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Calabi–Yau Frobenius algebras

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ABSTRACT

We define Calabi–Yau Frobenius algebras over arbitrary base commutative rings. We define a Hochschild analogue of Tate cohomology, and show that this "stable Hochschild cohomology" of periodic CY Frobenius algebras has a Batalin–Vilkovisky and Frobenius algebra structure. Such algebras include (centrally extended) preprojective algebras of (generalized) Dynkin quivers, and group algebras of classical periodic groups. We use this theory to compute (for the first time) the Hochschild cohomology of many algebras related to quivers, and to simplify the description of known results. Furthermore, we compute the maps on cohomology from extended Dynkin preprojective algebras to the Dynkin ones, which relates our CY property (for Frobenius algebras) to that of Ginzburg (for algebras of finite Hochschild dimension).

Contents

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1. Introduction

Frobenius algebras have wide-ranging applications to geometry, e.g., to TQFTs in [Moo01,Laz01, Cos07], and are closely related to the string topology Batalin–Vilkovisky (BV) algebra [CS99,CS04], as described in e.g. [Cos07,HL04,CV06,TZ06].

One motivation for this work is to explain the following algebraic phenomenon: the Hochschild cohomology of many interesting Frobenius algebras has a BV structure. For example, this is true for symmetric Frobenius algebras, and for preprojective algebras of Dynkin quivers (which are *not* symmetric).

To explain this, we define *Calabi–Yau* (*CY*) *Frobenius algebras*, whose Hochschild cohomology has not only the usual Gerstenhaber algebra structure, but a BV structure (at least when the algebra is also periodic, as we will explain). The CY Frobenius property is similar to the CY property of [Gin06], but not the same: any Frobenius algebra over $\mathbb{C}$ which is CY in the sense of [Gin06] must be a direct sum of matrix algebras.

The prototypical example of a CY Frobenius algebra A *of dimension* m ($CY(m)$) over a field $\mathbf{k}$ is one that has a resolution (for $A^e := A \otimes_{\mathbf{k}} A^{\mathrm{op}}$)

$$A^\vee \hookrightarrow P_m \to P_{m-1} \to \cdots \to P_0 \twoheadrightarrow A, \quad A^\vee := \mathrm{Hom}_{A^e}\big(A, A^e\big), \tag{1.0.1}$$

with each P_i a projective A-bimodule (the sequence must be exact). Our definition is a weakened version of this: $A^\vee \simeq \Omega^{m+1} A$ in the stable A-bimodule category (this notion is recalled in Appendix B).

Any symmetric Frobenius algebra has $A \cong A^\vee$, and so we say it is CY Frobenius of dimension -1. Our methods give a very short, simple proof of the fact that the Hochschild cohomology of such algebras is BV (Section 2.4), for which many (generally more complicated) proofs are given in e.g. [Cos07, Tra02,Men04,TZ06,Kau07b,Kau07a].

In [ES98b,ES98a,EE07], it was also noticed that the Hochschild cohomology of preprojective algebras of Dynkin quivers has a self-duality property: one has $HH^i(A) \cong HH^{5+6j-i}(A)^*$ for such algebras, with $i, 5+6j-i \geqslant 1$, and moreover one has the periodicity $HH^i(A) \cong HH^{i+6j}$ when $i, 6j+i \geqslant 1$ (here, j is an integer).

We explain this by introducing *stable Hochschild cohomology* $\underline{\mathrm{HH}}^\bullet(A)$, of Frobenius algebras A. This is a Hochschild analogue of Tate cohomology, which coincides with the usual Hochschild cohomology in positive degrees, and is defined using the stable module category by $\underline{\mathrm{HH}}^\bullet(A) = \underline{\mathrm{Hom}}(\Omega^\bullet A, A)$ (in the latter form, this was studied in various papers, e.g., [ES06]). We show that this is a $\mathbb{Z}$-graded ring, and prove it is graded-commutative. To our knowledge, this is the first time $\underline{\mathrm{Hom}}(\Omega^\bullet A, A)$ has been studied in this way. We also define the notion *stable Hochschild homology* $\underline{\mathrm{HH}}_\bullet(A)$, which coincides with the usual notion in positive degrees, and prove that $\underline{\mathrm{HH}}_\bullet(A)$ is a graded module over $\underline{\mathrm{HH}}^\bullet(A)$ using a natural *contraction* action, which generalizes contraction in the $\mathbb{Z}_{\geqslant 0}$-graded case. These results are in Theorem 2.1.15.

We then show that, in general, $\underline{\mathrm{HH}}_\bullet(A) \cong \underline{\mathrm{HH}}_{-1-\bullet}(A)^*$ (which makes the contraction operations *graded self-adjoint*), and that the $CY(m)$-Frobenius property produces dualities $\underline{\mathrm{HH}}_\bullet(A) \cong \underline{\mathrm{HH}}^{m-\bullet}(A)$. Put together, we obtain a *graded Frobenius algebra structure* on $\underline{\mathrm{HH}}^\bullet(A)$ (Theorem 2.3.27), explaining the aforementioned results of [ES98b,ES98a,EE07]. In particular, $\underline{\mathrm{HH}}^\bullet(A) \cong \underline{\mathrm{HH}}^{2m+1-\bullet}(A)^*$, as modules over $\underline{\mathrm{HH}}^0(A)$; in the preprojective algebra cases, $m = 2$, which explains the aforementioned duality.

Call a (Frobenius) algebra *periodic* if it has a periodic A-bimodule resolution (or, more generally, $A \simeq \Omega^n A$ in the stable bimodule category, for some n). We prove that *the stable Hochschild cohomology of periodic CY Frobenius algebras has a BV structure* (Theorem 2.3.64). More generally, for any periodic (not necessarily CY) Frobenius algebra, the structure of calculus [TT05,GDT89] on the pair $(\mathrm{HH}^\bullet, \mathrm{HH}_\bullet)$ extends to the $\mathbb{Z}$-graded setting $(\underline{\mathrm{HH}}^\bullet, \underline{\mathrm{HH}}_\bullet)$ (Theorem 2.3.47). Moreover, we show that, in the (centrally extended) preprojective cases, the BV structure and Frobenius algebra structure are compatible: the BV differential is graded self-adjoint—we call such an algebra a *BV Frobenius algebra*.

Additionally, all of the above work is done in the context not only of Frobenius algebras over a field, but over an arbitrary base commutative ring. Precisely, we use the notion of *Frobenius extensions of the first kind* [NT60,Kas61], which says that the algebra is projective over a base commutative ring $\mathbf{k}$ and that the duality is nondegenerate over this base. (Perhaps this could be generalized further to an A_∞-Frobenius property, but we do not do this here.)

New computational results (extending [ES98b,ES98a,EE07,Eu08,Eu07a]) concerning preprojective algebras include:

- The computation of Hochschild (co)homology of preprojective algebras of Dynkin type over $\mathbf{k} = \mathbb{Z}$ (in types D, E, the results [EE07,Eu08,Eu07a] are over characteristic-zero fields), and the proof that $\underline{\mathrm{HH}}^\bullet$ is *BV Frobenius* over any field (Theorem 3.2.7);
- The computation of cup product and BV structure on Hochschild (co)homology of *centrally extended preprojective algebras* [ER06] (the groups were computed over $\mathbb{C}$ in [Eu06]), which we show is *BV Frobenius* (Theorem 4.0.12);
- The computation of the induced maps from the Hochschild (co)homology in the **extended** Dynkin case to the Dynkin case by cutting off the extending vertex (Theorem 3.3.4). This explains the structure in the Dynkin case and elucidates the relationship between the usual Calabi–Yau and Calabi–Yau Frobenius properties (which is analogous to Euclidean vs. spherical geometry).

We also explain and simplify the cited known results.

Our other main example is the case of group algebras of finite groups, which are automatically Calabi–Yau Frobenius (since they are symmetric). We are interested in when these are periodic (and hence $\underline{\mathrm{HH}}^\bullet$ is BV and Frobenius, by Theorem 2.3.64). We show (Theorem 5.0.5) that the periodic (CY) Frobenius algebras are just the classical periodic algebras, i.e., those whose group cohomology is periodic (using classical results). In Appendices A and B, we also give an elementary topological proof that groups that act freely and simplicially on a sphere (such as finite subgroups of $SO(n)$) are periodic (CY) Frobenius.

1.1. Definitions and notation

Here we recall some standard definitions and state the notation we will use throughout.

All complexes will be assumed to have decreasing degree (i.e., they are chain complexes), unless otherwise specified.

Let us fix, once and for all, a commutative ring $\mathbf{k}$. When we say "algebra over $\mathbf{k}$," we mean an algebra A over $\mathbf{k} \to A$ such that the image of $\mathbf{k}$ is central in A. Bimodules over an algebra A over $\mathbf{k}$ will be assumed to be symmetric as $\mathbf{k}$-bimodules (i.e., A-bimodules mean $A^e := A \otimes_{\mathbf{k}} A^{\mathrm{op}}$-modules).

Notation 1.1.1. The category A-mod means *finitely-generated* A-modules, for any ring A. The category A-$\mathrm{mod}_{\mathbf{k}}$ means finitely-generated A-modules *which are finitely-generated projective as* $\mathbf{k}$*-modules*.

Notation 1.1.2. We will abbreviate "finitely-generated" as "fg."

By a "Frobenius algebra over a commutative ring $\mathbf{k}$," we will mean what is also known as a "Frobenius extension of the first kind" in the literature [NT60,Kas61], namely:

Definition 1.1.3. A Frobenius algebra A over a commutative ring $\mathbf{k}$ is a $\mathbf{k}$-algebra which is a fg projective $\mathbf{k}$-module, and which is equipped with a nondegenerate invariant inner product $(\,,\,)$, i.e.:

$$(ab, c) = (a, bc), \quad \forall a, b, c \in A;$$
$$(-, a) : A \to \mathrm{Hom}_{\mathbf{k}}(A, \mathbf{k}) \quad \text{is an isomorphism of } \mathbf{k}\text{-modules.} \tag{1.1.4}$$

Example 1.1.5. Any group algebra $\mathbf{k}[G]$ is Frobenius if G is finite, using the pairing $(g,h)=\delta_{g,h^{-1}}$ for $g,h\in G$. In fact, this algebra is symmetric (meaning $A\cong A^*$ as A-bimodules).

Example 1.1.6. The preprojective algebra Π_Q is known to be Frobenius if Q is Dynkin (cf. e.g., [ES98a, ER06]). It is not difficult to see (e.g., through explicit bases as in [Eu08]) that these are in fact Frobenius over $\mathbb{Z}$.

Example 1.1.7. Centrally extended preprojective algebras of Dynkin quivers were defined and proved to be Frobenius in [ER06] (working over $\mathbb{C}$).

Example 1.1.8. For any two Frobenius algebras A,B over $\mathbf{k}$, the algebra $A\otimes_{\mathbf{k}}B$ is also Frobenius. In particular, so is A^e.

We will mainly be interested in the cases where $\mathbf{k}$ is a field or $\mathbf{k}=\mathbb{Z}$, and A is a fg free $\mathbf{k}$-module. (These are generally known as "free Frobenius extensions of the first kind," cf. [NT60].)

We refer to Appendix B for some general results about Frobenius algebras and the stable module category over arbitrary commutative rings, which are direct generalizations of standard results in the case where $\mathbf{k}$ is a field. In particular, the results there justify the following definitions:

Definition 1.1.9. Let the projectively stable module category $\underline{A\text{-mod}}$ be the category whose objects are fg A-modules, and whose morphisms $\underline{\mathrm{Hom}}_A$ are given by

$$\underline{\mathrm{Hom}}_A(M,N):=\{f\in\mathrm{Hom}_A(M,N)\}\\ /\{\text{morphisms that factor through a projective } A\text{-module}\}. \tag{1.1.10}$$

Let $\underline{A\text{-mod}}_{\mathbf{k}}\subset\underline{A\text{-mod}}$ be the full subcategory of modules which are projective as $\mathbf{k}$-modules.

Definition 1.1.11. For any algebra A over $\mathbf{k}$, define the functor ${}^\vee: A\text{-mod}\to A^{\mathrm{op}}\text{-mod}$ by

$$M^\vee:=\mathrm{Hom}_A(M,A), \tag{1.1.12}$$

with the natural induced maps on morphisms.

Definition 1.1.13. For any Frobenius algebra A, let $\eta: A\xrightarrow{\sim}A$ be the **Nakayama automorphism** defined by

$$(a,b)=(\eta^{-1}(b),a),\quad \forall a,b\in A. \tag{1.1.14}$$

Definition 1.1.15. For any ($\mathbf{k}$-linear) automorphism $\phi: A\xrightarrow{\sim}A$, and any A-module M, let ${}_\phi M$ denote M with the twisted action given by precomposing $A\to\mathrm{End}_{\mathbf{k}}(M)$ by ϕ. Similarly, for any bimodule N, ${}_\phi N_\psi$ denotes twisting the left action by ϕ and the right action by ψ. If either of ϕ,ψ is the identity, we may omit it from the notation.

We have $A_\eta\cong A^*$ as A-bimodules (conversely, such an isomorphism is equivalent to $(-,-)$ with automorphism η).

Definition 1.1.16. Let A be a Frobenius algebra over $\mathbf{k}$ and M,N fg A-modules which are projective as $\mathbf{k}$-modules. For any integer $i\in\mathbb{Z}$, let us denote

$$\underline{\mathrm{Ext}}^i_A(M,N):=\underline{\mathrm{Hom}}_A(\Omega^iM,N). \tag{1.1.17}$$

Remark 1.1.18. When G is a finite group, and $\mathbf{k}$ is a field (of any characteristic), the cohomology groups $\underline{\mathrm{Ext}}^i_{\mathbf{k}[G]}(\mathbf{k}, M)$ are the Tate cohomology groups over $\mathbf{k}$ with coefficients in M. Indeed, we may compute $\underline{\mathrm{Ext}}^i_{\mathbf{k}[G]}(\mathbf{k}, M)$ by the complex $\mathrm{Hom}(P_\bullet, M)$ where $P_\bullet$ is any two-sided projective resolution of $\mathbf{k}$ (i.e., an exact (unbounded) complex of projectives such that the cokernel of $P_1 \to P_0$ is $\mathbf{k}$). This is one of the standard definitions of Tate cohomology (cf. e.g. [AM04, Definition 7.1]).

Remark 1.1.19. By the same token, it makes sense to define the Tate cohomology of any Hopf algebra H which is Frobenius over $\mathbf{k}$ by $\underline{\mathrm{Ext}}^i_H(\mathbf{k}, M)$, where $\mathbf{k}$ is the augmentation module. Note that, if $\mathbf{k}$ is a PID, then a Hopf algebra H over $\mathbf{k}$ is automatically Frobenius if it is fg projective as a $\mathbf{k}$-module [LS69].

Next, we recall the definition of a Gerstenhaber algebra.

Definition 1.1.20. A Gerstenhaber algebra $(\mathcal{V}^\bullet, \wedge, [\,,])$ over $\mathbf{k}$ is a $\mathbb{Z}$-graded supercommutative algebra $(\mathcal{V}, \wedge)$, together with a bracket $[\,,]: \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ of degree -1, such that the induced bracket of degree zero on the shifted graded $\mathbf{k}$-module $\mathcal{V}^{\bullet+1}$ is a Lie superbracket, satisfying the Leibniz identity,

$$[a \wedge b, c] = a \wedge [b, c] + (-1)^{mn} b \wedge [a, c], \quad a \in \mathcal{V}^m,\ b \in \mathcal{V}^n,\ c \in \mathcal{V}. \tag{1.1.21}$$

Finally, we recall the definition of a BV algebra.

Definition 1.1.22. A Batalin–Vilkovisky (BV) algebra $(\mathcal{V}^\bullet, \wedge, \Delta)$ is a $\mathbb{Z}$-graded supercommutative algebra $(\mathcal{V}^\bullet, \wedge)$ equipped with an operator $\Delta : \mathcal{V} \to \mathcal{V}$ of degree -1 such that $\Delta^2 = 0$, and such that the bracket $[\,,]$ defined by

$$(-1)^{m+1}[a, b] = \Delta(a \wedge b) - \Delta(a) \wedge b - (-1)^m a \wedge \Delta(b) + (-1)^{m+n} a \wedge b \wedge \Delta(1), \quad a \in \mathcal{V}^m,\ b \in \mathcal{V}^n, \tag{1.1.23}$$

endows $(\mathcal{V}^\bullet, \wedge, [\,,])$ with a Gerstenhaber algebra structure. Here, $1 \in \mathcal{V}^0$ is the algebra unit.

2. General theory

The goal of this section is to prove that periodic Calabi–Yau Frobenius algebras have BV and Frobenius structures on their Hochschild cohomology (Theorems 2.3.27, 2.3.64). Along the way, we will prove several general results about the Hochschild (co)homology of Frobenius algebras over an arbitrary base commutative ring. We also explain why our definition of Calabi–Yau Frobenius implies the CY condition of [ES06] (Theorem 2.3.21), and give a new proof that the Hochschild cohomology of symmetric algebras is BV (Section 2.4).

We will need a straightforward generalization of the stable module category to a version relative to $\mathbf{k}$, which we relegated to Appendix B.

2.1. Stable Hochschild (co)homology

In this section, we define and begin the study of the **stable Hochschild (co)homology**, by replacing Ext by $\underline{\mathrm{Ext}}$ in the definition. By Remark 1.1.18, this is the Hochschild version of Tate cohomology of finite groups.

For Hochschild homology, we will first need the notion of stable tensor product:

Definition 2.1.1. Let A be any algebra over $\mathbf{k}$ (projective as a symmetric $\mathbf{k}$-module). For any A^{op}-module M and any A-module N, such that M, N are fg projective as $\mathbf{k}$-modules, define

$$M \underline{\otimes}_A N := \big\{ f \in M \otimes_A N \bigm| f \in \operatorname{Ker}(M \otimes_A N \to M \otimes_A I), \\ \text{for any } \mathbf{k}\text{-split injection } N \hookrightarrow I, \text{ with } I \text{ relatively injective} \big\}. \tag{2.1.2}$$

The condition that $f \in \operatorname{Ker}(M \otimes_A N \to M \otimes_A I)$ for any particular $\mathbf{k}$-split injection $N \hookrightarrow I$ as above is equivalent to the condition holding for *all* such $\mathbf{k}$-split injections (cf. Appendix B).

Proposition 2.1.3. *If A is a Frobenius algebra, then the definition of $\underline{\otimes}_A$ is symmetric in the following sense*:

$$M \underline{\otimes}_A N \cong \big\{ f \in M \otimes_A N \bigm| f \in \operatorname{Ker}(M \otimes_A N \to J \otimes_A N), \\ \textit{for any } \mathbf{k}\textit{-split injection } M \hookrightarrow J, \textit{ with } J \textit{ a relatively injective } A^{\mathrm{op}}\textit{-module} \big\}. \tag{2.1.4}$$

Thus, one has

$$M \underline{\otimes}_A N \cong N \underline{\otimes}_{A^{\mathrm{op}}} M. \tag{2.1.5}$$

Proof. Fix $\mathbf{k}$-split injections $N \hookrightarrow I$, $M \hookrightarrow J$, for I, J relatively injective (= fg projective) A- and A^{op}-modules, respectively. Since I, J are projective, the maps $M \otimes_A I \to J \otimes_A I$ and $J \otimes_A N \to J \otimes_A I$ are injective. Hence, the kernel of $M \otimes_A N \to J \otimes_A N$ is the same as the kernel of $M \otimes_A N \to M \otimes_A I$ (both are the kernel of $M \otimes_A N \to J \otimes_A I$). $\square$

Definition 2.1.6. Let A be a Frobenius algebra over $\mathbf{k}$. For any A^{op}-module M, and any A-module N, both which are fg projective over $\mathbf{k}$, we define the ith stable Tor groups by

$$\underline{\operatorname{Tor}}_i^A(M, N) := M \underline{\otimes}_A \Omega^i N. \tag{2.1.7}$$

Proposition 2.1.8. *If A is Frobenius over $\mathbf{k}$, then*

$$\underline{\operatorname{Tor}}_i^A(M, N) \cong \operatorname{Tor}_i^A(M, N), \quad i \geqslant 1, \tag{2.1.9}$$

and moreover, the definition (2.1.7) *is symmetric in the following sense*:

$$M \underline{\otimes}_A \Omega^i N \cong \Omega^i M \underline{\otimes}_A N, \quad \forall i \in \mathbb{Z}. \tag{2.1.10}$$

Proof. The first statement follows similarly to Corollary B.0.14. Similarly, we find that $\Omega^i M \underline{\otimes}_A N \cong \operatorname{Tor}_i(M, N)$ for $i \geqslant 1$, which yields (2.1.10) for $i \geqslant 1$. The statement is tautological for $i = 0$. To extend to negative i, we may use the trick $(\Omega^{-i} M) \underline{\otimes}_A \Omega^i(\Omega^{-i} N) \cong \Omega^i(\Omega^{-i} M) \underline{\otimes}_A \Omega^{-i} N$. $\square$

Definition 2.1.11. Suppose that A is Frobenius over $\mathbf{k}$, and let M be any A-bimodule which is fg projective as a $\mathbf{k}$-module. The ith stable Hochschild (co)homology groups (which only depend on the stable equivalence class of M) are defined by

$$\underline{\mathrm{HH}}^i(A, M) := \underline{\operatorname{Ext}}^i_{A^e}(A, M), \qquad \underline{\mathrm{HH}}_i(A, M) := \underline{\operatorname{Tor}}_i^{A^e}(A, M). \tag{2.1.12}$$

Corollary 2.1.13. *With A, M as in the definition, $\underline{\mathrm{HH}}^i(A, M) \cong \mathrm{HH}^i(A, M)$ and $\underline{\mathrm{HH}}_i(A, M) \cong \mathrm{HH}_i(A, M)$ for $i \geqslant 1$.*

Remark 2.1.14. One could pose the definitions of stable Hochschild (co)homology and stable Ext and Tor when A is not Frobenius, but one probably wants Ω to be an autoequivalence to have a reasonable notion (e.g., if A is "relatively self-injective"; see Appendix B).

One has many algebraic structures attached to Hochschild cohomology and homology: put together, these form the structure of calculus (cf. e.g., [TT05]; see Definition 2.3.42 for the definition). This includes cup products $\cup$ for Hochschild cohomology, and contraction maps $HH^j(A) \otimes HH_\ell(A) \to HH_{\ell-j}(A)$ for $j \leqslant \ell$. For $f \in HH^j(A)$, we denote by $i_f : HH_\ell(A) \to HH_{\ell-j}(A)$ the corresponding contraction. We now show that the cup and contraction structures extend to the stable, $\mathbb{Z}$-graded setting.

