Journal of Mathematical Analysis and Applications **236,** 399–414 (1999)
Article ID jmaa.1999.6447, available online at http://www.idealibrary.com on IDEAL®

Operational Identities and Properties of Ordinary and Generalized Special Functions

G. Dattoli and A. Torre

and

S. Lorenzutta

ENEA, Dip. Innovazione, Divisione Fisica Applicata, Centro Ricerche Bologna, Via Don Fiammelli 2, 40129 Bologna, Italy

Submitted by William F. Ames

Received February 23, 1999

The theory of Hermite, Laguerre, and of the associated generating functions is reformulated within the framework of an operational formalism. This point of view provides more efficient tools which allow the straightforward derivation of a wealth of new and old identities. In this paper a central role is played by negative derivative operators and by their link with the Tricomi functions and the generalized Laguerre polynomials. © 1999 Academic Press

I. INTRODUCTION

In a number of previous papers [1, 2, 3] it has been shown that, by exploiting operatorial methods, many properties of ordinary and generalized special functions are easily derived and framed in a more general context. This approach has indeed allowed the derivation of the Burchnall identity [4] and of its extension to the two-variable two-index Hermite polynomials [2].

To give an example of how the method works, we consider the sum

$$\sigma(x,t) = \sum_{n=0}^{\infty} t^n H_n(x), \tag{1}$$

0022-247X/99 $30.00

where $H_n(x)$ denotes the ordinary Hermite polynomials with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \exp\left(xt - \frac{t^2}{2} \right). \tag{2}$$

The use of the identity

$$\left(x - \frac{d}{dx} \right)^n H_s(x) = H_{s+n}(x) \tag{3}$$

and of the fact that $H_0(x) = 1$, allows us to write Eq. (1) as

$$\sigma(x,t) = \sum_{n=0}^{\infty} t^n \left(x - \frac{d}{dx} \right)^n = \frac{1}{1 - t\left(x - \frac{d}{dx} \right)}. \tag{4}$$

The r.h.s. of Eq. (4) can be cast in the form

$$\frac{1}{1 - t\left(x - \frac{d}{dx} \right)} = \int_0^{\infty} e^{-s} e^{+st\left(x - \frac{d}{dx} \right)}\, ds, \tag{5}$$

and by using the disentanglement rule

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-1/2k}, \tag{6a}$$

which holds in the hypothesis that the operators $\hat{A}$ and $\hat{B}$ satisfy the commutation brackets

$$\left[\hat{A}, \hat{B} \right] = k, [\hat{A}, k] = [\hat{B}, k] = 0, \tag{6b}$$

we can write the integral on the r.h.s. of (5) as

$$\int_0^{\infty} e^{-s} e^{st\left(x - \frac{d}{dx} \right)}\, ds = \int_0^{\infty} e^{-\frac{1}{2}(st)^2} e^{-s(1-tx)} e^{-st\frac{d}{dx}}\, ds. \tag{7}$$

Since the exponential operator, containing the derivative, acts on the unity, we finally get

$$\begin{aligned} \sum_{n=0}^{\infty} t^n H_n(x) &= \int_0^{\infty} e^{-\frac{1}{2}(st)^2} e^{-s(1-tx)}\, ds \\ &= \frac{1}{t} \sqrt{\frac{\pi}{2}} \exp\left[\frac{(1-xt)^2}{2t^2} \right] \operatorname{erfc}\left[\frac{1-xt}{\sqrt{2}\ t} \right] \end{aligned} \tag{8}$$

which can be viewed as the Borel sum of $\sum_{n=0}^{\infty} t^u H_n(x)$.

In this paper we will see how previously unknown identities can be derived by using procedures involving various combination of operatorial identities. The plan of the paper is the following. In Section 2 we explore the properties of the Kampé de Fériet [5] Bell [6] polynomials and their link with generalized Bessel functions [7]. Sections 3 and 4 are devoted to the theory of Tricomi functions and of Laguerre polynomials, respectively. Finally Section 5 contains concluding remarks on the limits of the method and the application to other fields of research.

