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Some Remarks on Generalized Riemann Integral

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1. INTRODUCTION

Because of the desire to calculate the areas of elementary figures, a variety of integrals has been established, popular among which are the Lebesgue integral and the Riemann integral. It is clear that Riemann integral is fundamental in elementary calculus and it can be used to define and calculate many geometric and physical quantities, such as area, volume, and work. However, the Riemann integral has its limitations. The theory of the Lebesgue integral reveals that the Riemann integral is basically used for continuous functions. In fact, $f: [a, b] \to R$ is Riemann integrable iff f is continuous a.e. on $[a, b]$. Also, as is known, the convergence theorems for this integral are severely restricted. With the motivation of generalizing the Riemann integral so as to enlarge the class of integrable functions for which the convergence theorems hold, the Lebesgue integral was successfully established.

Generalizations of the Lebesgue integral, such as the Perron integral and special Denjoy integral, appeared later. The most interesting generalization is the generalized Riemann integral (GR integral for short), discovered by Kurzweil and Henstock independently, although it is equivalent to Perron and special Denjoy integrals. Contrary to the classical exposition of the Lebesgue integral which needs the concepts of measurable sets and measurable functions, before defining the integral, one can define the GR integral directly based on Riemann sums; therefore, the definition is constructive. Then, if one wishes, one can find all the Lebesgue measurable sets and measurable functions via the definition of GR integral. Furthermore, all the convergence theorems can be proved using the definition of the integral [1–3].

It is known that f is Lebesgue integrable iff both f and $|f|$ are GR integrable. Hence, the Lebesgue integral can be introduced through the GR integral, avoiding measure theory. Since the definition of the GR integral is very similar to that of the Riemann integral, one can easily grasp the

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powerful Lebesgue integral, as long as one has a background on the Riemann integral. However, as we illustrate in Section 4, to study the GR integrable functions which are not Lebesgue integrable is at most of marginal interest. We also show that the theory of differential and integral inequalities among GR integrable functions cannot go beyond Gronwall inequality due to the nature of this integral which lacks necessary operational properties.

For the convenience of presentation, we consider $f: [a, b] \to R$, throughout the paper, although most of the conclusions are valid for $f: \Omega \to R^m$, where $\Omega \subset R^n$ bounded or unbounded.

2. Historical Developments

By inverting the result of differentiating a known function, the Newton integral is defined to be

$$(\mathrm{N}) \int_a^b f(x)\, dx = F(b) - F(a), \tag{2.1}$$

where $F: [a, b] \to R$ is a primitive of f in $[a, b]$, that is, $F'(x) = f(x)$ for all $x \in [a, b]$. The more practical Riemann integral is defined as follows.

We say that f is Riemann integrable on $[a, b]$ and

$$(\mathrm{R}) \int_a^b f(x)\, dx = I \tag{2.2}$$

if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every partition of $[a, b]$

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{2.3}$$

with tags

$$x_{k-1} \leqslant t_k \leqslant x_k, \qquad k = 0, 1, 2, ..., n \tag{2.4}$$

we have

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon \tag{2.5}$$

whenever $x_k - x_{k-1} < \delta$ for $k = 1, 2, ..., n$.

Example 2.1. For each k, define $f_k: [0, 1] \to R$ by

$$f_k(x) = \begin{cases} 1 & \text{for} \quad x = j/k!,\ 0 \leqslant j \leqslant k! \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$\lim_{k \to \infty} f_k(x) = f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

However, $f(x)$ is not Riemann integrable, a defect of the Riemann integral mentioned in the Introduction.

The Lebesgue integral successfully overcomes these shortcomings. Further, the Newton–Leibnitz formula

$$F(x) - F(a) = (L) \int_a^x F'(t)\, dt, \tag{2.6}$$

which is valid iff F: $[a, b] \to R$ is absolutely continuous, shows that the theory of the Lebesgue integral makes it possible to reunite the two fundamental concepts of the integral, namely, that of the definite integral and that of the primitive, which appeared to be forever separated as soon as integration went outside the domain of continuous functions.