Theorem 2.1.15.

(i) *Let A be any Frobenius algebra over* $\mathbf{k}$. *Then one has a well-defined cup product on $\underline{\mathrm{HH}}^\bullet(A, A)$, giving the structure of an associative algebra, and extending the cup product on $HH^{\geqslant 1}(A, A)$.*
(ii) *One has a well-defined contraction operation $\underline{\mathrm{HH}}^j(A, A) \otimes \underline{\mathrm{HH}}_k(A, M) \to \underline{\mathrm{HH}}_{k-j}(A, M)$, which extends the usual contraction operation, and satisfies the relation*

$$i_f i_g(x) = i_{f \cup g}(x), \tag{2.1.16}$$

where $i_f(x)$ is the contraction of $f \in \underline{\mathrm{HH}}^\bullet(A, A)$ with $x \in \underline{\mathrm{HH}}_\bullet(A, M)$. ($M$ is any fg A-bimodule.)
(iii) *The algebra $\underline{\mathrm{HH}}^\bullet(A, A)$ is graded-commutative.*

Proof. (i) The cup product is easy to define: for $f \in \underline{\mathrm{HH}}^j(A, A)$ and $g \in \underline{\mathrm{HH}}^k(A, A)$, we have $f \in \underline{\mathrm{Hom}}(\Omega^j A, A)$ and $\Omega^j g \in \underline{\mathrm{Hom}}(\Omega^{j+k} A, \Omega^j A)$, so we may consider the composition

$$f \cup g := f \circ \Omega^j g = f \circ \Omega^{|f|} g, \tag{2.1.17}$$

where $|f|$ denotes the Hochschild cohomology degree. It follows immediately that the cup product is associative.

(ii) To define the contraction operation, note that $f \in \underline{\mathrm{HH}}^j(A, A) = \underline{\mathrm{Hom}}(\Omega^j A, A)$ induces a map

$$\Omega^j A \underline{\otimes}_{A^e} \Omega^k A \to A \underline{\otimes}_{A^e} \Omega^k A, \tag{2.1.18}$$

for all k. Applying the equivalence Ω, we obtain a map

$$A \underline{\otimes}_{A^e} \Omega^{j+k} A \to A \underline{\otimes}_{A^e} \Omega^k A, \tag{2.1.19}$$

for all $k \in \mathbb{Z}$. This is the desired map. We automatically get the intertwining property (2.1.16).

(iii) We have isomorphisms in the stable module category,

$$\Omega^j A \otimes_A \Omega^k A \xrightarrow{\sim} \Omega^{j+k} A, \tag{2.1.20}$$

which follow from the fact that $\Omega^j A$ may be considered as a projective left A-module for all j (using the sequence $\Omega^1 A \hookrightarrow A \otimes A \twoheadrightarrow A$, which is split as left A-modules). We will need the

Claim 2.1.21. *Let us use $\Omega^i A := (\Omega^1 A)^{\otimes_A i}$ for $i \geqslant 1$. Given $f \in \underline{\mathrm{Hom}}(\Omega^j A, \Omega^k A)$, we may form a representative of $\Omega f \in \underline{\mathrm{Hom}}(\Omega^{j+1} A, \Omega^{k+1} A)$ by either* $\mathrm{Id} \otimes_A f$ *or by* $(-1)^{j-k} f \otimes_A \mathrm{Id}$.

Proof. This follows from the fact that the stable module category is a suspended category as in [SA04] (since it is a full monoidal subquotient of the derived category which is closed under suspension). However, we give an explicit argument. The terms of the normalized bar resolution may be written as $A \otimes \Omega^\bullet A$, and if we construct this by splicing together sequences $\Omega^{n+1} A \hookrightarrow A \otimes \Omega^n A \twoheadrightarrow \Omega^n A$, it is easy to see that f lifts to $f \otimes_A \mathrm{Id}$. Writing the terms of the normalized bar resolution as $\Omega^\bullet A \otimes A$, we obtain the desired sign corrections of $(-1)^{j-k}$. $\square$

As a consequence, if we have $f \in \underline{\mathrm{HH}}^j(A,A)$, $g \in \underline{\mathrm{HH}}^k(A,A)$, then we may compute $f \cup g$ in two ways. First, if $j,k \geqslant 0$, then letting $f' \in \underline{\mathrm{Hom}}(\Omega^j A, A)$, $g' \in \underline{\mathrm{Hom}}(\Omega^k A, A)$, we may use either the formula $f' \otimes_A g'$ or $(-1)^{|f||g|} g' \otimes_A f'$. Similarly, if j,k are arbitrary, we take instead $f' \in \underline{\mathrm{Hom}}(\Omega^{a+j} A, \Omega^a A)$, $g' \in \underline{\mathrm{Hom}}(\Omega^{b+k} A, \Omega^b A)$ for $a+j, a, b+k, b \geqslant 0$. Then the same argument for the composition $\Omega^b f' \circ \Omega^{a+j} g'$ yields either $(-1)^{jb} f' \otimes_A g'$ or $(-1)^{(a+j)k} g' \otimes_A f'$. So, we obtain $(-1)^{jb} f \cup g = (-1)^{(b+k)j} g \cup f$, as desired. $\square$

Finally, we present a duality property for stable Hochschild homology, which will induce a Frobenius algebra structure on Hochschild cohomology for "Calabi–Yau Frobenius algebras." To do this, we first need to explain how to write standard complexes computing $\underline{\mathrm{HH}}_\bullet$ and $\underline{\mathrm{HH}}^\bullet$. More generally, we define these computing stable Ext and Tor.

Definition 2.1.22. For any Frobenius algebra A over $\mathbf{k}$ and any fg A-module M which is projective over $\mathbf{k}$, call a **two-sided** resolution of M, an exact $\mathbf{k}$-split complex of fg projective A-modules,

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots, \tag{2.1.23}$$

such that M is the cokernel of $P_1 \to P_0$ (and the kernel of $P_{-1} \to P_{-2}$):

$$P_1 \to P_0 \twoheadrightarrow M \hookrightarrow P_{-1} \to P_{-2}. \tag{2.1.24}$$

Definition 2.1.25. For any Frobenius algebra A, any fg left A-module M which is projective over $\mathbf{k}$, any fg right A-module N which is projective over $\mathbf{k}$, and any two-sided resolution $P_\bullet$ of M, define the associated ("standard") complex computing stable Tor,

$$C_\bullet^A(N, M) := N \otimes_A P_\bullet. \tag{2.1.26}$$

Similarly, if now M, N are both fg left A-modules which are $\mathbf{k}$-projective, we define the associated ("standard") complex computing stable Ext,

$$C_A^\bullet(M, N) := \mathrm{Hom}_A(P_\bullet, N). \tag{2.1.27}$$

We call the classes of the complexes $C^\bullet, C_\bullet$ in the (unbounded) derived category of fg $\mathbf{k}$-modules, the stable $M \underline{\otimes}_A^L N$ and $\underline{\mathrm{RHom}}(M, N)$.

Definition 2.1.28. For any two-sided A^e-resolution of A, we call $C_\bullet(A,A) := C_\bullet^{A^e}(A,A)$ and $C^\bullet(A,A) := C_{A^e}^\bullet(A,A)$ the associated "standard" complexes computing stable Hochschild homology and cohomology, respectively.

Remark 2.1.29. Note that we could have chosen to reserve the words "standard" for complexes resulting from the bar resolution of A (which can be completed to a two-sided resolution, as we will explain).

Theorem 2.1.30. *Let A be a Frobenius algebra over $\mathbf{k}$. We have the following duality*:

$$\mathbb{D}_{\mathbf{k}} : \underline{\mathrm{HH}}_i(A,A) \xrightarrow{\sim} \underline{\mathrm{HH}}_{-1-i}(A,A)^*, \quad \text{if } \mathbf{k} \text{ is a field}, \tag{2.1.31}$$

$$\mathbb{D}_{\mathbf{k}} : C_\bullet^{A^e}(A,A) \xrightarrow[\sim]{\text{q.i.}} C_{-1-\bullet}^{A^e}(A,A)^*, \quad \text{in general}. \tag{2.1.32}$$

Moreover, using (2.1.31)*, the contraction maps become graded self-adjoint*:

$$\mathbb{D}_{\mathbf{k}}(i_f x) = (-1)^{|f||x|} i_f^* \mathbb{D}_{\mathbf{k}}(x). \tag{2.1.33}$$

An easy extension of the theorem to coefficients in any bimodule M which is fg projective over $\mathbf{k}$ yields

$$\mathbb{D}_{\mathbf{k}} : \underline{\mathrm{HH}}_i(A, M) \xrightarrow{\sim} \underline{\mathrm{HH}}_{-1-i}\big(A, M^* \otimes_A A^\vee\big)^*, \qquad C_\bullet^{A^e}(A, A) \overset{\text{q.i.}}{\xrightarrow{\sim}} C_{-1-\bullet}^{A^e}\big(A, M^* \otimes_A A^\vee\big)^*. \tag{2.1.34}$$

To prove the theorem, the following easy identifications will be useful:

Lemma 2.1.35. *Let A be a Frobenius algebra over* $\mathbf{k}$*.*

(i) *For any left A-module M which is fg projective over* $\mathbf{k}$*,*

$$({}_\phi M)^* \cong \big(M^*\big)_\phi, \qquad ({}_\phi M)^\vee \cong \big(M^\vee\big)_\phi, \qquad {}_\phi A \otimes_A M \cong {}_\phi M. \tag{2.1.36}$$

Furthermore,

$$M^\vee \otimes_A A^* \cong M^* \cong \big(M^\vee\big)_\eta. \tag{2.1.37}$$

(ii) *As A-bimodules, we have*

$${}_\phi A_\psi \cong A_{\phi^{-1}\circ\psi}, \qquad A^\vee \cong A_{\eta^{-1}}, \qquad A^\vee \otimes_A A^* \cong A \cong A^* \otimes_A A^\vee. \tag{2.1.38}$$

Proof. (i) The first set of identities is immediate. For the second, we use (B.0.10):

$$M^\vee \otimes_A A^* \cong M^*_{\eta^{-1}} \otimes_A A^* \cong M^*. \tag{2.1.39}$$

(ii) The first identity is clear. Applying (2.1.37) to the case of bimodules, we have

$$A^* \otimes_A A^\vee \otimes_A A^* \cong A^\vee \otimes_{A^e} \big(A^e\big)^* \cong A^*, \tag{2.1.40}$$

which immediately gives the last identity, and hence the second. □

Proof of Theorem 2.1.30. Given any two-sided resolution $P_\bullet$ of A, the lemma shows that the exact complex $P^\vee_{-1-\bullet} \otimes_A A^*$ must be another two-sided resolution of A. Furthermore, we have

$$\big(P^\vee_{-1-\bullet} \otimes_A A^*\big) \otimes_{A^e} A \cong P^\vee_{-1-\bullet} \otimes_{A^e} A^* \cong \mathrm{Hom}_{A^e}\big(P_{-1-\bullet}, A^*\big) \cong \big(A \otimes_{A^e} P_{-1-\bullet}\big)^*, \tag{2.1.41}$$

where the last isomorphism uses the standard adjunction.

To show the graded self-adjoint property (2.1.33), we first note the following naturality: applying $(\tilde f^\vee \otimes_A \mathrm{Id}_{A^*}) \otimes_{A^e} \mathrm{Id}$ on the LHS of (2.1.41) (where $\tilde f$ is the lift of f to $P_\bullet$) is the same as applying $(\mathrm{Id} \otimes_{A^e} \tilde f)^*$ on the RHS. Now, the LHS can be replaced by the complex $\mathrm{Hom}_{A^e}(P_{-1-\bullet}, A^* \otimes_A P_\bullet)$ (which is the total complex, summing the $\bullet$ degrees). Applying $(\tilde f^\vee \otimes_A \mathrm{Id}_{A^*}) \otimes_{A^e} \mathrm{Id}$ becomes **right** composition with $\tilde f$. Using the same argument as in the proof of Theorem 2.1.15(iii), this is chain-homotopic to applying **left** composition with $(-1)^{|f|\cdot|-1-\bullet|}(\mathrm{Id}_{A^*} \otimes_A \tilde f)$. Now, via the quasi-isomorphism $A \otimes_{A^e} P_\bullet \simeq (P^\vee_{-1-\bullet} \otimes_A A^*) \otimes_{A^e} P_\bullet \simeq \mathrm{Hom}_{A^e}(P_{-1-\bullet}, A^* \otimes_A P_\bullet)$, applying $(-1)^{|f|\cdot|-1-\bullet|}\mathrm{Id}_{A^*} \otimes_A \tilde f$ gets carried to this map, which we showed is chain-homotopic to the map obtained from applying $(\mathrm{Id} \otimes_{A^e} \tilde{f})^*$ on the RHS of (2.1.41). Now, passing to homology, we get the desired graded self-adjointness. (In the Calabi–Yau Frobenius case which we will define, we can deduce this result more simply as in the proof of Theorem 2.3.27). □

2.2. Relative Serre duality for the stable module category

From now on, let A be a Frobenius algebra over $\mathbf{k}$. As was noticed in [ES06], when $\mathbf{k} =$ a field, the Auslander–Reiten homomorphisms give a Serre duality for the stable module category. We recall this and produce a relative version.

Notation 2.2.1. Let $\nu : A\text{-mod} \to A\text{-mod}$ be the **Nakayama functor**, $\nu = * \circ \vee$.

Note that, when $\mathbf{k}$ is a field, ν sends projectives to injectives, and vice-versa, and in fact induces an equivalence of categories {projective A-modules} $\leftrightarrow$ {injective A-modules}. Also, by (B.0.10), one has the following simple formula for νM (which will be useful later):

$$\nu M \cong A^* \otimes_A M \cong {}_{\eta^{-1}}M. \tag{2.2.2}$$

Proposition 2.2.3. *(See [ES06].) When $\mathbf{k}$ is a field, we have functorial isomorphisms*

$$\underline{\mathrm{Hom}}_A(M, N) \cong \underline{\mathrm{Hom}}_A(N, \Omega\nu M)^*. \tag{2.2.4}$$

The proof is based on the Auslander–Reiten formulas (cf., e.g. [ARS97,ASS06]).

In order to make proper sense of the Serre duality for arbitrary $\mathbf{k}$, it is necessary to replace the groups above by complexes and dual complexes. First, observe that the above isomorphism can be rewritten, by replacing M by $\Omega^i M$, as

$$\underline{\mathrm{Ext}}^i_A(M, N) \cong \underline{\mathrm{Ext}}^{-1-i}_A(N, \nu M)^*. \tag{2.2.5}$$

The following then gives a version of the above for general $\mathbf{k}$:

Theorem 2.2.6. *Let M, N be A-modules which are fg projective as $\mathbf{k}$-modules. Then, one has a functorial quasi-isomorphism in the derived category,*

$$C^\bullet_A(M, \nu N) \simeq C^{-1-\bullet}_A(N, M)^*. \tag{2.2.7}$$

Proof. Fix a two-sided projective resolution $P_\bullet$ of N. Applying ν, we obtain a resolution of $\nu(N)$. By Proposition B.0.4, $\mathrm{Hom}_A(M, \nu P_{-1-\bullet})$ may be used to compute $\mathrm{Ext}^\bullet_A(M, \nu N)$, since $\nu P_\bullet$ is a two-sided resolution of νN consisting of relatively injectives. Furthermore, for any two-sided resolution $Q_\bullet$ of M, we may obtain quasi-isomorphisms

$$C^\bullet_A(M, \nu N) = \mathrm{Hom}_A(Q_\bullet, \nu N) \xleftarrow{\sim} \mathrm{Hom}_A(Q_\bullet, \nu P_{-1-\bullet}) \xrightarrow{\sim} \mathrm{Hom}_A(M, \nu P_{-1-\bullet}). \tag{2.2.8}$$

Next, we show that $\mathrm{Hom}_A(M, \nu P_{-1-\bullet}) \simeq \mathrm{Hom}_A(P_{-1-\bullet}, M)^*$. For any module L, there is a functorial map $L^\vee \otimes_A M \to \mathrm{Hom}_A(L, M)$, which is an isomorphism if L is projective. Applying this map and its dual, we obtain

$$\mathrm{Hom}_A(P_{-1-\bullet}, M)^* \xrightarrow{\sim} \left(P^\vee_{-1-\bullet} \otimes_A M\right)^* \overset{\text{adj.}}{\cong} \mathrm{Hom}_A(M, \nu P_{-1-\bullet}), \tag{2.2.9}$$

the last map using adjunction. □

2.3. Calabi–Yau and periodic Frobenius algebras

We will modify the definition of Calabi–Yau algebra [Gin06] to suit Frobenius algebras. First, we recall this definition.

Definition 2.3.1. (See [Gin06].) An associative algebra A over a commutative ring $\mathbf{k}$ is called **Calabi–Yau** if it has finite Hochschild dimension, and one has a quasi-isomorphism in the derived category of A^e-modules,

$$f : A[d] \xrightarrow{\sim} \mathrm{RHom}_{A^e}(A, A \otimes A) \tag{2.3.2}$$

which is self-dual:

$$f^! \circ \iota = f[-d], \tag{2.3.3}$$

where, for any map $g : M \to N$ of A-bimodules, $g^! : \mathrm{RHom}_{A^e}(N, A^e) \to \mathrm{RHom}_{A^e}(M, A^e)$ is the natural map, and $\iota : A \to \mathrm{RHom}_{A^e}(A, \mathrm{RHom}_{A^e}(A, A \otimes A))$ is the natural map. Here $[-]$ denotes the shift in the derived category.

However, if A is Frobenius and of finite Hochschild dimension (part of being (usual) CY), then $\Omega^i A = 0$ for large enough i, and hence $A \simeq 0$ in the stable module category. Hence, $HH^i(A, M) = 0$ for all $i \geqslant 1$, so A has Hochschild dimension zero. That is, A is a projective A-bimodule (i.e., A is separable).

For a separable Frobenius algebra to additionally be Calabi–Yau of dimension zero, we require exactly that $A \cong A^\vee$ as A-bimodules. By Lemma 2.1.35, this is equivalent to $A^* \cong A$, i.e., A must be a symmetric, separable Frobenius algebra.

This is not general enough, so we replace this notion. Let us first restate the usual Calabi–Yau property in terms of a quasi-isomorphism of complexes:

$$\begin{tikzcd} 0 \arrow[r] & P_m \arrow[r] \arrow[d] & P_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_0 \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\ 0 \arrow[r] & P_0^\vee \arrow[r] & P_1^\vee \arrow[r] & \cdots \arrow[r] & P_m^\vee \arrow[r, two heads] & A. \end{tikzcd} \tag{2.3.4}$$

Here and from now on, the functor $^\vee$ will be in the category of A-bimodules, i.e., $M^\vee = \mathrm{Hom}_{A^e}(M, A \otimes A)$.

If now A becomes a Frobenius algebra, as explained earlier, we cannot have such a quasi-isomorphism (for $m \geqslant 1$): in fact, when dualizing the top sequence, we get something that begins with $A^\vee \hookrightarrow P_0^\vee \to P_1^\vee$. However, it still makes sense to ask for a commutative diagram, with exact rows, as follows:

$$\begin{tikzcd} A^\vee \arrow[r, hook] & P_m \arrow[r] \arrow[d] & P_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_0 \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\ A^\vee \arrow[r, hook] & P_0^\vee \arrow[r] & P_1^\vee \arrow[r] & \cdots \arrow[r] & P_m^\vee \arrow[r, two heads] & A. \end{tikzcd} \tag{2.3.5}$$

In fact, since the dual is automatically exact in the Frobenius case, by a standard result of homological algebra, such a diagram must automatically exist given a resolution as in the top row. So it is enough to ask for such a resolution.

On the level of the stable module category, having such a resolution implies the following condition:

Definition 2.3.6. A Frobenius algebra A over $\mathbf{k}$ is called **Calabi–Yau Frobenius of dimension** m if one has isomorphisms in the stable module category $\mathrm{Stab}_{\mathbf{k}}(A^e)$, for some $m \in \mathbb{Z}$:[1]

$$f : A^\vee \simeq \Omega^{m+1} A. \tag{2.3.7}$$

If there is more than one such m, then we pick the smallest nonnegative value of m (which exists because such algebras must be periodic as in the subsequent definition).

If additionally A has a grading, such that the above isomorphism is a graded isomorphism when composed with some shift (considering the stable module category to now be graded), then we say that A is a **graded Calabi–Yau Frobenius algebra**. More precisely, if $f : A^\vee(m') \simeq \Omega^{m+1} A$ is a graded isomorphism, where (ℓ) denotes the shift by ℓ with the new grading, then one says that A is **graded Calabi–Yau Frobenius with dimension** m **of shift** m'**.**

We remark that the above definition of Calabi–Yau Frobenius is (apparently) stronger than the notion of Calabi–Yau for self-injective algebras discussed in [ES06] for the case $\mathbf{k}$ is a field: see Theorem 2.3.21 and the comments thereafter.

Definition 2.3.8. A Frobenius algebra over $\mathbf{k}$ is called **periodic Frobenius of period** n for some $n > 0$ if one has (in $\mathrm{Stab}_{\mathbf{k}}(A^e)$)

$$g : A \simeq \Omega^n A, \tag{2.3.9}$$

and that is the smallest positive n for which one has such an isomorphism. If A has a grading, we define **graded periodic Frobenius of period** n **and shift** n' as before (if $g : A(n') \simeq \Omega^n A$ is a graded isomorphism in the stable module category).

Note that it makes sense to be Calabi–Yau Frobenius of **negative** dimension. In particular, *any symmetric Frobenius algebra is either Calabi–Yau Frobenius of dimension* -1, *or periodic Calabi–Yau Frobenius of dimension* $n-1$ *and period* n *for some* $n \geqslant 1$.

Also, any periodic Frobenius algebra must have even period, unless $2 \cdot \mathrm{Id} \simeq 0$ in the stable module category, e.g., $\mathrm{char}\, \mathbf{k} = 2$, as we will see in Theorem 2.3.47. In particular, the CY dimension must be odd for symmetric Frobenius algebras.