II. MULTIVARIABLE HERMITE POLYNOMIALS AND HERMITE–BESSEL FUNCTIONS

An example of Kampé de Fériet polynomials is provided by

$$H_n^{(3)}(x,y) = n! \sum_{r=0}^{[n/3]} \frac{y^r x^{n-3r}}{r!(n-3r)!} \tag{9}$$

which are specified by the generating function

$$e^{xt+yt^3} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(3)}(x,y), \tag{10}$$

and satisfy the identities

$$\frac{\partial}{\partial y} H_n^{(3)}(x,y) = \frac{\partial^3}{\partial x^3} H_n^{(3)}(x,y) \tag{11a}$$

and

$$\begin{aligned} H_{n+1}^{(3)}(x,y) &= \left(x + 3y\frac{\partial^2}{\partial x^2} \right) H_n^{(3)}(x,y) \\ \frac{\partial}{\partial x} H_n^{(3)}(x,y) &= n H_{n-1}^{(3)}(x,y). \end{aligned} \tag{11b}$$

From Eq. (11a), since $H_n^{(3)}(x,0) = x^n$, it also follows that

$$H_n^{(3)}(x,y) = e^{y\frac{\partial^3}{\partial x^3}} x^n. \tag{11c}$$

Furthermore, according to Eq. (11b), the $H_n^{(3)}(x, y)$ are said to be quasi-monomials under the action of the operators

$$\hat{M} = x + 3y\frac{\partial^2}{\partial x^2}$$
$$\hat{\mathscr{D}}_x = \frac{\partial}{\partial x} \tag{12}$$

The above property allows us to derive the identity

$$H_{n+m}^{(3)}(x, y) = \left(x + 3y\frac{\partial^2}{\partial x^2}\right)^m H_n^{(3)}(x, y), \tag{13}$$

which can be exploited to investigate further properties of $H_n^{(3)}(x, y)$. The first example we discuss is the generating function

$$S^{(3)}(x, y; t \mid m) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+m}^{(3)}(x, y), \tag{14}$$

which, for the ordinary case, reduces to the well known Rainville generating function, namely

$$S^{(2)}\left(x, -\frac{1}{2}; t \mid m\right) = \exp\left(xt - \frac{t^2}{2}\right) H_m(x - t). \tag{15}$$

By exploiting the identity (13) we can write Eq. (14) as

$$S^{(3)}(x, y; t \mid m) = \exp\left[t\left(x + 3y\frac{\partial^2}{\partial x^2}\right)\right] H_m^{(3)}(x, y) \tag{16}$$

We can now introduce the operators

$$\hat{A} = 3ty\frac{\partial^2}{\partial x^2}$$
$$\hat{B} = tx \tag{17}$$

and note that

$$\left[\hat{A}, \hat{B}\right] = m\hat{A}^{1/2}, \quad m = 2\sqrt{3}\,y^{1/2}t^{3/2}. \tag{18}$$

We can therefore use the decoupling identity [1]

$$e^{\hat{A}+\hat{B}} = e^{\frac{m^2}{12}} e^{-\frac{m}{2}\hat{A}^{1/2}+\hat{A}} e^{\hat{B}}, \tag{19}$$

to disentangle the exponentials on the r.h.s. of (16), and write

$$S^{(3)}(x, y; t \mid m) = e^{yt^3} e^{-3t^2 y \frac{\partial}{\partial x} + 3ty \frac{\partial^2}{\partial x^2}} e^{tx} H_m^{(3)}(x, y). \tag{20}$$

The use of the identity

$$e^{\alpha \frac{d^m}{dx^m}} f(x) = f\left(x + m\alpha \frac{d^{m-1}}{dx^{m-1}}\right) e^{\alpha \frac{d^m}{dx^m}}, \tag{21}$$

finally yields

$$S^{(3)}(x, y; t \mid m) = e^{xt+yt^3} e^{3t^2 y \frac{\partial}{\partial x} + 3ty \frac{\partial^2}{\partial x^2}} H_m^{(3)}(x, y). \tag{22}$$

