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Perturbed Proximal Point Algorithms for Generalized Quasivariational Inclusions*

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In this paper, we study a class of generalized quasivariational inclusions. By using the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert space, we have established an existence theorem of solutions for generalized quasivariational inclusions, suggesting a new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions which strongly converge to the exact solution of the generalized quasivariational inclusions. As special cases, some known results in this field are also discussed. © 1997 Academic Press

1. INTRODUCTION

Variational inequality theory and complementarity problem theory have become very effective and powerful tools for studying a wide range of problems arising in mechanics, mathematical programming, optimization and control problems, equilibrium theory of economics, management science operations research, and other branches of mathematics and engineering sciences. In recent years, the classical variational inequality and complementarity problem have been extended and generalized in many different directions. Various quasi (implicit) variational inequalities and quasi (implicit) complementarity problems are very important generalizations of these classical problems. These were introduced and studied by Bensoussan and Lions [3], Bensoussan, Gourst, and Lions [2], Baiocchi and Capelo [1], Mosco [24], Pang [32, 33], Noor [25–28], Isac [20–22], Siddiqi

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and Ansari [38, 39], Ding and Tarafdar [12, 13], Gao and Yao [17], and Zeng [40]. Harker and Pang [18] and Noor, Noor, and Rassias [30] provide excellent surveys on the developments of the classical variational inequalities and complementarity problems in finite dimensional Euclidean spaces and infinite dimensional Hilbert spaces.

Another important and useful generalization of the classical variational inequalities and complementarity problems are the generalized quasi (implicit) variational inequalities and generalized quasi (implicit) complementarity problems introduced and studied by Browder [4], Rockafeller [35], Saigal [36], Fang and Peterson [16], Fang [15], Chan and Pang [5], Siddiqi and Ansari [37], Ding [6–9], Ding and Tan [11], Ding and Deng [10], and Ding and Tarafdar [14].

In a recent paper [19], Hassouni and Moudafi introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusion.

In this paper, we shall introduce and study a class of generalized quasivariational inclusions. By applying the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert spaces, it is shown that the quasivariational inclusion problems are equivalent to the fixed point problems. A new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions which strongly converge to the exact solution of the generalized quasivariational inclusion are proposed and analysed. As special cases, some known results in the field are also discussed.

2. PRELIMINARIES

Let H be a Hilbert space endowed with a norm $\|\cdot\|$ and a inner product $\langle\cdot,\cdot\rangle$. Let $T, A: H \to 2^H$ be set-valued mappings, $g: H \to H$ be a single-valued mapping, and $\phi: H \times H \to \mathbf{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y): H \to \mathbf{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $g(H) \cap \operatorname{dom} \partial\phi(\cdot, y) \neq \emptyset$ for each $y \in H$. Then the problem of finding $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that $g(x) \in \operatorname{dom} \partial\phi(\cdot, x)$ and

$$\langle u - v, y - g(x)\rangle \geq \phi(g(x), x) - \phi(y, x), \qquad \forall y \in H, \quad (2.1)$$

is called the generalized quasivariational inclusion problem (GQVIP (T, A, g, ϕ)).

Special Cases. (1) If $\phi(x, y) = \phi(x)$ for all $y \in H$ and T and A are both single-valued mappings, then the problem (2.1) reduces to the variational inclusion problem (1.1) considered by Hassouni and Moudafi [19],

(2) If K is a given closed convex subset of H and $\phi = I_K$ is the indicator function of K,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then the problem (2.1) reduces to the generalized strongly nonlinear variational inequality problems, i.e., find $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that $g(x) \in K$ and

$$\langle u - v, y - g(x) \rangle \geq 0, \qquad \forall y \in K. \tag{2.2}$$

(3) If $K: H \to 2^H$ is a set-valued mapping such that each $K(x)$ is a closed convex subset of H (or $K(x) = m(x) + K$ where $m: H \to H$ and K is a closed convex subset of H) and for each fixed $y \in H$, $\phi(\cdot, y) = I_{K(y)}(\cdot)$ is the indicator function of $K(y)$,

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{otherwise}, \end{cases}$$

then the problem (2.1) reduces to the generalized strongly nonlinear quasivariational inequality problems, i.e., find $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that $g(x) \in K(x)$ and

$$\langle u - v, y - g(x) \rangle \geq 0, \qquad \forall y \in K(x). \tag{2.3}$$

For the recent extension and generalization of the problems (2.2) and (2.3), see Noor [29].