Example 2.3.10. The preprojective algebras of ADE Dynkin quivers are periodic Calabi–Yau Frobenius (using [EE07]) of dimension 2 and shift 2, and of period 6 and shift $2h$ (twice the Coxeter number). The essential ingredient is the Schofield resolution [RS] (cf., e.g., [EE07]), with $R = \mathbf{k}^I$, where I is the vertex set:

$$A^\vee(2) \hookrightarrow A \otimes_R A(2) \to A \otimes_R V \otimes_R A \to A \otimes_R A \twoheadrightarrow A. \tag{2.3.11}$$

Here, V is the free $\mathbf{k}$-module spanned by the edges of the quiver. This completes to a periodic projective resolution of length 6, since the Nakayama automorphism has order 2 (for more details, see Proposition 2.3.15). The fact that the shift is $2h$ follows from the fact that $A^\vee(2) \cong A_\eta(h)$, which amounts to the fact that the degree of the image of $\mathrm{Id} \in A \otimes_R A^*$ under the isomorphism

[1] The self-duality property required for Calabi–Yau algebras (2.3.3) can still be imposed here: this would be the condition that $f : A^\vee \simeq \Omega^{m+1} A$ satisfies $f^\vee = \Omega^{-m-1} f$. We do not need this.

$A \otimes_R A^* \cong A \otimes_R A$ from the pairing, is $h-2$ (i.e., $h-2$ is the degree of the product of any basis element with its dual basis element).

We note that an important part of the above is showing that Π_Q is free (equivalently, projective) over $\mathbb{Z}$; this follows from explicit $\mathbb{Z}$-bases (such as those in [Eu08], which one may verify is integral; types A, D over $\mathbb{Z}$ are also in [Sch07, §4.2.2]).

Example 2.3.12. Similarly, the centrally extended preprojective algebras [ER06], over $\mathbf{k} = \mathbb{C}$, are periodic Calabi–Yau Frobenius with dimension 3 (of shift 4) and period 4 (of shift $2h$), using [Eu06]. These are symmetric, i.e., have trivial Nakayama automorphism. Note that these are not, in general, torsion-free over $\mathbb{Z}$ (cf. Section 4), hence not Frobenius over $\mathbb{Z}$, although the definition may be modified to correct this.

Example 2.3.13. Similarly to the Dynkin case, one may consider preprojective algebras of *generalized* Dynkin type: this refers to preprojective algebras of type T_n (otherwise known as L_n) which can be obtained from $\Pi_{A_{2n}}$ by passing to fixed points under the Nakayama automorphism, $\Pi_{T_n} := (\Pi_{A_{2n}})^\eta$. In other words, this is associated to a graph of T_n type.

In [Eu07b] (cf. [BES07]), using a variant of the Schofield resolution [RS], it is proved that Π_{T_n} is periodic Calabi–Yau Frobenius of dimension 5 (and shift $h+2$) and period 6 (and shift $2h$). Also, in [Eu07b], a correction to results of [ES98a] in type A is given.

Example 2.3.14. (See [BBK02].) The trivial extension algebras of path algebras of Dynkin quivers are periodic Calabi–Yau Frobenius (in fact, symmetric) of dimension $2h-3$ and period $2h-2$, where h is the Coxeter number. These are "almost-Koszul dual" to the preprojective algebras; see [BBK02].

For Calabi–Yau Frobenius algebras, being periodic Frobenius is closely related to having finite-order Nakayama automorphism:

Proposition 2.3.15. *If A is Calabi–Yau Frobenius of dimension $\neq -1$, then the following are equivalent*:

(i) *A is periodic Frobenius*;
(ii) *For some $p > 0$, one has*

$$A \simeq A_{\eta^p} \text{ in the stable bimodule category}^2 \tag{2.3.16}$$

(*e.g., if η^p is inner*).

In the situation that the above are satisfied, then the Calabi–Yau dimension m, the period n, and the smallest $p > 0$ such that (2.3.16) *holds, are related by*

$$n = p \cdot \gcd(n, m+1), \tag{2.3.17}$$

and $r = \gcd(n, m+1)$ is the smallest positive integer such that $\Omega^r A \simeq A_{\eta^k}$ for some integer k.

In particular, a Calabi–Yau Frobenius algebra of dimension $\leqslant -2$ must have infinite-order Nakayama automorphism (since if it were periodic, the CY dimension is defined to be nonnegative). Being dimension -1 is a special case, consisting of stably symmetric Frobenius algebras that are not periodic.

[2] Note that, if we had defined Calabi–Yau and periodic Frobenius algebras using complexes rather than the stable module category, then this would be replaced by an honest isomorphism of bimodules, $A \cong A_{\eta^p}$, and hence η^p would have to be inner.

Proof. (ii) implies (i). We have $\Omega^{m+1} A \simeq A^\vee \cong A_{\eta^{-1}}$. For any p such that (2.3.16) holds, we have $A \simeq A_{\eta^{-p}} \simeq \Omega^{p\cdot(m+1)} A$, yielding (i).

(i) implies (ii). As in the previous paragraph, we deduce that $\Omega^{k\cdot(m+1)} A \simeq A_{\eta^{-k}}$. If $\Omega^n A \simeq A$, then we would deduce that $A \simeq \Omega^{-n\cdot(m+1)} A \simeq A_{\eta^n}$, yielding (ii).

To obtain (2.3.17), let $r > 0$ be the smallest positive integer such that $\Omega^r A \simeq A_{\eta^k}$ for some integer k. It follows that $r \mid (m+1)$ and $r \mid n$. Since $r \mid (m+1)$, it must be that k is relatively prime to p, otherwise $A_{\eta^{k\cdot \frac{m+1}{r}}} \simeq \Omega^{m+1} A \simeq A_{\eta^{-1}}$ would contradict the minimality of p. Similarly, we deduce that $n = p \cdot r$, and $r = \gcd(n, m+1)$. $\square$

Also, we have the following growth criterion:

Notation 2.3.18. For any fg $\mathbf{k}$-module M, let $g(M) = g_{\mathbf{k}}(M) \geqslant 0$ denote its minimal number of generators.

Proposition 2.3.19. *Suppose that* $\mathbf{k}$ *has finite Krull dimension* (*or just that its maximal ideal spectrum has finite dimension*)*. If A is Calabi–Yau Frobenius of dimension $\neq -1$, or if A is periodic Frobenius, then for any fg A-modules M, N which are projective as* $\mathbf{k}$*-modules, there is a positive integer $p \geqslant 1$ such that, for all $i \in \mathbb{Z}$, $\underline{\mathrm{Ext}}^i(M, N)$ and $\underline{\mathrm{Tor}}^i(M^*, N)$ are generated by at most p generators over* $\mathbf{k}$*. We may take $p = ag(M)\cdot g(N) + b$ for some $a, b \geqslant 0$ depending only on A and* $\mathbf{k}$*.*

Moreover, for any fg A-bimodule L which is projective as a $\mathbf{k}$*-module, there is a positive integer $q \geqslant 1$ such that $\underline{\mathrm{HH}}^i(A, L), \underline{\mathrm{HH}}_i(A, L)$ are generated by at most q generators, for all $i \in \mathbb{Z}$. Again, we may take $q = a' \cdot g(L) + b'$ for some $a', b' \geqslant 0$ depending only on A and* $\mathbf{k}$*.*

Proof. Without loss of generality, assume that the tautological map $\mathbf{k} \to A$ is injective. Under either assumption of the proposition (A is CY Frobenius of dim $\neq -1$ or A is periodic Frobenius), we have $\Omega^r A \simeq A_\phi$ for some $r \geqslant 1$ and some automorphism ϕ of A. Thus, for all integers s, we may write $s = r \cdot k + r'$ for some $0 \leqslant r' < r$, and then $\Omega^s_{A^e} A \simeq \Omega^{r'}_{A^e} A_{\phi^k}$. Since $\Omega^s_A M \simeq \Omega^s_{A^e} A \otimes_A M$, we have

$$\underline{\mathrm{Ext}}^s(M, N) \cong \underline{\mathrm{Hom}}\big(\Omega^s_{A^e} A \otimes_A M, N\big) \cong \underline{\mathrm{Hom}}\big({}_{\phi^{-k}}\Omega^{r'}_{A^e} A \otimes_A M, N\big). \tag{2.3.20}$$

Furthermore, using the normalized bar resolution of A (twisted by automorphisms of A), we may take $\Omega^s_{A^e} A$ to be isomorphic, as a right A-module, to $(A/\mathbf{k})^{\otimes s} \otimes A$. Then, localizing $\mathbf{k}$ at any prime ideal, $\underline{\mathrm{Hom}}({}_{\phi^{-k}}\Omega^{r'}_{A^e} A \otimes_A M, N)$ can have rank at most $(g(A)-1)^{r'} \cdot g(M) \cdot g(N)$. By Theorem 1 of [Swa67], $\underline{\mathrm{Hom}}({}_{\phi^{-k}}\Omega^{r'}_{A^e} A \otimes_A M, N)$ can then be generated by at most $(g(A)-1)^{r'} \cdot g(M) \cdot g(N) + d$ elements, where d is the dimension of the maximal ideal spectrum of $\mathbf{k}$.

We can apply the same idea for $\underline{\mathrm{Tor}}^s(M, N)$. For $\underline{\mathrm{HH}}^j(A, L)$ and $\underline{\mathrm{HH}}_j(A, L)$ where L is now an A-bimodule, we now use that these are given by $\underline{\mathrm{Hom}}_{A^e}(\Omega^j A, L)$ and $\Omega^j A \underline{\otimes}_{A^e} L$, and apply the same argument as above. $\square$

In particular, the above proposition rules out tensor products of periodic Frobenius algebras over a field (which are not separable) from being periodic or Calabi–Yau Frobenius of dimension $\neq -1$, by the Künneth theorem.

The following theorems also have graded versions (by incorporating the shifts in the definitions), but we omit them for simplicity.

Theorem 2.3.21.

(i) *For any Calabi–Yau Frobenius algebra A of dimension m, and any fg A-modules M, N which are projective over* $\mathbf{k}$*, one has isomorphisms* (*functorial in M, N*)

$$\underline{\mathrm{Hom}}_A(M,N) \cong \underline{\mathrm{Hom}}_A\big(N, \Omega^{-m}M\big)^*, \quad \text{if } \mathbf{k} \text{ is a field,} \tag{2.3.22}$$

$$\underline{\mathrm{RHom}}_A^\bullet(M,N) \simeq \underline{\mathrm{RHom}}_A^{m-\bullet}(N,M)^*, \quad \text{generally.} \tag{2.3.23}$$

That is, Ω^{-m} is a (right) Serre functor for the stable module category relative to $\mathbf{k}$.

(ii) *For any periodic Frobenius algebra A of period n, and any fg A-modules M, N, one has isomorphisms*

$$\underline{\mathrm{Hom}}_A(M,N) \cong \underline{\mathrm{Hom}}_A\big(\Omega^n M, N\big), \qquad \underline{\mathrm{RHom}}_A^\bullet(M,N) \simeq \underline{\mathrm{RHom}}_A^{\bullet+n}(M,N). \tag{2.3.24}$$

As a corollary, we see that any Calabi–Yau Frobenius algebra over a field is also Calabi–Yau in the sense of [ES06]. We do not know if the reverse implication holds.

Proof. (i) By (2.2.2) (which holds for arbitrary $\mathbf{k}$) and (2.1.38), we have a stable equivalence

$$M \simeq \Omega^{m+1}\nu(M), \quad \text{i.e.,} \quad \nu M \simeq \Omega^{-m-1}M. \tag{2.3.25}$$

Since $\Omega\nu$ already provides a right Serre functor in the senses needed for (i) (using Theorem 2.2.6), we now know that Ω^{-m} also provides such a right Serre functor (note that the above stable equivalences are clearly functorial).

(ii) This is easy. □

Remark 2.3.26. For general Frobenius algebras, one can still say that $\underline{\mathrm{Hom}}(M,N) \cong \underline{\mathrm{Hom}}(N, A^* \otimes_A \Omega M)^*$ (by (2.2.2) and Theorem 2.2.6), so that the Serre functor involves shifting and twisting by an invertible (under $\otimes_A$) bimodule. Similarly, the following results on Hochschild (co)homology have analogues for arbitrary Frobenius algebras using twisting (by powers of η) as well as shifting. For this, one considers the bigraded algebra $\underline{\mathrm{Hom}}_{A^e}^\bullet(A, A_{\eta^{\bullet'}})$ and its bigraded module $\Omega^\bullet A \underline{\otimes}_{A^e} A_{\eta^{\bullet'}}$. We do not need this for our examples. However, it might be interesting to try to apply this formalism to finite-dimensional Hopf algebras (analogously to [BZ08]).

We now present results on stable Hochschild cohomology of periodic and CY Frobenius algebras:

Theorem 2.3.27. *Let A be a Calabi–Yau Frobenius algebra of dimension m, and M any A-bimodule.*

(i) *One has isomorphisms and quasi-isomorphisms*

$$\mathbb{D}: \underline{\mathrm{HH}}^\bullet(A,M) \cong \underline{\mathrm{HH}}_{m-\bullet}(A,M), \qquad \underline{\mathrm{RHom}}_{A^e}^\bullet(A,M) \simeq \big(A \underline{\otimes}_{A^e}^L M\big)_{m-\bullet}. \tag{2.3.28}$$

For $M = A$, these isomorphisms intertwine cup product with contraction:

$$\mathbb{D} \circ i_f(x) = f \cup \mathbb{D}(x), \quad \forall x \in \underline{\mathrm{HH}}_\bullet(A,A),\ f \in \underline{\mathrm{HH}}^\bullet(A,A). \tag{2.3.29}$$

(ii) *Let M be fg projective over* $\mathbf{k}$. *One has*

$$\underline{\mathrm{HH}}^\bullet(A,M) \cong \underline{\mathrm{HH}}^{m-\bullet}\big(A, M^*\big)^*, \quad \text{if } \mathbf{k} \text{ is a field,} \tag{2.3.30}$$

$$\underline{\mathrm{RHom}}_{A^e}^\bullet(A,M) \simeq \underline{\mathrm{RHom}}_{A^e}^{m-\bullet}\big(A, M^*\big)^*, \quad \text{generally.} \tag{2.3.31}$$

In the case $M = A$, we may rewrite this, respectively, as

$$\underline{\mathrm{HH}}^\bullet(A,A) \cong \underline{\mathrm{HH}}^{(2m+1)-\bullet}(A,A)^*, \qquad \underline{\mathrm{RHom}}_{A^e}^\bullet(A,A) \simeq \underline{\mathrm{RHom}}_{A^e}^{(2m+1)-\bullet}(A,A)^*. \tag{2.3.32}$$

(iii) *The induced pairing*

$$\underline{\mathrm{HH}}^\bullet(A, A) \otimes \underline{\mathrm{HH}}^{(2m+1)-\bullet}(A, A) \to \mathbf{k} \tag{2.3.33}$$

is invariant with respect to cup product:

$$(f, g \cup h) = (f \cup g, h), \quad |f| + |g| + |h| = 2m + 1, \tag{2.3.34}$$

and is nondegenerate if $\mathbf{k}$ *is a field. Moreover, for all* $\mathbf{k}$, *one has a nondegenerate invariant pairing in the derived category* (*of degree* $-(2m+1)$),[3]

$$\underline{\mathrm{RHom}}_{A^e}(A, A) \otimes \underline{\mathrm{RHom}}_{A^e}(A, A) \to \mathbf{k}, \tag{2.3.35}$$

inducing (2.3.32).

In other words, $C^\bullet$ *is a Frobenius algebra in the derived category of* $\mathbf{k}$*-modules, and if* $\mathbf{k}$ *is a field,* $\underline{\mathrm{HH}}^\bullet$ *is a graded Frobenius algebra over* $\mathbf{k}$ (*using a definition that only requires finite-generation in each degree*).

We also remark that (2.3.34) and (2.3.29), together with the graded commutativity of Theorem 2.1.15(iii), give another proof of the graded self-adjointness of i^*_- (2.1.33) in this case.

Proof. (i) Let us pick a two-sided resolution of A:

$$\cdots \to P_2 \to P_1 \to P_0 \twoheadrightarrow A \hookrightarrow P_{-1} \to P_{-2} \to \cdots, \tag{2.3.36}$$

so that, removing the A, the complex $P_\bullet$ is an exact complex of projectives. Since $\Omega^{m+1}A \simeq A^\vee$ (in the stable module category), we may form chain maps between the two obtained resolutions,

$$P_{\bullet+(m+1)} \leftrightarrow P^\vee_{-1-\bullet}, \tag{2.3.37}$$

such that their composition on either side induces identity maps on the level of $\underline{\mathrm{Hom}}(\Omega^i A, \Omega^i A)$ and $\underline{\mathrm{Hom}}(\Omega^i A^\vee, \Omega^i A^\vee)$ for all $i \in \mathbb{Z}$. As a result, upon applying the functor $A \otimes_{A^e} -$, we obtain quasi-isomorphisms of the resulting complexes,

$$P_{\bullet+(m+1)} \otimes_{A^e} M \xrightarrow{\sim} P^\vee_{-1-\bullet} \otimes_{A^e} M. \tag{2.3.38}$$

However, it is clear that these are standard complexes computing $\underline{\mathrm{HH}}_{m+1+\bullet}(A, M)$ and $\underline{\mathrm{HH}}^{-1-\bullet}(A, M)$, which is what we needed.

To prove the intertwining property (2.3.29), we note that, for $f \in \underline{\mathrm{HH}}^j(A, A)$, applying i_f in the LHS of (2.3.29) is the same as applying the corresponding element of $\underline{\mathrm{Hom}}(A, \Omega^{-j}A)$ to the $M = A$ in (2.3.38). Similarly, applying $f \cup -$ to the RHS is post-composing with f, which is applying the same element of $\underline{\mathrm{Hom}}(A, \Omega^{-j}A)$ to the A in the RHS of (2.3.38).

(ii) This follows from part (i) and Theorem 2.1.30. To fix the signs (so as to obtain the Frobenius property in part (iii)), we use for the duality $x \mapsto (-1)^{|x|\cdot m}\mathbb{D}^* \circ \mathbb{D}_{\mathbf{k}} \circ \mathbb{D}(x)$, rather than only $\mathbb{D}^* \circ \mathbb{D}_{\mathbf{k}} \circ \mathbb{D}$. Alternatively, we can use $\nu A = A^{\vee *} \simeq A_{\eta^2} \simeq \Omega^{-2(m+1)}A$ and Theorem 2.2.6, and similarly for arbitrary M, which will give the desired property.

(iii) For this, we will use functoriality of the isomorphisms in (ii). (The result also follows from Theorem 2.1.30(iii).)

[3] Here, a pairing $X \otimes Y \to \mathbf{k}$ is nondegenerate if it induces a (quasi-)isomorphism $X \simeq Y^*$, or equivalently, $Y \simeq X^*$.

In the case that $\mathbf{k}$ is a field, the first isomorphism of (2.3.32) comes from the functorial isomorphisms of (2.2.7) (in the case $\mathbf{k}$ is a field, we can rewrite this as $\underline{\mathrm{Hom}}(M,N) \xrightarrow{\sim} \underline{\mathrm{Hom}}(N, \nu\Omega M)^*$). Thus, we have the commutative square (for any $g \in \underline{\mathrm{HH}}^j(A,A)$)

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{A^e}(A, \Omega^{-i}A) \arrow[r, "\sim"] \arrow[d, "\Omega^{-i-j}g\circ"] & \underline{\mathrm{Hom}}_{A^e}(\Omega^{-i}A, \Omega^{-(2m+1)}A)^* \arrow[d, "(\circ\Omega^{-i}g)^*"] \\ \underline{\mathrm{Hom}}_{A^e}(A, \Omega^{-i-j}A) \arrow[r, "\sim"] & \underline{\mathrm{Hom}}_{A^e}(\Omega^{-i-j}A, \Omega^{-(2m+1)}A)^* \end{tikzcd} \tag{2.3.39}$$

which gives exactly the invariance property needed. On the level of complexes, we may make the desired statement as follows. Let $P_\bullet$, as in (2.3.36), be a two-sided resolution of A. Then, we have the following sequence, where the isomorphisms mean quasi-isomorphisms, and all complexes are total complexes graded by $\bullet$:

$$C^\bullet \xleftarrow{\sim} \mathrm{Hom}_{A^e}(P_\bullet, P_{-\bullet}) \xrightarrow{\sim} \mathrm{Hom}_{A^e}(P_{-\bullet}, \nu P_{1+\bullet})^* \xrightarrow{\sim} \mathrm{Hom}_{A^e}(P_{-\bullet}, P_{\bullet-(2m+1)})^* \xrightarrow{\sim} \big(C^{(2m+1)-\bullet}\big)^*. \tag{2.3.40}$$

Next, for any $j \in \mathbb{Z}$ and any $g \in \mathrm{Hom}_{A^e}(P_{\bullet+j}, P_\bullet)$, we have the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{A^e}(P_\bullet, P_{-\bullet}) \arrow[r, "\sim"] \arrow[d, "g\circ"] & \mathrm{Hom}_{A^e}(P_{-\bullet}, P_{\bullet-(2m+1)}A)^* \arrow[d, "(\circ g)^*"] \\ \mathrm{Hom}_{A^e}(P_\bullet, P_{-\bullet-j}) \arrow[r, "\sim"] & \mathrm{Hom}_{A^e}(P_{-\bullet-j}, P_{\bullet-(2m+1)})^* \end{tikzcd} \tag{2.3.41}$$

which gives the desired result. The pairing $C^\bullet \otimes C^\bullet \to \mathbf{k}$ in the derived category is given by replacing $C^\bullet$ with $\mathrm{Hom}_{A^e}(P_\bullet, P_{-\bullet})$ and using composition. $\square$

We now need to recall the definition of calculus:

Definition 2.3.42. (See [Tsy04, p. 93].) A precalculus is a pair of a Gerstenhaber algebra $(\mathcal{V}^\bullet, \wedge, [\,,\,])$ and a graded vector space $\mathcal{W}^\bullet$ together with:

(1) A module structure $\iota_- : \mathcal{V}^\bullet \otimes \mathcal{W}^{-\bullet} \to \mathcal{W}^{-\bullet}$ of the graded commutative algebra $\mathcal{V}^\bullet$ on $\mathcal{W}^{-\bullet}$;
(2) An action $\mathcal{L}_-$ of the graded Lie algebra $\mathcal{V}^{\bullet+1}$ on $\mathcal{W}^{-\bullet}$, which satisfies the following compatibility conditions:

$$\iota_a\mathcal{L}_b - (-1)^{|a|(|b|+1)}\mathcal{L}_b\iota_a = \iota_{[a,b]}, \tag{2.3.43}$$

$$\mathcal{L}_{a\wedge b} = \mathcal{L}_a\iota_b + (-1)^{|a|}\iota_a\mathcal{L}_b. \tag{2.3.44}$$

A calculus is a precalculus $(\mathcal{V}^\bullet, \mathcal{W}^\bullet, [\,,\,], \wedge, \iota_-, \mathcal{L}_-)$ together with a differential d of degree 1 on $\mathcal{W}^\bullet$ satisfying the Cartan identity:

$$\mathcal{L}_a = d\iota_a - (-1)^{|a|}\iota_a d. \tag{2.3.45}$$

It is a result of [GDT89] that, for any associative algebra A, the collection

$$\big(\mathrm{HH}^\bullet(A,A), \mathrm{HH}_\bullet(A,A), \{\,,\,\}, \cup, i_-, \mathcal{L}_-, B\big) \tag{2.3.46}$$

is a calculus, where B is the Connes differential, $\{,\}$ the Gerstenhaber bracket, and $\mathcal{L}_-$ the Lie derivative operation.