The original definition of the Lebesgue integral is as follows. Define Lebesgue measurable sets first, and then Lebesgue measurable functions. Suppose that f: $[a, b] \to R$ is Lebesgue measurable and $-M < f(x) < M$ on $[a, b]$. Then f is said to be Lebesgue integrable on $[a, b]$ and

$$(\mathrm{L}) \int_a^b f(x)\, dx = I \tag{2.7}$$

if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every partition of $[a, b]$

$$-M = y_0 < y_1 < \cdots < y_n = M \tag{2.8}$$

with $\max_{0 \leqslant i \leqslant n-1} (y_{i+1} - y_i) < \delta$, we have

$$\left| I - \sum_{i=1}^{n} \xi_i m(E_i) \right| < \varepsilon, \tag{2.9}$$

where $y_{i-1} \leqslant \xi_i \leqslant y_i$, $E_i = \{x \in [a, b]\colon y_{i-1} < f(x) \leqslant y_i\}$, and m denotes the Lebesgue measure.

In general, f as well as its domain may be unbounded; we can truncate the domain and range of f in this case and then use the above definition and limit process to give the general definition.

For every Lebesgue measurable f, $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ $f^- = \max\{-f, 0\}$ are Lebesgue measurable; and for every Lebesgue measurable $g \geqslant 0$, there is an increasing sequence of step functions which converges to g almost everywhere. The above observation enables us to define the Lebesgue integral, avoiding measure theory, as follows.

$M \subset R$ is said to be null if, for every $\varepsilon > 0$, there is a sequence of open sets $\{(a_k, b_k)\}$ such that $M \subset \bigcup_{k=1}^{\infty} (a_k, b_k)$ and $\sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon$. $f: R \to R$ is said to be Lebesgue integrable iff there are increasing sequences of step functions, $\{f_k\}$, $\{g_k\}$, such that $[f_k(x) - g_k(x)] \to f(x)$ as $k \to \infty$ for all x but a null set and

$$\lim_{k \to \infty} (\mathrm{R}) \int_{-\infty}^{\infty} f_k(x)\, dx < \infty, \qquad \lim_{k \to \infty} (\mathrm{R}) \int_{-\infty}^{\infty} g_k(x)\, dx < \infty.$$

(Note that the step function is always Riemann integrable.) We then define

$$(\mathrm{L}) \int_{-\infty}^{\infty} f(x)\, dx = \lim_{k \to \infty} (\mathrm{R}) \int_{-\infty}^{\infty} f_k(x)\, dx - \lim_{k \to \infty} (\mathrm{R}) \int_{-\infty}^{\infty} g_k(x)\, dx. \qquad (2.10)$$

Evidently, this definition is descriptive. See [9].

EXAMPLE 2.2. Consider the function

$$F(x) = \begin{cases} x^2 \sin 1/x^2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Then

$$F'(x) = f(x) = \begin{cases} 2x \sin 1/x^2 - 2/x \cos 1/x^2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Since $f(x)$ is Newton but not Lebesgue integrable, Newton and Lebesgue integrals do not have any inclusive relations. To introduce an integral more general than either the Newton or the Lebesgue integral, we first give some notation.

For f: $[a, b] \to R$, F: $[a, b] \to R$ is termed the major (minor) function of f if, at every $x \in [a, b]$,

$$-\infty \neq \underline{D}F(x) \geqslant f(x) \qquad (+\infty \neq \bar{D}F(x) \leqslant f(x)), \qquad (2.11)$$

where $\underline{D}$ and $\bar{D}$ denote the lower and upper derivatives, respectively. Then

$$\bar{H}(a, b) = \inf\{F(b) - F(a)\}, \qquad (2.12)$$

for all major functions of f, is the upper Perron integral, while

$$\underline{H}(a, b) = \sup\{F(b) - F(a)\}, \qquad (2.13)$$

for all minor functions of f, is the lower Perron integral. f is said to be

Perron integrable in $[a, b]$ if f has both major and minor functions on $[a, b]$ and $\underline{H}(a, b) = \bar{H}(a, b)$; in this case we define

$$(\mathrm{P}) \int_a^b f(x)\, dx = \underline{H}(a, b) = \bar{H}(a, b). \tag{2.14}$$