The action of the exponential operator, on the polynomial $H_m^{(3)}(x, y)$ can be specified by noting that the second-order Kampé de Fériet polynomials satisfy the identity

$$e^{y \frac{\partial^2}{\partial x^2}} x^n = H_n^{(2)}(x, y), \tag{23a}$$

and that

$$e^{r \frac{d}{dx}} f(x) = f(x + r). \tag{23b}$$

We can therefore conclude that

$$S^{(3)}(x, y; t \mid m) = e^{xt+yt^3} H_m^{(3,2)}(x + 3t^2 y, 3ty, y), \tag{24}$$

where $H_n^{(3,2)}(x, \omega, y)$ denotes the polynomials

$$H_n^{(3,2)}(x, \omega, y) = n! \sum_{r=0}^{[n/3]} \frac{y^r H_{n-3r}^{(2)}(x, \omega)}{r!(n-3r)!}, \tag{25}$$

which belong to those of Bell type and can also be defined through the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(3,2)}(x, \omega, y) = \exp\left[xt + \omega t^2 + yt^3\right]. \tag{26}$$

Let us now consider the possibility of deriving a relation of the type (8), involving Kampé de Fériet polynomials, namely

$$\sigma^{(3)}(x, y; t) = \sum_{n=0}^{\infty} t^n H_n^{(3)}(x, y). \tag{27}$$

By taking into account the identity (13) and by following the same procedure, leading to Eq. (5), we can write the Borel sum associated with the series (27)

$$\sigma^{(3)}(x, y; t) = \int_0^{\infty} e^{-s} e^{st\left(x + 3y\frac{\partial^2}{\partial x^2}\right)} \, ds, \tag{28}$$

and use the disentanglement rule (19) to end up with (valid for $|tx| < 1$ and $ty < 0$)

$$\sigma^{(3)}(x, y; t) = \int_0^{\infty} e^{-s(1-tx)} e^{s^3 y t^3} \, ds. \tag{29}$$

The function on the r.h.s. cannot be defined in terms of known function as in the case of Eq. (8), where we have exploited Gaussian and error functions. We can however introduce the function

$$\tau(x, y) = \int_0^{\infty} e^{-s^3 y} e^{-sx} \, ds. \tag{30}$$

Defined for $(x, y) > 0$ and satisfying the differential equation

$$\begin{gathered} \frac{\partial}{\partial y} \tau(x, y) = \frac{\partial^3}{\partial x^3} \tau(x, y) \\ \tau(x, 0) = \frac{1}{x}, \; \tau(0, y) = \frac{1}{3y} \Gamma(1/3), \end{gathered} \tag{31}$$

which allows us to conclude

$$\sigma^{(3)}(x, y; t) = \tau\left[(1 - tx), -t^3 y\right]. \tag{32}$$

The polynomials defined by the generating functions (26) are quasi-monomials too as the $H_n^{(3)}(x, y)$, with the only difference being the multiplication operator should be replaced by

$$\hat{M} = x + 2\omega \frac{\partial}{\partial x} + 3y \frac{\partial^2}{\partial x^2}. \tag{33}$$

By exploiting the same procedure leading to Eq. (24), we can prove the further identity

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+m}^{(3,2)}(x, \omega, y) = e^{xt+\omega t^2+yt^3} H_m^{(3,2)}(x + 3t^2 y + 2\omega t, \omega + 3ty, y). \tag{34}$$

According to the above examples it is evident that we can exploit the identities, associated to exponential operators, to derive new relations for conventional and generalized polynomials. For instance by comparing Eqs. (24) and (34) it immediately follows that $H_n^{(3,2)}(x,0,y) = H_n^{(3)}(x,y)$.

Before closing this section we want to touch further on an important point linked to the concept of Hermite-Bessel functions. It has been shown, in Ref. [1], that this class of functions is a direct consequence of the quasimonomiality principle and that they can be "generated" by replacing $\hat{M}$ with x in the generating function of ordinary Bessel function.

To generate Hermite−Bessel functions associated with the Bell-type polynomial $H_n^{(3,2)}(x, \omega, y)$ we introduce the generating function (see Eq. 33)

$$\begin{aligned} G(x, \omega, y; t) &= \exp\left[\frac{\hat{M}}{2}\left(t - \frac{1}{t}\right)\right] \\ &= \exp\left[\frac{1}{2}\left(x + 2\omega\frac{\partial}{\partial x} + 3y\frac{\partial^2}{\partial x^2}\right)\left(t - \frac{1}{t}\right)\right]. \end{aligned} \tag{35}$$

By exploiting (35) and the already quoted decoupling procedure, we can write

$$\begin{aligned} G(x, \omega, y; t) &= \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{\omega}{4}\left(t - \frac{1}{t}\right)^2 + \frac{y}{8}\left(t - \frac{1}{t}\right)^3\right] \\ &= \sum_{n=-\infty}^{\infty} t^n \, {}_{H^{(3,2)}}J_n(x, \omega, y), \end{aligned} \tag{36}$$

where ${}_{H^{(3,2)}}J_n(\cdot)$ is defined by

$${}_{H^{(3,2)}}J_n(x, \omega, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}^{(3,2)}(x, \omega, y)}{r!(n+r)!2^{n+2r}}. \tag{37}$$

The properties and the usefulness of this class of functions will be discussed in the concluding section.

