J. Differential Equations 249 (2010) 1213–1240

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonlinear stability of periodic traveling wave solutions of systems of viscous conservation laws in the generic case

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ARTICLE INFO

Article history:
Received 25 January 2010
Revised 19 April 2010

MSC:
35B35

Keywords:
Periodic traveling waves
Bloch decomposition
Modulated waves

ABSTRACT

Extending previous results of Oh–Zumbrun and Johnson–Zumbrun, we show that spectral stability implies linearized and nonlinear stability of spatially periodic traveling wave solutions of viscous systems of conservation laws for systems of generic type, removing a restrictive assumption that wave speed be constant to first order along the manifold of nearby periodic solutions. Key to our analysis is a nonlinear cancellation estimate observed by Johnson and Zumbrun, along with a detailed understanding of the Whitham averaged system. The latter motivates a careful analysis of the Bloch perturbation expansion near zero frequency and suggests factoring out an appropriate translational modulation of the underlying wave, allowing us to derive the sharpened low-frequency estimates needed to close the nonlinear iteration arguments.

1. Introduction

Nonclassical viscous conservation laws arising in multiphase fluid and solid mechanics exhibit a rich variety of traveling wave phenomena, including homoclinic (pulse-type) and periodic solutions along with the standard heteroclinic (shock, or front-type) solutions [5,1,29,22,23]. Here, we investigate stability of spatially periodic traveling waves: specifically, sufficient conditions for stability of the wave.

In previous work [21,13], we showed that strong spectral stability in the sense of Schneider [26, 25,27] implies linearized and nonlinear $L^1 \cap H^K \to L^\infty$ stability in all dimensions $d \geqslant 1$. However, as

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[1] Research of M.A.J. was partially supported by an NSF Postdoctoral Fellowship under NSF grant DMS-0902192.
[2] Research of K.Z. was partially supported under NSF grants No. DMS-0300487 and DMS-0801745.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.04.015

pointed out in [22,28], the conditions of Schneider are *nongeneric in the conservation law setting*, implying the restrictive condition that wave speed be constant to first order along the manifold of nearby periodic solutions. Indeed, it was shown in [23] that failure of this condition implies a degradation in the decay rates of the Green function of the linearized equations about the periodic wave, suggesting that nonlinear stability would be unlikely in the general (nonstationary wave speed) case in dimension $d = 1$.

In this paper, we show that these difficulties are only apparent, and that, somewhat surprisingly, spectral stability implies nonlinear stability even if this additional condition on wave speeds is dropped. More precisely, we show that small $L^1 \cap H^s$ perturbations of a planar periodic solution $u(x,t) \equiv \bar{u}(x_1)$ (without loss of generality taken stationary) converge at Gaussian rate in L^p, $p \geqslant 2$, to a modulation

$$\bar{u}\big(x_1 - \psi(x,t)\big) \tag{1.1}$$

of the unperturbed wave, where $x = (x_1, \tilde{x})$, $\tilde{x} = (x_2, \dots, x_d)$, and ψ is a scalar function whose x- and t-gradients decay at Gaussian rate in all L^p, $p \geqslant 2$, but which itself decays more slowly by a factor $t^{1/2}$; in particular, ψ is merely bounded in L^∞ for dimension $d = 1$.

In proving this result, we make crucial use of the tools developed in [21,13], in particular, a key nonlinear cancellation argument of [13]. Our methods derive further inspiration from the rich literature concerning the nonlinear stability of shock waves of viscous conservation laws to which we connect by remarks throughout this paper: see [1,5,29,6,7,30,17,32,9,18,31,33,19]. However, we emphasize that the analysis presented is completely self-contained, and the reader may follow or ignore these references as desired. In particular, while having an understanding of these references may assist in making our analysis more transparent it is by no means necessary even to be familiar with the said references.

Similarly, we point out that the main motivation for our treatment of the generic case considered here is a detailed understanding of the Whitham averaged system studied in [28,20,14,15] corresponding to the underlying periodic wave. In particular, the key new observation making possible the treatment of the generic case is a rescaling of the Bloch perturbation expansion about frequency $\xi = 0$, motivated by relations to the Whitham averaged system; see Section 2. Throughout the text, we will make several further remarks concerning the connection to the Whitham system. However, as with the shock stability literature mentioned above, our analysis can be completely understood without any reference at all to the underlying Whitham averaged system and hence readers may ignore these remarks as desired.

It was shown in [28,20] that the low-frequency dispersion relation near zero of the linearized operator about a periodic solution $\bar{u}$ agrees to first order with that of the linearization about a constant state of the Whitham averaged system

$$\begin{aligned} \partial_t M + \sum_j \partial_{x_j} F^j &= 0, \\ \partial_t(\Omega N) + \nabla_x(\Omega S) &= 0, \end{aligned} \tag{1.2}$$

formally governing slowly modulated solutions

$$u(x,t) = \bar{u}^{a(\varepsilon x, \varepsilon t)}\big(\Psi(x,t)\big) + O(\varepsilon), \quad \varepsilon \to 0 \tag{1.3}$$

presumed to describe large spatio-temporal behavior x, $t \gg 1$, where $\bar{u}^a(\cdot)$ as in (1.8) parametrizes the set of nearby periodic solutions, $M \in \mathbb{R}^n$ denotes the average over one period, F^j the average of an associated flux, $\Omega = |\nabla_x \Psi| \in \mathbb{R}^1$ the frequency, $S = -\Psi_t / |\nabla_x \Psi| \in \mathbb{R}^1$ the speed s, and $N = \nabla_x \Psi / |\nabla_x \Psi| \in \mathbb{R}^d$ the normal ν associated with nearby periodic waves, with an additional constraint

$$\operatorname{curl}(\Omega N) = \operatorname{curl}(\nabla_x \Psi) \equiv 0. \tag{1.4}$$

As noted in [28,20], this implies both that the eigenvalues $\lambda_j(\xi)$ bifurcating from $\lambda = 0$ at $\xi = 0$ are C^1 along rays through the origin, and that weak hyperbolicity (reality of characteristics of (1.2)–(1.4)) is necessary for spectral or linearized stability.

As noted in [15], there is a deeper analogy between the low-frequency linearized dispersion relation and the Whitham averaged system at the structural level, suggesting a useful rescaling of the low-frequency perturbation problem described in Section 2. It is this intuition that motivates our derivation of sharp low-frequency estimates crucial to the analysis of nonlinear stability. With these estimates in place, the rest of the argument goes exactly as in [13,21].