Theorem 2.3.47. *Let A be any periodic Frobenius algebra of period n. Then, n must be even if $2\cdot\underline{\mathrm{HH}}^\bullet(A,A)\neq 0$, and*:

(i) *One has isomorphisms and quasi-isomorphisms*

$$\underline{\mathrm{HH}}^\bullet(A,A)\cong\underline{\mathrm{HH}}^{\bullet+n}(A,A),\qquad \underline{\mathrm{HH}}_\bullet(A,A)\cong\underline{\mathrm{HH}}_{\bullet+n}(A,A),\tag{2.3.48}$$

$$C^\bullet_{A^e}(A,A)\xrightarrow{\sim}C^{\bullet+n}_{A^e}(A,A),\tag{2.3.49}$$

and similarly a quasi-isomorphism between the standard complex computing stable Hochschild homology and its shift by n. Moreover, the isomorphisms may be induced by cup product on the left with the element $1'\in\underline{\mathrm{HH}}^n(A,A)$ representing the given isomorphism $\Omega^nA\xrightarrow{\sim}A$, and by the contraction $i_{1'}$.

(ii) *The stable Hochschild cohomology is a Gerstenhaber algebra, which extends the Gerstenhaber structure on usual Hochschild cohomology.*

(iii) *The stable Hochschild cohomology and stable Hochschild homology form the structure of a calculus, extending the usual calculus structure.*

Proof. (i) The isomorphisms follow as in previous proofs from the stable module isomorphism $1'$: $A\simeq\Omega^nA$. To show that they are induced by cup product or contraction with $1'\in\underline{\mathrm{HH}}^n(A,A)$, let us construct a projective resolution[4] of A such that

$$\Omega^jA\cong\left(\Omega^1A\right)^{\otimes_A j}.\tag{2.3.50}$$

Following the proof of Theorem 2.1.30(iii), we construct this from any exact sequence $\Omega^1A\hookrightarrow P\twoheadrightarrow A$ such that P is a fg projective A-bimodule and Ω^1A is an A-bimodule which is projective as a left and right A-module (separately), by splicing together $(\Omega^1A)^{\otimes_A j}\hookrightarrow P\otimes_A(\Omega^1A)^{\otimes_A(j-1)}\twoheadrightarrow(\Omega^1A)^{\otimes_A(j-1)}$ for all $j\geqslant 1$. If we construct these inductively by tensoring on the left by Ω^1A, then we see that the sequences are all exact since Ω^1A is a projective right A-module; also, $P\otimes_A(\Omega^1A)^{\otimes_A(j-1)}$ is a projective A^e-module because the result is obvious in the case that P is a free A^e-module.

Now, given $f\in\underline{\mathrm{Hom}}_{A^e}(\Omega^jA,\Omega^kA)$, we may construct $\Omega f\in\underline{\mathrm{Hom}}_{A^e}(\Omega^{j+1}A,\Omega^{k+1}A)$ by applying $\Omega^1A\otimes_A -$, by construction of the above resolution. On the other hand, the isomorphism $\underline{\mathrm{HH}}^i(A,A)\cong\underline{\mathrm{HH}}^{i+n}(A,A)$ is given by the stable module isomorphism $1':\Omega^nA\simeq A$. That is, we use the stable module isomorphism $\Omega^{n+i}A\simeq\Omega^iA$, which by the above is $1'\otimes_A\mathrm{Id}:\Omega^nA\otimes_A\Omega^iA\to A\otimes_A\Omega^iA$, so $\underline{\mathrm{HH}}^i(A,A)\cong\underline{\mathrm{HH}}^{i+m}(A,A)$ is given by cup product on the left with $1'$.

Now, since cup product with $1'$ induces an isomorphism, we must have a right inverse $(1')^{-1}$ such that $1'\cup(1')^{-1}=\mathrm{Id}$, and hence it is a two-sided inverse by Theorem 2.1.15(i), (iii):

$$\begin{aligned}\mathrm{Id}&=\left(1'\cup(1')^{-1}\right)\cup\left(1'\cup(1')^{-1}\right)=1'\cup\left((1')^{-1}\cup 1'\right)\cup(1')^{-1}\\&=\left((1')^{-1}\cup 1'\right)\cup\left(1'\cup(1')^{-1}\right)=\left((1')^{-1}\cup 1'\right).\end{aligned}\tag{2.3.51}$$

Thus, we also deduce the statement at the beginning of the theorem, that either $|1'|$ is even, or $2\cdot\mathrm{Id}=0$. (This can also be deduced from the fact that $2\cdot(1'\cup 1')=0$ if $|1'|$ is odd.)

(ii), (iii) To deduce this, we use that the desired structures exist in positive degree and satisfy the necessary axioms. Using the element $1'$ from part (i) (and Theorem 2.1.15), this result follows from the following general proposition (see also the comments after the statement). □

[4] This is isomorphic to the normalized bar resolution.

Proposition 2.3.52. *Let $(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$ be a (pre)calculus and $z \in \mathcal{V}$ a homogeneous element. Then, there is a unique extension of the calculus structure to the localization $(\mathcal{V}[z^{-1}]^\bullet, \mathcal{V}[z^{-1}]^\bullet \otimes_{\mathcal{V}^\bullet} \mathcal{W}^\bullet)$, where by convention, $z^{-1} \wedge z = 1$.*

Now, if A is a periodic Frobenius algebra, with a homogeneous element $1' \in \underline{\mathrm{HH}}^{\geqslant 1}$ inducing the periodicity, then we claim that $(\underline{\mathrm{HH}}^\bullet(A), \underline{\mathrm{HH}}_\bullet(A)) = (\underline{\mathrm{HH}}^{\geqslant 0}(A)[(1')^{-1}], \underline{\mathrm{HH}}_{\geqslant 0}(A)[(1')^{-1}])$. There is clearly a map $(\underline{\mathrm{HH}}^{\geqslant 0}(A))[(1')^{-1}] \to \underline{\mathrm{HH}}^\bullet(A)$, which is an isomorphism in nonnegative degrees, and must therefore be an isomorphism. As a result, we deduce that the calculus structure on $(\underline{\mathrm{HH}}^{\geqslant 0}(A), \underline{\mathrm{HH}}_{\geqslant 0}(A))$ extends uniquely to a calculus structure on $(\underline{\mathrm{HH}}^\bullet(A), \underline{\mathrm{HH}}_\bullet(A))$.

We remark that this calculus is not periodic in a trivial way: it is not true that the Lie derivatives or B must commute with $i_{1'}$. However, one can write formulas for all the operations in terms of operations on degrees $0, 1, 2, \ldots, |1'| - 1$ and involving $1'$.

Proof of Proposition 2.3.52. We use the notation of Definition 2.3.42, since we are discussing general calculi and not only the Hochschild calculus. By definition, $\mathcal{V}[z^{-1}]$ is graded commutative. Furthermore, note that, since $z \wedge z = (-1)^{|z|} z \wedge z$, either $|z|$ is even, or $\mathcal{V}[z^{-1}]$ is an algebra over $\mathbb{Z}/2$. Either way, z is central in $\mathcal{V}[z^{-1}]$ (not merely graded-central), and we can omit any mention of $(-1)^{|z|}$.

Let $\phi : \mathcal{V} \to \mathcal{V}[z^{-1}]$ denote the localization map. Note that, if $f \in \ker(\phi)$, i.e., $f \wedge z^k = 0$ for some $k \geqslant 0$, then $\{f, g\} \wedge z^{k+1} = 0$ by the Leibniz rule, so $\{f, g\} \in \ker(\phi)$. Let $\psi : \mathcal{W} \to \mathcal{W}[z^{-1}] := \mathcal{V}[z^{-1}] \otimes_{\mathcal{V}} \mathcal{W}$ denote the base-change map. Then, we have $x \in \ker(\psi)$ iff $\iota_{z^k}(x) = 0$ for some $k \geqslant 1$, and similarly to the above, we deduce that $\mathcal{L}_a(x), d(x) \in \ker(\psi)$ for all $a \in \mathcal{V}$ using the calculus identities. Similarly, for any $y \in \mathcal{W}$, and any $f \in \ker(\phi)$, we have $\mathcal{L}_f(y) \in \ker(\psi)$. Thus, it makes sense to speak about the calculus structures as being defined on $(\phi(\mathcal{V}), \psi(\mathcal{W}))$, and our goal is to extend the structure to $(\mathcal{V}[z^{-1}], \mathcal{W}[z^{-1}])$ and verify that the calculus identities are still satisfied.

For operators, we will use $[-,-]$ to denote the *graded* commutator: $[\alpha, \beta] := \alpha \circ \beta - (-1)^{|\alpha||\beta|} \beta \circ \alpha$. For example, $[\mathcal{L}_a, \iota_b] := \mathcal{L}_a \circ \iota_b - (-1)^{(|a|-1)|b|} \iota_b \circ \mathcal{L}_a$.

We extend the Gerstenhaber bracket from $\phi(\mathcal{V})$ as follows:

$$\left[z^{-1}, g\right] := -z^{-2} \wedge [z, g], \tag{2.3.53}$$

together with skew-symmetry and the Leibniz rule. We must check that this yields a well-defined bracket, which amounts to the computation

$$\begin{aligned} \left[(f \wedge z) \wedge z^{-1}, g\right] &:= z^{-1} \wedge [f \wedge z, g] + (f \wedge z) \wedge \left[z^{-1}, g\right] \\ &= [f, g] + z^{-1} \wedge f \wedge [z, g] - \left(f \wedge z^{-1}\right) \wedge [z, g] = [f, g]. \end{aligned} \tag{2.3.54}$$

It is easy to check that this yields a Gerstenhaber bracket, and we omit this.

We extend the Lie derivative $\mathcal{L}_-$ to $(\mathcal{V}[z^{-1}]^\bullet, \mathcal{W}[z^{-1}]^\bullet)$ as follows. For $f \in \phi(\mathcal{V})$, we extend the operation $\mathcal{L}_f$ to $\mathcal{W}[z^{-1}]$ using (2.3.43), with $a := f$ and $b := z$, using the same procedure as above. To define the derivative $\mathcal{L}_{f \wedge z^{-1}}$ we use (2.3.44) with $a := f$ and $b := z$. It is straightforward that this is well defined.

We must verify that the above gives a precalculus. We know that the identities are satisfied when everything is in $(\phi(\mathcal{V}), \psi(\mathcal{W}))$. Thus, to verify that (2.3.43) holds, we need to show that the LHS (graded)-commutes with ι_z. This follows because ι_b (graded)-commutes with ι_z, and $[\mathcal{L}_a, \iota_z] = \iota_{[a,z]}$, which graded-commutes with ι_b. We may then inductively show that (2.3.43) holds: first, if it holds for any (a, b), it must hold replacing a by $a \wedge z^{\wedge j}$ for any $j \in \mathbb{Z}$ by construction. Then, inductively, if the identity holds for (a, b), we may deduce that it holds for $(a, b \wedge z^{-1})$ using our definition of $\mathcal{L}_{b \wedge z^{-1}}$ (which is based on (2.3.44)).

To verify that (2.3.44) holds, it is enough to show that the identity for $(a, b \wedge z)$ implies the identity for (a, b). We may first prove this for $a \in \phi(\mathcal{V})$, and then prove it for all a using the identity for (z, a) and hence for (a, z). That is, it suffices to prove that the identity for $(a, b \wedge z)$ and (a, z) implies the identity for (a, b). We have

$$\begin{aligned}\mathcal{L}_{a\wedge b}\iota_z + (-1)^{|a|+|b|}\iota_{a\wedge b}\mathcal{L}_z = \mathcal{L}_{a\wedge b\wedge z} &= \mathcal{L}_{a\wedge z}\iota_b + (-1)^{|a|}\iota_{a\wedge z}\mathcal{L}_b \\ &= \mathcal{L}_a\iota_b\iota_z + (-1)^{|a|}\iota_a\mathcal{L}_z\iota_b + (-1)^{|a|}\iota_{a\wedge z}\mathcal{L}_b \\ &= \mathcal{L}_a\iota_b\iota_z + (-1)^{|a|+|b|}(\iota_{a\wedge b}\mathcal{L}_z - \iota_a\iota_{[b,z]}) + (-1)^{|a|}(\iota_a\iota_{[z,b]} + \iota_a\mathcal{L}_b\iota_z),\end{aligned} \tag{2.3.55}$$

which gives the identity upon cancelling the two inner terms and multiplying on the right by $\iota_{(z)^{-1}}$.

We can similarly verify that $\mathcal{L}_-$ gives a Lie action. As before, it suffices to show that the identity $\mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y]$ for $(a, b\wedge z)$ and (a, z) implies the identity for (a, b). Since z is invertible, it suffices to verify that $[\mathcal{L}_a, \mathcal{L}_b]\iota_z = \mathcal{L}_{[a,b]}\iota_z$. Since (using the Leibniz rule and the fact that z is central)

$$\mathcal{L}_{[a,b\wedge z]} + (-1)^{(|a|+1)(|b|+1)}\mathcal{L}_{b\wedge[z,a]} = \mathcal{L}_{[a,b]\wedge z} = \mathcal{L}_{[a,b]}\iota_z + (-1)^{|a|+|b|+1}\iota_{[a,b]}\mathcal{L}_z, \tag{2.3.56}$$

it suffices to verify that the LHS equals the RHS after substituting the desired identity $\mathcal{L}_{[a,b]}\iota_z = [\mathcal{L}_a, \mathcal{L}_b]\iota_z$. That is, it suffices to prove that

$$\mathcal{L}_{[a,b\wedge z]} + (-1)^{(|a|+1)(|b|+1)}\mathcal{L}_{b\wedge[z,a]} = [\mathcal{L}_a, \mathcal{L}_b]\iota_z + (-1)^{|a|+|b|+1}\iota_{[a,b]}\mathcal{L}_z. \tag{2.3.57}$$

We have

$$\text{LHS} = \big[\mathcal{L}_a, \mathcal{L}_b\iota_z + (-1)^{|b|}\iota_b\mathcal{L}_z\big] + (-1)^{(|a|+1)(|b|+1)}\mathcal{L}_b\iota_{[z,a]} - (-1)^{|a|(|b|+1)}\iota_b[\mathcal{L}_z, \mathcal{L}_a], \tag{2.3.58}$$

$$\text{RHS} = [\mathcal{L}_a, \mathcal{L}_b]\iota_z + (-1)^{|b|}\mathcal{L}_a\iota_b\mathcal{L}_z - (-1)^{|a||b|}\iota_b\mathcal{L}_a\mathcal{L}_z, \tag{2.3.59}$$

from which (2.3.57) follows by expanding $\iota_{[z,a]} = \iota_z\mathcal{L}_a - \mathcal{L}_a\iota_z$ in the first line, and making two pairwise cancellations.

Next, we have to extend differential d. For this, we use the Cartan identity (2.3.45), with $a := z$ (or a power of z). We need to check that, with this definition, (2.3.45) holds, and that $d^2 = 0$. We will show that (2.3.45) holds applied to any element $b \in \mathcal{W}[z^{-1}]$. First, we show that (2.3.45) must hold when $b \in \psi(\mathcal{W})$. This amounts to the statement that

$$\iota_{[z,b]} = \big[\iota_z, [d, \iota_b]\big]. \tag{2.3.60}$$

We simplify the RHS as

$$\big[\iota_z, [d, \iota_b]\big] = \big[[\iota_z, d], \iota_b\big] \overset{\text{def.}}{=} -(-1)^{|z|}[\mathcal{L}_z, \iota_b] = (-1)^{|z|+|b|(|z|+1)}\iota_{[b,z]} = \iota_{[z,b]}, \tag{2.3.61}$$

as desired (the first step used $[\iota_z, \iota_b] = 0$ by graded-commutativity). Next, we show that (2.3.45) must hold for all b. This amounts to the statement that

$$[d, \iota_{a\wedge z}] = [d, \iota_a]\iota_z + (-1)^{|a|}\iota_a[d, \iota_z], \tag{2.3.62}$$

which follows immediately using $\iota_{a\wedge z} = \iota_a\iota_z$.

Finally, to show that $d^2 = 0$, it suffices to show that $[d^2, \iota_z] = 0$, i.e., $d\mathcal{L}_z + \mathcal{L}_z d = 0$. Actually, we know that this identity holds when applied to $\psi(\mathcal{W})$, so it is enough to show that $[d\mathcal{L}_z + \mathcal{L}_z d, \iota_z] = 0$. We have

$$[d\mathcal{L}_z + \mathcal{L}_z d, \iota_z] = -d\iota_{[z,z]} + \mathcal{L}_z^2 + \mathcal{L}_z^2 - \iota_{[z,z]}d = -\mathcal{L}_{[z,z]} + 2\mathcal{L}_z^2 = 0, \tag{2.3.63}$$

using at the end the fact that $\mathcal{L}_-$ is an action. □

Adding the CY Frobenius condition, we obtain the

Theorem 2.3.64.

(i) *Let A be any periodic Calabi–Yau Frobenius algebra. Then, the stable Hochschild cohomology is a BV algebra, with BV differential obtained by the duality* (2.3.28) *from the Connes differential. That is, the differential $\Delta := \mathbb{D} \circ B \circ \mathbb{D}$, where $\mathbb{D}$ is the duality* (2.3.28) *and B the Connes differential, satisfies* (1.1.23);

(ii) *If A is only CY(m) Frobenius* (*and not necessarily periodic*), *and $m \geqslant 1$, then we may still define $\Delta := \mathbb{D} \circ B \circ \mathbb{D}$ in degrees $\leqslant m$, and* (1.1.23) *is satisfied when $|a|, |b| \geqslant 0$ and $1 \leqslant |a| + |b| \leqslant m$.*

Proof. (i) The proof is similar to the proof of Theorem 3.4.3 in [Gin06]. Namely, using (2.3.29) and (2.3.43), we have

$$\{a, b\} = \mathbb{D} i_{\{a,b\}} \mathbb{D}(1) = a \cup \mathbb{D}\mathcal{L}_b\mathbb{D}(1) - (-1)^{|a|(|b|+1)}\mathbb{D}\mathcal{L}_b\mathbb{D}(a). \tag{2.3.65}$$

Now, using (2.3.45) and (2.3.44), we have

$$\mathrm{RHS} = a \cup \Delta(b) - (-1)^{|b|} a \cup b \cup \Delta(1) - (-1)^{|a|(|b|+1)}\Delta(b \cup a) + (-1)^{|a||b|} b \cup \Delta(a). \tag{2.3.66}$$

Using graded-commutativity, this immediately gives (1.1.23).

(ii) The above proof goes through in the general Calabi–Yau Frobenius case in the degrees indicated. Note that we needed $|a|, |b| \geqslant 0$ because we used (2.3.43) applied to a and b. □

Remark 2.3.67. In fact, for any graded-commutative algebra $\mathcal{V}^\bullet$, giving the structure of calculus using $\mathcal{W}^\bullet := \mathcal{V}^{m-\bullet}$ which satisfies the intertwining property (2.3.29) (for $\mathbb{D}$ the tautological isomorphism) is equivalent to giving $\mathcal{V}^\bullet$ a BV algebra structure. The above theorem showed that calculus + duality gives BV; the other direction is as follows: The intertwining property (2.3.29) uniquely specifies what the module structure of $\mathcal{V}^\bullet$ on $\mathcal{W}^\bullet$ is, and the differential then gives $\mathcal{L}$. One may then deduce the remaining identities from $\Delta^2 = 0$ and the BV identity (1.1.23). The identity $\mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y]$ says precisely that Δ^2 is a derivation[5]; (2.3.43) says Δ is a differential operator of order $\leqslant 2$, together with the BV identity (i.e., that $a \otimes b \mapsto (-1)^{|a|+1}[a, b]$ is the principal symbol of Δ). Then, (2.3.44) is a consequence of (2.3.45) (and it is a proof that Δ being an operator of order $\leqslant 2$ yields the Leibniz rule for its principal symbol).

Put together, any periodic Calabi–Yau Frobenius algebra has Hochschild cohomology which is a BV algebra and a Frobenius algebra (in the derived category), and together with Hochschild homology forms a periodic calculus (together with an isomorphism between the two that intertwines cup product with contraction). Moreover, the shift functor Ω^{-m} is a (right) Serre functor for the category of fg left modules.

In the case of (generalized, centrally extended) preprojective algebras (Examples 2.3.10, 2.3.12, 2.3.13) over $\mathbb{C}$ (or any field for the ordinary preprojective algebras), the formula for the extension of B is quite simple, as we will prove

$$B_\bullet = (-1)^\bullet \mathbb{D}_k \circ B^*_{-2-\bullet} \circ \mathbb{D}_k, \tag{2.3.68}$$

$$B_{-1} = 0. \tag{2.3.69}$$

This says that B is **graded self-adjoint**. As a consequence, so is the BV differential Δ. This motivates the

[5] It is worth remarking, by comparison, that, for a graded-commutative algebra with an odd differential operator Δ of order $\leqslant 2$, the Jacobi identity for its principal symbol $\pm[\,,\,]$ says that Δ^2 is a differential operator of order $\leqslant 2$ (so being a derivation is between this and $\Delta^2 = 0$). Skew-symmetry of $[\,,\,]$ is automatic.

Definition 2.3.70. A BV Frobenius algebra is a $\mathbb{Z}$-graded Frobenius algebra H (= a $\mathbb{Z}$-graded algebra whose graded components are fg projective over $\mathbf{k}$, and with a homogeneous invariant inner product of some fixed degree) together with a graded self-adjoint differential $\Delta_\bullet : H_\bullet \to H_{\bullet-1}$ of degree -1, i.e.,

$$\Delta_\bullet = (-1)^\bullet \Delta_\bullet^\dagger, \tag{2.3.71}$$

with $\Delta_\bullet^\dagger$ the adjoint of $\Delta_\bullet$.

