On Impulsive Semidynamical Systems

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1. INTRODUCTION

It is now realized that the theory of impulsive differential equations is an important area of investigation. Moreover, the theory of such equations is much richer than the corresponding theory of differential equations [1, 2]. In this paper we isolate the ideas involved in the theory of impulsive differential equations and initiate the study of impulsive semidynamical systems. For allied results on discontinuous dynamical systems see [3, 4].

2. SEMIDYANMALICAL SYSTEM

We begin with the definition of a semidynamical system.

DEFINITION 1. A triple $(X, \pi, R_+)$ is said to be a semidynamical system if $X$ is a metric space, $R_+$ is the set of all non-negative reals, and $\pi: X \times R_+ \to X$ is a continuous function such that

(i) $\pi(x, 0) = x$ for all $x \in X$, and

(ii) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in R_+$.

We sometimes denote a semidynamical system $(X, \pi, R_+)$ by $(X, \pi)$. If instead of $R_+$ we have $R$, then the triple $(X, \pi, R)$ is said to be a dynamical system, where $R$ is the set of all reals.

For any $x \in X$, the function $\pi_x: R_+ \to X$ defined by $\pi_x(t) = \pi(x, t)$ is clearly continuous and we call $\pi_x$ the trajectory of $x$. The set

$C^+(x) = [\pi(x, t) : t \in R_+]$

is called the positive orbit of $x$. Also, we set

$C^+(x, r) = [\pi(x, t) : 0 \leq t < r]$.

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For $t < 0$ we denote the backward trajectory of $x$ by $\pi(x, t) = \{y \in X : \pi(y, t) = x\}$ and by $C^-(x) = \bigcup \{\pi(x, t) : t \leq 0\}$. For any set $M \subset X$ let $M^+(x) = C^+(x) \cap M - \{x\}$, $M^-(x) = C^-(x) \cap M - \{x\}$, and $M(x) = M^+(x) \cup M^-(x)$.

**Definition 2.** An impulsive semidynamical system $(X, \pi; M, I)$ consists of a semidynamical system $(X, \pi)$ together with a nonempty closed subset $M$ of $X$ and a continuous function $I : M \to X$ such that the following property holds:

For any $x \in M$ there exists an $\varepsilon > 0$ such that $\pi(x, t) \notin M$ for all $t$, $0 < |t| < \varepsilon$.

(2.1)

Throughout the paper we write $N = I(M)$, and for any $x \in X$, $I(x) = x^+$.

**Definition 3.** Let $(X, \pi; M, I)$ be an impulsive semidynamical system and let $x \in X$. The trajectory of $x$ in $(X, \pi; M, I)$ is a function $\tilde{t}_x$ defined on a subset $[0, s)$ of $\mathbb{R}$ to $X$ inductively as follows: Set $x = x_0$. If $M^+(x_0) = \emptyset$, then $\tilde{t}_x(t) = \pi(x, t)$ for all $t \in \mathbb{R}$. If $M^+(x_0) \neq \emptyset$, then by (2.1) in Definition 2 there exists a positive $s_0 \in \mathbb{R}^+$ such that $\pi(x_0, s_0) = x_1 \in M$ and $\pi(x_0, t) \notin M$ for $0 < t < s_0$. We define $\tilde{t}_x$ on $[0, s_0]$ by

$$\tilde{t}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0, \\ x_1^+, & t = s_0. \end{cases}$$

To complete the definition of $\tilde{t}_x$ in this case, we continue the above process starting at $x_1^+$. Thus, either $M^+(x_1^+) = \emptyset$ and we define $\tilde{t}(t) = \pi(x_1^+, t - s_0)$ for $t \geq s_0$ and $s = \infty$, or else $M^+(x_1^+) \neq \emptyset$, which implies the existence of an $s_1 > 0$ as before, and we define