1.1. Equations and assumptions

Throughout this paper, we consider a parabolic system of conservation laws of the form

$$u_t + \sum_j f^j(u)_{x_j} = \Delta_x u, \tag{1.5}$$

where $u \in \mathcal{U}(\text{open}) \in \mathbb{R}^n$, $f^j \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $d \geqslant 1$, $t \in \mathbb{R}^+$. Such equations are known to exhibit a variety of traveling wave phenomena, including heteroclinic/homoclinic solutions as well as spatially periodic solutions; see [1,22,5,29] for examples of solutions of each type for the van der Waals equations of phase-transitional elasticity and for a class of quadratic-flux equations modeling local behavior of multi-phase flow in oil recovery.

We assume the existence of an X-periodic traveling wave solution of (1.5) of the form

$$u = \bar{u}(x \cdot \nu - st), \tag{1.6}$$

with $\nu \in S^{d-1}$. Clearly, any such solution must satisfy the traveling wave ODE

$$\bar{u}'' = \left(\sum_j \nu_j f^j(\bar{u}) \right)' - s\bar{u}',$$

with boundary conditions $\bar{u}(0) = \bar{u}(X) =: u_0$. Integrating once, we obtain a first-order profile equation

$$\bar{u}' = \sum_j \nu_j f^j(\bar{u}) - s\bar{u} - q, \tag{1.7}$$

where $(u_0, q, s, \nu, X) \equiv \text{const}$. Without loss of generality, we take $\nu = e_1$ and $s = 0$, so that $\bar{u} = \bar{u}(x_1)$ represents a stationary solution depending only on x_1.

In order to ensure the existence of periodic solutions of (1.7), we follow [28,20,21] and make the following natural assumptions:

(H1) $f^j \in C^{K+1}$, $K \geqslant [d/2] + 4$.
(H2) The map $H : \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times S^{d-1} \times \mathbb{R}^n \to \mathbb{R}^n$ taking $(X; w, s, \nu, q) \mapsto u(X; w, s, \nu, q) - w$ is full rank at $(\bar{X}; \bar{u}(0), 0, e_1, \bar{q})$, where $u(\cdot;\cdot)$ is the solution operator of (1.7).

By the Implicit Function Theorem, conditions (H1)–(H2) imply that the set of periodic solutions in the vicinity of $\bar{u}$ form a smooth $(n + d + 1)$-dimensional manifold

$$\left\{ \bar{u}^a\big(x \cdot \nu(a) - \alpha - s(a)t\big) \right\}, \tag{1.8}$$

with $\alpha \in \mathbb{R}$ corresponding to translation and $a \in \mathbb{R}^{n+d}$. With these assumptions in hand, we now begin our study of the stability of a given periodic solution of (1.7).

1.1.1. Linearized equations

As with any nonlinear stability analysis, we begin by considering the linearization of (1.5) about the fixed periodic standing wave solution $\bar{u}$ depending only on the variable x_1. Without loss of generality, we assume that $\bar{u}$ is 1-periodic, i.e. that $\bar{u}(x_1+1)=\bar{u}(x_1)$ for all $x_1 \in \mathbb{R}$. Considering nearby solutions of the form

$$\bar{u}(x)+\varepsilon v(x,t)+\mathcal{O}\big(\varepsilon^2\big),$$

where $|\varepsilon| \ll 1$ and $v(\cdot,t) \in L^2(\mathbb{R})$, corresponding to spatially localized perturbations, we see that v satisfies the linear equation

$$v_t = Lv := \Delta_x v - \sum \big(A^j v\big)_{x_j}, \tag{1.9}$$

where coefficients $A^j := Df^j(\bar{u})$ are now periodic functions of x_1. As the underlying solution $\bar{u}$ depends only on x_1, Eq. (1.9) is clearly autonomous in the transverse coordinate $\tilde{x}=(x_2,\dots,x_d)$. Taking the Fourier transform in $\tilde{x}$ then, we obtain

$$\hat{v}_t = L_{\tilde{\xi}}\hat{v} = \hat{v}_{x_1,x_1} - \big(A^1\hat{v}\big)_{x_1} - i\sum_{j\neq 1} A^j\xi_j\hat{v} - \sum_{j\neq 1}\xi_j^2\hat{v}, \tag{1.10}$$

where $\tilde{\xi}=(\xi_2,\dots,\xi_d)$ is the transverse frequency vector. Seeking solutions of the form $\hat{v}(x_1,\tilde{\xi},t)=e^{\lambda t}\hat{v}(x_1,\tilde{\xi})$, it is clear that the stability of $\bar{u}$ requires a detailed analysis of the spectrum of the operator $L_{\tilde{\xi}}$. A particularly useful way to characterize the spectrum of $L_{\tilde{\xi}}$ is to perform a Bloch decomposition of the corresponding spectral problem; a procedure which we now describe in detail.

1.1.2. Bloch–Fourier decomposition and stability conditions

As coefficients of $L_{\tilde{\xi}}$ are 1-periodic, Floquet theory implies the L^2 spectrum is purely continuous and corresponds to the union of the L^∞ eigenvalues corresponding to considering the linearized operator with boundary conditions $v(x+T)=e^{i\kappa}v(x)$ for all $x\in\mathbb{R}$, where $\kappa\in[-\pi,\pi]$ is referred to as the Floquet exponent and is uniquely defined mod 2π. In particular, $\mu\in\sigma(L_{\tilde{\xi}})$ if and only if the spatially periodic spectral problem $L_{\tilde{\xi}}\hat{v}=\lambda\hat{v}$ admits a bounded eigenfunction of the form

$$w(x_1,\xi,\lambda) := e^{i\xi_1 x_1} q(x_1,\xi_1,\tilde{\xi},\lambda), \tag{1.11}$$

where $\xi=(\xi_1,\tilde{\xi})\in\mathbb{R}^d$ and q is a 1-periodic function of x_1. Substituting the ansatz (1.11) into the spectral problem

$$L_{\tilde{\xi}} w = \lambda w$$

motivates the use of the Fourier–Bloch decomposition of the spectral problem. To this end, we follow [4,26,25,27] and define the family of operators

$$L_\xi = e^{-i\xi_1 x_1} L_{\tilde{\xi}} e^{i\xi_1 x_1} \tag{1.12}$$

operating on the class of L^2 periodic functions on $[0,1]$; the (L^2) spectrum of $L_{\tilde{\xi}}$ is equal to the union of the spectra of all L_ξ with ξ_1 real with associated eigenfunctions given by (1.11) where q is a 1-periodic eigenfunction of L_ξ. By continuity in ξ of the spectrum, and discreteness of the spectrum of the elliptic operators L_ξ on the compact domain $[0,1]$, we have that the spectra of L_ξ may be described as the union of countably many continuous surfaces $\lambda_j(\xi)$.