It is known that if we consider only absolutely continuous major and minor functions of f in (2.12) and (2.13), the resultant integral is exactly the Lebesgue integral. Hence, the Perron integral includes both Lebesgue and Newton integrals. Also, requiring all the major and minor functions to be continuous in (2.12) and (2.13) will yield the same Perron integral. Suppose that F, G are any major and minor functions of f, respectively; then $F(b) - F(a) \geqslant G(b) - G(a)$. Therefore, f is Perron integrable in $[a, b]$ iff, for any $\varepsilon > 0$, there exist a major function F and a minor function G of F on $[a, b]$ such that $F(b) - F(a) < G(b) - G(a) + \varepsilon$. It then becomes clear that the Perron integral may be regarded as a synthesis of two fundamental concepts of integration: one corresponding to the idea of a definite integral as a limit of certain approximating sums, and the other to that of an indefinite integral understood as a primitive function (see [7]). The following result is known [7].

THEOREM 2.1. *A measurable function, which has on $[a, b]$ at least one continuous major function and one continuous minor function, is necessarily Perron integrable on $[a, b]$.*

Another way to generalize the Lebesgue integral is to replace the absolute continuity requirement on F in (2.6) by a weaker one, and, accordingly, a more general integral on the right hand side of (2.6) should be considered. This idea leads to the special Denjoy integral. For defining this integral, we need some preparation.

For a function F: $[a, b] \to R$ and a strictly increasing U: $[a, b] \to R$, we may define the four Dini derivatives of F with respect to U at x. For example, D^+F_U the upper-right Dini derivative with respect to U, is defined by

$$D^+F_U(x) = \lim_{y \to x+} \sup \frac{F(y) - F(x)}{U(y) - U(x)}. \tag{2.15}$$

For any F: $E \subset R \to R$, we introduce

DEFINITION 2.1. The weak variation $V(F; E)$ and strong variation $V_*(F; E)$ of F on E are defined by

$$V(F; E) = \sup \sum |F(b_i) - F(a_i)|, \qquad V_*(F; E) = \sup \sum O(F; I_i), \tag{2.16}$$

where $I_i = [a_i, b_i]$ and $\{[a_i, b_i]\}$ is any sequence of non-overlapping intervals whose end-points belong to E, and $O(F; I_i)$ denotes the oscillation of F on I_i. If $V(F; E)$ $(V_*(F; E)) < +\infty$, the function F is said to be of bounded variation in the wide (restricted) sense on E, or VB (VB_*) on E. F is said to be of generalized bounded variation in the wide (restricted) sense on E, or simply VBG (VBG_*) on E, if E is the sum of a sequence of sets on each of which F is VB (VB_*).

DEFINITION 2.2. F is said to be absolutely continuous in the wide (restricted) sense on E, or simply AC (AC_*) on E, if to each $\varepsilon > 0$ there corresponds an $\eta > 0$ such that for every sequence of non-overlapping intervals $\{I_i = [a_i, b_i]\}$ whose end-points belong to E, $\sum (b_i - a_i) < \eta$ implies $\sum |F(b_i) - F(a_i)| < \varepsilon (\sum O(F; I_i) < \varepsilon)$. F is said to be generalized absolutely continuous in the wide (restricted) sense on E, or simply ACG (ACG_*) on E, if F is continuous on E and E is the sum of a sequence of sets on each of which F is AC (AC_*).

The basic relations among them are given in [7] by

THEOREM 2.2. *In order that F be $AC_*(ACG_*)$ on a bounded closed E, it is necessary and sufficient that F be both VB_* and AC (VBG_* and ACG) on E. F is ACG_* on an open interval I iff there is a strictly increasing and absolutely continuous function with respect to which F has its Dini derivatives finite at every point of I.*

For the difference between ACG and ACG_* functions, it is known that a VBG_* (hence ACG_*) function on E is almost everywhere finitely differentiable on E and this is not true for ACG functions, in general.