$$\tilde{t}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leq t < s_0 + s_1, \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2 = \pi(x_1^+, s_0 + s_1)$. This process either ends after a finite number of steps, if $M^+(x_1^+) = \emptyset$ for some value of $n$, or $M^+(x_1^+) \neq \emptyset$, $n = 1, 2, \ldots$, and the process continues indefinitely. This gives rise to either a finite or an infinite sequence $\{x_n\}$ of points of $X$, and with each $x_n$ is associated a real number $s_n$ and the impulse $x_n^+$, where $\pi(x_n^+, s_n) = x_{n+1}$. We shall denote the number of elements $\{x_n\}$ by $N(x)$. The interval of definition of $\tilde{t}_x$ is clearly $[0, s)$, where $s = \sum s_n$. This completes the definition of the trajectory $\tilde{t}_x$. We shall denote the points of discontinuity of $\tilde{t}_x$ by $\{x_n^+\}$ and call $x_n^+$ an impulsive point of $x$.

We define a function $\Phi$ from $X$ into the extended positive reals $\mathbb{R}^+ \cup \{\infty\}$ as follows: Let $x \in X$. If $M^+(x) = \emptyset$ we set $\Phi(x) = \infty$, otherwise...
$M^+(x) \neq \phi$ and we set $\Phi(x) = s$, where $\pi(x, t) \notin M$ for $0 < t < s$ but $\pi(x, s) \in M$.

**Definition 4.** A trajectory $\pi_x$ in $(X, \pi, M, I)$ is said to be periodic of period $r$ and order $k$ if there exist positive integers $m \geq 1$ and $k \geq 1$ such that $k$ is the smallest integer for which $x_m^+ = x_{m+k}^+$ and $r = \sum_{j=m+1}^{m+k-1} s_j$.

Clearly the trajectory $\pi_x$ is continuous if either $M^+(x) = \phi$, or for each $n$, $x_n = x_n^+$; otherwise, it has discontinuities at a finite or an infinite number of its impulsive points $x_n^+$. At any such point, however, $\pi_x$ is continuous from the right. From the point of view of an impulsive semidynamical system, the trajectories that are of interest are those with an infinite number of discontinuities and with $[0, \infty)$ as the interval of definition. We call them infinite trajectories.

Let $(X, \pi)$ be a semidynamical system and $x \in X$. Then the positive prolongation set of $x$ is $D^+(x) = \{ y \in X: \pi(x_n, t_n) \to y, \text{ where } x_n \to x \text{ and } \{t_n\} \text{ in any sequence in } R^+ \}$, the positive limit set $L^+(x) = \{ y \in X: \pi(x, t_n) \to y \text{ for } t_n \to \infty \}$, and $x$ is of Ch $0^+$ if $D^+(x) = C^+(x)$.

Given an impulsive semidynamical system, we say that a point is of Ch $0^+$ if it is of Ch $0^+$ in the underlying semidynamical system. Also, for an infinite trajectory $\pi_x$, the limit set is defined by

$\bar{L}^+(x) = \{ y: \pi_x(t_n) \to y \text{ for } t_n \to \infty \}$.

**Remark 1.** From the definition, it is clear that

$\bar{L}^+(x) = \limsup_{x \to n} \phi(x_n) \Phi(x_n))$.

Consequently, $\bar{L}^+(x)$ is a closed set.

**Lemma 1.** Let $\pi_x$ be an infinite trajectory in $(X, \pi; M, I)$ and $x_n^+ \to y \in X$. Then $S = \limsup_{n \to \infty} \Phi(x_n^+)$ has only a finite number of elements in any bounded subset of $R^+$.

**Proof.** Suppose that an infinite subset $\{r_n\}$ of $S$ lies in a bounded subset of $R^+$. Let $\{s_{nk}\}$ be a subsequence of $\{s_n\}$, $s_n = \Phi(x_n^+)$, converging to $r_k$ for $k = 1, 2, \ldots$. Assume, without loss of generality, that $r_k \to r < \infty$.