This leads to the

Question 2.3.72. *Are the formulas* (2.3.68), (2.3.69) *valid for any periodic Calabi–Yau Frobenius algebra*?

More generally:

Question 2.3.73. *For which Frobenius algebras A does* (2.3.68), (2.3.69) *define an extension of the usual calculus to* $(\underline{\mathrm{HH}}^\bullet, \underline{\mathrm{HH}}_\bullet)$?

If the answer to Question 2.3.72 is positive, then the stable Hochschild cohomology of any periodic CY Frobenius algebra is BV Frobenius. If the answer to 2.3.73 includes CY Frobenius algebras, then one does not need the periodicity assumption.

To answer the above questions, we suggest to work on the level of standard Hochschild chains. Consider the two-sided resolution $\hat{N}_\bullet$ of A given by $N_\bullet \twoheadrightarrow A \hookrightarrow (N_\bullet^\vee \otimes_A A^*)$, where $N_\bullet$ is the normalized bar resolution. Then, the chain complex $\hat{N}_\bullet \otimes_{A^e} A$ has the form

$$\cdots \to A \otimes \bar{A} \otimes \bar{A} \to A \otimes \bar{A} \to A \to A^* \to A^* \otimes (\bar{A})^* \to A^* \otimes (\bar{A})^* \otimes (\bar{A})^* \to \cdots. \tag{2.3.74}$$

Then, the two-sided Connes differential $\hat{B}$ should be given, on chains, by

$$\hat{B}_i := \begin{cases} B, & i \geqslant 0, \\ 0, & i = -1, \\ (-1)^i B^*, & i \leqslant -2. \end{cases} \tag{2.3.75}$$

We hope to address this in a future paper.

2.4. Hochschild cohomology of symmetric algebras is BV

In this subsection, we give a simple proof that a symmetric Frobenius algebra over an arbitrary base ring $\mathbf{k}$ has **ordinary** Hochschild cohomology which is BV. This is based on using the formulas (2.3.68), (2.3.69) to extend B (since the algebra need not be periodic, we cannot use Theorem 2.3.64). Actually, we show this more generally for “stably symmetric” algebras, i.e., $A \simeq A^\vee$.

First, let us extend the Lie derivative operation to all of $(\underline{\mathrm{HH}}^\bullet, \underline{\mathrm{HH}}_\bullet)$, in the spirit of (2.3.68), (2.3.69):

$$\mathcal{L}_b := \hat{B} i_b - (-1)^{|b|} i_b \hat{B}. \tag{2.4.1}$$

We claim that $(i_a \mathcal{L}_b - (-1)^{|a|(|b|+1)} \mathcal{L}_b i_a)(x) = i_{\{a,b\}}(x)$, when $a, b \in \underline{\mathrm{HH}}^{\geqslant 0}(A, A)$ and $x \in \underline{\mathrm{HH}}_{\leqslant -1}(A, A)$. We have (using graded commutators)

$$\mathcal{L}_b(x) = (-1)^{|x|\cdot(|b|+1)} \mathbb{D}_{\mathbf{k}} (-1)^{|b|} \big(B^* i_b^* - i_b^* B^*\big) \mathbb{D}_k(x) = -(-1)^{|x|\cdot(|b|+1)} \mathbb{D}_{\mathbf{k}} \big[i_b^*, B^*\big] \mathbb{D}_{\mathbf{k}}(x). \tag{2.4.2}$$

The signs above use the identity $c \cdot |x| + d \cdot (|x| + c) = d \cdot |x| + c \cdot (|x| + d) = (c + d) \cdot |x| + c \cdot d$ (setting $c = 1$, $d = |b|$ or vice-versa). We then obtain

$$\begin{aligned}[i_a, \mathcal{L}_b](x) &= (-1)^{|x|\cdot(|a|+|b|+1)} \mathbb{D}_{\mathbf{k}}\big[\big[i_b^*, B^*\big], i_a^*\big]\mathbb{D}_{\mathbf{k}}(x) = (-1)^{|x|\cdot(|a|+|b|+1)} \mathbb{D}_{\mathbf{k}}\big(\big[i_a, [B, i_b]\big]\big)^* \mathbb{D}_{\mathbf{k}}(x) \\ &= (-1)^{|x|\cdot(|a|+|b|+1)} \mathbb{D}_{\mathbf{k}} i^*_{\{a,b\}} \mathbb{D}_{\mathbf{k}}(x) = i_{\{a,b\}}(x). \qquad (2.4.3)\end{aligned}$$

But, as in the proof of Theorem 2.3.64, this identity immediately gives the BV identity (1.1.23), letting $x = \mathbb{D}(1) \in \underline{\mathrm{HH}}^{-1}(A, A)$. Note also that, by definition, $\hat{B}\mathbb{D}(1) = 0$, so that $\Delta(1) = 0$.

3. Hochschild (co)homology of ADE preprojective algebras over any base

As mentioned in Example 2.3.10, the preprojective algebra of a quiver of type ADE is a periodic Calabi–Yau Frobenius algebra. In this section, we explicitly describe its Hochschild (co)homology over an arbitrary base (including positive-characteristic fields). In characteristic zero, this has already been done in [EE07,Eu06]. We also review and simplify the algebraic structures on Hochschild (co)homology, and prove that $\underline{\mathrm{HH}}^\bullet$ is BV Frobenius (verifying Question 2.3.72 in this case). Finally, in Section 3.3, we explicitly describe the maps relating the Hochschild (co)homology of Dynkin preprojective algebras with the extended Dynkin case; in the latter case, the Hochschild cohomology groups were described over $\mathbf{k} = \mathbb{C}$ in [CBEG07] and later over $\mathbb{Z}$ in [Sch07].

The new theorems proved here are: Theorem 3.2.7, which computes the Hochschild (co)homology of the Dynkin preprojective algebras over $\mathbb{Z}$ and proves that it is a BV Frobenius algebra, and Theorem 3.3.4, which compares the Hochschild cohomology of the Dynkin and extended Dynkin preprojective algebras over $\mathbb{Z}$. We also restate in a simplified form two theorems from [EE07,Eu07a] (Theorems 3.1.2 and 3.1.13).

We first recall the definition of the preprojective algebras Π_Q of ADE type.

Notation 3.0.1. We will use $[n]$ to denote the degree-n-component of a graded vector space (discarding other degrees), and (n) to denote shifting a graded vector space by degree n. In particular, $A[m]$ is a graded vector space concentrated in degree m, and $(A(n))[m] = (A[m-n])(n)$. The vector space $A[m](-m)$ is concentrated in degree zero.

Notation 3.0.2. We will use $\deg(z)$ to denote the degree of an element z in a graded algebra or module. This is to distinguish with Hochschild degree, where we denote $|a| = m$ if $a \in \underline{\mathrm{HH}}^m(A, M)$ (so $\deg(a)$ would denote the degree with respect to the grading on A and M). The notation $[m]$, (m) refer to the $\deg(-)$ grading, and never to Hochschild degree.

Let $\mathbf{k}$ be a commutative ring. Let Q be a quiver of ADE type with vertex set I. By convention, Q also denotes the edges of the quiver. Let $\overline{Q} := Q \sqcup Q^*$ be the double quiver, where $Q^* := \{a^* \mid a \in Q\}$ is the quiver obtained by reversing all arrows (a^* is the reverse of a).

Let $P_Q, P_{\overline{Q}}$ be the associated path algebras over $\mathbf{k}$, and let $\Pi_Q := P_{\overline{Q}}/((r))$ with $r := \sum_{a \in Q}[a, a^*]$. Let e_i denote the image of the vertex i for any $i \in I$.

Recall from Example 2.3.10 that Π_Q is a Frobenius algebra over $\mathbf{k}$ (in fact, periodic Calabi–Yau). Let $(\,,\,)$ denote an invariant inner product, and let η be the Nakayama automorphism of Π_Q, so that $(x, y) = (\eta^{-1}(y), x)$. Recall [RS,ES98a,Eu06] that we may choose $(\,,\,), \eta$ such that

$$\eta(e_i) = e_{\bar{\eta}(i)}, \quad \text{defining } \bar{\eta}: I \to I \text{ by} \qquad (3.0.3)$$

$$\alpha_{\bar{\eta}(i)} = -w_0(\alpha_i); \qquad (3.0.4)$$

here, w_0 is the longest element of the Weyl group of the root system attached to Q, and $\alpha_i, i \in I$ are the roots. Furthermore, η may be uniquely chosen to act on $\overline{Q} \subset P_{\overline{Q}}$ so that

$$\eta(Q^*) \subset \overline{Q} \quad \text{and} \quad \eta(Q) \subset \begin{cases} -\overline{Q}, & \text{if } Q \text{ is of type } D, E, \\ \overline{Q}, & \text{if } Q \text{ is of type } A, \end{cases} \tag{3.0.5}$$

since Q is a tree. As a consequence of these formulas, we see that η and $\bar{\eta}$ are involutions. We remark that η is always nontrivial, even though, for D_{2n+1}, E_7 and E_8, $\bar{\eta}$ becomes trivial. (Except, over characteristic 2, η is trivial for D_{2n+1}, E_7, and E_8.)

Let $m_1, m_2, \ldots, m_{|I|}$ be the exponents of the root system attached to Q, in increasing order. Let $h := m_{|I|} + 1$ be the Coxeter number.

Recall that the Hilbert series of a $\mathbb{Z}_{\geqslant 0}$-graded vector space M is defined to be $h(M;t) := \sum_{m \geqslant 0} \dim M[m] t^m$. If E is a $\mathbb{Z}_{\geqslant 0}$-graded $\mathbf{k}^I$-module for any field $\mathbf{k}$, then we define the matrix-valued Hilbert series of E, $h(E;t)$, by $h(E;t)_{ij} := \sum_{m \geqslant 0} \dim E[m]_{i,j} t^d$, where $E[m]_{i,j} := e_i E[m] e_j$, where $e_i, e_j \in \mathbf{k}^I$ are the idempotents corresponding to the vertices $i, j \in I$.

3.1. *Reminder of characteristic zero results*

Let $A := \Pi_Q$, and assume that $\mathbf{k}$ is a characteristic-zero field. We may then describe the Hochschild homology, $HH_*(A)$, and cohomology, $HH^*(A)$, as follows:

Definition 3.1.1. (See [EE07].) Let $U := (HH^0(A)[< h-2])(2)$, $L := HH^0(A)[h-2](-(h-2))$, $K := HH^2(A)(2)$, and $Y := HH^6(A)[-h-2](h+2)$. Also, let $P \in \operatorname{End}(\mathbf{k}^I)$ be the permutation matrix corresponding to the involution $\bar{\eta}$. Let $I^{\bar{\eta}}$ be the set of vertices fixed by $\bar{\eta}$.

Theorem 3.1.2 *(Restated). (See [EE07].)*

(i) *U has Hilbert series*

$$h(U;t) = \sum_{i:\, m_i < \frac{h}{2}} t^{2m_i}. \tag{3.1.3}$$

(ii) *We have natural isomorphisms $K \cong \ker(P+1)$ and $L \cong \mathbf{k}^{I^{\bar{\eta}}}$.*
(iii) *As graded vector spaces, one has*

$$\underline{\mathrm{HH}}^0(A,A) \cong U(-2) \oplus Y(h-2), \qquad \underline{\mathrm{HH}}^1(A,A) \cong U(-2), \qquad \underline{\mathrm{HH}}^2(A,A) \cong K(-2), \tag{3.1.4}$$

$$\underline{\mathrm{HH}}^{6+i}(A,A) \cong \underline{\mathrm{HH}}^i(A,A)(-2h), \qquad \underline{\mathrm{HH}}^i(A,A)(2) \cong \big(\underline{\mathrm{HH}}^{5-i}(A,A)(2)\big)^*, \tag{3.1.5}$$

$$\underline{\mathrm{HH}}_i(A,A) \cong \underline{\mathrm{HH}}^{2-i}(A,A)(2), \tag{3.1.6}$$

$$HH^0(A) \cong U(-2) \oplus L(h-2), \qquad HH_0(A) \cong \mathbf{k}^I. \tag{3.1.7}$$

3.1.1. *The cup product*

Let us summarize also the cup product structure, which was computed in [ES98a,Eu08]. We explain it using our language and results. In view of the first isomorphism of (3.1.5) and Theorem 2.3.47, it is enough to consider cup products among elements of Hochschild degrees between 0 and 5.

Since the Calabi–Yau Frobenius dimension is 2 of shift 2, by Theorem 2.3.27, the Hochschild cohomology is a Frobenius algebra with pairing of Hochschild degree -5 (meaning, $(f,g) \neq 0$ implies that $|f| + |g| = 5$ in Hochschild degree), and of graded degree 4 (meaning, in graded degree, $(f,g) \neq 0$ implies $\deg(f) + \deg(g) = -4$). In particular, for all $i \in \mathbb{Z}$, the composition

$$\underline{\mathrm{HH}}^i(A,A) \otimes \underline{\mathrm{HH}}^{5-i}(A,A) \xrightarrow{\cup} \underline{\mathrm{HH}}^5(A,A) \xrightarrow{(\,,\mathrm{Id})} \mathbf{k} \tag{3.1.8}$$

is a perfect pairing of graded degree 4, the same as the second isomorphism of (3.1.5).

Moreover, if $|f| + |g| + |h| = 5$ (in Hochschild degree), using the graded-commutativity of cup product, we have

$$(f \cup g, h) = (-1)^{|g|\cdot|h|}(f \cup h, g) = (-1)^{|f|\cdot(|g|+|h|)}(g \cup h, f), \tag{3.1.9}$$

and since the pairing is perfect, we see that knowing the cup product $\underline{\mathrm{HH}}^{|f|} \otimes \underline{\mathrm{HH}}^{|g|} \to \underline{\mathrm{HH}}^{5-|h|}$ determines the cup product in the other two pairs of Hochschild degrees, $(|f|, |h|)$ and $(|g|, |h|)$. That is, we may divide the cup products into the unordered triples summing to 5 modulo 6:

$$(0,0,5), (0,1,4), (0,2,3), (1,1,3), (1,2,2), (1,5,5), (2,4,5), (3,3,5), (3,4,4), \tag{3.1.10}$$

and the cup product in any fixed two Hochschild degrees of a triple determines the other two pairs of cup products.

The first triple above corresponds to multiplication in the center $Z(A)$, via the quotient $Z(A) \twoheadrightarrow \underline{\mathrm{HH}}^0(A, A)$ which performs $U(-2) \oplus L(h+2) \twoheadrightarrow U(-2) \oplus Y(h+2)$ (see [ES98a,Eu08] for an explicit computation of this multiplication). Then, the next two triples describe $\underline{\mathrm{HH}}^1(A, A)$ and $\underline{\mathrm{HH}}^2(A, A)$ as $\underline{\mathrm{HH}}^0(A, A)$-modules. As explained in [ES98a,Eu08], $\underline{\mathrm{HH}}^1(A, A)$ is cyclic as an $\underline{\mathrm{HH}}^0(A, A)$-module (generated by the Euler vector field), and since K is concentrated in graded degree zero, the structure on $\underline{\mathrm{HH}}^2(A, A)$ is the obvious (trivial) one: it is a $\mathbf{k}$-vector space.

The cup products between Hochschild degrees $(1, 1)$ and $(3, 3)$ are trivial for graded degree reasons. For types D, E, A_{2n+1}, the cup product is also trivial in Hochschild degrees $(3, 4)$ for degree reasons, and in degrees $(2, 4)$ by an argument using the BV identity (see [Eu08]—the argument only shows that the cup product is h-torsion, and in fact it appears to be nontrivial for type D_{2n+1} in characteristic two). In type A_{2n}, the cup product between degrees $(3, 4)$ and $(2, 4)$ is nontrivial and can be explicitly computed (see [ES98a]; see also the similar type T case in [Eu07b]). When nontrivial, the products between degrees $(3, 4)$ and $(2, 4)$ are only between the lowest possible degrees (so, it reduces to a pairing of vector spaces concentrated in bottom degree, which in fact has rank one since the bottom-degree part in Hochschild degree 4 has dimension one).

This leaves only the cup products $(1, 2, 2)$ and $(1, 5, 5)$. These are best described as cup products

$$\underline{\mathrm{HH}}^2(A, A) \otimes \underline{\mathrm{HH}}^2(A, A) \to \underline{\mathrm{HH}}^4(A, A) \cong \underline{\mathrm{HH}}^1(A, A)^*, \tag{3.1.11}$$

$$\underline{\mathrm{HH}}^5(A, A) \otimes \underline{\mathrm{HH}}^5(A, A) \to \underline{\mathrm{HH}}^4(A, A) \cong \underline{\mathrm{HH}}^1(A, A)^*. \tag{3.1.12}$$

Here, we obtain a nondegenerate symmetric bilinear pairing α on K, and a symplectic pairing β on Y, respectively (one must obtain some symmetric and skew-symmetric bilinear pairings on K and Y, respectively, since K and Y are concentrated in degree zero, and U has $\mathbf{k}$-dimension equal to one in each graded degree; nondegeneracy is then a result of explicit computations in [ES98a,Eu08] ([Eu08, Theorem 4.0.8] for types D, E; throughout [ES98a, part II] for type A)).

3.1.2. The Connes and BV differentials

Using the dualities and intertwining properties, one immediately obtains the contraction maps. It remains only to compute the Connes differential, which yields the BV differential by duality, and then using the Cartan and BV identities, one immediately computes the Lie derivatives and Gerstenhaber bracket.

We reprint the Connes differential from [Eu07a]. Let $(z_k), (\omega_k)$ be homogeneous bases for $U(-2), Y(h-2) \subset \underline{\mathrm{HH}}^0(A, A)$, respectively, with $\deg(z_k) = k$. Let $(\theta_k) \subset U(-2) \subset \underline{\mathrm{HH}}^1(A, A)$ be a homogeneous basis for $\underline{\mathrm{HH}}^1(A, A)$ with $\deg(\theta_k) = k$, and let $(f_k) \subset K \subset \underline{\mathrm{HH}}^2(A, A)$ be a basis. We will write $(f_k^*), (\theta_k^*), (z_k^*), (\omega_k^*)$ for the dual bases of $\underline{\mathrm{HH}}^3(A, A), \underline{\mathrm{HH}}^4(A, A)$, and $\underline{\mathrm{HH}}^5(A, A)$. (Using dual notation is where we diverge from [Eu08,Eu07a]). Let us abusively identify these elements with their images under periodicity and $\mathbb{D}$. So, for example, θ_k may denote the corresponding element in any group $\underline{\mathrm{HH}}^{1+6s}$ or $\underline{\mathrm{HH}}_{1+6s}$ for any $s \in \mathbb{Z}$. (This also differs from the notation of [Eu08,Eu07a].)

Theorem 3.1.13. *(See [Eu07a, Theorem 5.0.10].) The Connes differential $B_\bullet : \mathrm{HH}_\bullet(A, A) \to \mathrm{HH}_{\bullet+1}(A, A)$ is given by*

$$0 = B_{6s} = B_{4+6s} = B_{2+6s}(U) = B_{3+6s}(Y^*), \tag{3.1.14}$$

$$B_{1+6s}(\theta_k) = \left(1 + \frac{k}{2} + sh\right) z_k, \qquad B_{3+6s}(z_k^*) = \left((s+1)h - 1 - \frac{k}{2}\right)\theta_k^*, \tag{3.1.15}$$

$$B_{2+6s}(\omega_k) = \left(\frac{1}{2} + s\right) h\beta^{-1}(\omega_k), \tag{3.1.16}$$

$$B_{5+6s}(f_k^*) = (s+1)h\alpha^{-1}(f_k^*), \tag{3.1.17}$$

where in the first line, $B_{2+6s}(U)$ means the image of the summand of $U(2hs)$ under B_{2+6s}, and similarly for $B_{3+6s}(Y^*)$.

Hence, the same formulas are valid for $\underline{\mathrm{HH}}_\bullet$ where now $s \in \mathbb{Z}$ is arbitrary. As a consequence of writing it this way, using the symmetry and skew-symmetry of α, β, respectively, it is easy to verify the (new)

Corollary 3.1.18. *The Connes differential B is graded self-adjoint with respect to the duality $\mathbb{D}_{\mathbf{k}}$. Hence, the same is true for the BV differential Δ.*

Each line of (3.1.14)–(3.1.17) verifies $B_i = (-1)^i B^*_{-2-i}$ for the concerned summands of HH_i.

Remark 3.1.19. In the generalized Dynkin case of type T_n (Example 2.3.13), a similar observation to the above, together with the computation of B found in [Eu07b], shows that B is graded self-adjoint in the T_n case, and hence so is Δ, i.e., $\underline{\mathrm{HH}}^\bullet$ is BV Frobenius.

3.2. Extension of results to $\mathbb{Z}$ and arbitrary characteristic

Now, we explain the general $\mathbb{Z}$-structure of Hochschild (co)homology. Note that, by the Universal Coefficient Theorem, one may immediately deduce the $\mathbf{k}$-module structure from this for any $\mathbf{k}$; we explain it for fields $\mathbb{F}_p$ (with p prime) to see the duality. We will also see that the stable Hochschild cohomology $\underline{\mathrm{HH}}^\bullet$ is BV Frobenius over *any* base field, in Theorem 3.2.7(v) below, and give the complete structure of Δ over any field.

Definition 3.2.1. We define (and redefine) the vector spaces T, U, K, K', Y, Y', L by

$$K := \underline{\mathrm{HH}}_0(A, A)[0], \qquad K' := \mathrm{Torsion}\big(\underline{\mathrm{HH}}_{-1}(A, A)[0]\big), \qquad T := \underline{\mathrm{HH}}_0(A, A)[>0], \tag{3.2.2}$$

$$Y := \big(\underline{\mathrm{HH}}^{-1}(A, A)[h-2]\big)\big(-(h-2)\big), \qquad Y' := \mathrm{Torsion}\big(\underline{\mathrm{HH}}^0(A, A)[h-2]\big)\big(-(h-2)\big), \tag{3.2.3}$$

$$U(-2) := \underline{\mathrm{HH}}^0(A, A)\big[<(h-2)\big], \qquad L := \mathrm{HH}^0(A, A)[h-2]\big(-(h-2)\big). \tag{3.2.4}$$

Let T^* (abusively) denote the graded $\mathbb{Z}$-module

$$T^* := H^1\big(T^{L*}\big) \cong \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z}), \tag{3.2.5}$$

where $L*$ denotes the derived dual, and H^1 the first cohomology.