In brief, the problem (2.1) is the most general and unifying form of various extended classes of variational inequalities, variational inclusions, and complementarity problems. For the iterative methods, application, and formulation, see [1–40].

In order to prove our main theorems, we need the following concepts and results; see Pascali and Sburlan [34].

DEFINITION 2.1. Let X be a Banach space with the dual space X^* and let $\phi: X \to \mathbf{R} \cup \{+\infty\}$ be a proper functional. ϕ is said to be subdifferential at a point $x \in X$ if there exists an $f^* \in X^*$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, y - x \rangle, \qquad \forall y \in X,$$

where f^* is called a subgradient of ϕ at x. The set of all subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi: X \to 2^{X^*}$ defined by

$$\partial\phi(x) = \{f^* \in X^*: \phi(y) - \phi(x) \geq \langle f^*, y - x \rangle, \forall y \in X\}$$

is said to be the subdifferential of ϕ.

DEFINITION 2.2. Let H be a Hilbert space and let $G: H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G: H \to H$ defined by

$$J_\rho^G(x) = (I + \rho G)^{-1}(x), \qquad \forall x \in H$$

is said to be the resolvent operator of G where I is the identity mapping on H.

LEMMA 2.1. *Let X be a reflexive Banach space endowed with a strictly convex norm and $\phi: X \to \mathbf{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then $\partial\phi: X \to 2^{X^*}$ is a maximal monotone mapping.*

LEMMA 2.2. *Let $G: H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^G: H \to H$ of G is nonexpansive, i.e., for all $x, y \in H$,*

$$\left\| J_\rho^G(x) - J_\rho^G(y) \right\| \leq \|x - y\|.$$

DEFINITION 2.3. A mapping $g: H \to H$ is said to be

(i) γ-strongly monotone if there exists a constant $\gamma > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \gamma \|x - y\|^2, \qquad \forall x, y \in H;$$

(ii) σ-Lipschitz continuous if there exists a constant $\sigma \geq 0$ such that

$$\|g(x) - g(y)\| \leq \sigma \|x - y\|, \qquad \forall x, y \in H.$$

DEFINITION 2.4. A set-valued mapping $T: H \to 2^H$ is said to be

(i) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle u - v, x - y \rangle \geq \alpha \|x - y\|^2, \qquad \forall x, y \in H, u \in T(x), \text{ and } v \in T(y);$$

(ii) β-Lipschitz continuous if there exists a constant $\beta \geq 0$ such that

$$\delta(T(x), T(y)) \leq \beta \|x - y\|, \qquad \forall x, y \in H,$$

where $\delta(A, B) = \sup\{\|a - b\|: a \in A,\ b \in B\}$, $\forall A, B \in 2^H$.

3. MAIN RESULTS

In this section, we shall prove an existence theorem of solutions for the GQVIP (T, A, g, ϕ) (2.1) and suggest a new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions of the problem (2.1). Then we show that the sequence of approximate solutions strongly converges to the exact solution of the problem (2.1).

We first transfer the problem (2.1) in a fixed point problem.

THEOREM 3.1. *(x^*, u^*, v^*) is a solution of the problem* (2.1) *if and only if (x^*, u^*, v^*) satisfies the relation*

$$g(x) = J_\rho^{\partial\phi(\cdot, x)}(g(x) - \rho(u - v)), \qquad \forall x \in H, \tag{3.1}$$

where $\rho > 0$ is a constant, $J_\rho^{\partial\phi(\cdot, x)} = (I + \rho\partial\phi(\cdot, x))^{-1}$ is the resolvent operator of $\partial\phi(\cdot, x)$, and I is the identity mapping on H.