Then

$\pi(x_m^+, s_{nk}) = z_{nk} \to z_k = \pi(y, r_k) \in C^+(y) \cap M$

and $z_k$ converges to $\pi(y, r) \in C^+(y) \cap M$. But this contradicts property (2.1) of Definition 2 and the proof is complete.
Lemma 2. Let \( \hat{\pi}_x \) be an infinite trajectory in \( (X, \pi; M, I) \). Suppose that \( x_n^+ \to y \) and \( \Phi(x_n^+) = s_n \to \infty \). Then \( \overline{C}^+(y) \subseteq \liminf_{n \to \infty} C^+(x_n^+, s_n) \). If, furthermore, \( y \) is of \( \text{Ch } 0^+ \), then \( \limsup_{n \to \infty} C^+(x_n^+, s_n) \subseteq C^+(y) \). Thus, if \( y \) is of \( \text{Ch } 0^+ \), \( \overline{L}^+(x) = \overline{C}^+(y) \).

Proof. Let \( z \in C^+(y) \). Then \( z = \pi(y, r) \) for some \( r \geq 0 \). Since \( s_n \to \infty \), there exists an \( n_0 \) such that \( s_n \geq r \) for \( n \geq n_0 \). Hence

\[ z_n = \pi(x_n^+, r) \in C^+(x_n^+, s_n), \quad n > n_0 \]

and converges to \( \pi(y, r) \), implying that \( z \in \liminf_{n \to \infty} C^+(x_n^+, s_n) \). Thus \( C^+(y) \subseteq \liminf_{n \to \infty} C(x_n^+, s_n) \), which, being closed, contains \( \overline{C}^+(y) \). This proves the first part.

Now suppose that \( z \in \limsup_{n \to \infty} C^+(x_n^+, s_n) \). Then \( z_k = \pi(x_n^+, r_k) \), \( 0 \leq r_k < s_n \), converges to \( z \). Since \( x_n^+ \to y \), \( z \in D^+(y) \) and since \( y \) is of \( \text{Ch } 0^+ \), \( z \in \overline{C}^+(y) \). Thus \( \limsup_{n \to \infty} C(x_n^+, s_n) \subseteq \overline{C}^+(y) \). The last part follows from Remark 1 and the foregoing results that imply

\[ \overline{C}^+(y) \subseteq \liminf_{n \to \infty} C^+(x_n^+, s_n) \subseteq \limsup_{n \to \infty} C^+(x_n^+, s_n) \subseteq \overline{C}^+(y). \]

Example 1. Consider the differential system

\[ x' = \sin y, \quad y' = \cos^2 y, \]

whose solutions are \( x + c = \sec y, \quad y = k\pi + \pi/2, \quad k = 0, \pm 1, \pm 2, \ldots \). The trajectories of the semidynamical system are then given in Fig. 1.

Let \( M \) and \( N \) be as in Fig. 1 and let \( I \) be the homeomorphism mapping the line segment \( PP' \) onto \( QQ' \), \( Q^+ = P \). Then \( x_n^+ \to P \), but \( P \) is not a point of \( \text{Ch } 0^+ \). It is easy to see that Lemma 2 does not hold.

The proof of Lemma 2 also yields the next result.
Lemma 3. Let $\pi_x$ be an infinite trajectory in $(X, \pi; M, I)$. Suppose that $x_n^+ \to y$ and $\Phi(x_n^+) \to s < \infty$. Then $L^+(x) = C(y, s) = \lim_{n \to \infty} C^+(x_n^+, s_n)$.

Lemma 4. Let $\pi_x$ be an infinite trajectory in $(X, \pi; M, I)$. Let $x_n^+ \to y$ and $S = \lim sup_{n \to \infty} \Phi(x_n^+)$ be bounded. Then $L^+(x) = C^+(y, r)$ for some $r < \infty$.

Proof: By Lemma 1, $S$ is finite, say $S = [r_1, r_2, ..., r_m]$. For each $r_k$ there exists a subsequence $(s_n)$ of $(x_n)$, $\Phi(x_n) = s_n$, converging to $r_k$. Hence, by Lemma 3, it follows that $\lim_{n \to \infty} C^+(x_n^+, s_n) = C^+(y, r_k)$.

Thus if $r = \max[r_k : k = 1, 2, ..., m]$, then

$$\lim_{n \to \infty} C(x_n^+, s_n) = L^+(x) = C^+(y, r).$$

We are now in a position to prove the following results.