$f\colon [a, b] \to R$ is said to be special Denjoy integrable, if there is an ACG_* function F on $[a, b]$ with $F'(x) = f(x)$ a.e. on $[a, b]$, and in this case we define the special Denjoy integral by

$$(\text{D}_*) \int_a^b f(x)\, dx = F(b) - F(a). \tag{2.17}$$

In general, we have

$$F(x) - F(a) = (\text{D}_*) \int_a^x F'(t)\, dt \tag{2.18}$$

iff F is ACG_*. By comparison of (2.18) with (2.6), the intention stated below Theorem 2.1 is realized. Evidently, the special Denjoy integral is more general than the Lebesgue integral, since F from Example 2.2 is ACG_* but not absolutely continuous. In fact, the special Denjoy integral and Perron integral are equivalent (see [7]). The monotone convergence theorem among special Denjoy integrable functions is obviously true since $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_n \leqslant \cdots$ implies that $f_n - f_1$ is non-negative, hence Lebesgue integrable, and

$$\begin{aligned}
&\lim_{n\to\infty} (\mathrm{D}_*) \int_a^b f_n(x)\,dx \\
&\quad= \lim_{n\to\infty} (\mathrm{D}_*) \int_a^b [f_n(x) - f_1(x)]\,dx \\
&\qquad+ (\mathrm{D}_*) \int_a^b f_1(x)\,dx \\
&\quad= \lim_{n\to\infty} (\mathrm{L}) \int_a^b [f_n(x) - f_1(x)]\,dx + (\mathrm{D})_* \int_a^b f_1(x)\,dx \\
&\quad= (\mathrm{L}) \int_a^b [f_\infty(x) - f_1(x)]\,dx + (\mathrm{D}_*) \int_a^b f_1(x)\,dx \\
&\quad= (\mathrm{D}_*) \int_a^b [f_\infty(x) - f_1(x)]\,dx + (\mathrm{D}_*) \int_a^b f_1(x)\,dx \\
&\quad= (\mathrm{D}_*) \int_a^b \lim_{n\to\infty} f_n(x)\,dx.
\end{aligned}$$

3. Generalized Riemann Integral

We have already seen in Section 2 that the Perron–Denjoy integral is more general than the Lebesgue integral, and Perron's approach does not even need the help of any notions of measurability. However, the definitions of these two integrals are by no means easier than that of Lebesgue since both Perron's and Denjoy's definitions are descriptive. Kurzweil and Henstock successfully gave a constructive definition for the so-called generalized Riemann integral (or Henstock integral, or Riemann-complete integral), which is equivalent to the Perron–Denjoy integral, and hence, it can be utilized to define the Lebesgue integral constructively.

As we know, the Riemann integral has the limitation that convergence theorems are severely restricted. This is because the class of Riemann integrable functions is very small, as Example 2.1 shows. Suppose $f(x) \geqslant 0$ on $[a, b]$; then the integral of f over $[a, b]$ should be understood as the area in some sense of the region S under the graph of f. Hence, each term $f(t_k)(x_k - x_{k-1})$ in (2.5) should be a good approximation to the area of the strip S_k under the graph and between $x = x_{k-1}$, $x = x_k$. In the definition of the Riemann integral, we usually choose $\{x_k\}$ first and then require (2.5) for any $t_k \in [x_{k-1}, x_k]$ as long as $x_k - x_{k-1} < \delta$ for $k = 1, 2, ..., n$. This can be impossible if f is very steep in $[x_{k-1}, x_k]$ and $x_k - x_{k-1}$ is relatively large since we have many choices for t_k from $[x_{k-1}, x_k]$. This is exactly the reason why many functions are not Riemann integrable. In other words, the local behavior of f should be considered in the definition in order to generalize the Riemann integral.

Contrary to the classical process, choose t_k first and then, according to the behavior of f around t_k, decide how close x_{k-1} and x_k should be to t_k so that $f(t_k)(x_k - x_{k-1})$ is a good approximation of S_k. Overall, for any $t \in [a, b]$, choose an open interval $\delta(t)$ containing t such that $f(t)(v-u)$ is a good approximation to the area of the strip under the graph between $x = u$ and $x = v$, whenever $t \in [u, v] \subset \delta(t)$. This idea leads to the following definition of the generalized Riemann integral.

f: $[a, b] \to R$ is said to be generalized Riemann integrable and

$$(\mathrm{GR}) \int_a^b f(x)\, dx = I \tag{3.1}$$

if for each $\varepsilon > 0$ there is a function δ: $[a, b] \to (0, \infty)$ such that for any partition of $[a, b]$ with tags $\{t_k\}$,

$$a = x_0 \leqslant t_1 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1} \leqslant t_n \leqslant x_n = b \tag{3.2}$$

satisfying

$$x_k - x_{k-1} < \delta(t_k) \tag{3.3}$$

for each k, it follows that

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon. \tag{3.4}$$

It is easy to prove that for any given δ: $[a, b] \to (0, \infty)$, a partition of $[a, b]$ satisfying (3.2) and (3.3) is always available, and hence the GR integral is unique if it exists.