Continuing this functional setup, we recall that any function $u \in L^2(\mathbb{R}^d)$ admits the *Bloch–Fourier representation*

$$u(x) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x} \hat{u}(\xi, x_1)\, d\xi_1\, d\tilde{\xi}, \tag{1.13}$$

where $\hat{u}(\xi, x_1) := \sum_k e^{2\pi i k x_1} \hat{\hat{u}}(\xi_1 + 2\pi k, \tilde{\xi})$ are periodic functions of period $X = 1$. Notice here we are deviating from conventional notation as $\hat{u}$ corresponds to the Bloch–Fourier transform while $\hat{\hat{u}}$ corresponds to the Fourier transform of u in the full variable x. By Parseval's identity, the Bloch–Fourier transform $u(x) \to \hat{u}(\xi, x_1)$ is an isometry in $L^2(\mathbb{R}^d)$, i.e.

$$\|u\|_{L^2(x)} = \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} \int_0^1 \left|\hat{u}(\xi_1, \xi, x_1)\right|^2 dx_1\, d\tilde{\xi}\, d\xi_1 =: \|\hat{u}\|_{L^2(\xi; L^2(x_1))}, \tag{1.14}$$

where $L^2(x_1)$ is taken on $[0, 1]$ and $L^2(\xi)$ on $[-\pi, \pi] \times \mathbb{R}^{d-1}$. Moreover, the Bloch–Fourier transform diagonalizes the periodic-coefficient operator L, yielding the *inverse Bloch–Fourier transform representation*

$$e^{Lt} u_0 = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x} e^{L_\xi t} \hat{u}_0(\xi, x_1)\, d\xi_1\, d\tilde{\xi} \tag{1.15}$$

relating behavior of the linearized system to that of the diagonal operators L_ξ.

We now discuss the spectral stability of the underlying solution $\bar{u}(x_1)$. To begin, notice by the translation invariance of (1.5) the function $\bar{u}'(x)$ is a 1-periodic solution of the differential equation $Lv = 0$. Hence, it follows that $\lambda = 0$ is a 1-periodic eigenvalue of the linear operator L_0. Moreover, due to the divergence form of the governing equation (1.5) along with the linearized existence theory and assumption (H2), the zero eigenspace of L_0 is at least $(n+1)$-dimensional, see [28]. In our analysis, we will assume these considerations account for all 1-periodic null-directions of L_0. More precisely, following [21], we assume along with (H1)–(H2) the following *strong spectral stability* conditions:

(D1) $\sigma(L_\xi) \subset \{\operatorname{Re}\lambda < 0\}$ for $\xi \neq 0$.
(D2) $\operatorname{Re}\sigma(L_\xi) \leqslant -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}^d$ and $|\xi|$ sufficiently small.
(D3′) $\lambda = 0$ is an eigenvalue of L_0 of multiplicity exactly $n+1$.

As shown in [20] using Evans function methods, (H1)–(H2) and (D1)–(D3′) imply that there exist $n+1$ smooth eigenvalues

$$\lambda_j(\xi) = -ia_j(\xi) + o\big(|\xi|\big) \tag{1.16}$$

of L_ξ bifurcating from $\lambda = 0$ at $\xi = 0$, where $-ia_j$ are homogeneous degree one functions; for an alternative, more direct proof, see Lemma 2.1 below.[3] Moreover, as in [21], we make the further nondegeneracy hypothesis:

(H3) The functions $a_j(\xi)$ in (1.16) are distinct.

[3] Note: the Evans function is not used anywhere in this paper.

The functions a_j may be seen to be the characteristics associated with the Whitham averaged system (1.2)–(1.4) linearized about the values of M, S, N, Ω associated with the background wave $\bar{u}$; see [20,21]. Thus, (D1) implies weak hyperbolicity of (1.2)–(1.4) (reality of a_j), while (H3) corresponds to strict hyperbolicity.

Remark 1.1. Condition (D3′) is a weakened version of the condition (D3) of [21,13] that $\lambda = 0$ be a *semisimple* eigenvalue of L_0 of minimal multiplicity $n+1$, which implies [22,23,28] the special property that wave speed be stationary at $\bar{u}$ along the manifold of nearby periodic solutions. The stronger conditions (D1)–(D3) are exactly the spectral assumptions of [26,25,27] introduced by Schneider in the reaction–diffusion case. Conditions (D1)–(D3) (resp. (D1)–(D3′)) correspond to "dissipativity" of the large-time behavior of the linearized system [26,25,27].

Remark 1.2. The periodic solutions discussed in [22] of the van der Waals equations of phase-transitional elasticity in Lagrangian coordinates were of "quasi-Hamiltonian" type, satisfying (D3). However, as pointed out by Serre [28], considered as solutions of the same equations written in Eulerian coordinates, these traveling waves have variable wave speeds, satisfying the generic condition (D3′) but not (D3). Note that the Lagrangian formulation is not available in multi-dimensions, so that this generic situation is indeed the more relevant for the physical application to phase-transitional flow.

1.2. Main result

With these preliminaries, we can now state our main result.

Theorem 1.3. *Let $\bar{u}(x_1)$ be a periodic standing wave solution of* (1.5) *and let $\tilde{u}(x,t)$ be any solution of* (1.5) *such that $\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0}$ is sufficiently small. Then assuming* (H1)–(H3) *and* (D1)–(D3′)*, there exists a constant $C>0$ and a function $\psi(\cdot,t)\in W^{K,\infty}(\mathbb{R}^d)$ such that for all $t\geqslant 0$, $p\geqslant 2$, and $d\geqslant 1$ we have the estimates*

$$\begin{aligned}
\left\|\tilde{u}-\bar{u}(\cdot-\psi)\right\|_{L^p(\mathbb{R}^d)}(t) &\leqslant C(1+t)^{-\frac{d}{2}(1-1/p)}\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0},\\
\left\|\tilde{u}-\bar{u}(\cdot-\psi)\right\|_{H^K(\mathbb{R}^d)}(t) &\leqslant C(1+t)^{-\frac{d}{4}}\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0},\\
\left\|(\psi_t,\psi_x)(\cdot,t)\right\|_{L^p(\mathbb{R}^d)}(t) &\leqslant C(1+t)^{-\frac{d}{2}(1-1/p)}\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0},\\
\left\|(\psi_t,\psi_x)(\cdot,t)\right\|_{H^K(\mathbb{R}^d)} &\leqslant C(1+t)^{-\frac{d}{4}}\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0}.
\end{aligned}\tag{1.17}$$

Moreover, assuming further that $p=\infty$ and $d=1$, or $p>2$ and $d=2$, or $p\geqslant 2$ and $d\geqslant 3$, we have the estimates

$$\|\tilde{u}-\bar{u}\|_{L^p(\mathbb{R}^d)}(t),\ \left\|\psi(\cdot,t)\right\|_{L^p(\mathbb{R}^d)} \leqslant C(1+t)^{-\frac{d}{2}(1-1/p)+\frac{1}{2}}\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0}\tag{1.18}$$

for all $t\geqslant 0$. In particular, $\bar{u}$ is nonlinearly bounded $L^1\cap H^K\to L^\infty$ stable for $d\geqslant 1$, nonlinearly asymptotically $L^1\cap H^K\to L^\infty$ stable for $d\geqslant 2$, and nonlinearly asymptotically $L^1\cap H^K\to H^K$ stable for $d\geqslant 3$ with estimate

$$\|\tilde{u}-\bar{u}\|_{H^K(\mathbb{R}^d)}(t),\ \left\|\psi(\cdot,t)\right\|_{H^K(\mathbb{R}^d)} \leqslant C(1+t)^{-\frac{d}{4}+\frac{1}{2}}\|\tilde{u}-\bar{u}\|_{L^1(\mathbb{R}^d)\cap H^K(\mathbb{R}^d)}|_{t=0}\tag{1.19}$$

for all $t\geqslant 0$.