Note that, after tensoring by $\mathbb{C}$, the vector spaces U, K, Y, L above become (isomorphic to) the ones defined previously. Also, K, K', Y, Y', L are all concentrated in degree zero (by construction), hence the shifts in the definition. On the other hand, U, T, T^* live in multiple degrees. Also, note that, by e.g. the Universal Coefficient Theorem, when torsion appears in the cohomology over $\mathbb{Z}$, then if we work instead over $\mathbb{F}_p$, the torsion is tensored by $\mathbb{F}_p$ and replicated in degree one lower (one higher for homology). The free part carries over (tensored by $\mathbb{F}_p$ of course).

Notation 3.2.6. Let the **bad primes** consist of 2 for $Q = D_n$, 2, 3 for $Q = E_6$ or E_7, and 2, 3, 5 for $Q = E_8$, and no others.

Theorem 3.2.7.

(i) *The modules T, T^* are finite with torsion elements of order equal to bad primes, and K', Y' are finite with torsion of order dividing the Coxeter number h. The modules K, Y, L, U are free over $\mathbb{Z}$.*
(ii) *We have*

$$\underline{\mathrm{HH}}^0(A, A) \cong U(-2) \oplus Y(h-2) \oplus Y'(h-2), \qquad \underline{\mathrm{HH}}^1(A, A) \cong U(-2), \tag{3.2.8}$$

$$\underline{\mathrm{HH}}^2(A, A) \cong K(-2) \oplus T(-2), \qquad \underline{\mathrm{HH}}^3(A, A) \cong K(-2) \oplus K'(-2), \tag{3.2.9}$$

$$\underline{\mathrm{HH}}^4(A, A) \cong U^*(-2) \oplus T^*(-2), \qquad \underline{\mathrm{HH}}^5(A, A) \cong U^*(-2) \oplus Y(-h-2), \tag{3.2.10}$$

$$\underline{\mathrm{HH}}_i(A, A) \cong \underline{\mathrm{HH}}^{2-i}(A, A)(2), \qquad \underline{\mathrm{HH}}^{6+i} \cong \underline{\mathrm{HH}}^i(-2h), \tag{3.2.11}$$

$$\mathrm{HH}^0(A, A) \cong U(-2) \oplus L(h-2), \qquad \mathrm{HH}_0(A, A) \cong \mathbb{Z}^I \oplus T. \tag{3.2.12}$$

(iii) *For any prime p, letting $M_p := M \otimes \mathbb{F}_p$ for all M, we have*

$$\underline{\mathrm{HH}}^0(A_p, A_p) \cong U_p(-2) \oplus Y_p(h-2) \oplus Y'_p(h-2), \qquad \underline{\mathrm{HH}}^1(A_p, A_p) \cong U_p(-2) \oplus T_p(-2), \tag{3.2.13}$$

$$\underline{\mathrm{HH}}^2(A_p, A_p) \cong K_p(-2) \oplus K'_p(-2) \oplus T_p(-2), \qquad \underline{\mathrm{HH}}^{5-i}(A_p, A_p)(2) \cong \big(\underline{\mathrm{HH}}_i(A_p, A_p)(2)\big)^*, \tag{3.2.14}$$

$$\underline{\mathrm{HH}}_i(A_p, A_p) \cong \underline{\mathrm{HH}}^{2-i}(A_p, A_p)(2), \qquad \underline{\mathrm{HH}}^{6+i} \cong \underline{\mathrm{HH}}^i(-2h), \tag{3.2.15}$$

$$\mathrm{HH}^0(A_p, A_p) \cong U_p(-2) \oplus L_p(h-2), \qquad \mathrm{HH}_0(A_p, A_p) \cong \mathbb{F}_p^I \oplus T_p. \tag{3.2.16}$$

(iv) *For any (bad) prime p, T_p^* is cyclic as an $\underline{\mathrm{HH}}^0(A_p, A_p)$-module.*
(v) *The BV differential Δ (and hence the Connes differential) is graded self-adjoint over any field* **k**. *In particular, for* $\mathbf{k} = \mathbb{F}_p$, *$\Delta$ is zero restricted to any summand K'_p, Y'_p, T_p, T_p^*.*

The remainder of this subsection is devoted to the proof of the theorem.

Lemma 3.2.17. *The center of A does not increase over positive characteristic. That is, the map $Z(A) \otimes \mathbb{F}_p \to Z(A \otimes \mathbb{F}_p)$ is an isomorphism for all ADE quivers.*

We omit the proof of the above lemma, which was done using MAGMA for type E, and a straightforward explicit computation in the A_n, D_n cases, using bases in terms of paths in the quiver. Note that the lemma is actually true for **all** quivers, since the non-Dynkin case is proved in [CBEG07,Sch07].

Corollary 3.2.18. *The groups* $\mathrm{HH}^1(A, A)$, $\mathrm{HH}^5(A, A)$, $\mathrm{HH}_1(A, A)$, *and* $\mathrm{HH}_3(A, A)$ *are all free $\mathbb{Z}$-modules.*

Proof. It is enough to show that they are torsion-free. For HH^1, this follows from the fact that the differential d_0 in $C^\bullet(A, A)$ must have saturated image (otherwise $\mathrm{HH}^0(A, A)$ would increase in some positive characteristic); alternatively, this is a consequence of the universal coefficient theorem. For

$\mathrm{HH}^5(A,A)$, we use the derived duality $\mathbb{D}_{\mathbf{k}} : C^\bullet(A,A) \xrightarrow{\sim} C^{5-\bullet}(A,A)^*$, so that the differential d_0 corresponds to d_4^* in the latter (again, we could also use the universal coefficient theorem). Then, the duality $\mathbb{D}$ of dimension 2 and the periodicity 6 gives the results for Hochschild homology: $\mathrm{HH}^j \cong \mathrm{HH}_{6n+2-j}$ whenever $j, 6n+2-j$ are both positive. $\square$

Using the duality $\mathbb{D}$ and the periodicity by period 6, it remains only to compute the torsion of $\mathrm{HH}^2(A,A), \mathrm{HH}^3(A,A), \mathrm{HH}^4(A,A)$, and $\mathrm{HH}^6(A,A)$. Using the duality $\mathbb{D}_{\mathbf{k}}$, HH^2 and HH^4 must have dual torsion, so it is really enough to compute in degrees $2, 3$, and 6. We will see that the torsion in these degrees will be nontrivial, but only in bad primes (2 for D_n, $2, 3$ for E_6 and E_7, and $2, 3, 5$ for E_8), and primes dividing the Coxeter number h.

First, by the duality $\mathbb{D}$, the torsion of $\underline{\mathrm{HH}}_0(A,A)$ and $\underline{\mathrm{HH}}^2(A,A)$ are isomorphic. Since $\mathrm{HH}_0(A,A)$ has no torsion in degree zero, and the inclusion $\underline{\mathrm{HH}}_0(A,A) \subset \mathrm{HH}_0(A,A)$ is full in nonzero degrees, the torsion of $\underline{\mathrm{HH}}_0(A,A)$ (and hence of $\underline{\mathrm{HH}}_2(A,A)$) is the same as that of $\mathrm{HH}_0(A,A)$. The latter was computed in [Sch07], and we collect results for convenience:

Proposition 3.2.19. *(See [Sch07, Theorem 4.2.60].) The module* $\mathrm{HH}_0(A,A) \cong \mathbb{Z}^I \oplus T$, *where* $T := \mathrm{HH}_0(A,A)_+$ *is finite and given as follows*:

- *For* $Q = A_n$, $T = 0$,
- *For* $Q = D_n$,

$$T \cong \bigoplus_{4|m,\ 0<m\leqslant 2(n-2)} \mathbb{Z}/2(m). \tag{3.2.20}$$

- *For* $Q = E_n$, *T is a (finite) direct sum of shifted copies of* $\mathbb{Z}/2$ *and* $\mathbb{Z}/3$, *and in the case* $n = 8$, *also of* $\mathbb{Z}/5$. *In particular*:

$$T_{E_6} \cong \mathbb{Z}/2(4) \oplus \mathbb{Z}/3(6), \tag{3.2.21}$$

$$T_{E_7} \cong \big(\mathbb{Z}/2(4) \oplus \mathbb{Z}/2(8) \oplus \mathbb{Z}/2(16)\big) \oplus \mathbb{Z}/3(6), \tag{3.2.22}$$

$$T_{E_8} \cong \big(\mathbb{Z}/2(4) \oplus \mathbb{Z}/2(8) \oplus \mathbb{Z}/2(16) \oplus \mathbb{Z}/2(28)\big) \oplus \big(\mathbb{Z}/3(6) \oplus \mathbb{Z}/3(18)\big) \oplus \mathbb{Z}/5(10). \tag{3.2.23}$$

Moreover, for any Q and any bad prime p, there exists a top-degree torsion element, $r_{p,\mathrm{top}}$, such that all homogeneous p-torsion elements $[x]$ have the property that $[x \cdot z] = r_{p,\mathrm{top}}$ for some homogeneous central element $z \in \mathrm{HH}^0(\Pi_Q, \Pi_Q)$.

We deduce immediately that $T^* \otimes \mathbb{F}_p$ is cyclic under dual contraction, and hence (by (2.1.33)) also under contraction. By the intertwining property (2.3.29), we deduce part (iv) of Theorem 3.2.7.

It remains only to compute the torsion of $\underline{\mathrm{HH}}^3(A,A)$ and $\underline{\mathrm{HH}}^6(A,A)$, i.e., to compute K' and Y' and verify that there is no other torsion. We will use some results of [Eu08] for this, but let us explain them using our language. Using the formulation of Definition 3.2.1 and the (normalized) bar complex, it suffices to compute the torsion of the cokernels of the maps

$$C_0(A,A) \cong A \xrightarrow{d_0} A^* \cong C_{-1}(A,A), \qquad C^{-1}(A,A) \cong A_{\eta^{-1}} \xrightarrow{d^{-1}} A \cong C^0(A,A). \tag{3.2.24}$$

To express these maps, write $\mathrm{Id} = \sum_i x_i^* \otimes x_i \in A^* \otimes A$. Then, the map $A \otimes A \to A^* \otimes A$ in the normalized bar resolution is given by $(x \otimes y) \mapsto xy \cdot \mathrm{Id}$. So, the "conorm" differential d_0 in (3.2.24) must be given by $y \mapsto \sum_i x_i y x_i^* \in A^*$. Since we may assume $\deg(x_i) = -\deg(x_i^*)$, the image can only be nonzero if $\deg(y) = 0$. Similarly, the "norm" differential d^{-1} is given by $d^{-1}(y) = \sum_i x_i^* y x_i \in A$, now viewing x_i^* as an element of A via $(\,,\,)$. Here also, only $\deg(y) = 0$ need be considered. We deduce

Proposition 3.2.25.

(i) *The conorm, d_0, and norm, d^{-1}, maps are given by*

$$d_0(y) = \sum_i x_i y x_i^* \in A^*, \qquad d^{-1}(y) = \sum_i x_i^* y x_i \in A, \tag{3.2.26}$$

where

$$\mathrm{Id} = \sum_i x_i^* \otimes x_i \in A^* \otimes A. \tag{3.2.27}$$

The image of the conorm and norm maps d_0, d^{-1} must lie in the top degree of A^*, A, respectively.

(ii) *Identifying A^* with A using $(\,,)$, and letting $\omega_i \in e_i A e_{\bar{\eta}(i)}$ denote the image of $e_i^* \in A^*$, we have*

$$d_0(e_i) = \sum_{j \in I} \mathrm{tr}(\mathrm{Id}|_{e_j A e_i})\omega_j, \qquad d^{-1}(e_i) = \sum_{j \in I} \delta_{j,\bar{\eta}(j)}\delta_{i,\bar{\eta}(i)}\mathrm{tr}(\eta|_{e_i A e_j})\omega_j. \tag{3.2.28}$$

Proof. Part (i) has already been proved and is more generally true for any (graded) Frobenius algebra A. We show part (ii), using the fact that $A[0] \cong \mathbf{k}^I$ as a subalgebra, with η acting by the permutation $\bar{\eta}$. Let $f : A \to \mathbf{k}$ be the function such that $(a, b) = f(ab)$. We have that $f(e_i, \omega_i) = f(\omega_i) = 1$ for all i. Thus, to compute $d_0(e_i)$, it is enough to find $f(e_j d_0(e_i)) = f(\sum_\ell (e_j x_\ell e_i x_\ell^*))$ for all $j \in I$. This is the same as $\sum_\ell (e_j x_\ell e_i, x_\ell^*)$, which is the trace of the projection $A \twoheadrightarrow e_j A e_i$. Similarly, we have

$$\sum_\ell \bigl(e_j x_\ell^* e_i, x_\ell\bigr) = \sum_\ell \bigl(\eta(e_i x_\ell)e_j, x_\ell^*\bigr) = \mathrm{tr}\bigl(x \mapsto e_{\bar{\eta}(i)}\eta(x)e_j\bigr) = \mathrm{tr}(\eta|_{e_i A e_j})\delta_{j,\bar{\eta}(j)}\delta_{i,\bar{\eta}(i)}. \qquad \square \tag{3.2.29}$$

We note that in [Eu08], sums such as (3.2.26) (with x_i a basis) are used to describe K and Y; the proposition above explains their origin through norm and conorm maps. In particular, bases are not needed, and under the connectivity assumption $A[0] \cong \mathbf{k}^I$ of (ii), one can re-express the sum as a trace. We believe that the necessity of using such formulas to describe usual Hochschild cohomology gives further justification for studying stable Hochschild cohomology.

Using the proposition, to compute K' and Y', it suffices to compute two matrices: an $I \times I$-matrix, H_Q, whose entries are $(H_Q)_{ij} = \mathrm{tr}(\mathrm{Id}_{e_i A e_j})$, and an $I^{\bar{\eta}} \times I^{\bar{\eta}}$-matrix, H_Q^η, whose entries are $(H_Q^\eta)_{ij} = \mathrm{tr}(\eta_{e_i A e_j})$, where $I^{\bar{\eta}} := \{i \in I\colon\ \bar{\eta}(i) = i\}$. These matrices were computed in the D, E cases in [Eu08]. In fact, H_Q itself was originally computed for all Dynkin cases in [MOV06]: since A is free over $\mathbf{k}$, $H_Q = h(A; 1)$ is the Hilbert series matrix $h(A; t)$ evaluated at $t = 1$. In [MOV06] is the following formula for $h(A; t)$:

$$h(A; t) = \bigl(1 + Pt^h\bigr)\bigl(1 - Ct + t^2\bigr)^{-1}. \tag{3.2.30}$$

So $H_Q = (1 + P)(2 - C)^{-1}$. These are then easy to compute. It is also not difficult to compute H_Q^η, which we omit, since the D, E cases are already in [Eu08], and the A case is easy. We obtain the following (for $\mathbf{k} = \mathbb{Z}$):

Proposition 3.2.31. *K' and Y' are zero if $Q = A_n$, and otherwise are given by*

$$K' \cong \begin{cases} \mathbb{Z}/2)^{\oplus 2\lfloor \frac{n}{2} \rfloor - 2}, & Q = D_n, \\ (\mathbb{Z}/2)^{\oplus 2}, & Q = E_6, \\ (\mathbb{Z}/2)^{\oplus 6}, & Q = E_7, \\ (\mathbb{Z}/2)^{\oplus 8}, & Q = E_8, \end{cases} \tag{3.2.32}$$

$$Y' \cong \begin{cases} \mathbb{Z}/2, & Q = D_n, \quad n \text{ even}, \\ \mathbb{Z}/(n-1) = \mathbb{Z}/(h/2), & Q = D_n, \quad n \text{ odd}, \\ \mathbb{Z}/3, & Q = E_7, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2.33}$$

This proves (i) of Theorem 3.2.7. At this point, (ii) and (iii) are immediate from the dualities and the Universal Coefficient Theorem.

It remains to prove part (v). Since we already know (by Corollary 3.1.18) the result for characteristic zero, it suffices to take $\mathbf{k} = \mathbb{F}_p$ for some prime p. As before, let $1 \in \underline{\mathrm{HH}}^0$ be the identity, and $1' \in \underline{\mathrm{HH}}^6$ induce the periodicity. First, we note that $\Delta(1) = 0$ and $\Delta(1') = 0$ because this is true over $\mathbb{Z}$ (by [Eu07a]), and moreover that $\{(1')^i, (1')^j\} = 0$ for all i, j for the same reason. Hence, $L_{1'}$ is also graded self-adjoint, and it suffices to verify that $(\Delta(a), b) = (-1)^{|a|}(a, \Delta(b))$, when $0 \leqslant |a|, |b| \leqslant 5$, and hence either $|a| + |b| = 6$ or $|a| = |b| = 0$. To do this, we will show that Δ kills summands of the form K'_p, Y'_p, T_p and T^*_p (the second statement of (v)).

It is clear, for graded degree reasons, that Δ kills summands of the form Y', K', T, T^* over $k = \mathbb{Z}$. It remains to show that the new summands appearing over $\mathbb{F}_p$ are also killed. For K' in $\underline{\mathrm{HH}}^2$, this is true for degree reasons and the fact that the kernel of Δ on $\underline{\mathrm{HH}}^1[0]$ is zero (from the characteristic-zero case), using that $\Delta^2 = 0$. For degree reasons, the summand of $T^*_p(-2)$ in $\underline{\mathrm{HH}}^3$ and the summand of $Y'(-h-2)$ in $\underline{\mathrm{HH}}^5$ must be killed.

It remains only to show that the summand of $T_p(-2)$ in $\underline{\mathrm{HH}}^1$ is killed. Note first that there is an element $HEu \in \underline{\mathrm{HH}}^1(\Pi_Q)$, the "half-Euler vector field," whose action on closed paths (which must have even length since Q is a tree) is to multiply by half the path-length. It follows from [Eu08] (cf. [Sch05], §10 for the extended Dynkin case and the following subsection) that HEu generates $\underline{\mathrm{HH}}^1(\Pi_Q)$ as an $\underline{\mathrm{HH}}^0(\Pi_Q)$-module in the case that $\mathbf{k} = \mathbb{Z}$. Now, over $\mathbf{k} = \mathbb{F}_p$, $U(-2) \otimes \mathbb{F}_p$ is isomorphic to a direct summand $U_p(-2)$ of $\underline{\mathrm{HH}}^1(\Pi_Q)$. From the argument in [Eu07a], we know that the operator $\Delta|_{\underline{\mathrm{HH}}^1(A,A)}$ acts on $U_p(-2)$ by $\Delta(zHEu) = (\frac{\deg(z)}{2} + 1)z$, for all $z \in \underline{\mathrm{HH}}^0(\Pi_Q)[< h-2]$.

We claim that the Lie derivative L_{HEu} acts on $\underline{\mathrm{HH}}_\bullet$ by multiplication by half the graded degree. For $\mathrm{HH}_\bullet$, this follows from the explicit formula for the Lie derivative (as argued in [Eu07a]), and then this extends easily to $\underline{\mathrm{HH}}_\bullet$ by taking the unique extension guaranteed by Proposition 2.3.52 (using the construction given in the proof). It follows from this that, under the Gerstenhaber bracket, $\mathrm{ad}(HEu)$ acts on $\underline{\mathrm{HH}}^\bullet$ by multiplication by half the graded degree.

From the identity (1.1.23) and the fact that $\Delta(1) = 0$, we have

$$\Delta(HEu \cup x) - \Delta(HEu) \cup x + HEu \cup \Delta(x) = \{HEu, x\} = \frac{\deg(x)}{2} x, \tag{3.2.34}$$

for any homogeneous $x \in \underline{\mathrm{HH}}^1$. Since $\Delta(HEu) = 1$ and $\Delta|_{\underline{\mathrm{HH}}^2} = 0$, we obtain

$$HEu \cup \Delta(x) = \left(\frac{\deg(x)}{2} + 1\right) x. \tag{3.2.35}$$

For any choice of splitting of T_p in $\underline{\mathrm{HH}}^1$, the structure of T as given in Proposition 3.2.19 shows that $\frac{\deg(x)}{2} + 1 = 0$ as an element of $\mathbb{F}_p$, for all $x \in T_p$. Hence, we have

$$HEu \cup \Delta(x) = 0. \tag{3.2.36}$$

Since, for degree reasons, $\Delta(x)$ is in the $U_p(-2)$-summand of $\underline{\mathrm{HH}}^0$, we deduce that $\Delta(x) = 0$. This completes the proof of Theorem 3.2.7.

3.3. The maps between the extended Dynkin and Dynkin preprojective algebras

In this subsection, we will interpret K', Y', and T in terms of the preprojective algebras of the corresponding extended Dynkin quivers. For this, we use the projection $\pi : \Pi_{\tilde{Q}} \twoheadrightarrow \Pi_Q$ and functoriality of Hochschild homology. We need to recall a few results from [Sch07] first.

Let Q be an ADE quiver and let $\tilde{Q}$ be the corresponding extended Dynkin quiver. By [CBEG07, Sch07], we know that $\Pi_{\tilde{Q}}$ is (ordinary) Calabi–Yau of dimension 2, and in particular has Hochschild dimension 2. Let $Z_{\tilde{Q}} := \mathrm{HH}^0(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}})$ and $Z_{\tilde{Q},+} := Z_{\tilde{Q}}[\geqslant 1]$. The ring $Z_{\tilde{Q}}$ is closely related to the Kleinian singularity ring: one has $Z_{\tilde{Q}} \otimes \mathbb{C} \cong \mathbb{C}[x, y]^{\Gamma}$ where Γ is the group corresponding to $\tilde{Q}$ under the McKay correspondence, and one can even replace $\mathbb{C}$ with $\mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$. There is a standard integral presentation of $\mathbb{C}[x, y]^{\Gamma}$ which actually describes $Z_{\tilde{Q}}$ over $\mathbb{Z}$ (see, e.g., [Sch07, Propositions 6.4.2, 7.4.1, and 8.4.1]).