EXAMPLE 3.1. We are going to prove (GR) $\int_0^1 f(x)\, dx = 0$ for the f of Example 2.1. Enumerate all the rationals in $[0, 1]$ as (r_k) and define

$$\delta(t) = \begin{cases} 1 & \text{for } t \text{ irrational} \\ \varepsilon/2^k & \text{for } t = r_k,\ k = 1, 2, \ldots. \end{cases}$$

For any partition of $[0, 1]$ satisfying (3.2) and (3.3) with respect to so-defined δ, we have

$$\left|\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})\right|$$

$$= \sum{}' f(t_k)(x_k - x_{k-1}) + \sum{}'' f(t_k)(x_k - x_{k-1})$$

$$< \sum_{k=1}^{\infty} f(r_k)(x_k - x_{k-1})$$

$$< \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon,$$

where $\sum'$ denotes the sum for all rational t_k and $\sum''$ for all irrational t_k.

THEOREM 3.1 [4]. *For $F, f\colon [a, b] \to R$, suppose F is continuous and $F'(x) = f(x)$ on $[a, b]$, except at most a countable set. Then f is generalized Riemann integrable on $[a, b]$, and*

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{3.5}$$

It is known that the generalized Riemann integral and Perron–Denjoy integral are equivalent. Hence for an integrable function f, the function must be measurable, finitely valued almost everywhere, $F(x) = (\mathrm{GR}) \int_a^x f(t)\, dt$ is ACG_*, and $F'(x) = f(x)$ a.e. on $[a, b]$.

It is proved [4] that $f\colon [a, b] \to R$ is Lebesgue integral iff both $|f|$ and f are GR integrable. Hence we can equivalently define the Lebesgue measure just through the elementary definition of the GR integral as follows.

For any bounded $E \subset R$, let $E \subset [a, b]$ and

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

If $\chi_E(x)$ is GR integrable on $[a, b]$, we say that E is a Lebesgue measurable set with measure $m(E) = (\mathrm{GR}) \int_a^b \chi_E(x)\, dx$. For any $E \subset R$, let $E_n = E \cap [-n, n]$. If E_N is Lebesgue measurable for all n, we say that E is Lebesgue measurable with measure $m(E) = \lim_{n \to \infty} m(E_n)$. In this manner, we can actually give the whole theory of the Lebesgue integral. It is interesting to point out that if we enlarge the collection of tagged partitions in the definition of the GR integral by allowing that t_k can be outside of $[x_{k-1}, x_k]$ for every k, then the same process will give exactly the Lebesgue integral. See [5].

4. Properties of GR Integral and Integrable Functions

The basic properties are included in the following two theorems. Suppose F, f, g: $[a,b] \to R$, $C \in (a, b)$, α, β are constants. As a convention, we omit GR in front of integral signs.

THEOREM 4.1. (a) *If f, g are GR integrable, so is $\alpha f + \beta g$ and*

$$\int_a^b (\alpha f + \beta g)(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx. \tag{4.1}$$

(b) *f is integrable on $[a, b]$ iff it is integrable on $[a, c]$ for any $c \in (a, b)$ and $\lim_{c \to b^-} \int_a^c f(x)\, dx$ exists. $\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx$ if integrable.*

(c) *f is GR integrable on $[a, b]$ iff it is integrable on $[a, c]$ and $[c, d]$. Moreover,*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \tag{4.2}$$

(d) *If F is continuous and $F'(x) = f(x)$ on $[a, b]$, except at most a countable set, then $|f(x)|$ is GR integrable on $[a, b]$ iff F is of bounded variation on $[a, b]$. Moreover,*

$$\int_a^b |f(x)|\, dx = \bigvee_a^b (F). \tag{4.3}$$

(e) *If f is GR integrable and g is of bounded variation on $[a, b]$, then fg is GR integrable on $[a, b]$. Moreover,*

$$\int_a^b g(x) f(x)\, dx = g(b)F(b) - g(a)F(a) - (S) \int_a^b F(x)\, dg(x), \tag{4.4}$$

where $F(x) = (GR) \int_a^x f(t)\, dt$ and (S) denotes the Stieltjes integral.