Remark 1.4. In Theorem 1.3, derivatives in $x \in \mathbb{R}^d$ for $d \geqslant 2$ refer to total derivatives. Moreover, unless specified by an appropriate index, throughout this paper derivatives in spatial variable x will always refer to the total derivative of the function.

Remark 1.5. Although $L^1 \cap H^K \to H^k$ asymptotic stability for $d \geqslant 3$ does not follow directly from (1.18), the nonlinear damping estimate in Proposition 4.6 shows that L^2 decay implies H^K decay, from which the estimate (1.19) clearly follows.

In dimension one, Theorem 1.3 asserts only bounded $L^1 \cap H^K \to L^\infty$ stability, a very weak notion of stability. Moreover, the bounds (1.17)–(1.18) agree for dimension $d = 1$ with those obtained in [13] in the stationary wave speed case that (D3) holds in place of (D3′), but for higher dimensions are weaker by roughly a factor of $t^{1/2}$.

Remark 1.6. In dimension $d = 1$, it is straightforward to show that the results of Theorem 1.3 extend to all $1 \leqslant p \leqslant \infty$ using the pointwise techniques of [23]; see Remark 3.8.

1.3. Discussion and open problems

The proof of Theorem 1.3 largely completes the line of investigation carried out in [23,28,20,21, 13], showing that spectral stability implies linear and nonlinear stability of planar spatially periodic traveling waves. The corresponding spectral stability problem has been studied analytically in [22,28, 20], yielding various necessary conditions, and by a numerical Evans function investigation in [22]. An interesting direction for further study would be more systematic numerical investigation along the lines of [10,3,12,11,2] in the viscous shock wave case. A second interesting open problem would be to extend the results for planar waves to the case of solutions with multiple periods, as considered in the reaction–diffusion setting in [26,25,27].

The key to the nonlinear analysis in critical dimensions $d = 1, 2$, as in [13,26,25,27], is to subtract out a slower-decaying part of the solution described by an appropriate modulation equation and show that the residual decays sufficiently rapidly to close a nonlinear iteration. However, we should note that the modulated approximation $\bar{u}(x_1 - \psi(x,t))$ in (1.1) is *not* the full ansatz $\bar{u}^a(\Psi(x,t))$ suggested by (1.3), but only the translational part not involving perturbations a in the profile. (See [20] for a derivation of (1.2)–(1.4).) That is, we don't need to separate out all variations along the manifold of periodic solutions, but only the special variations connected with translation invariance.

This can be understood heuristically by the observation that (1.2) indicates that variables a and $\nabla_x \Psi$ are roughly comparable, which would suggest, by the diffusive behavior[4] $|\Psi| \gg |\nabla_x \Psi|$, that a is negligible with respect to Ψ. Indeed, this heuristic argument translates rigorously to our ultimate computation of linearized behavior leading to the final result; see Section 2 and Remark 2.3. In this respect, the connection to the Whitham system is somewhat clearer in the generic case considered here than in the quasi-Hamiltonian case treated previously in [23,21,13].[5]

It would be interesting to better understand the connection between the Whitham averaged system (or suitable higher-order correction) and behavior at the nonlinear level, as explored at the linear level in [20,21,14,15]. As discussed further in [20], another interesting problem would be to try to rigorously justify the WKB expansion for the related vanishing viscosity problem, in the spirit of [6,7].

2. Spectral preparation

As a starting point in our analysis, we analyze under assumptions (H1)–(H3), (D1), and (D3′) the structure of the null-space of the operator L_ξ for $0 < |\xi| \ll 1$. By (D3′), the generalized null-space

[4] As evidenced by the heat equation.

[5] In the degenerate case that the stronger condition (D3) holds, i.e., wave speed is stationary at $\bar{u}$, the situation is somewhat more complicated, and these relations break down; see [13] for further discussion.

corresponding to $\xi = 0$ consists of $n+1$ functions. The next lemma describes the way in which these eigenfunctions of L_0 bifurcate in ξ.

Lemma 2.1. *Assuming* (H1)–(H3), (D1), *and* (D3′), *the eigenvalues* $\lambda_j(\xi/|\xi|, |\xi|)$ *of* L_ξ *are analytic functions of* $\xi/|\xi|$ *and* $|\xi|$. *Suppose further that* 0 *is a non-semisimple eigenvalue of* L_0, *i.e.,* (D3′) *holds, but not* (D3). *Then the Jordan structure of the zero eigenspace of* L_0 *consists of an* n*-dimensional kernel and a single Jordan chain of height* 2. *In particular,* $\bar{u}'$ *spans the right eigendirection lying at the base of the Jordan chain while the left kernel of* L_0 *coincides with the* n*-dimensional subspace of constant functions. Moreover, for* $|\xi|$ *sufficiently small, there exist right and left eigenfunctions* $q_j(\xi/|\xi|, |\xi|, \cdot)$ *and* $\tilde{q}_j(\xi/|\xi|, |\xi|, \cdot)$ *of* L_ξ *associated with* λ_j *of form* $q_j = \sum_{k=1}^{n+1} \beta_{j,k} v_k$ *and* $\tilde{q}_j = \sum_{k=1}^{n+1} \tilde{\beta}_{j,k} \tilde{v}_k$, *where* $\{v_j\}$ *and* $\{\tilde{v}_j\}$ *are dual bases of the total eigenspace of* L_ξ *associated with sufficiently small eigenvalues, analytic in* $\omega = \xi/|\xi|$ *and* $|\xi|$, *with* $\tilde{v}_j(\omega; 0)$ *constant for* $j \neq n$ *and* $v_n(\omega; 0) \equiv \bar{u}'(\cdot)$; $\tilde{\beta}_{j,1}, \dots, \tilde{\beta}_{j,n-1}, |\xi|^{-1}\tilde{\beta}_{j,n}, \tilde{\beta}_{j,n+1}$ *and* $\beta_{j,1}, \dots, \beta_{j,n-1}, |\xi|\beta_{j,n}, \beta_{j,n+1}$ *are analytic in* $\xi/|\xi|$, $|\xi|$; *and* $\langle \tilde{q}_j, q_k \rangle = \delta_j^k$.