Proof. Let (x^*, u^*, v^*) satisfy the relation (3.1), that is,

$$g(x^*) = J_\rho^{\partial\phi(\cdot, x^*)}(g(x^*) - \rho(u^* - v^*)).$$

The equality holds if and only if

$$v^* - u^* \in \partial\phi(\cdot, x^*)(g(x^*)),$$

by the definition of $J_\rho^{\partial\phi(\cdot, x^*)}$. The relation holds if and only if

$$\phi(y, x^*) - \phi(g(x^*), x^*) \geq \langle v^* - u^*, y - g(x^*)\rangle, \qquad \forall y \in H,$$

by the definition of the subdifferential $\partial\phi(\cdot, x^*)$. Hence (x^*, u^*, v^*) is the solution of

$$\langle u^* - v^*, y - g(x^*)\rangle \geq \phi(g(x^*), x^*) - \phi(y, x^*), \qquad \forall y \in H. \quad \blacksquare$$

Remark 3.1. From Theorem 3.1, we see that the quasivariational inclusion (2.1) is equivalent to the fixed point problem (3.1). Equation (3.1) can be written as

$$x = x - g(x) + J_\rho^{\partial\phi(\cdot, x)}[g(x) - \rho(u - v)]. \tag{3.2}$$

This fixed point formulation enables us to suggest the following algorithms.

ALGORITHM 3.1. For any given $x_0 \in H, \overline{u_0} \in T(x_0)$, and $\overline{v_0} \in A(x_0)$, let

$$y_0 = (1 - \beta_0)x_0 + \beta_0\Big[x_0 - g(x_0) + J_\rho^{\partial\phi(\cdot, x_0)}\big(g(x_0) - \rho(\overline{u_0} - \overline{v_0})\big)\Big].$$

Take any fixed $u_0 \in T(y_0)$ and $v_0 \in A(y_0)$, and let

$$x_1 = (1 - \alpha_0)x_0 + \alpha_0\left[y_0 - g(y_0) + J_\rho^{\partial\phi(\cdot, y_0)}\left(g(y_0) - \rho(u_0 - v_0)\right)\right].$$

Continuing this way, we can define sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty$ as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n\left[y_n - g(y_n) + J_\rho^{\partial\phi(\cdot, y_n)}\left(g(y_n) - \rho(u_n - v_n)\right)\right], \\ y_n &= (1 - \beta_n)x_n + \beta_n\left[x_n - g(x_n) + J_\rho^{\partial\phi(\cdot, x_n)}\left(g(x_n) - \rho\left(\overline{u_n} - \overline{v_n}\right)\right)\right], \end{aligned} \tag{3.3}$$

for $n = 0, 1, \ldots,$ where $u_n \in T(y_n)$, $v_n \in A(y_n)$, $\overline{u_n} \in T(x_n)$, and $\overline{v_n} \in A(x_n)$ can be chosen arbitrarily, $0 \leq \alpha_n, \beta_n \leq 1$, $\sum_{n=0}^\infty \alpha_n$ diverges, and $\rho > 0$ is a constant.

Using fixed point formulation (3.2), we have the following Algorithm.

ALGORITHM 3.2. For any given $x_0 \in H$, compute the sequences $\{x_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty$ by the iterative schemes

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\phi(\cdot, x_n)}\left[g(x_0) - \rho(u_n - v_n)\right], \tag{3.4}$$

for $n = 0, 1, \ldots,$ where $u_n \in T(x_n)$ and $v_n \in A(x_n)$ can be chosen arbitrarily and $\rho > 0$ is a constant.

To perturb the Algorithm 3.2, we first add, in the right-hand side of (3.4), an error e_n to take into account a possible inexact computation of the proximal point and we consider another perturbation by replacing ϕ in (3.4) by ϕ_n, where each $\phi_n: H \times H \to \mathbf{R} \cup \{+\infty\}$ is such that for each fixed $y \in H$, $\phi_n(\cdot, y)$ is a proper convex lower semicontinuous function on H and the sequence $\{\phi_n\}$ approximates ϕ on $H \times H$. Then we obtain the following perturbed proximal point algorithm.

ALGORITHM 3.3. For any given $x_0 \in H$, compute the sequences $\{x_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty$ by the iterative schemes

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\phi_n(\cdot, x_n)}\left(g(x_n) - \rho(u_n - v_n)\right) + e_n, \tag{3.5}$$

where $\{e_n\}_{n=0}^\infty$ is an error sequence in H, $u_n \in T(x_n)$ and $v_n \in A(x_n)$ can be chosen arbitrarily, and $\rho > 0$ is a constant.