(f) *Suppose that f and g are GR integrable. Let $F(x) = \int_a^x f(t)\, dt$ and $G(x) = \int_a^x g(t)\, dx$. Then fG is GR integrable iff Fg is. Moreover,*

$$\int_a^b f(x)G(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b F(x)\, g(x)\, dx. \tag{4.5}$$

(g) *f is GR integrable iff there exists an ACG_* function F on $[a, b]$ such that $F'(x) = f(x)$ a.e. on $[a, b]$. Furthermore,*

$$\int_a^x f(t)\, dt = F(x) - F(a). \tag{4.6}$$

THEOREM 4.2. *Suppose f_n: $[a, b] \to R$ are all GR integrable and non-decreasing for almost all $x \in [a, b]$. Let $f(x) = \lim_{n \to \infty} f_n(x)$; then f is GR integrable iff $\lim_{n \to \infty} \int_a^b f_n(x)\,dx < +\infty$. Moreover,*

$$\int_a f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx. \tag{4.7}$$

Evidently, the dominated convergence theorem of GR integrable functions is no more than that of Lebesgue integrable functions because of the relation between these two integrals. For the proofs of these two theorems, see [4, 7, 8].

We say that g: $[a, b] \to R$ is a GR multiplier if fg is GR integrable on $[a, b]$ as long as f is. Theorem 4.1(e) shows that any function of bounded variation is a GR multiplier. The following example convinces us that this might be the best general result since even an ACG_* function may not be a GR multiplier. This means, in particular, that fG and Fg from Theorem 4.1(f) may not be integrable in general.

EXAMPLE 4.1. Define

$$F(x) = \begin{cases} x \cos \pi/x, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

then $F(x)$ is continuous and $F'(x) = \cos \pi/x + (\pi/x) \sin \pi/x$ for $x \neq 0$. Hence $F'(x)$ is GR integrable on $[0, \frac{1}{2}]$ and so is $f(x) = (\pi/x) \sin \pi/x$. Obviously, $\lim_{\varepsilon \to 0+} \int_\varepsilon^{1/2} |f(x)|\,dx = +\infty$; therefore, we can find a C^1 function ϕ: $[0, \frac{1}{2}] \to [0, 1]$ with the properties that $\phi(0) = 0$, strictly increasing and such that $\lim_{\varepsilon \to 0+} \int_\varepsilon^{1/2} \phi(x)\,|f(x)|\,dx = +\infty$. Note that $f(x) > 0$ in $(1/(2k+1), 1/2k)$ and $f(x) < 0$ in $(1/(2k+2), 1/(2k+1))$ for $k = 1, 2, 3, ...$; it is then not difficult to define a C^1 function ψ: $(0, \frac{1}{2}] \to R$ such that $|\psi(x)| \leqslant \phi(x)$ for $x \in (0, \frac{1}{2}]$ and

$$\int_{I_k} \psi(x)\,f(x)\,dx \geqslant \int_{I_k} \phi(x)\,|f(x)|\,dx - \frac{1}{2^k} \qquad \text{for any} \quad I_k = \left[\frac{1}{k+1}, \frac{1}{k}\right],$$

$k = 1, 2, ...$. By defining $\psi(0) = 0$, we see that ψ is continuous on $[0, \frac{1}{2}]$ and ACG_*. However, $\lim_{\varepsilon \to 0+} \sup \int_\varepsilon^{1/2} \psi(x)\,f(x)\,dx = +\infty$. Hence ψf is not GR integrable on $[0, \frac{1}{2}]$ by Theorem 4.1(b).

