm of $E$ is not order continuous and $E \neq \{0\}$. Choose $u \in E^+$ such that $\|u\| = 1$. Hence, it follows from Theorem 39.3 of Zaanen [14] that there exists $\varphi \in (E')^+$ such that $\|\varphi\| = 1$ and $\varphi(u) = \|u\| = 1$.

Next, since the norm of $F$ is not order continuous, there exists some $y \in F^+$ and there exists a disjoint sequence $(y_n) \subset [0, y]$ which does not converge to zero in norm.

We consider the operator $T : E \to F$ defined by the following

$$T(x) = \varphi(x)y$$

for each $x \in E$.

It is clear that $T$ is positive and compact (because its rank is one) and hence $T$ is Dunford–Pettis. But $T$ is not $L$-weakly compact. In fact, since $\|u\| = 1$ and $T(u) = \varphi(u)y = y$ then $y \in T(B_E)$. As $(y_n) \subset [0, y]$, we conclude that $(y_n)$ is a disjoint sequence in the solid hull of $T(B_E)$. Hence, if $T$ is $L$-weakly compact then $\|y_n\| \to 0$ as $n \to \infty$, which is a contradiction. So (1) $\Rightarrow$ (2).

The implication (2)(i) $\Rightarrow$ (1) is clear.

(2)(ii) $\Rightarrow$ (1) In this situation, it follows from Corollary 3.2 that each operator $T : E \to F$ is $L$-weakly compact.

(2)(iii) $\Rightarrow$ (1) Let $T : E \to F$ be a positive Dunford–Pettis operator. As the norm of $E'$ is order continuous, it follows from Theorem 2.1 that $T$ is $M$-weakly compact. Now, as the norm of $F$ is order continuous, Corollary 3.6.14 of Meyer-Nieberg [9] implies that $T$ is $L$-weakly compact. \[\square\]

As a consequence of Theorem 3.3, we obtain the following characterization:

**Corollary 3.4.** Let $E$ be a Banach lattice. Then the following assertions are equivalent:

1. Each positive Dunford–Pettis operator $T : E \to E$ is $L$-weakly compact.
2. The norms of $E$ and $E'$ are order continuous.

As a consequence of Theorem 2.7 of Aliprantis and Burkinshaw [2] and Corollary 3.4, we obtain.

**Corollary 3.5.** Let $E$ be a Banach lattice. If each positive Dunford–Pettis operator $T : E \to E$ is $L$-weakly compact, then each positive Dunford–Pettis operator $T : E \to E$ is compact.
Finally, as a consequence of Theorems 2.1 and 3.3, we obtain:

**Corollary 3.6.** Let $E$ be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:

1. Each positive Dunford–Pettis operator $T : E \to E$ is $L$-weakly compact.
2. Each positive Dunford–Pettis operator $T : E \to E$ is $M$-weakly compact.

**References**