Now we show the existence of solutions of the GQVIP (T, A, g, ϕ) (2.1).

THEOREM 3.2. *Let $T: H \to 2^H$ be α-strongly monotone and β-Lipschitz continuous, $A: H \to 2^H$ be γ-Lipschitz continuous, $g: H \to H$ be λ-strongly* *monotone and σ-Lipschitz continuous, and $\phi: H \times H \to \mathbf{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper convex lower semicontinuous function on H, $g(H) \cap \operatorname{dom} \partial\phi(\cdot, y) \neq \emptyset$ and for each $x, y, z \in H$,*

$$\left\| J_\rho^{\partial\phi(\cdot, x)}(z) - J_\rho^{\partial\phi(\cdot, y)}(z) \right\| \leq \mu \|x - y\|.$$

Suppose there exists a constant $\rho > 0$ such that

$$k = \mu + 2\sqrt{1 - 2\lambda + \sigma^2} < 1,$$

$$\alpha > \gamma(1 - k) + \sqrt{(\beta^2 - \gamma^2)k(2 - k)},$$

$$\left| \rho - \frac{\alpha + \gamma(k-1)}{\beta^2 - \gamma^2} \right| < \frac{\sqrt{(\alpha + \gamma(k-1))^2 - (\beta^2 - \gamma^2)k(2-k)}}{\beta^2 - \gamma^2}. \tag{3.6}$$

Then the GQVIP (T, A, g, ϕ) (2.1) has a solution (x^*, u^*, v^*).

Proof. By Theorem 3.1, it is suffice to prove that there exist $x^* \in H$, $u^* \in T(x^*)$, and $v^* \in A(x^*)$ such that (3.1) holds. Define a set-valued mapping $F: H \to 2^H$ by

$$F(x) = \bigcup_{u \in T(x)} \bigcup_{v \in A(x)} \left[x - g(x) + J_\rho^{\partial\phi(\cdot, x)}(g(x) - \rho(u - v)) \right], \quad \forall x \in H.$$

For arbitrary $x, y \in H$, $a \in F(x)$, and $b \in F(y)$, there exist $u_1 \in T(x)$, $v_1 \in A(x)$, $u_2 \in T(y)$, and $v_2 \in A(y)$ such that

$$a = x - g(x) + J_\rho^{\partial\phi(\cdot, x)}(g(x) - \rho(u_1 - v_1)),$$

$$b = y - g(y) + J_\rho^{\partial\phi(\cdot, y)}(g(y) - \rho(u_2 - v_2)).$$

By the assumption of ϕ and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\|a - b\| &\\
&\leq \|x - y - (g(x) - g(y))\| \\
&\quad + \left\| J_\rho^{\partial\phi(\cdot, x)}(g(x) - \rho(u_1 - v_1)) - J_\rho^{\partial\phi(\cdot, x)}(g(y) - \rho(u_2 - v_2)) \right\| \\
&\quad + \left\| J_\rho^{\partial\phi(\cdot, x)}(g(y) - \rho(u_2 - v_2)) - J_\rho^{\partial\phi(\cdot, y)}(g(y) - \rho(u_2 - v_2)) \right\| \\
&\leq 2\|x - y - (g(x) - g(y))\| + \|x - y - \rho(u_1 - u_2)\| \\
&\quad + \rho\|v_1 - v_2\| + \mu\|x - y\|.
\end{aligned}$$