EXAMPLE 4.2. Assume that g: $R \to R$ and x: $[a, b] \to R$. It is in general nonsense to ask the GR integrability of $g(x(t))$ in case one of them is only GR integrable. Let $g(s) = |s|$, which is Lipschitzian, and $x(t)$ be GR integrable but not Lebesgue integrable; then $g(x(t))$ is not GR integrable. Let $g(s) = s^{-1/2}$, which is GR integrable, $x(t) = t^2$, which is Lipschitzian; $g(x(t))$ is not GR integrable on $[0, 1]$, however.

In conclusion, unlike Lebesgue integrable functions, GR integrable functions lack some important operational properties which are necessary in the theories of differential and integral equations. Because of this, it is clear that the importance of the GR integral is only to provide an elementary and constructive treatment of the theory of the Lebesgue integral.

Finally, we give the following (GR) integral inequality which is a generalization of [6]. As Example 4.2 reveals, we can consider only linear type problems when GR integrals are involved.

THEOREM 4.3. (Generalized Gronwall's Inequality). *Assume* ϕ, α, β: $[a, b] \to R$ *such that* ϕ *is GR integrable,* α *is* ACG_*, β *is of bounded variation, and* $\alpha'(t) \geqslant 0$, $\beta(t) \geqslant 0$ *a.e.* $[a, b]$. *Then*

$$\phi(t) \leqslant \alpha(t) + \int_a^t \beta(s)\phi(s)\, ds, \qquad a \leqslant t \leqslant b \tag{4.8}$$

implies

$$\phi(t) \leqslant \alpha(t) \exp\left(\int_a^t \beta(s)\, ds\right), \qquad a \leqslant t \leqslant b. \tag{4.9}$$

If (4.8) *holds a.e., then so does* (4.9).

Proof. Let $R(t) = \int_a^t \beta(s)\phi(s)\, ds$, which is well-defined by Theorem 4.1(e). Then

$$R'(t) = \beta(t)\phi(t) \leqslant \beta(t)\alpha(t) + \beta(t)R(t) \qquad \text{a.e.} \tag{4.10}$$

This implies

$$R'(t) - \beta(t)R(t) \leqslant \beta(t)\alpha(t) \qquad \text{a.e.} \tag{4.11}$$

and hence

$$\left[R(t) \exp\left(-\int_a^t \beta(s)\, ds\right)\right]' \leqslant \beta(t)\alpha(t) \exp\left(-\int_a^t \beta(s)\, ds\right) \qquad \text{a.e.} \tag{4.12}$$

Since $\exp(-\int_a^t \beta(s)\, ds)$ is Lipschitzian and $R(t)$ is ACG_*, it is then easy to prove that $R(t)\exp(-\int_a^t \beta(s)\, ds)$ is also ACG_*. Hence, by Theorem 4.1(g) one can integrate (4.12), obtaining

$$R(t) \exp\left(-\int_a^t \beta(s)\, ds\right) \leqslant \int_a^t \beta(s)\alpha(s) \exp\left(-\int_a^s \beta(u)\, du\right) ds. \tag{4.13}$$

Thus

$$R(t) \leqslant \int_a^t \beta(s)\alpha(s) \exp\left(\int_s^t \beta(u)\,du\right) ds. \tag{4.14}$$

Let $F = \alpha$, $G = -\exp(\int_s^t \beta(u)\,du)$ in Theorem 4.1(f); then $f = \alpha'$ and $g = \beta(s)\exp(\int_s^t \beta(u)\,du)$. Applying (4.5) to (4.14) we get

$$R(t) \leqslant -\alpha(t) + \alpha(a)\exp\left(\int_a^t \beta(s)\,ds\right) + \int_a^t \alpha'(s)\exp\left(\int_s^t \beta(u)\,du\right) ds. \tag{4.15}$$

Consequently,

$$\begin{aligned} R(t) &\leqslant -\alpha(t) + \exp\left(\int_a^t \beta(s)\,ds\right)\left[\alpha(a) + \int_a^t \alpha'(s)\,ds\right] \\ &= -\alpha(t) + \alpha(t)\exp\left(\int_a^t \beta(s)\,ds\right), \end{aligned}$$

since $\alpha(t)$ is ACG_*. Finally,

$$\phi(t) \leqslant \alpha(t) + R(t) \leqslant \alpha(t)\exp\left(\int_a^t \beta(s)\,ds\right), \qquad t \geqslant a,$$

and the proof is complete.

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