Remark 2.2. Notice that the results of Lemma 2.1 are somewhat unexpected in the general case that $\lambda = 0$ is a non-semisimple eigenvalue of L_0. Indeed, it is well known that eigenvalues bifurcating from a non-trivial Jordan block typically do so in a nonanalytic fashion. The fact that analyticity prevails in our situation is a consequence of the very special structure of the left and right generalized null-spaces of the unperturbed operator L_0, and the special forms of the equations considered. See Section 2.1 for further discussion/motivation.

Proof of Lemma 2.1. Recall that L_ξ as an elliptic second-order operator on bounded domain has spectrum consisting of isolated eigenvalues of finite multiplicity. Expanding

$$L_\xi = L_0 + |\xi| L^1_{\xi/|\xi|} + |\xi|^2 L^2_{\xi/|\xi|} \tag{2.1}$$

for each fixed angle $\hat{\xi} := \xi/|\xi|$, consider the continuous family of spectral perturbation problems in $|\xi|$ indexed by angle $\omega = \xi/|\xi|$ about the eigenvalue $\lambda = 0$ of L_0.

Because 0 is an isolated eigenvalue of L_0, the associated total right and left eigenprojections P_0 and $\tilde{P}_0$ perturb analytically in both ω and $|\xi|$, giving projection P_ξ and $\tilde{P}_\xi$ [16]. These yield in standard fashion (for example, by projecting appropriately chosen fixed subspaces) locally analytic right and left bases $\{v_j\}$ and $\{\tilde{v}_j\}$ of the associated total eigenspaces given by the range of P_ξ, $\tilde{P}_\xi$.

Defining $V = (v_1, \dots, v_{n+1})$ and $\tilde{V} = (\tilde{v}_1, \dots, \tilde{v}_{n+1})^*$, $*$ denoting the matrix adjoint, we may convert the infinite-dimensional perturbation problem (2.1) near $|\xi| = 0$ into the $(n+1) \times (n+1)$ matrix perturbation problem

$$M_\xi = M_0 + |\xi| M_1 + |\xi|^2 M_2 + O\big(|\xi|^3\big), \tag{2.2}$$

where $M_\xi(\omega, |\xi|) := \langle \tilde{V}^*_\xi, L_\xi V_\xi \rangle$ and $\langle \cdot, \cdot \rangle$ refers to the $L^2(x_1)$ inner product on $[0, X]$. That is, the eigenvalues $\lambda_j(\xi)$ lying near 0 of L_ξ coincide with the eigenvalues of M_ξ, and the associated right and left eigenfunctions of L_ξ are

$$f_j = V w_j \quad \text{and} \quad \tilde{f}_j = \tilde{w}_j \tilde{V}^*, \tag{2.3}$$

respectively, where w_j and $\tilde{w}_j$ are the associated right and left eigenvectors of M_ξ.

Case (i). If $\lambda = 0$ is a semisimple eigenvalue of L_0, then $M_0 = 0$, and (2.2) reduces to the simpler perturbation problem $\check{M}_\xi := |\xi|^{-1} M_\xi = M_1 + |\xi| M_2$ studied in [21,13], which $\lambda_j(\xi) = |\xi| \check{\lambda}_j(\xi)$, $\check{\lambda}_j(\xi)$ denoting the eigenvalues of $\check{M}_\xi$. Since the $\check{\lambda}_j$ are continuous, the λ_j are differentiable at $|\xi| = 0$ in the parameter $|\xi|$ as asserted in the introduction. Moreover, by (H3), the eigenvalues $\check{\lambda}_j(0)$ of $M_1 = \check{M}_0$ are distinct, and so they perturb analytically in ω, $|\xi|$, as do the associated right and left eigenvectors.

Case (ii). Hereafter, assume that $\lambda = 0$ is a non-semisimple eigenvalue of L_0, so that M_0 is nilpotent but nonzero, possessing a non-trivial associated Jordan chain. By translation invariance[6] of (1.5) in the variable x_1, we have $L_0\bar{u}'(x_1) = 0$ so that $\bar{u}'$ is in the right kernel of L_0. Moreover, as the n-dimensional subspace of constant functions by direct computation lie in the kernel of $L_0^* = (\partial_{x_1}^2 + A_1^*\partial_{x_1})$, where $A_1(x_1) := df^1(\bar{u}(x_1))$, we have that the $(n+1)$-dimensional zero eigenspace of L_0 consists precisely of an n-dimensional kernel and a single Jordan chain of height two.

Now, recall the assumption (H2) that $H : \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times S^{d-1} \times \mathbb{R}^n \to \mathbb{R}^n$ taking $(X; a, s, \nu, q) \mapsto u(X; a, s, \nu, q) - a$ is full rank at $(\bar{X}; \bar{u}(0), 0, e_1, \bar{q})$, where $u(\cdot;\cdot)$ is the solution operator of profile ODE (1.7). The fact that $\ker L_0$ is n-dimensional implies that the restriction $\check{H}$ taking $(w, q) \mapsto u(X; w, s, \nu, q) - w$ for fixed (X, ν, s) is also full rank, i.e., H is full rank with respect to the specific parameters (X, s, ν). Applying the Implicit Function Theorem and counting dimensions, we find that the set of periodic solutions, i.e., the inverse image of zero under map H local to $\bar{u}$ is a smooth $(n + d + 1)$-dimensional manifold $\{\bar{u}^a(x \cdot \nu(a) - \alpha - s(a)t)\}$, with $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^{n+d}$. Moreover, $d + 1$ dimensions may be parametrized by (X, s, ν), or without loss of generality $(a_1, \dots, a_{d+1}) = (X, s, \nu)$.

Fixing (X, ν) and $(a_{d+2}, \dots, a_{n+d+1})$, and varying s, we find by differentiation of (1.7) that $f_* := -\partial_s\bar{u}$ satisfies[7] the generalized eigenfunction equation

$$L_0 f_* = \bar{u}'.$$

Thus, $\bar{u}'$ spans the eigendirection lying at the base of the Jordan chain, with the generalized zero-eigenfunction of L_0 corresponding to variations in speed along the manifold of periodic solutions about $\bar{u}$. Without loss of generality, therefore, we may take the functions $\tilde{v}_1, \dots, \tilde{v}_{n-1}$ and $\tilde{v}_{n+1}$ to be constant at $|\xi| = 0$, i.e., depending only on $\omega = \xi/|\xi|$ and not x_1, and $v_n \equiv \bar{u}'$ at $|\xi| = 0$ independent of ω.