Since T and g are both strongly monotone and Lipschitz continuous and A is Lipschitz continuous, by using the technique of Noor [25], we have

$$\|x - y - (g(x) - g(y))\| \leq \sqrt{1 - 2\lambda + \sigma^2}\, \|x - y\|,$$

$$\|x - y - \rho(u_1 - u_2)\| \leq \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\, \|x - y\|,$$

$$\|v_1 - v_2\| \leq \delta(A(x), A(y)) \leq \gamma\, \|x - y\|.$$

It follows that

$$\begin{aligned} \delta(F(x), F(y)) &\leq \Big[2\sqrt{1 - 2\lambda + \sigma^2} + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2} \\ &\quad + \rho\gamma + \mu\Big]\|x - y\| \\ &= [k + t(\rho) + \rho\gamma]\|x - y\| = \theta\, \|x - y\|, \end{aligned} \tag{3.7}$$

where $k = 2\sqrt{1 - 2\lambda + \sigma^2} + \mu, t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}$, and $\theta = k + t(\rho) + \rho\gamma$. By the condition (3.6), we have $\theta < 1$. It follows from the condition 3.7 and Theorem 3.1 of Siddiqi and Ansari [37] that F has a fixed $x^* \in H$. By the definition of F, there exist $u^* \in T(x^*)$ and $v^* \in A(x^*)$ such that

$$g(x^*) = J_\rho^{\partial\phi(\cdot, x^*)}(g(x^*) - \rho(u^* - v^*)).$$

Therefore (x^*, u^*, v^*) is a solution of GQVIP (T, A, g, ϕ) (2.1). ∎

In the following, we shall show the convergence of the Algorithms 3.1 and 3.3.

THEOREM 3.3. *Let H, T, A, g, and ϕ satisfy all conditions in Theorem* 3.2. *If the condition* (3.6) *is also satisfied, then the iterative sequences* $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, *and* $\{v_n\}_{n=0}^{\infty}$ *defined in the Algorithm* 3.1 *strongly converge to* x^*, u^*, *and* v^*, *respectively, and* (x^*, u^*, v^*) *is a solution of the GQVIP* (T, A, g, ϕ) (2.1).

Proof. By Theorem 3.2, the GQVIP (T, A, g, ϕ) (2.1) has a solution (x^*, u^*, v^*). From Theorem 3.1 we have $x^* \in H$, $u^* \in T(x^*)$, $v^* \in A(x^*)$, and for all $n \geq 0$,

$$\begin{aligned} x^* &= x^* - g(x^*) + J_\rho^{\partial\phi(\cdot, x^*)}(g(x^*) - \rho(u^* - v^*)) \\ &= (1 - \alpha_n)x^* + \alpha_n\Big[x^* - g(x^*) \\ &\qquad + J_\rho^{\partial\phi(\cdot, x^*)}(g(x^*) - \rho(u^* - v^*))\Big] \\ &= (1 - \beta_n)x^* + \beta_n\Big[x^* - g(x^*) \\ &\qquad + J_\rho^{\partial\phi(\cdot, x^*)}(g(x^*) - \rho(u^* - v^*))\Big]. \end{aligned}$$

By the algorithm 3.1, using a similar argument as in the proof of Theorem 3.2, we obtain

$$\|x_n - x^* - (g(x_n) - g(x^*))\| \le \sqrt{1 - 2\lambda + \sigma^2}\,\|x_n - x^*\|,$$

$$\|x_n - x^* - \rho(\overline{u_n} - u^*)\| \le \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\,\|x_n - x^*\|,$$

$$\|y_n - x^* - (g(y_n) - g(x^*))\| \le \sqrt{1 - 2\lambda + \sigma^2}\,\|y_n - x^*\|,$$

$$\|y_n - x^* - \rho(u_n - u^*)\| \le \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\,\|y_n - x^*\|.$$