By [Sch07, §10.1], we know that, as graded $\mathbb{Z}$-modules,

$$\mathrm{HH}_0(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}}) \cong \mathbb{Z}^I \oplus Z_{\tilde{Q},+} \oplus T, \tag{3.3.1}$$

$$\mathrm{HH}_1(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}}) \cong Z_{\tilde{Q}}(2) \oplus Z_{\tilde{Q},+}, \tag{3.3.2}$$

$$\mathrm{HH}_2(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}}) \cong Z_{\tilde{Q}}(2), \qquad \mathrm{HH}^i \cong \mathrm{HH}_{2-i}. \tag{3.3.3}$$

Theorem 3.3.4. *The induced maps $\pi_{*,i} : \mathrm{HH}_i(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}}) \to \mathrm{HH}_i(\Pi_Q, \Pi_Q)$ are given as follows*:

(0) *$\pi_{*,0}$ is an isomorphism on $\mathbb{Z}^I \oplus T$, and kills $Z_{\tilde{Q},+}$*;
(1) *$\pi_{*,1}$ is a surjection $Z_{\tilde{Q}}(2) \twoheadrightarrow U$, with kernel the elements of degree $\geqslant h$ (and killing the second factor, $Z_{\tilde{Q},+}$)*;
(2) *$\pi_{*,2} : Z_{\tilde{Q}}(2) \to U \oplus Y(h) \oplus Y'(h)$ is a surjection onto $U \oplus Y'(h)$, killing $Z_{\tilde{Q}}[> h-2]$, and sending $Z_{\tilde{Q}}[h-2]$ onto torsion.*

Moreover, these maps give rise to maps $\mathrm{HH}^i(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}}) \to \mathrm{HH}^i(\Pi_Q, \Pi_Q)$ for $i \in \{0, 1\}$, which describe the image of central elements, and describe descent of outer derivations, related to the above by $\mathbb{D}$. On HH^0, the map $Z_{\tilde{Q}}[h-2] \to L(h-2)$ maps isomorphically to the saturation of the kernel of $L(h-2) \twoheadrightarrow (Y(h-2) \oplus Y'(h-2))$ (i.e., the kernel of $L(h-2) \twoheadrightarrow Y(h-2)$).

Proof. Part (0) is a consequence of [Sch07, Theorem 4.2.60].

(1) Let us prove this by computing instead the map on HH^1; there are duality isomorphisms $\mathrm{HH}^1 \xrightarrow{\sim} \mathrm{HH}_1$ which must commute with π_* because they are realized by explicit maps of chain complexes expressed in terms of the quiver. Recall also that $\mathrm{HH}^1(A, A)$ is the space of outer derivations (derivations of A modulo inner derivations).

We claim that the outer derivations descend from $\Pi_{\tilde{Q}}$ to Π_Q. To see this, we use the explicit description of them for $\Pi_{\tilde{Q}}$ from [Sch07, §10.2]: the outer derivations are realized by certain $\mathbb{Q}$-linear combinations of the half-Euler vector field (multiplying by the degree in Q, setting degree in Q^* to be zero), and maps $\phi_x : y \mapsto \{x, y\}$, using the Poisson bracket $\{,\}$ induced by the necklace Lie bracket on HH_0 (or the Poisson structure on $\mathbb{C}[x, y]^{\Gamma}$). The latter was shown to make sense as a map $\mathrm{HH}_0(\Pi_{Q'}, \Pi_{Q'}) \otimes \Pi_{Q'} \to \Pi_{Q'}$ in [Sch07, §5.2], for any quiver Q', and the half-Euler vector field obviously makes sense. Also, although HH^1 consists of some fractions of sums of these outer derivations, clearly an outer derivation is a multiple of some integer on $\Pi_{\tilde{Q}}$ only if the same is true in Π_Q.

Next, we claim that all outer derivations on Π_Q are obtained in this way. This is an immediate consequence of the fact that $\mathrm{HH}^0(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}})$ surjects to $U(-2)$, since $\mathrm{HH}^1(\Pi_Q, \Pi_Q)$ is $U(-2)$ times the half-Euler vector field mentioned above (cf. [Eu08, Proposition 8.0.4] and [ES98a, II]). We thus deduce the desired statement, and (1).

Next, we prove (2). We note that this is equivalent to the desired statement on the level of HH^0, i.e., for the map $Z_{\tilde{Q}} \to Z_Q$, by virtue of the duality maps $\mathbb{D} : \mathrm{HH}^0(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}}) \xrightarrow{\sim} \mathrm{HH}_2(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}})$ and $\mathbb{D} : \mathrm{HH}^0(\Pi_Q, \Pi_Q) \twoheadrightarrow \mathrm{HH}_2(\Pi_Q, \Pi_Q)$, where the latter is the quotient $L \twoheadrightarrow (Y \oplus Y')$ on L, and the identity on U.

To prove (2), we use the fact that the Connes differential, and hence the BV differential, are functorial. For the extended Dynkin side, by [Sch07, Theorem 10.3.1], the BV differential $\Delta : \mathrm{HH}^1 \to \mathrm{HH}^0$ is the map sending $z \cdot HEu$ to $(HEu + 1)z$, where HEu is the half-Euler vector field, and $z \in HH^1$; Δ kills the derivations related to the Poisson bracket as above. In other words, the map $B : \mathrm{HH}_1 \to HH_2$ sends $(z, w) \in Z_{\tilde{Q}}(2) \oplus (Z_{\tilde{Q}})_+$ to $(HEu + 1)(z) \in Z_{\tilde{Q}}$ (here we ignored the shift by two in applying HEu). On the Dynkin side, the Connes differential is also given by $z \mapsto (HEu + 1)(z)$, for $z \in U$. Hence, by functoriality of the Connes differential, we deduce that $\pi_{*,2}$ is as described in degrees $< h$, and in degree h, has to at least map to Y' (there can be $h/2$-torsion on account of the $(HEu + 1)$).

To complete the argument, it suffices to prove the surjectivity to Y': in terms of HH^0, we have to show that the map $Z_{\tilde{Q}}[h-2] \to L(h-2)$ maps surjectively to the kernel of $L(h-2) \twoheadrightarrow Y(h-2)$. For this, we can perform a relatively easy explicit computation, showing that the elements from [Sch07] map to the saturation of the column span of H_Q^{η}. For A_n, D_n this is straightforward; for E_6, E_8, there is nothing to show; and for E_7, where $h - 2 = 16$, this alternatively follows from Proposition 7.3.3 of [Eu06] (which computes the square of an element $z_8 \in Z_{\tilde{Q}}[8]$: this turns out to be the needed element which spans the rank-one kernel of $L \twoheadrightarrow Y$. But, we already know that $Z_{\tilde{Q}}[8] \xrightarrow{\sim} Z_Q[8]$ by the above.) $\square$

Remark 3.3.5. The above gives an alternative (integral) computation of the algebra structure on $\mathrm{HH}^0(\Pi_Q, \Pi_Q)$ given in [Eu08, §7]: this must be obtained from truncating the "Kleinian singularity" algebra $\mathrm{HH}^0(\Pi_{\tilde{Q}}, \Pi_{\tilde{Q}})$ at degrees $\leqslant h - 2$, and composing with the kernel map $\ker(L \twoheadrightarrow Y) \hookrightarrow L$. The asserted relation to the Kleinian singularity $\mathbb{C}^2/\Gamma$ associated to Q is that $Z_{\tilde{Q}} \cong e_{i_0} \Pi_{\tilde{Q}} e_{i_0}$ [Sch05, Theorem 10.1.1], where i_0 is the extending vertex of $\tilde{Q}$, and that $e_{i_0} \Pi_{\tilde{Q}} e_{i_0} \otimes \mathbb{C} \cong \mathbb{C}[x, y]^{\Gamma}$ (alternatively, instead of by $\mathbb{C}$, one can tensor by any ring containing $\frac{1}{|\Gamma|}$ and $|\Gamma|$th roots of unity).

Remark 3.3.6. In [Sch07], the exact structure of T turned out to be mandated by the requirement that, for non-Dynkin, non-extended Dynkin quivers $\hat{Q} \supsetneq \tilde{Q} \supsetneq Q$, the torsion of $\mathrm{HH}_0(\Pi_{\hat{Q}}, \Pi_{\hat{Q}})$ is $\mathbb{Z}/p$ in degrees $2p^{\ell}$ for all primes p and all $\ell \geqslant 1$, and these are generated by elements of the form $\frac{1}{p}[r^{p^{\ell}}]$ (where $r = \sum_{a \in Q} aa^* - a^*a$ is the relation). The specific structure of the torsion in the Dynkin and extended Dynkin cases compensates for the fact that $Z_{\tilde{Q}}$ is missing some degrees that would otherwise be necessary to produce the torsion of $\mathrm{HH}_0(\Pi_{\hat{Q}}, \Pi_{\hat{Q}})$ (using the description in [Sch07, Theorem 4.2.30], of torsion elements of $\mathrm{HH}_0(\Pi_{\hat{Q}}, \Pi_{\hat{Q}})$ not coming from $\mathrm{HH}_0(\Pi_Q, \Pi_Q)$ as cyclic products of elements $r_{\tilde{Q}}$ with elements of $Z_{\tilde{Q}}$.)

4. Hochschild (co)homology of centrally extended preprojective algebras

In this section, we compute the BV structure on the Hochschild cohomology of centrally extended preprojective algebras A over $\mathbf{k} = \mathbb{C}$, and verify that the BV differential is graded self-adjoint (hence, the Hochschild cohomology is a BV Frobenius algebra). From this, the structure of calculus on $(\underline{\mathrm{HH}}^\bullet(A), \underline{\mathrm{HH}}_\bullet(A))$ easily follows as in Remark 2.3.67, using the duality $\mathbb{D}$ (we omit the explicit formulas).

As before, let Q be a quiver of ADE type with vertex set I. In [ER06], the centrally extended preprojective algebra Π_Q^{μ} is defined as a central extension of Π_Q, in terms of a parameter $\mu \in \mathbf{k}^I$. We assume that μ is a **regular weight**, i.e., if $\mu = \sum_{i \in I} \mu_i \cdot e_i$ for $\{e_i\} \subset \mathbf{k}^I$ the idempotents corresponding to I, then $(\sum_i \mu_i \omega_i, \alpha) \neq 0$ for any root α of the root system attached to Q, with ω_i the fundamental weights. Explicitly, for all $\alpha = \sum_{i \in I} \alpha_i e_i \in \mathbb{Z}^I$ such that $\sum_{i \in I} \alpha_i^2 - \sum_{a \in Q} \alpha_{h(a)} \alpha_{t(a)} = 1$, we have $\sum_i \alpha_i \mu_i \neq 0$.

Let $P_{\overline{Q}}[z]$ be the algebra of polynomials in the central parameter z with coefficients in $P_{\overline{Q}}$. Then we define

$$\Pi_Q^\mu := P_{\overline{Q}}[z] \Big/ \Big\langle z \cdot \mu - \sum_{a \in Q} [a, a^*] \Big\rangle. \tag{4.0.1}$$

This is a graded algebra with $\deg(e) = \deg(e^*) = 1$, $\deg(z) = 2$, for all $e \in Q$. Now, let $A := \Pi_Q^\mu$, and let Z be the center of A. Let h denote the Coxeter number of Q. Let $A_+ := A[\geqslant 1]$ denote the part of positive degree, and let $A_{\mathrm{top}} := A[2h-2]$ denote the part of A of top degree.

In [ER06], it is proved that Π^μ is Frobenius over $\mathbf{k} = \mathbb{C}$. We note that, over $\mathbf{k} = \mathbb{Z}$, this is not, in general, true. For example, for $Q = A_2$ and $\mu = \rho = \sum_{i \in I} e_i$, we obtain that $\Pi_{A_2}^\rho \cong P_{\overline{Q}}/2P_{\overline{Q}}[\geqslant 3]$, which is not fg or projective over $\mathbb{Z}$. However, certain parameters $\mu \in \mathbb{Z}^I$ should yield a Frobenius algebra, and we hope to explore this in a future paper. Namely, these parameters should be those such that $(\sum_{i \in I} \mu_i \omega_i, \alpha) = \pm 1$ for all roots α; more generally, for $\Pi_Q^\mu \otimes \mathbb{F}_p$ to be Frobenius over p, the condition should be that $(\sum_{i \in I} \mu_i \omega_i, \alpha)$ is not a multiple of p. We hope to explore this in a future paper.

For the rest of this section, let us take $\mathbf{k} := \mathbb{C}$ and assume that μ is regular ($(\sum_i \mu_i \omega_i, \alpha) \neq 0$).

There is a periodic resolution of A of period 4 [Eu06, §3], and A is a symmetric algebra, so we immediately deduce (as stated in Example 2.3.12) that A is a periodic Calabi–Yau Frobenius algebra of dimension 3 (of shift 4) and period 4 (of shift $2h$).

Theorem 4.0.2. *(See [Eu06].) The Hochschild cohomology groups of A over $\mathbf{k} := \mathbb{C}$ are given by (for $n \geqslant 0$):*

$$HH^{4n+1}(A) \cong \big(Z \cap \mu^{-1}[A, A]\big)(-2nh-2) \cong zZ(-2nh-2), \tag{4.0.3}$$

$$HH^{4n+2}(A) \cong A/\big([A, A] + \mu Z\big)(-2nh-2), \tag{4.0.4}$$

$$HH^{4n+3}(A) \cong A_+/[A, A](-2nh-4), \tag{4.0.5}$$

$$HH^{4n+4}(A) \cong Z/A_{\mathrm{top}}(-2(n+1)h). \tag{4.0.6}$$

From the periodicity, we immediately deduce the groups $\underline{\mathrm{HH}}^\bullet$, by allowing n to be an arbitrary integer in the above. The fact that $\underline{\mathrm{HH}}^i \cong (\underline{\mathrm{HH}}^{3-i})^*$ says that the nondegenerate trace pairing [ELR07] induces nondegenerate pairings

$$\big(Z \cap \mu^{-1}[A, A]\big)(-2) \otimes A/\big([A, A] + \mu Z\big) \to \mathbf{k}, \tag{4.0.7}$$

$$A_+/[A, A](-4) \otimes Z/A_{\mathrm{top}} \to \mathbf{k}. \tag{4.0.8}$$

To describe the cup products, as before, it suffices to describe the product between two degrees for every triple of integers between 0 and 3, which sums to 3 modulo 4:

$$(0, 0, 3), (0, 1, 2), (1, 1, 1), (1, 3, 3), (2, 2, 3). \tag{4.0.9}$$

Recall from [ELR07,Eu06] the Hilbert series for these graded vector spaces, using again $m_1 < \ldots < m_{|I|} = h$ to denote the exponents of the root system (note that the sets $\{h - m_i\} = \{m_i\}$ are identical):

$$h\big(\underline{\mathrm{HH}}^0(A); t\big) = h\big(\underline{\mathrm{HH}}^1(A); t\big) = \sum_{i=1}^r \big(t^{2m_i - 2} + t^{2m_i} + \cdots + t^{2h-6}\big), \tag{4.0.10}$$

$$h\big(\underline{\mathrm{HH}}^2(A); t\big) = h\big(\underline{\mathrm{HH}}^3(A); t\big) = \sum_{i=1}^r \big(t^{-2} + 1 + \cdots + t^{2m_i - 6}\big). \tag{4.0.11}$$

Theorem 4.0.12.

(i) *As modules over* $\underline{\mathrm{HH}}^0(A)$, *we have* $\underline{\mathrm{HH}}^1(A) \cong \underline{\mathrm{HH}}^0(A)$ *and* $\underline{\mathrm{HH}}^2(A) \cong \underline{\mathrm{HH}}^3(A) \cong (\underline{\mathrm{HH}}^0(A))^*$.
(ii) *All of the cup products* $\underline{\mathrm{HH}}^i(A) \otimes \underline{\mathrm{HH}}^j(A) \to \underline{\mathrm{HH}}^{i+j}(A)$ *for* $1 \leqslant i \leqslant j \leqslant 3$ *are zero except for* $\underline{\mathrm{HH}}^1(A) \otimes \underline{\mathrm{HH}}^2(A) \to \underline{\mathrm{HH}}^3(A)$, *which, using the identifications of* (i), *is the canonical map* $\underline{\mathrm{HH}}^0(A) \otimes (\underline{\mathrm{HH}}^0(A))^* \to (\underline{\mathrm{HH}}^0(A))^*$.

Proof. (i) From [Eu06, p. 10] it follows easily that $\underline{\mathrm{HH}}^1(A)$ is of the desired form. Since there must be a unique (up to scaling) element in degree zero, we can use the explicit isomorphism $\underline{\mathrm{HH}}^0(A) \xrightarrow{\sim} \underline{\mathrm{HH}}^1(A)$, $z \mapsto z \cdot Eu$ where Eu is the Euler vector field. Then, the statements about $\underline{\mathrm{HH}}^2(A), \underline{\mathrm{HH}}^3(A)$ follow immediately from Theorem 2.3.27. (We may even show compatibility with the duality pairings defined in [Eu06] using the trace map of [ELR07], by a simple computation along the lines of [Eu08].)

(ii) For graded degree reasons, using (4.0.10), (4.0.11), and the fact that $\underline{\mathrm{HH}}^4(A) \cong \underline{\mathrm{HH}}^0(A)(-2h)$, the triples $(1,3,3)$ and $(2,2,3)$ of multiplications (4.0.9) are zero. Then, $\underline{\mathrm{HH}}^1(A) \cup \underline{\mathrm{HH}}^1(A) = 0$ since $Eu \cup Eu = 0$, by graded-commutativity. The final statement then follows from Theorem 2.3.27. $\square$

We now describe explicitly the Connes and BV differentials. For this, we fix the isomorphism $\mathbb{D} : \underline{\mathrm{HH}}_\bullet(A) \cong \underline{\mathrm{HH}}^{3-\bullet}(A)$ of Theorem 2.3.27, and use the elements $Eu \in \underline{\mathrm{HH}}^{4m+1}(A)$, $Eu^* \in \underline{\mathrm{HH}}^{4m+2}(A)$, and $1^* \in \underline{\mathrm{HH}}^{4m+3}(A)$. Here, the notation is a bit abusive, since really $Eu^* \in \underline{\mathrm{HH}}^6(A)$, $1^* \in \underline{\mathrm{HH}}^7(A)$ using Theorem 2.3.27, but we identify these elements with their images under the periodicity. We describe all elements of $\underline{\mathrm{HH}}^{4m+2}(A), \underline{\mathrm{HH}}^{4m+3}(A)$ by $Eu^*/z, 1^*/z$ for $z \in \underline{\mathrm{HH}}^0(A)$, which refers to the unique elements so that $z \cup Eu^*/z = Eu^*$ and $z \cup 1^*/z = 1^*$.

Theorem 4.0.13. *With the above identifications, the BV differential is given by, for all* $m \in \mathbb{Z}$,

$$\Delta_{2m} = 0, \tag{4.0.14}$$

$$\Delta_{4m+1}(zEu) = \big(\deg(z) + 4 - 2hm\big) z \cdot (1')^{\cup m}, \tag{4.0.15}$$

$$\Delta_{4m+3}\big(1^*/z\big) = \big(2h(1-m) - 4 - \deg(z)\big) Eu^*/z. \tag{4.0.16}$$

In particular, Δ *is graded self-adjoint, i.e.,* $\underline{\mathrm{HH}}^\bullet$ *is BV Frobenius.*

Proof. We use the Cartan identity (2.3.45) in the case $a = Eu$: $Bi_{Eu} + i_{Eu}B = \mathcal{L}_{Eu}$. Also, it is easy to check (as in e.g. [Eu07a]) that $\mathcal{L}_{Eu}(f) = \deg(f) \cdot f$ for all $f \in \mathrm{HH}_\bullet(A)$ (and hence for $\underline{\mathrm{HH}}_\bullet(A)$ as well). From this (using (4.0.10), (4.0.11), the fact that the Calabi–Yau shift is 4, and the vanishing of $\underline{\mathrm{HH}}^1(A) \cup \underline{\mathrm{HH}}^3(A)$) we compute $B|_{\mathrm{HH}_0(A)}$:

$$\mathbb{D}\big(Eu \cup \Delta\big(1^*/z\big)\big) = (Bi_{Eu} + i_{Eu}B)\big(\mathbb{D}\big(1^*/z\big)\big) = \big(2h - 4 - \deg(z)\big) \cdot 1^*/z, \tag{4.0.17}$$

which implies that $\Delta_3(1^*/z) = (2h - 2 - \deg(z)) Eu^*/z$, using Theorem 4.0.12(ii). Then, $\Delta^2 = 0$ implies that $\Delta_2 = 0$.

Inductively, we claim that $\Delta_{2m} = 0$ and Δ_{2m+1} is an isomorphism, for all $m \leqslant 1$. Assume that $\Delta_{2m} = 0$ and Δ_{2m+1} is an isomorphism. We need only show that Δ_{2m-1} is also an isomorphism. This follows from $\mathcal{L}_{Eu} = B_{3-2m} i_{Eu} + i_{Eu} B_{4-2m} = i_{Eu} B_{4-2m}$, together with the fact that $\mathcal{L}_{Eu}$ and i_{Eu} are isomorphisms (here, we use that the degrees of $\underline{\mathrm{HH}}_0(A)$ are between 0 and $2h-6$). At the same time, we may deduce the desired formulas (since $\mathcal{L}_{Eu}$ multiplies by degree).

It remains only to prove that the formulas still hold in Hochschild degrees > 3. To show this, we consider the formula

$$\mathcal{L}_{1'} = Bi_{1'} - i_{1'}B, \tag{4.0.18}$$

for $1' \in \underline{\mathrm{HH}}^4(A)$ the periodicity element. From this, we may compute that, applied to degrees $\leqslant 3$, $\mathcal{L}_{1'}$ kills even degrees, and on odd degrees acts by $\mathcal{L}_{1'}(zEu) = 2hz(1')^m$ (where $|zEu| = 4m+1$), and $\mathcal{L}_{1'}(1^*/z) = 2hEu^*/z$. Since $\mathcal{L}_{1'}$ is a derivation and $\mathcal{L}_{1'}(1') = 0$, we deduce the desired result. □

5. Periodic group algebras of finite groups

As mentioned already, for any finite group G, the group algebra $\mathbf{k}[G]$ is Frobenius, and in fact symmetric (hence, Calabi–Yau Frobenius). It is natural to ask when such group algebras are periodic.

Certainly, if $\mathbf{k}[G]$ is periodic, then its Hochschild cohomology is periodic. It is well known that one has the following formula for Hochschild cohomology, as an abstract graded $\mathbf{k}$-module:

$$\mathrm{HH}^\bullet\big(\mathbf{k}[G], \mathbf{k}[G]\big) = \bigoplus_{\substack{\text{conjugacy classes } C_i \\ \text{with representative } c_i \in C_i}} H^\bullet\big(Z_G(c_i), \mathbf{k}\big), \tag{5.0.1}$$

where $Z_G(c_i)$ is the centralizer of c_i in G, and the $H^\bullet(H, \mathbf{k})$ denotes the *group cohomology* of H with coefficients in $\mathbf{k}$. (For an explanation, see Proposition 5.0.6, where we give a refined version.)