Recalling from [13] the fact that

$$\langle c, L^1\bar{u}'\rangle = \left\langle c, \left(\omega_1(2\partial_{x_1} - A_1) - \sum_{j\neq 1}\omega_j A_j\right)\bar{u}'\right\rangle = \left\langle c, \omega_1\partial_{x_1}^2\bar{u} - \sum_{j\neq 1}\omega_j\partial_{x_1} f^j(\bar{u})\right\rangle \equiv 0$$

for any constant function c, where again $\langle\cdot,\cdot\rangle$ denotes $L^2(x_1)$ inner product on the interval $x_1 \in [0, X]$, and $A_j := df^j(\bar{u}(\cdot))$, we find under this normalization that (2.2) has the special structure

$$M_0 = \begin{pmatrix} 0_{(n-1)\times(n-1)} & 0_{n-1} & 0_{n-1} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} * & 0_{n-1} & * \\ * & * & * \\ * & 0 & * \end{pmatrix}. \tag{2.4}$$

Now, rescaling (2.2) as

$$\check{M}_\xi := |\xi|^{-1} S(\xi) M_\xi S(\xi)^{-1}, \tag{2.5}$$

where

$$S := \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & |\xi| & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.6}$$

we obtain

$$\check{M}_\xi = \check{M}_0 + |\xi|\check{M}_1 + O\big(|\xi|^2\big), \tag{2.7}$$

[6] Alternatively, this may be seen by differentiating the profile equation (1.7) with respect to x_1.

[7] Note that function f_* is X-periodic, and hence in the domain of L_0 since we have fixed the period X.

where $\check{M}_j = \check{M}_j(\omega)$ like the original M_j are analytic matrix-valued functions of ω, and the eigenvalues $m_j(\xi) = m_j(\omega; |\xi|)$ of $\hat{M}_\xi$ are $|\xi|^{-1}\lambda_j(\xi)$.

As the eigenvalues m_j of $\check{M}_\xi$ are continuous, the eigenvalues $\lambda_j(\xi) = |\xi| m_j$ are differentiable at $|\xi| = 0$ as asserted in the introduction. Moreover, by (H3), the eigenvalues $\check{\lambda}_j(0)$ of $\check{M}_0$ are distinct, and so they perturb analytically in ω, $|\xi|$, as do the associated right and left eigenvectors z_j and $\tilde{z}_j$. Undoing the rescaling (2.5), and recalling (2.3), we obtain the result. □

Remark 2.3. Note that the nth coordinate of vectors $w \in \mathbb{C}^{n+1}$ in the perturbation problem (2.2) corresponds as the coefficient of $\bar{u}'$ to variations Ψ in displacement. Thus, rescaling (2.5) amounts to substituting for Ψ the variable $|\xi|\Psi \sim \Psi_x$ of the Whitham averaged system (1.2).

2.1. Motivation/connection with the Whitham averaged equations

While not necessary for the understanding of the above analysis, it may be helpful to mention that the motivation for the preceding perturbation result comes from consideration of the relation between the Whitham (WKB) equations (1.2) and the Bloch expansion at $\xi = 0$. See for example the much more detailed study of [15] in the context of generalized KdV equations.

As a model for this relation in the simplest, one-dimensional case $d = 1$, consider a linear constant-coefficient system of PDE's of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t + \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x = \begin{pmatrix} \theta_1 & 0 \\ \gamma & \theta_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{xx} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.8}$$

where α, β, θ_1, and θ_2 are constants, with Fourier symbol corresponding to the general matrix perturbation problem

$$M(\xi) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + i\xi \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix} - \xi^2 \begin{pmatrix} \theta_1 & * \\ \gamma & \theta_2 \end{pmatrix} + \mathcal{O}\big(|\xi|^3\big) \tag{2.9}$$

for $|\xi| \ll 1$, associated with analytic bifurcation of eigenvalues of a 2×2 Jordan block. Problem (2.9) models behavior of the key Jordan block in (2.4) associated with the translational mode and displacement ψ as described in Remark 2.2, while (2.8) is a realization of the corresponding Fourier symbol modeling expected low-frequency behavior of the linearized equations about the wave.

What then is the analogy to the Whitham system (1.2), and how can we see the correspondence between analytic bifurcation of eigenvalues of $M(\xi)$ and hyperbolic structure of (1.2)?

The answer comes from going beyond comparison of eigenvalues to consideration of the structure of eigenprojections. Specifically, "balancing" by rescaling $M(\xi)$ by the matrix $T(\xi) = \operatorname{diag}(i\xi, 1)$ yields the equivalent system

$$\widetilde{M}(\xi) = (i\xi)^{-1} T(\xi) M(\xi) T(\xi)^{-1} = i\xi \begin{pmatrix} \alpha & 1 \\ \gamma & \beta \end{pmatrix} - \xi^2 \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} + \mathcal{O}\big(|\xi|^3\big)$$

(generically) not involving a Jordan block, from which one may easily analyze spectrum of $M(\xi)$ bifurcating from the $\xi = 0$ state. In particular, one directly observes analyticity of the bifurcating eigenvalues so long as the eigenvalues of $\left(\begin{smallmatrix} \alpha & 1 \\ \gamma & \beta \end{smallmatrix}\right)$ are simple.

Performing an analogous change of coordinates on (2.8), noting that multiplication by $i\xi$ corresponds to differentiation in x, we obtain

$$\begin{pmatrix} y_{1,x} \\ y_2 \end{pmatrix}_t + \begin{pmatrix} \alpha & 1 \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} y_{1,x} \\ y_2 \end{pmatrix}_x = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \begin{pmatrix} y_{1,x} \\ y_2 \end{pmatrix}_{xx}, \tag{2.10}$$

which to first order is of hyperbolic form. Indeed, this can be recognized as analogous to the Whitham averaged system, with variable $y_{1,x}$ corresponding to frequency $\Omega = \psi_x$ and y_2 corresponding to remaining variables M; see again Remark 2.2.

While the computations in this small example may seem trivial, they form the core of the preceding proof, which in turn is the foundation for all of our analysis to follow. Note that *we do not in the end need a detailed description of the Whitham system in our analysis, but only the basic structure/scaling that it suggests in the spectral problem.*

3. Linear estimates

Next, we use Lemma 2.1 along with the spectral stability assumption (D2) to obtain various pointwise (in time) bounds on the solution operator e^{Lt} corresponding to the linearized problem (1.9). Our strategy is to treat the high- and low-frequency parts of the full solution operator $S(t)$ separately since, as is typical, the low-frequency analysis is considerably more delicate than the corresponding high-frequency analysis.