III. TRICOMI FUNCTIONS AND ASSOCIATED GENERALIZATIONS

In the following we will make great use of the operator $\hat{\mathscr{D}}_x^{-1}$, which is the inverse of the derivative operator. Strictly speaking, $\hat{\mathscr{D}}_x^{-1}$ being an indefinite integral, we should specify the limit of integration; we will however assume that the lower limit of integration is zero and we will use the notation ${}_b\hat{\mathscr{D}}_x^{-1}$ to indicate b as lower limit of integration. By exploiting the notion of Cauchy repeated integral, we can write

$$\hat{\mathscr{D}}_x^{-n} f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\, dt, \tag{38}$$

and it is also easy to realize that

$$e^{-\hat{\mathscr{D}}_x^{-1}} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \hat{D}_x^{-r} = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}. \tag{39}$$

The function on the r.h.s. of Eqs. (39) is a 0th order Tricomi function [7] (T.F.), in particular we have

$$\hat{\mathscr{D}}_x^{-m} e^{-\hat{\mathscr{D}}_x^{-1}} = x^m C_m(x), \quad C_m(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{x^r}{(r+m)!}. \tag{40}$$

The T.F. can be generated through

$$\sum_{m=-\infty}^{+\infty} t^m C_m(x) = \exp\left[t - \frac{x}{t}\right], \tag{41}$$

which can be exploited to state the following (well known identity) between Tricomi and ordinary Bessel functions

$$C_m(x) = x^{-m/2} J_m(2\sqrt{x}). \tag{42}$$

A fairly straightforward consequence of the Eqs. (39) and (41) is the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} x^{n+m} C_{n+m}(x) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{\mathscr{D}}_x^{-(n+m)} \exp\left(-\hat{\mathscr{D}}_x^{-1}\right) \\ &= \hat{\mathscr{D}}_x^{-m} \exp\left[(t-1)\hat{\mathscr{D}}_x^{-1}\right] \\ &= x^m C_m[-(t-1)x]. \end{aligned} \tag{43}$$

Along with $C_m(x)$ we introduce the associated Hermite version, denoted by ${}_HC_m(x, y)$, and specified by the generating function

$$\sum_{n=-\infty}^{+\infty} t^n \, {}_HC_n(x, y) = \exp\left[t - \frac{x}{t} + \frac{y}{t^2}\right], \tag{44a}$$

and by the series expansion

$${}_HC_m(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x, y)}{r!(m+r)!}. \tag{44b}$$

By exploiting the same argument of the previous section, it is evident that ${}_HC_m(x, y)$ satisfies the properties

$${}_HC_m(x, y)|_{y=0} = C_m(x) \tag{45a}$$

and

$${}_HC_m(x, y) = e^{y\frac{d^2}{dx^2}} C_m(x). \tag{45b}$$

The Hermite−Tricomi functions, involving higher order polynomials, will be discussed in the forthcoming sections.

IV. LAGUERRE POLYNOMIALS AND OPERATORIAL IDENTITIES

Laguerre polynomials have been shown to be quasimonomials under the action of the operators

$$\begin{aligned} \hat{M} &= 1 - \hat{\mathscr{D}}_x^{-1} \\ \hat{P} &= -\left(x\hat{\mathscr{D}}_x^2 + \hat{\mathscr{D}}_x\right), \end{aligned} \tag{46}$$

which provide a realization of the Weyl group. It is indeed, easily proved that

$$\left[\hat{P}, \hat{M}\right] = \hat{1}. \tag{47}$$

The quasimonomiality property has been exploited to show that ordinary Laguerre polynomials can be explicitly constructed by using the relation