Hence, in order for the Hochschild cohomology to be periodic, it must be that the numbers of generators of the cohomology groups of G are bounded. Let us now set $\mathbf{k} := \mathbb{Z}$. Then, the classical Suzuki–Zassenhaus theorem classifies all such groups. These groups are those such that all abelian subgroups are cyclic, and they fall into six explicit families (cf. [AM04, p. 150]). Moreover, these all have periodic group cohomology. Since this property is preserved under taking subgroups, we deduce the (probably well known)

Proposition 5.0.2. *The group algebra $\mathbb{Z}[G]$ of a finite group G has periodic Hochschild cohomology iff all abelian subgroups of G are cyclic. For such groups, $\mathbf{k}[G]$ has periodic Hochschild cohomology (relative to $\mathbf{k}$), for all commutative rings $\mathbf{k}$.*

We would like to know if such group algebras are in fact periodic Calabi–Yau Frobenius algebras (since they are symmetric, it is enough to check if they are periodic Frobenius). This is stronger than having periodic Hochschild cohomology, since we actually need $\Omega^n\mathbf{k}[G] \simeq \mathbf{k}[G]$ for some $n \geqslant 1$. This would be satisfied if we could show that such $\mathbf{k}[G]$ have periodic resolutions.

Fortunately, there is a very similar classical result of Swan (which also used the (mod-p) classification of periodic groups):

Theorem 5.0.3. *(See [Swa60].) Let $R := \mathbb{Z}[S^{-1}]$ for some set S of primes. Let G be a finite group. Then, there is a periodic resolution of R as an $R[G]$ module, i.e.,*

$$0 \to R \hookrightarrow P_{n-1} \to P_{n-2} \to \cdots \to P_0 \twoheadrightarrow R \to 0, \tag{5.0.4}$$

iff G has periodic group cohomology with coefficients in R.

We deduce the following:

Theorem 5.0.5. *The group algebra $\mathbb{Z}[G]$ is periodic Calabi–Yau Frobenius iff G has the property that all abelian subgroups are cyclic. Equivalently, $\mathbb{Z}[G]$ is periodic CY Frobenius of period n iff its Hochschild cohomology (or the group cohomology of G) is periodic of period n.*

Proof. Using the above results, it is enough to show how to explicitly pass between a projective resolution of $\mathbf{k}$ as a $\mathbf{k}[G]$-module, and a projective resolution of $\mathbf{k}[G]$ as a $\mathbf{k}[G]^e$-module, in such a way as to preserve periodicity of a given period. More generally, we prove the following proposition. □

Proposition 5.0.6. *Let $H = (H, \mu, \Delta, \eta, \epsilon, S)$ be any Hopf algebra over $\mathbf{k}$ which is a projective $\mathbf{k}$-module. Let $H_\Delta := (1 \otimes S) \circ \Delta(H) \subset H \otimes H^{\mathrm{op}}$. Then, given any projective resolution $P_\bullet$ of $\mathbf{k}$ as an H-module, $\mathrm{Ind}_{H_\Delta}^{H^e} P_\bullet$ is a projective resolution of H as an H-bimodule, which is split as a complex of left H-modules.*

Conversely, if $Q_\bullet$ is a resolution of H as an H-bimodule which is split as a complex of right H-modules, then $Q_\bullet \otimes_H \mathbf{k}$ is a projective resolution of $\mathbf{k}$ as an H-module.

To remove the left–right asymmetry, note that a sequence of (projective) H-bimodules which are split as left H-modules can have the H-bimodule action twisted to $a \star m \star b := S(b)mS(a)$ to make them split as right H-modules rather than as left H-modules. If we apply $\otimes_H \mathbf{k}$ to the twisted version of H, we still obtain $\mathbf{k}$.

Proof. We only prove the first statement, since the last one is easy. The proof is essentially a refinement of the usual proof of (5.0.1). We claim that (1) $\mathrm{Ind}_{H_\Delta}^{H \otimes H^{\mathrm{op}}} \mathbf{k} \cong H$ as H-bimodules, and (2) with the left $(H \otimes 1)$ and right H_Δ actions, H^e is isomorphic as an H-bimodule to $H \otimes H$ with the usual outer H-bimodule structure. By part (2) of the claim, and the fact that H is projective over $\mathbf{k}$, we deduce that $H \otimes H^{\mathrm{op}}$ is a projective H_Δ module. Since $\mathrm{Ind}_{H_\Delta}^{H \otimes H^{\mathrm{op}}}$ is the functor $H^e \otimes_{H_\Delta} -$, we obtain the desired result.

To prove the claim, consider the $\mathbf{k}$-linear maps

$$\phi : H \otimes H \to H \otimes H^{\mathrm{op}}, \qquad \phi(g \otimes h) = g \cdot (1 \otimes S) \circ \Delta(h), \tag{5.0.7}$$

$$\psi : H \otimes H^{\mathrm{op}} \to H \otimes H, \qquad \psi(g \otimes h) = g \cdot (S \otimes 1) \circ \Delta\big(S^{-1}h\big). \tag{5.0.8}$$

By the antipode identity, coassociativity, and the counit condition, $\phi \circ \psi = \mathrm{Id} = \psi \circ \phi$. On the other hand, ϕ intertwines the right H-module structure on $H \otimes H$ with the right $H_\Delta = H$-module structure on $H \otimes H^{\mathrm{op}}$, and ψ intertwines in the opposite direction; also, both intertwine the standard left H-module structure. So, we obtain part (2). Part (1) then follows by explicit (easy) computation. □

Corollary 5.0.9. *A Hopf algebra has a periodic bimodule resolution which is split as a complex of right modules iff its augmentation module has a periodic left module resolution. A Hopf algebra which is a Frobenius algebra is periodic Frobenius iff its augmentation module $\mathbf{k}$ satisfies $\Omega^n \mathbf{k} \simeq \mathbf{k}$ in the stable left-module category.*

As remarked earlier, if $\mathbf{k}$ is a PID, any Hopf algebra which is fg as a $\mathbf{k}$-module is automatically Frobenius [LS69], so we can remove the Frobenius assumption from the corollary in this case (replacing with fg projective over $\mathbf{k}$).

Proof. The first assertion follows immediately from the proposition. For the second, we show that one has a stable module equivalence $\Omega^n \mathbf{k} \overset{H}{\simeq} \mathbf{k}$ iff one has a stable bimodule equivalence $\Omega^n H \overset{H^e}{\simeq} H$. We can use the functors $- \underline{\otimes}_H \mathbf{k}$, $H^e \underline{\otimes}_{H_\Delta} -$ to achieve this. □

The periodic groups described by the theorem include all groups which act freely on spheres:

Theorem 5.0.10. *Let G be any finite group which acts freely and orientation-preserving on a sphere S^ℓ with $\ell \geqslant 1$. Then, for some $r \geqslant 1$, the group algebra $\mathbf{k}[G]$ is a periodic Calabi–Yau Frobenius algebra, of dimension $\frac{\ell+1}{r} - 1$ and period $\frac{\ell+1}{r}$.*

This theorem follows from e.g. Lemma 6.2 of [AM04] (a spectral sequence argument showing that group cohomology is periodic in this case), by using Theorem 5.0.5. We give, however, a simple topological proof in Appendix A, in the case when G acts cellularly on a finite CW complex homeomorphic to S^ℓ.

Corollary 5.0.11. *For any finite subgroup $G < SO(\ell+1) := SO(\ell+1, \mathbb{R})$ for any $\ell \geqslant 1$, $\mathbf{k}[G]$ is periodic Calabi–Yau Frobenius.*

We note that this corollary may also be deduced from the version of the theorem proved in Appendix A, where G acts cellularly. To do this, we form a CW decomposition of $S^{\ell+1}$ by geodesic codimension-one slices, fixed under the orbit of G, which separate a given point x from all of its orbits under G.

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Appendix A. Finite groups acting freely on spheres

In this appendix, we provide an elementary proof of Theorem 5.0.10 in the case that G acts cellularly on a finite CW complex homeomorphic to S^ℓ. Our proof avoids the use of the classification of periodic groups, and is purely topological.

The main idea of the proof is to construct a bimodule resolution of $\mathbf{k}[G]$ by constructing a CW complex which is homotopic to G, and which admits a free action of $G \times G^{\mathrm{op}}$, such that the induced $G \times G^{\mathrm{op}}$-module structure on the cellular homology, $\mathbf{k}[G]$, is the standard bimodule structure. The CW complex will have finitely many cells of each dimension, and the resulting cellular chain complex will be periodic. Thus, this complex yields a periodic bimodule resolution of $\mathbf{k}[G]$.

To do this, we need the following simple topological lemma:

Notation A.0.1. Let D^n, S^n denote the n-dimensional disc and sphere, respectively.

Lemma A.0.2. *Let $m, n \geqslant 0$ be any integers. Consider the topological space $X := D^{m+1} \times S^n$, and let $f : \partial X \to S^m$ be the attaching map which is the first projection of $S^m \times S^n$ to S^m. Then, the glued topological space $X \cup_f S^m$ is homeomorphic to S^{n+m+1}.*

In the special case that $n = 0$, the above construction is the standard way to build S^{m+1} out of S^m: we attach two hemispheres D^{m+1} to S^m placed at the equator.

Proof. Let us view $S^{n+m+1} \subseteq \mathbb{R}^{n+m+2}$ as the unit sphere. Let $Y^\circ \subset S^{n+m+1}$ be the subset

$$Y^\circ := \left\{ (x_0, \dots, x_{n+m+1}) \in S^{n+m+1} \subset \mathbb{R}^{n+m+2}\colon \sum_{i=0}^{m} |x_i|^2 < 1 \right\}. \tag{A.0.3}$$

Now, Y° is an open subset of S^{n+m+1} homeomorphic to $D^{m+1} \times S^n$, under the map

$$D^{m+1} \times S^n \ni (\mathbf{x}, \mathbf{y}) \mapsto \left(\mathbf{x}, \frac{1}{\sqrt{1 - \|x\|^2}} \mathbf{y} \right). \tag{A.0.4}$$

The complement of Y° is the subset $S^m \times \{0\} \subset S^{n+m+1}$. That is, $\partial(Y^\circ) \cong S^m$, and the attaching map $S^{m+1} \times S^n \to \partial(Y^\circ)$ is f. $\square$

In fact, we will not only use the statement of the lemma, but (a cellular version of) the explicit homeomorphism given in the proof.

Proof of the cellular version of Theorem 5.0.10. We will construct the topological space $X := S^\infty \times_{S^\infty/G} S^\infty$ as an explicit CW complex with finite n-skeleta, whose associated cellular chain complex is $(\ell+1)$-periodic. Moreover, the group $G \times G$ will act freely and cellularly (since G is finite, this means that the group sends each d-dimensional cell onto another d-dimensional cell). This gives the desired result by the remarks at the beginning of the subsection.

Let Y denote the finite CW complex with $Y \cong S^\ell$ that we are given. First, we construct from this a topological space $Z \cong D^{\ell+1}$ that extends the action, viewing Y as the boundary of Z. To do this, let $Z := (Y \times [0,1])/(Y \times \{0\})$, and let G act in the obvious way (preserving the second component). We will think of Z as the solid unit disc in $\mathbb{R}^{\ell+1}$ and of Y as its boundary. We will also view Z as a mere topological space isomorphic to $D^{\ell+1}$, i.e., a single $\ell+1$-cell with a G-action, for the purpose of constructing complexes.

Now, set $W_\ell := Y$. We inductively construct a copy W of S^∞ by attaching (viewing Z^k on the LHS as a single $k(\ell+1)$-cell):

$$Z \times Y \cong D^{\ell+1} \times Y \xrightarrow{\text{att.}} Y = W_\ell, \tag{A.0.5}$$

$$Z^2 \times Y \cong D^{2(\ell+1)} \times Y \xrightarrow{\text{att.}} W_{(\ell+1)+\ell}, \tag{A.0.6}$$

$$Z^3 \times Y \cong D^{3(\ell+1)} \times Y \xrightarrow{\text{att.}} W_{2(\ell+1)+\ell}, \ldots, \tag{A.0.7}$$

where "att." means an attaching map (so NOT a map of topological spaces). We define these attaching maps to be the composition of the first projection $S^{k(\ell+1)-1} \times Y \twoheadrightarrow S^{k(\ell+1)-1}$ with the homeomorphism $S^{k(\ell+1)-1} \xrightarrow{\sim} W_{(k-1)(\ell+1)+\ell} = W_{k(\ell+1)-1}$, which exists inductively by Lemma A.0.2. Constructed this way, each homeomorphism $S^{k(\ell+1)+\ell} \xrightarrow{\sim} W_{k(\ell+1)+\ell}$ has image in the same sum of top cells of $Z^k \times Y$: it is the sum of the ℓ cells of Y which make up $S^\ell \xrightarrow{\sim} Y$.

Thus, on the level of chains, if we label the cells of W_∞ in degree $k(\ell+1)+p$ by $c_{k,p} = Z^k \times c_{0,p}$, where $c_{0,p}$ are the cells of Y for $0 \leqslant p \leqslant \ell$, the complex $C_\bullet(W_\infty)$ is a periodic free complex.

By construction, W_∞ has a free action of G. Now, finally, set $X := W_\infty \times G$, and let us view X as the homeomorphic space $W_\infty \times_{W_\infty/G} W_\infty$, and equip X with the resulting free cellular action of $G \times G$. We then have that X is a topological space whose homology is $\mathbf{k}[G]$ with the usual bimodule action. We deduce that $C_\bullet(X)$ is a free periodic resolution of $\mathbf{k}[G]$. $\square$

Appendix B. Frobenius algebras over general commutative base rings

In this appendix, we will extend some results known for Frobenius algebras over fields to a relative context, using e.g. [ARS97] as a reference for the usual results.

We first deduce a relative self-injectivity for A.

Notation B.0.1. For any $\mathbf{k}$-algebra A which is fg projective as a $\mathbf{k}$-module, let $*$ denote the functor $* : \mathrm{Hom}_{\mathbf{k}}(-,\mathbf{k}) : A\text{-mod} \to A^{\mathrm{op}}\text{-mod}$.

Definition B.0.2. Call an A-module I which is fg projective over $\mathbf{k}$ "injective relative to $\mathbf{k}$" or "relatively injective" if, for any A-modules M, N which are projective over $\mathbf{k}$, and any injection $M \hookrightarrow N$ whose cokernel is a projective $\mathbf{k}$-module (i.e., the injection is $\mathbf{k}$-split), there exists a unique dotted arrow completing any diagram of the following form:

$$\begin{tikzcd} I & \\ M \arrow[u] \arrow[r, hook] & N. \arrow[ul, dashed] \end{tikzcd} \tag{B.0.3}$$

The following helps explain the meaning of relative injectivity:

Proposition B.0.4. *Let A be any algebra over a commutative ring $\mathbf{k}$. If M is any module over A which is fg projective as a $\mathbf{k}$-module, then any relatively injective A-module is acyclic for the functor* $\mathrm{Hom}_A(M,-)$. (*I.e., such I satisfy* $\mathrm{Ext}^i_A(M,I)=0$ *for all $i \geqslant 1$.*)

Proof. Let us take a projective resolution $P_\bullet$ of M, i.e., $\cdots \to P_1 \to P_0 \twoheadrightarrow M$. Since M is projective over $\mathbf{k}$, the resolution is $\mathbf{k}$-split. Thus, the relative injectivity property will guarantee that there is no positive homology of $\mathrm{Hom}(P_\bullet, I)$. $\square$

The main use of "relative to $\mathbf{k}$" is to make the dualization $*: M \mapsto \mathrm{Hom}_{\mathbf{k}}(M,\mathbf{k})$ behave well:

Lemma B.0.5. *Assume A is a $\mathbf{k}$-algebra which is a fg projective $\mathbf{k}$-module. Then*:

(i) *The dualization $*: A$-mod $\to A^{\mathrm{op}}$-mod is a contravariant functor which sends projective modules to modules which are injective relative to $\mathbf{k}$, and vice-versa.*
(ii) *There is a functorial isomorphism $M^{**} \cong M$, for M fg projective over $\mathbf{k}$. That is, the restriction of $*$ to fg A-modules which are projective over $\mathbf{k}$ (in both the domain and codomain) is an anti-equivalence of categories, and $* \circ * \simeq \mathrm{Id}$.*

The proof is just as in the case where $\mathbf{k}$ is a field, so we omit it.

Corollary B.0.6. *If A is as in the lemma, then A-mod has enough relatively injectives in the following sense: for every fg A-module M which is projective over $\mathbf{k}$, there exists a relatively injective module I and a $\mathbf{k}$-split injection $M \hookrightarrow I$.*

Proof. For any $\mathbf{k}$-projective $M \in A$-mod, pick a surjection $P \twoheadrightarrow M^*$ in the category A^{op}-mod. This is $\mathbf{k}$-split. Then, dualizing, one obtains a $\mathbf{k}$-split injection $M \hookrightarrow P^*$, and P^* is relatively injective. $\square$

As a corollary, we also deduce the relative self-injectivity for Frobenius algebras:

Corollary B.0.7. *If A is a Frobenius algebra over $\mathbf{k}$, then A is relatively injective as an A-module. Moreover, all projectives are relatively injective and vice-versa.*

Proof. We know that A^* is isomorphic to A as an A-module (using the invariance and nondegeneracy of the pairing). Now, as an A^{op}-module, A is projective; hence A^* is relatively injective as an A-module.

For the last statement, we use the fact that all projectives are direct summands of free modules, and hence are direct summands of relatively injective modules, and are hence relatively injective. For the converse, for any relatively injective module M, M^* is a projective A^{op}-module, hence a relatively injective A^{op}-module, and hence a projective A-module. $\square$

Next we show that the other duality $\vee$ also behaves well for Frobenius algebras in the relative context. All of the following results are more generally true for *relatively self-injective* algebras, which we define as algebras A satisfying the conclusion of Corollary B.0.6: they are fg projective over $\mathbf{k}$ and relatively injective (equivalently, A^* is projective, i.e., all projectives are relatively injective and vice-versa). The same proofs apply.

Proposition B.0.8. *Suppose A is Frobenius over $\mathbf{k}$. Then*:

(i) *The functor $\vee$ restricts to a functor on full subcategories*:

$$\{\text{fg } A\text{-modules which are projective over } \mathbf{k}\} \leftrightarrow \big\{\text{fg } A^{\mathrm{op}}\text{-modules which are projective over } \mathbf{k}\big\}, \tag{B.0.9}$$

and $\vee \circ \vee \simeq \mathrm{Id}$ on these subcategories.
(ii) *The functor $\vee$ preserves **exact** $\mathbf{k}$-split complexes of fg projective $\mathbf{k}$-modules.*

Proof. (i) First, let us show that, if $M \in A$-mod is fg projective over $\mathbf{k}$, then so is $M^\vee$:

$$M^\vee = \mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_A\big(M, (A^*)^*\big) = \mathrm{Hom}_A\big(M, \mathrm{Hom}_{\mathbf{k}}(A^*, \mathbf{k})\big)$$
$$\overset{\text{adj.}}{\cong} \mathrm{Hom}_{\mathbf{k}}\big(A^* \otimes_A M, \mathbf{k}\big) = \big(A^* \otimes_A M\big)^* \cong ({}_{\eta^{-1}}M)^*, \tag{B.0.10}$$

which is fg projective over $\mathbf{k}$. Moreover, this is functorial in M, and we deduce that $\vee \circ \vee \simeq \mathrm{Id}$.

(ii) For bounded-below complexes, this follows from the fact (Proposition B.0.4) that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \geqslant 1$. Then, for any unbounded $\mathbf{k}$-split exact complex of projectives, we truncate at an arbitrary place. □

Corollary B.0.11. *Let A be a Frobenius algebra over $\mathbf{k}$. We have mutually inverse autoequivalences $\Omega, \Omega^{-1} : \underline{A\text{-mod}}_{\mathbf{k}} \to \underline{A\text{-mod}}_{\mathbf{k}}$, which yield exact sequences*

$$0 \to \Omega M \hookrightarrow P \twoheadrightarrow M \to 0, \tag{B.0.12}$$

with P a fg projective A module, for all fg A-modules M which are projective over $\mathbf{k}$.

Proof. Choose, for every module M, a sequence (B.0.12), and similarly a sequence of the form

$$0 \to M \hookrightarrow I \twoheadrightarrow \Omega^{-1}(M) \to 0, \tag{B.0.13}$$

with I satisfying the relative injectivity property. (To obtain such a sequence, form a sequence of the form (B.0.12) for M^* in the category A^{op}-mod, and then dualize.) Then, using the fact that a map $M \to N$ factors through a specific injection $M \hookrightarrow I$ for I relatively injective iff it factors through any other injection into a relatively injective (both are true iff M factors through all injections), it is straightforward to finish using the same arguments as in the case when $\mathbf{k}$ is a field (see, e.g., [ARS97, ASS06]). □

Corollary B.0.14. *Let A be any Frobenius algebra over $\mathbf{k}$. For any degree $i \geqslant 1$, and any fg A-modules M, N which are projective over $\mathbf{k}$,*

$$\mathrm{Ext}^i_A(M, N) \cong \underline{\mathrm{Hom}}_A\big(\Omega^i M, N\big). \tag{B.0.15}$$

Proof. $\mathrm{Ext}^i_A(M, N)$ can be computed using a projective resolution of M,

$$P_{i+1} \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_1} P_0 \twoheadrightarrow M, \tag{B.0.16}$$

by taking the quotient

$$\big\{f \in \mathrm{Hom}_{A^e}(P_i, N) \bigm| f \circ d_{i+1} = 0\big\} / \{g \circ d_i\}_{g \in \mathrm{Hom}_{A^e}(P_{i-1}, N)}. \tag{B.0.17}$$

Now, factor $P_i \to P_{i-1}$ as $P_i \twoheadrightarrow \Omega^i M \hookrightarrow P_{i-1}$ (for $\Omega^i M = \operatorname{im} d_i \cong \operatorname{coker} d_{i+1}$), so that we may now write $\mathrm{Ext}^i_A(M, N)$ as $\mathrm{Hom}_A(\Omega^i M, N)/\{$morphisms factoring through $\Omega^i M \hookrightarrow P_{i-1}\}$. Then, by the observation from the proof of Corollary B.0.11 that morphisms factor through $\Omega^i M \hookrightarrow P_{i-1}$ iff they factor through any other injection to an injective A-bimodule relative to $\mathbf{k}$ (and injective A-bimodules relative to $\mathbf{k}$ are the same as fg projective A-bimodules by Corollary B.0.7), we obtain the desired result. □

Remark B.0.18. When **k** is a field, the above actually endows the stable module category with the structure of a triangulated category, which is the quotient of the derived category of fg A-modules by finite complexes of projective A-modules. However, this is NOT true for general **k** (we had to restrict to the non-abelian subcategory of **k**-projectives before doing anything).

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