To this end, notice by standard spectral perturbation theory [16], the total eigenprojection $P(\xi)$ onto the eigenspace of L_ξ associated with the eigenvalues $\lambda_j(\xi)$, $j = 1, \dots, n+1$, described in the previous section is well defined and analytic in ξ for ξ sufficiently small, since these (by discreteness of the spectra of L_ξ) are separated at $\xi = 0$ from the rest of the spectrum of L_0. Moreover, by assumption (D2) there exists an $\varepsilon > 0$ such that $\Re\lambda_j(\xi) \leqslant -\theta|\xi|^2$ for $0 < |\xi| < 2\varepsilon$. With this choice ε, we introduce a smooth cutoff function $\phi(\xi)$ that is identically one for $|\xi| \leqslant \varepsilon$ and identically zero for $|\xi| \geqslant 2\varepsilon$. Moreover, we split the solution operator $S(t) := e^{Lt}$ into a low-frequency part

$$S^I(t)u_0 := \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi(\xi)P(\xi)e^{L_\xi t}\hat{u}_0(\xi, x_1)\, d\xi_1\, d\tilde{\xi} \tag{3.1}$$

and the associated high-frequency part

$$S^{II}(t)u_0 := \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\big(I - \phi(\xi)P(\xi)\big)e^{L_\xi t}\hat{u}_0(\xi, x_1)\, d\xi_1\, d\tilde{\xi}, \tag{3.2}$$

by which one may readily check that $S(t) = (S^I(t) + S^{II}(t))$. As the low-frequency analysis is more delicate, we begin by obtaining $L^2 \to L^p$ bounds on the operator $S^{II}(t)$.

3.1. High-frequency bounds

Using the fact that L_ξ is a sectorial operator, and the spectral separation of $\lambda_j(\xi)$ from the remaining spectra of L_ξ, standard semi-group theory [8,24] implies the exponential decay bounds

$$\begin{aligned}
\big\|e^{L_\xi t}\big(I - \phi(\xi)P(\xi)\big)f\big\|_{L^2([0,X])} &\leqslant Ce^{-\theta t}\|f\|_{L^2([0,X])},\\
\big\|e^{L_\xi t}\big(I - \phi(\xi)P(\xi)\big)\partial_{x_1}^m f\big\|_{L^2([0,X])} &\leqslant Ct^{-\frac{m}{2}}e^{-\theta t}\|f\|_{L^2([0,X])},\\
\big\|\partial_{x_1}^m e^{L_\xi t}\big(I - \phi(\xi)P(\xi)\big)f\big\|_{L^2([0,X])} &\leqslant Ct^{-\frac{m}{2}}e^{-\theta t}\|f\|_{L^2([0,X])}
\end{aligned} \tag{3.3}$$

for θ, $C > 0$, and $0 \leqslant m \leqslant K$ (K as in (H1)). Together with (1.14), these give immediately the following estimates.

Proposition 3.1. *(See [21, Proposition 6.1].) Under assumptions* (H1)–(H3) *and* (D1)–(D2)*, for some θ, $C > 0$, and all $t > 0$, $2 \leqslant p \leqslant \infty$, $0 \leqslant l \leqslant K+1$, $0 \leqslant m \leqslant K$,*

$$\begin{aligned}
&\left\|\partial_x^l S^{II}(t) f\right\|_{L^2(x)}, \left\|S^{II}(t)\partial_x^l f\right\|_{L^2(x)} \leqslant Ct^{-\frac{l}{2}} e^{-\theta t}\|f\|_{L^2(x)},\\
&\left\|\partial_x^m S^{II}(t) f\right\|_{L^p(x)}, \left\|S^{II}(t)\partial_x^m f\right\|_{L^p(x)} \leqslant Ct^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{m}{2}} e^{-\theta t}\|f\|_{L^2(x)},
\end{aligned} \tag{3.4}$$

where, again, derivatives in $x \in \mathbb{R}^d$ refer to total derivatives.

Remark 3.2. Notice since Proposition 3.1 concerns only the high-frequency solution operator S^{II}, neither of the assumptions (D3′) or (D3) concerning the structure of the null-space of L_0 are necessary here.

Proof of Proposition 3.1. Using (1.14) and (3.3), the triangle inequality immediately implies the bound

$$\begin{aligned}
\left\|\partial_x^l S^{II}(t) f\right\|_{L^2(\mathbb{R}^d)} &\leqslant \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} \left\|\partial_x^l\big(1-\phi(\xi)P(\xi)\big)e^{L_\xi t}\hat{f}(\xi,\cdot)\right\|_{L^2(x;[0,1])} d\tilde{\xi}\, d\xi_1\\
&\leqslant Ct^{-l/2}e^{-\theta t}\int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} \left\|\hat{f}(\xi,\cdot)\right\|_{L^2(x;[0,1])} d\tilde{\xi}\, d\xi_1\\
&\leqslant Ct^{-l/2}e^{-\theta t}\|f\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which yields the first result.

Next, we prove the second inequality for derivatives in x_1 with $p=\infty$ and $m=0$. To this end, notice that by applying (1.14) in x_1 and the Hausdorff–Young inequality $\|h\|_{L^\infty(\mathbb{R}^{d-1})} \leqslant \|\hat{h}\|_{L^1(\mathbb{R}^{d-1})}$ in the transverse coordinate $\tilde{x}$, where here $\hat{h}$ denotes the standard Fourier transform in $\tilde{x}$, we have

$$\left\|S^{II}(t) f\right\|_{L^\infty(\tilde{x};L^2(x_1))} \leqslant Ct^{-\frac{d-1}{4}} e^{-\theta t}\|f\|_{L^2(\mathbb{R}^d)}$$

and

$$\left\|\partial_{x_1} S^{II}(t) f\right\|_{L^\infty(\tilde{x};L^2(x_1))} \leqslant Ct^{-\frac{d-1}{4}-\frac{1}{2}} e^{-\theta t}\|f\|_{L^2(\mathbb{R}^d)}.$$

Thus, we have by Sobolev embedding

$$\begin{aligned}
\left\|S^{II}(t) f\right\|_{L^\infty(\mathbb{R}^d)} &\leqslant C\big(\left\|S^{II}(t) f\right\|_{L^\infty(\tilde{x};L^2(x_1))}\cdot\left\|\partial_{x_1} S^{II}(t) f\right\|_{L^\infty(\tilde{x};L^2(x_1))}\big)^{1/2}\\
&\leqslant Ct^{-d/4}e^{-\theta t}\|f\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

as claimed. Therefore, the result for derivatives in x_1 with $m=0$ and general $2 \leqslant p \leqslant \infty$ follows by L^p-interpolation. A similar argument applies for $1 \leqslant m \leqslant K$.