Thus, by the Algorithm 3.1, the assumption of ϕ, and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\|y_n - x^*\| \\
&\quad = \Big\|(1-\beta_n)x_n + \beta_n\big[x_n - g(x_n) + J_\rho^{\partial\phi(\cdot,x_n)}\big(g(x_n) - \rho(\overline{u_n} - \overline{v_n})\big)\big] \\
&\qquad -(1-\beta_n)x^* - \beta_n\big[x^* - g(x^*) \\
&\qquad\qquad + J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho(u^* - v^*))\big]\Big\| \\
&\quad \le (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\qquad + \beta_n\Big\|J_\rho^{\partial\phi(\cdot,x_n)}\big(g(x_n) - \rho(\overline{u_n} - \overline{v_n})\big) \\
&\qquad\qquad - J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x_n) - \rho(\overline{u_n} - \overline{v_n})\big)\Big\| \\
&\qquad + \beta_n\Big\|J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x_n) - \rho(\overline{u_n} - \overline{v_n})\big) \\
&\qquad\qquad - J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho(u^* - v^*))\Big\| \\
&\quad \le (1-\beta_n)\|x_n - x^*\| + 2\beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\qquad + \beta_n\|x_n - x^* - \rho(\overline{u_n} - u^*)\| + \beta_n\rho\|\overline{v_n} - v^*\| \\
&\qquad + \beta_n\mu\|x_n - x^*\| \\
&\quad \le (1-\beta_n)\|x_n - x^*\| + \beta_n k\|x_n - x^*\| \\
&\qquad + \beta_n t(\rho)\|x_n - x^*\| + \beta_n\rho\gamma\|x_n - x^*\| \\
&\quad \le (1-\beta_n)\|x_n - x^*\| + \beta_n\theta\|x_n - x^*\| \\
&\quad \le \|x_n - x^*\|.
\end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| = \Big\|&(1-\alpha_n)x_n \\
&+ \alpha_n\big[y_n - g(y_n) + J^{\partial\phi(\cdot,y_n)} - \rho\big(g(y_n) - \rho(u_n - v_n)\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&-(1-\alpha_n)x^* - \alpha_n\Big[x^* - g(x^*) \\
&\qquad\qquad + J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho(u^* - v^*)\big)\Big]\Big\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\|. \qquad (3.9)
\end{aligned}
$$

It follows from (3.8) and (3.9) that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|x_n - x^*\| \\
&= \big[1 - (1-\theta)\alpha_n\big]\|x_n - x^*\| \\
&\le \prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\|x_0 - x^*\|.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{\infty}[1-(1-\theta)\alpha_i]=0$. Hence the sequence $\{x_n\}$ strongly converges to x^*. By (3.8), the sequence $\{y_n\}$ also strongly converges to x^*. Since $u_n \in T(x_n)$, $u^* \in T(x^*)$, and T is β-Lipschitz continuous, we have

$$\|u_n - u^*\| \le \beta\|y_n - x^*\| \to 0,$$

and hence the sequence $\{u_n\}$ strongly converges to u^*. Similarly, we can show that the sequence $\{v_n\}$ strongly converges to v^*. This completes the proof. ▮

THEOREM 3.4. *Let $T: H \to 2^H$ be α-strongly monotone and β-Lipschitz continuous, $A: H \to 2^H$ be γ-Lipschitz continuous, $g: H \to H$ be λ-strongly monotone and σ-Lipschitz continuous, and $\phi, \phi_n: H \times H \to \mathbf{R} \cup \{+\infty\}$, $n = 1, 2, \dots,$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ and each $\phi_n(\cdot, y)$ are both proper convex lower semicontinuous functions on H, $g(H) \cap$* dom $\partial\phi(\cdot, y) \neq \emptyset$, *and for each $x, y, z \in H$ and for all $n \ge 1$,*

$$\Big\|J_\rho^{\partial\phi_n(\cdot,x)}(z) - J_\rho^{\partial\phi_n(\cdot,y)}(z)\Big\| \le \mu\|x - y\|.$$

Assume $\lim_{m\to\infty}\|J_\rho^{\partial\phi_n(\cdot,y)}(z) - J_\rho^{\partial\phi(\cdot,y)}(z)\| = 0$ *for all* $y, z \in H$, $\lim_{n\to\infty}\|e_n\| = 0$, *and there exists a constant $\rho > 0$ such that the condition* (3.6) *in Theorem* 3.2 *holds. Then the iterative $\{x_n\}$, $\{u_n\}$, and $\{v_n\}$ defined in the Algorithm* 3.3 *strongly converges to x^*, u^*, and v^*, respectively, and (x^*, u^*, v^*) is a solution of the GQVIP (T, A, g, ϕ)* (2.1).