$$L_n(x) = \left(1 - \hat{\mathscr{D}}_x^{-1}\right)^n. \tag{48}$$

By manipulating (48), we can obtain the alternative definition

$$L_n(x) = \left(\hat{\mathscr{D}}_x - 1\right)^n \frac{x^n}{n!}, \tag{49}$$

and by recalling the ordinary definition of associated Laguerre polynomials

$$L_n^{(m)}(x) = (-1)^m \hat{\mathscr{D}}_x^m L_{n+m}(x) \tag{50}$$

we can use Eqs. (48) or (49) to write Eq. (50) in the operatorial form

$$L_n^{(m)}(x) = \left(1 - \hat{\mathscr{D}}_x\right)^m \left(1 - \hat{\mathscr{D}}_x^{-1}\right)^n = \left(1 - \hat{\mathscr{D}}_x\right)^{m+n} \frac{(-1)^n x^n}{n!} \tag{51}$$

which holds for m not-necessarily integer.

These last results can be exploited in a number of ways, and provide a useful tool to frame old and new generating functions in a more systematic context.

As a first example we consider the generating function

$$S(x;t,u) = \sum_{m,n=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} L_n^{(m)}(x), \tag{52}$$

which according to Eqs. (51) and (39) can be written as

$$S(x;t,u) = e^{t(1-\hat{\mathscr{D}}_x)} e^{u(1-\hat{\mathscr{D}}_x^{-1})} = e^{t+u} C_0((x-t)u). \tag{53}$$

Furthermore a straightforward application of the same method, yields [8]

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(m)}(x) = e^t \exp\left[-t\hat{\mathscr{D}}_x\right] L_n(x) = e^t L_n(x-t), \tag{54}$$

which can also be easily generalized, we find indeed

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} L_n^{(2m)}(x) = e^{t(1-\hat{\mathscr{D}}_x)^2} L_n(x) = e^t\, {}_H L_n(x-2t,t), \tag{55}$$

where ${}_H L_m(x, y)$ are Hermite–Laguerre polynomials. These polynomials are obtained by replacing $x^n \to H_n(x, y)$ in the definition of the Laguerre polynomial. For further comments the reader is addressed to the concluding section.

Other important generating functions are linked to generalized sum reported in Ref. [8].

We note indeed that a straightforward consequence of Eq. (51) is

$$L_n^{(-n)}(x) = \frac{(-1)^n}{n!} x^n, \tag{56}$$

which can be exploited to get the summation rules [8]

$$\sum_{n=0}^{\infty} t^n L_n^{(-n+m)}(x) = (1+t)^m e^{-xt}, \ |t| < 1 \tag{57a}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(-n+m)}(x) = \sum_{s=0}^{m} t^s \binom{m}{s} C_s(xt). \tag{57b}$$

It is also easily realized that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(m+n)}(x) = e^{2t} \sum_{s=0}^{m} t^s \binom{m}{s} C_s[(x-t)t], \tag{58a}$$

and that possible generalization can be written in the form

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{(2n)}(x) &= \exp\left[t\left(3 - 3\hat{\mathscr{D}}_x + \hat{\mathscr{D}}_x^2 - \hat{\mathscr{D}}_x^{-1}\right)\right] \\ &= e^{3t} {}_H C_0\left((x-3t)t, t^3\right), \end{aligned} \tag{58b}$$

where ${}_H C_0(x, y)$ is the Hermite–Tricomi function introduced in the previous section. Before concluding this section we remark that the generating function [8]

$$\sum_{n=0}^{\infty} L_m^{(n-m)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(1 - \hat{\mathscr{D}}_x\right)^n \frac{(-1)^m x^m}{m!} = e^t \frac{(t-x)^m}{m!}, \tag{59a}$$

can be generalized, thus getting, e.g.,

$$\sum_{n=0}^{\infty} L_{rm}^{(n-m)}(x) \frac{t^n}{n!} = e^t \sum_{s=0}^{(r-1)m} \binom{(r-1)m}{s} L_{rm-s}^{-rm+s}(x-t). \tag{59b}$$

The examples we have provided in this section, yield perhaps a further feeling on the usefulness of the operatorial identities exploited to deal with the theory of Laguerre polynomials. Further speculations will be discussed in the forthcoming section.