Theorem 1. Let $\pi_x$ be an infinite trajectory in $(X, \pi; M, I)$ and let

$$\lim sup_{n \to \infty} x_n^+ = F = \{y_k : k = 1, 2, ..., m, 1 < m < \infty\}.$$

Suppose that whenever $x_n^+ \to y_k$, $\Phi(x_n^+) \to \Phi(y_k)$ and $\{x_n^+\}$ is sequentially compact. Then, for any $y \in F$, $\pi_y$ is periodic of order $m$ and period $\sum_{k=1}^m \Phi(y_k)$.

Proof. Let $x_{nk} \to y_k$, $k = 1, 2, ..., m$. Since $\Phi(x_{nk}^+) \to \Phi(y_k)$,

$$x_{nk+1} = \pi(x_{nk}^+, \Phi(x_{nk}^+)) \in M$$

converges to a point $z = g(y) \in F$.

From the continuity of the functions $\pi$ and $I$ it is clear that $g$ is a well defined function of $F$ into itself. Since $F$ is finite, given $y^+ \in F$, there is an integer $l > 0$ such that $y^+ = y_0^+$, $y_0^+ = g(y_0^+)$, ..., $y_l^+ = g^{l-1}(y_0^+)$ are all distinct and $g^l(y_0^+) = y_0^+$. We wish to show that $F = \{y_0^+, ..., y_l^+\}$.

Let $x_{nk} \to y_0^+$. Then clearly, by the definition of $g$, $x_{nk+i} \to y_i^+$, $0 < i < l$, and for each $i$, $0 < i < l-1$, there exists an $e_i > 0$ such that $U(y_i^+, e_i) \cap F = \{y_i^+\}$; and for $0 < i < l-1$, if $x_{nk+i} \in U(y_i^+, e_i)$, then $x_{nk+i} \in U(y_{i+1}^+, e_{i+1})$. Let $k_0$ be such that for $k > k_0$, $x_{nk} \in U(y_0^+, e_0)$; then clearly $x_{nk+i} \in U(y_i^+, e_i) < 0 < i < l$. However, $x_{nk+i} \to y_i^+$ implies that

$x_{nk+i} = \pi(x_{nk+i-1}^+, \Phi(x_{nk+i-1})) \to \pi(y_{i-1}^+, \Phi(y_{i-1}))$

and

$I(x_{nk+i}) = x_{nk+i}^+ \to I(y_i) = y_i^+.$
Therefore, there exists a $k_1$ such that if $k \geq k_1$, $x_{m_1}^+ \in U(y_{0}^+, \varepsilon_0)$ for all $k \geq k_1$.

Now let $m$ be any positive integer and $m \geq m_{k_1}$. Then $m = n_{k_1} + nl + j$, for some integer $n \geq 0$ and $0 < j < l$. It is clear by induction on $n$ that $x_{m}^+ \in U(y_j^+, \varepsilon_j)$. Consequently, $F = \{y_0^+, \ldots, y_{l-1}^+\}$. The rest of the theorem follows from Definition 4.

**Theorem 2.** Let $\tilde{\pi}$ be an infinite trajectory. Suppose that $x_n^+ \to y$ and $\Phi(x_n^+) = s, \to \Phi(y) = s$. Then

(i) if $0 < s < \infty$, $\tilde{\pi}_y$ is periodic of period $s$ and order 1;

(ii) if $s = \infty$, then $\tilde{\pi}_y = \pi_v$; and

(iii) if $s = 0$, then $y = y^+$.

**Proof.** Clearly, $\pi(x_n^+, s_n) \to \pi(y, s) = z \in M$. Since $\pi(x_n^+, s_n) = x_{n+1}$, it follows from the continuity of $I$ that $x_{n+1}^+ \to z^+$. Hence $z^+ = y$. Since by assumption $s = \Phi(y)$, $\tilde{\pi}_y$ is periodic of period $s$ and order 1 proving (i). The proofs of (ii) and (iii) are immediate.

One can prove similarly the next result.