Proof. By Theorem 3.2, the GQVIP (T, A, g, ϕ) (2.1) has a solution (x^*, u^*, v^*) such that $u^* \in T(x^*)$, $v^* \in A(x^*)$, and

$$x^* = x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho(u^* - v^*)\big).$$

By setting $h(x^*) = g(x^*) - \rho(u^* - v^*)$ and by using the Algorithm 3.3 and the assumptions of ϕ and ϕ_n, $n = 1, 2, \dots,$ we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \Big\| x_n - g(x_n) + J_\rho^{\partial\phi_n(\cdot,\, x_n)}(g(x_n) - \rho(u_n - v_n)) + e_n \\
&\qquad - x^* + g(x^*) - J_\rho^{\partial\phi(\cdot,\, x^*)}(g(x^*) - \rho(u^* - v^*)) \Big\| \\
&\leq \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \Big\| J_\rho^{\partial\phi_n(\cdot,\, x_n)}(g(x_n) - \rho(u_n - v_n)) \\
&\qquad - J_\rho^{\partial\phi_n(\cdot,\, x_n)}(g(x^*) - \rho(u^* - v^*)) \Big\| \\
&\quad + \Big\| J_\rho^{\partial\phi_n(\cdot,\, x_n)}(g(x^*) - \rho(u^* - v^*)) \\
&\qquad - J_\rho^{\partial\phi_n(\cdot,\, x^*)}(g(x^*) - \rho(u^* - v^*)) \Big\| \\
&\quad + \Big\| J_\rho^{\partial\phi_n(\cdot,\, x^*)}(g(x^*) - (u^* - v^*)) \\
&\qquad - J_\rho^{\partial\phi(\cdot,\, x^*)}(g(x^*) - \rho(u^* - v^*)) \Big\| + \|e_n\| \\
&\leq 2 \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \| x_n - x^* - \rho(u_n - u^*) \| + \rho \|v_n - v^*\| \\
&\quad + \mu \|x_n - x^*\| \\
&\quad + \Big\| J^{\partial\phi_n(\cdot,\, x^*)} - \rho(h(x^*)) - J_\rho^{\partial\phi(\cdot,\, x^*)}(h(x^*)) \Big\| + \|e_n\| \\
&\leq (k + t(\rho) - \rho\gamma) \|x_n - x^*\| + \Big\| J_\rho^{\partial\phi_n(\cdot,\, x^*)}(h(x^*)) \\
&\qquad - J_\rho^{\partial\phi(\cdot,\, x^*)}(h(x^*)) \Big\| + \|e_n\| \\
&\leq \theta \|x_n - x^*\| + \varepsilon_n,
\end{aligned} \tag{3.10}$$

where $k = \mu + 2\sqrt{1 - 2\lambda + \sigma^2}$, $t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}$, $\theta = k + t(\rho) + \rho\gamma$, and $\varepsilon_n = \|J_\rho^{\partial\phi_n(\cdot,\, x^*)}(h(x^*)) - J_\rho^{\partial\phi(\cdot,\, x^*)}(h(x^*))\| + \|e_n\|$. By the condition (3.6) in Theorem 3.2, we have $\theta < 1$. It follows from (3.10) that

$$\|x_{n+1} - x^*\| \leq \theta^{n+1} \|x_0 - x^*\| + \sum_{i=1}^{n} \theta^i \varepsilon_{n+1-i}.$$

Since $\varepsilon_n \to 0$ by the assumption, it follows from Orgeta and Rheinboldt [31, p. 338] that

$$\lim_{n \to \infty} \|x_{n+1} - x^*\| = 0,$$

and hence the sequence $\{x_n\}$ strongly converges to x^*. Since $u_n \in T(x_n)$, $v_n \in A(x_n)$, $u^* \in T(x^*)$, $v^* \in A(x^*)$, we have

$$\|u_n - u^*\| \le \delta\big(T(u_n), T(u^*)\big) \le \beta \|x_n - x^*\|,$$
$$\|v_n - v^*\| \le \delta\big(A(v_n), A(v^*)\big) \le \gamma \|x_n - x^*\|.$$

It follows that the sequences $\{u_n\}$ and $\{v_n\}$ also strongly converge to u^* and v^*, respectively. This completes the proof. ∎

Remark 3.2. If $\phi(x, y) = \phi(x)$ for all $y \in H$ and T and A are both single-valued mappings, then Theorem 3.4 reduces to Theorem 2.1 of Hassouni and Moudafi [19]. The Remark in [19] is still applicable for our Theorem 3.4.

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