V. CONCLUDING REMARKS

In the previous section we have exploited the operator $\hat{\mathscr{D}}_x^{-1}$ and we have seen that it plays a crucial role to state the properties of Tricomi function and Laguerre polynomials. We have also shown that its use makes fairly transparent a number of properties of the Laguerre polynomials which would be obtained in a less direct way by exploiting more conventional means. Further comments on the usefulness of negative derivative operators could be useful. Let us therefore consider the following integro-differential equation

$$\begin{aligned} \frac{\partial}{\partial t} f(x,t) &= -\int_0^x dx' f(x',t) \\ f(x,0) &= g(x), \end{aligned} \tag{60}$$

whose solution can be formally written as

$$f(x,t) = \exp\left[-t\hat{D}_x^{-1}\right] g(x). \tag{61}$$

If $g(x)$ can be expanded in series around $x = 0$, we find

$$\begin{aligned} f(x,t) &= \exp\left[-t\hat{D}_x^{-1}\right] \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \sum_{s=0}^{\infty} \frac{(-t)^s x^{n+s}}{s!(n+s)!} \\ &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n C_n(xt). \end{aligned} \tag{62}$$

The solution of the integro-differential equation (60) can be therefore expressed as a series expansion in terms of Tricomi functions.

Let us now consider the operator $\hat{\mathscr{D}}_x^{-m}$ whose exponentiation yields

$$\exp\left[-t\hat{\mathscr{D}}_x^{-m}\right] = \sum_{s=0}^{\infty} \frac{(-t)^s}{s!} \frac{x^{ms}}{(ms)!}. \tag{63}$$

The function on the r.h.s. of Eq. (63) is the 0th order Wright function specified by the series

$$J_n^{(m)}(x^m) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{x^{ms}}{(n+ms)!}. \tag{64}$$

The solution of

$$\begin{aligned} \frac{\partial}{\partial t} f(x,t) &= \frac{-1}{(m-1)!}\int_0^x (x-\xi)^{m-1} f(\xi,t)\,d\xi \\ f(x,0) &= g(x), \end{aligned} \tag{65}$$

can be therefore written as

$$f(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n J_n^{(m)}(tx^m) = {}_{J^{(m)}}g(x;t). \tag{66}$$

The notation ${}_{J^{(m)}}g(x;t)$ denotes a function having the same MacLaurin expansion of $g(x)$ with the only difference that x^n is replaced by $x^n J_n^{(m)}(x^m t)$.

In the previous section we have considered generating functions which do not involve products of Laguerre polynomials. In this section we fill the gap by presenting the following example

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^n n! t^n L_n^{(m-n)}(x) L_n^{(m-n)}(y) \\ &\quad = \left(1-\hat{\mathscr{D}}_x\right)^m \left(1-\hat{\mathscr{D}}_y\right)^m \exp(-xyt) \\ &\quad = \left(1-\hat{\mathscr{D}}_x\right)^m (1+xt)^m \exp(-xyt). \end{aligned} \tag{67}$$

This last example is a proof of the implications contained into the use of negative derivative operators. We can however provide a further example aimed at clarifying how far one can go by exploiting the above formalism, and by proving that the definition (51) holds for noninteger values of m too.

To this aim we consider the following operatorial definition of Hermite polynomials (see Section I and Refs. 1–5)

$$e^{-1/4 \frac{d^2}{dx^2}} (2x)^n = \overline{H}_n(x), \left(\overline{H}_n(x) = n! \sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s}}{s!(n-2s)!} \right), \tag{68}$$

and

$$e^{-1/4(2\hat{\mathscr{D}}_x + 4x\hat{\mathscr{D}}_x^2)} (4x)^n = \overline{H}_{2n}(\sqrt{x}). \tag{69}$$

By exploiting the decoupling rule [1]

$$e^{\hat{A}+\hat{B}} = (1+m\hat{A})^{1/m} e^{\hat{B}}, \tag{70}$$

which holds if $[\hat{A}, \hat{B}] = m\hat{A}^2$, we can write Eq. (69) as

$$\left(1 - \frac{d}{dx}\right)^{1/2} e^{-x\frac{d^2}{dx^2}}(4x)^n = \overline{H}_{2n}(\sqrt{x}), \tag{71a}$$

or what is the same

$$\left(1 - \frac{d}{dx}\right)^{-1/2}\left[\left(1 - \frac{d}{dx}\right)e^{-x\frac{d^2}{dx^2}}(4x)^n\right] = \overline{H}_{2n}(\sqrt{x}). \tag{71b}$$