**Theorem 3.** Let $\tilde{\pi}_y$ be an infinite trajectory. Suppose that $x_n^+ \to y$ and $S = \limsup_{n \to \infty} s_n$, $s_n = \Phi(x_n^+)$. Let $s = \inf S$. Then

(i) if $s > 0$ and $\Phi(y) = s$, $\tilde{\pi}_y$ is periodic of period $s$ and order 1;

(ii) if $s = 0$, $y^+ = y$.

**Example 2.** Consider the semidynamical system $(X, \pi)$, where $X = \mathbb{R}^2$ and $\pi((x, y), t) = (x, y + t)$, $t \in \mathbb{R}_+$. Let $M$ and $N$ be the closed sets in $\mathbb{R}^2$ as shown in Fig. 2 and let $I$ map $M$ onto $N$ homeomorphically so
that \( I(B) = F \) and \( I(D) = H \). Observe that \( x_n^+ \to F \) and \( s_n \) are constants converging to \( s \). But \( s \) is not the time without impulse of the point \( F \), i.e., \( \Phi(F) \neq s \). Note also that \( \bar{\pi}_F \) is not a periodic trajectory. This shows the necessity of the assumption that \( s \) is the time without impulse of \( y \) in Theorems 1 and 2.

The foregoing results on periodicity have used the continuity of the function \( \Phi \) extensively. Example 2 shows that even in the simplest of situations the placement of \( M \) in the phase space can easily destroy the continuity of \( \Phi \). Hence we shall next concentrate on discussing the continuity of \( \Phi \). We need the following definitions.

**Definition 5.** Let \( (X, \pi, R) \) be a dynamical system. An open set \( V \) in \( X \) is called a tube if there exist an \( r > 0 \) and a subset \( S \subset V \) such that

(i) \( \pi(x, t) \in V \) for \( x \in S \) and \(-r < t < r \), and

(ii) for each \( x \in V \) there is a unique \( r(x) \), \( |r(x)| < r \) such that \( \pi(x, r(x)) \in S \).

It is clear that \( V = \{ \pi(x, t) : x \in S, -r < t < r \} \). The set \( S \) is called an \( (r, V) \)-section of the tube \( V \). It is well known that if \( x_0 \in X \) is not a rest point, that is, \( \pi(x_0, t) \neq x_0 \) for all \( t \), then there exists a tube containing \( x_0 \).

**Definition 6.** Let \( (X, \pi, R) \) be a dynamical system. A subset \( M \subset X \) is said to be well placed in \( X \) if every point \( x \in M \) lies in some tube whose section lies in \( M \).

**Theorem 4.** Let \( (X, \pi, R) \) be a dynamical system. Let \( M \) be a closed and well placed subset of \( X \). Suppose that \( (X, \pi; M, I) \) is an impulsive semidynamical system. Then \( \Phi \) is continuous on \( X \).

**Proof.** It is easy to see that \( \Phi \) is lower semicontinuous and hence it is enough to show that \( \Phi \) is upper semicontinuous at \( x \) in case \( \Phi(x) = s < \infty \).

Let \( \varepsilon > 0 \). Since \( \pi(x, s) = y \in M \), there exist an \( r \), \( 0 < r < \varepsilon \), a tube \( V \), and a set \( S \subset M \), \( y \in S \) such that \( S \) is an \( (r, V) \)-section of the tube \( V \). By continuity of \( \pi \), there exists an open set \( U \) containing \( x \) such that \( \pi(U \times s) \subset V \). Hence, for any \( z \in U \), \( z_0 = \pi(z, s) \in V \), and therefore there exists an \( r(z_0) \) such that \( |r(z_0)| < r < \varepsilon \), and \( \pi(z_0, r(z_0)) = \pi(z, s + r(z_0)) \in S \subset M \). Thus \( \Phi(z) < s + \varepsilon \) for any \( z \in V \) and the proof is complete.

We shall next give a characterization of \( \bar{\mathcal{I}}^+(x) \).

**Lemma 5.** Let \( \bar{\pi}_x \) be an infinite trajectory in an impulsive semidynamical system \( (X, R^+; M, I) \) and \( \Phi \) be continuous on \( F = \limsup_{n \to \infty} x_n^+ \). Then for any \( y \in F \), \( y_n^+ \in F \), \( n = 1, 2, \ldots \).
Proof. Let $x_{n,k}^+ \to y$. Then $\Phi(x_{n,k}^+) \to \Phi(y)$ and $\Phi(x_{n,k}^+) \to \Phi(x_{n,k}^+) \to \Phi(y)$. Hence $n(x_{n,k}^+, \Phi(x_{n,k}^+)) = x_{n,k+1} \in M$ and $\{x_{n,k+1}\}$ converges to $y_1 = n(y_1, \Phi(y))$. Thus $\{x_{n,k+1}\}$ converges to $y_1^+$. The proof now follows by induction.

**Theorem 5.** Let $\pi_x$ be an infinite trajectory. Let the set
$$[x_n^+: n = 1, 2, \ldots]$$
have a compact closure and $\{s_n\}$ be bounded, $s_n = \Phi(x_n^+)$. If $\Phi$ is continuous on $F = \lim sup_{n \to \infty} x_n^+$, then
$$\tilde{L}^+(x) = \bigcup \left[ C^+(y; \Phi(y)) : y \in F \right].$$

Proof. It is easy to see that $C^+(y; \Phi(y)) \subset \tilde{L}^+(x)$ for all $y \in F$, noting that $\Phi(y)$ is finite for all $y \in F$. Conversely, let $z \in \tilde{L}^+(x)$. Then
$$z = \lim_{k \to \infty} \pi(x_{n,k}, t_k) = \lim_{k \to \infty} z_k, \quad 0 \leq t_k < s_{n,k},$$
We may assume, since $\{s_n\}$ is bounded, that $t_k \to t_0$, and since $[x_n^+: n = 1, 2, \ldots]$ has compact closure, $x_{n,k}^+ \to y \in F$. Then, clearly, by continuity of $\Phi$, $0 \leq t_0 \leq \Phi(y)$ and
$$z = \pi(y, t_0) \in C^+(y, \Phi(y)).$$

This completes the proof.

We remark that Theorem 5 is valid if the underlying impulsive semidynamical system is a dynamical system as in Theorem 4 without any assumption on $\{x_n^+\}$.

In Example 1, note that for $x_0$, the sequence $\{s_n\}$ is not bounded, $F = \{p\}$. $\tilde{L}^+(x_0) = C^+(p) \cup \{(x, \pi/2) : x \in R\}$, and therefore Theorem 5 is not satisfied.

**Theorem 6.** Let $\tilde{x}_x$ be an infinite trajectory in an impulsive semidynamical system $(X, R^+; M, I)$, let the set $[x_n^+: n = 1, 2, \ldots]$ have a compact closure, and let $\Phi$ be continuous on $F = \lim sup_{n \to \infty} x_n^+$. If each point of $F$ is a point of $Ch 0^+$, then $\tilde{L}^+(x) = \bigcup \{ C^+(y; \Phi(y)) : y \in F \}$.

Proof. Let $z \in L^+(x)$. Then $z = \lim_{k \to \infty} \pi(x_{n,k}^+, t_k), 0 \leq t_k < s_{n,k}$. Assume, without loss of generality, that $x_{n,k}^+ \to y \in F$. If $\{t_k\}$ is bounded, then assuming that $t_k \to t$, by continuity of $\Phi$, $t \leq s = \Phi(y)$. Therefore, $z = \pi(y, s) \in C^+(y; s) \subset \tilde{C}^+(y)$. In case $t_k \to \infty$, $z \in \tilde{D}^+(y)$. Consequently, as $y$ is of $Ch 0^+$, $z \in \tilde{C}^+(y) = \tilde{C}^+(y)$. The last equality follows from the fact that $\phi(y) = \infty$. This proves that $\tilde{L}^+(x) \subset \bigcup \{ \tilde{C}^+(y) : y \in F \}$. The converse follows easily from Lemma 6.
REFERENCES