By using Eq. (71b) and the definition (51), we easily infer the (well known) identity

$$L_n^{(-1/2)}(x) = \frac{(-1)^n}{2^{2n}n!}\overline{H}_{2n}(\sqrt{x}). \tag{72}$$

We can reconsider the above problem from a more general point of view, by starting from the generating function

$$s^{(2)}[x, y; t] = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2n}(x, y) \tag{73}$$

which according to the discussion of Section II can be written as

$$\begin{aligned} s^{(2)}[x, y; t] &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(x + 2y\frac{\partial}{\partial x}\right)^{2n} \\ &= \exp\left[t\left(x + 2y\frac{\partial}{\partial x}\right)^2\right] = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-s^2(1-4yt)}e^{2s\sqrt{t}x}\, ds. \end{aligned} \tag{74}$$

The last Gaussian integral yields

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2n}(x, y) = \frac{1}{\sqrt{1 - 4yt}} \exp\left(\frac{x^2 t}{1 - 4yt}\right) \tag{75}$$

so that we finally obtain

$$H_{2n}(x, y) = n!(4y)^n L_n^{(-1/2)}\left(-\frac{x^2}{4y}\right). \tag{76}$$

General procedures of this type could also be exploited, to establish the link between higher order Hermite and Laguerre polynomials. Before closing the present paper we go back to the Hermite–Bessel functions, whose usefulness in theory of electromagnetic processes induced by relativistically moving charges has been clarified in Ref. [7].

We want to, however, emphasize that by setting $t = e^{i\varphi}$, in Eq. (36) we get

$$\sum_{n=-\infty}^{+\infty} e^{in\varphi}\, {}_{H^{(3,2)}}J_n(x, \omega, y) = \exp\left[ix \sin(\varphi) - \omega \sin^2(\varphi) - iy \sin^3(\varphi)\right], \tag{77}$$

which is the Jacobi–Anger expansion associated with the above type of Hermite–Bessel function, and clarifies that they can be exploited in radiation problems going beyond the simple dipole approximation.

Finally, let us note that on account of the quasimonomiality properties of the $H_n^{(3,2)}(\cdots)$ polynomials we can easily state that the Hermite–Bessel function ${}_{H^{(3,2)}}J_n(x, \omega, y)$ satisfies the 6th order differential equation

$$\left(\hat{M}\hat{\mathscr{D}}_x\hat{M}\hat{\mathscr{D}}_x + \hat{M}^2 - n^2\right){}_{H^{(3,2)}}J_n(\cdots) = 0, \tag{78}$$

where $\hat{M}$ is provided by Eq. (33).

The topics we have touched on in this paper have, perhaps, given the feeling that the problems associated with the interplay between special functions, conventional or generalized, and operatorial identities is so wide that it cannot be treated in the space of a paper. We believe, however, that the examples we have discussed yield a clear idea of the flexibility and usefulness of the proposed methods.

ACKNOWLEDGMENTS

The kind interest and help by Prof. H. M. Srivastava is gratefully acknowledged.

REFERENCES

1. G. Dattoli, P. L. Ottaviani, A. Torre, and L. Vazquez, Evolution operator equations: Integration with algebraic and finite difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory, *Rivista Nuovo Cimento* **20** (1997), 1–133.
2. G. Dattoli, S. Lorenzutta, and A. Torre, Miscellaneous identities of generalized Hermite polynomials, *Matematiche* **LII** (1997), 337–343.
3. G. Dattoli, A. Torre, and M. Carpanese, Operational rules and arbitrary order Hermite generating functions, *J. Math. Anal. Appl.* (1998), to be published.
4. J. L. Burchnall, *Quart. J. Math. Oxford Series (2)* **12** (1941), 9–11.

5. See, e.g., P. Appell and J. Kampé-de Fériet, ''Fonctions Hypergéométriques et Hypersphériques, polynomes d'Hermite,'' Gauthier-Villars, Paris, 1926.
6. E. T. Bell, *Ann. of Math.* **35** (1934), 258–277.
7. G. Dattoli and A. Torre, ''Theory and Applications of Generalized Bessel Functions,'' ARACNE, Rome, 1996.
8. See, e.g., H. Srivastava and M. Manocha, ''A Treatise on Generating Functions,'' Ellis Horwood, New York, 1984.