Finally, the second inequality for derivatives in $\tilde{x}$ with $m=0$ follows from the inverse Fourier transform, Eq. (3.2), and the large $|\xi|$ bound

$$\left\|e^{L_\xi t} f\right\|_{L^2(x_1)} \leqslant e^{-\theta|\tilde{\xi}|^2 t}\|f\|_{L^2(x_1)}, \quad |\xi| \text{ sufficiently large},$$

which easily follows from Parseval's identity and the fact that L_ξ is a relatively compact perturbation of $\partial_x^2 - |\xi|^2$. Thus, by the above estimate we have

$$\begin{aligned}\left\|e^{Lt}\partial_{\tilde{x}}f\right\|_{L^2(x)} &\leqslant C\left\|e^{L_\xi t}|\tilde{\xi}|\hat{f}\right\|_{L^2(x_1,\xi)}\\ &\leqslant C\sup_{\xi}\left(e^{-\theta|\tilde{\xi}|^2 t}|\xi|\right)\|\hat{f}\|_{L^2(x_1,\xi)}\\ &\leqslant Ct^{-1/2}e^{-\theta t}\|f\|_{L^2(x)}.\end{aligned}$$

L^∞ bounds follow by Hausdorff–Young inequality and Sobolev embedding as in the previous case, whence the result for $m=0$ and $2\leqslant p\leqslant\infty$ follows by L^p-interpolation. A similar argument applies for $1\leqslant m\leqslant K$. $\square$

3.2. Low-frequency bounds

As noted above, analysis of the solution operator at low-frequency is considerably more complicated than the (almost trivial) high-frequency bounds outlined above. To aid in our analysis then, we denote by

$$G^I(x,t;y) := S^I(t)\delta_y(x) \tag{3.5}$$

the Green kernel associated with S^I, and

$$\left[G^I_\xi(x_1,t;y_1)\right] := \phi(\xi)P(\xi)e^{L_\xi t}\left[\delta_{y_1}(x_1)\right] \tag{3.6}$$

the corresponding kernel appearing within the Bloch–Fourier representation of G^I, where the brackets on $[G_\xi]$ and $[\delta_y]$ denote the periodic extensions of these functions onto the whole line. Our first step is to provide a useful representation for G^I which incorporates the spectral assumptions (D1)–(D3′). Using Lemma 2.1 then, we have the following descriptions of G^I and $[G^I_\xi]$, in terms of the spectral expansion of L_ξ near $|\xi|=0$.

Proposition 3.3. *(See [21, Proposition 6.2].) Under assumptions* (H1)–(H3) *and* (D1)–(D3′),

$$\begin{aligned}\left[G^I_\xi(x_1,t;y_1)\right] &= \phi(\xi)\sum_{j=1}^{n+1}e^{\lambda_j(\xi)t}q_j(\xi,x_1)\tilde{q}_j(\xi,y_1)^*,\\ G^I(x,t;y) &= \left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d}e^{i\xi\cdot(x-y)}\left[G^I_\xi(x_1,t;y_1)\right]d\xi\\ &= \left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d}e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j=1}^{n+1}e^{\lambda_j(\xi)t}q_j(\xi,x_1)\tilde{q}_j(\xi,y_1)^*\,d\xi,\end{aligned} \tag{3.7}$$

where $*$ denotes matrix adjoint, or complex conjugate transpose, $q_j(\xi,\cdot)$ and $\tilde{q}_j(\xi,\cdot)$ are right and left eigenfunctions of L_ξ associated with eigenvalues $\lambda_j(\xi)$ defined in (1.16)*, normalized so that $\langle\tilde{q}_j,q_j\rangle\equiv 1$.*

Proof. Relation (3.7)(i) is immediate from the spectral decomposition of elliptic operators on finite domains, and the fact that λ_j are distinct for $|\xi|>0$ sufficiently small, by (H3). Substituting (3.5) into (3.1) and computing

$$\widehat{\delta_y}(\xi,x_1) = \sum_k e^{2\pi ikx_1}\widehat{\delta_y}(\xi+2\pi ke_1) = \sum_k e^{2\pi ikx_1}e^{-i\xi\cdot y-2\pi iky_1} = e^{-i\xi\cdot y}\left[\delta_{y_1}(x_1)\right], \tag{3.8}$$

where the second and third equalities follow from the fact that the Fourier transform (either continuous or discrete) of the delta-function is unity, we obtain

$$G^l(x,t;y) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi P(\xi) e^{L_\xi t}\widehat{\delta_y}(\xi, x_1)\, d\xi$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(x-y)}\phi P(\xi) e^{L_\xi t}\big[\delta_{y_1}(x_1)\big]\, d\xi,$$

yielding (3.7)(ii) by (3.6) and the fact that ϕ is supported on $[-\pi,\pi]$. □

We now state our main result for this section, which utilizes the spectral representation of G^l and $[G^l_\xi]$ described in Proposition 3.3 to factor the low-frequency Green kernel into a leading order piece (corresponding to translations) plus a faster decaying residual. Underlying this decomposition is the fundamental relation

$$G(x,t;y) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(x-y)}\big[G_\xi(x_1,t;y_1)\big]\, d\xi, \tag{3.9}$$

which serves as the crux of the low-frequency analysis in the present context as well as that of [23, 13].

Proposition 3.4. *Under assumptions* (H1)–(H3) *and* (D1)–(D3′), *the low-frequency Green function* $G^l(x,t;y)$ *of* (3.5) *can be decomposed as* $G^l = E + \tilde{G}^l$ *with*

$$E = \bar{u}'(x)e(x,t;y), \tag{3.10}$$

where, for some $C > 0$, *all* $t > 0$,

$$\sup_y \big\|\tilde{G}^l(\cdot,t,;y)\big\|_{L^p(x)} \leqslant C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})},$$

$$\sup_y \big\|\partial_y^r \tilde{G}^l(\cdot,t,;y)\big\|_{L^p(x)}, \sup_y \big\|\partial_t^r \tilde{G}^l(\cdot,t,;y)\big\|_{L^p(x)} \leqslant C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \tag{3.11}$$

for $p \geqslant 2$, $1 \leqslant r \leqslant 2$. *Moreover, for* $p \geqslant 2$, $0 \leqslant j, l$, $j + l \leqslant K$, $1 \leqslant r \leqslant 2$ *we have*

$$\sup_y \big\|\partial_x^j \partial_t^l \partial_y^r e(\cdot,t,;y)\big\|_{L^p(x)} \leqslant C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{(j+l)}{2}-\frac{1}{2}} \tag{3.12}$$

and similarly, for $0 \leqslant j, l$, $j + l \leqslant K$,

$$\sup_y \big\|\tilde{\partial}_x^j \partial_t^l e(\cdot,t,;y)\big\|_{L^p(x)} \leqslant C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{(j+l)}{2}} \tag{3.13}$$

provided that $p \geqslant 2$ *and* $j + l \geqslant 1$ *or* $d \geqslant 3$, *or* $p = \infty$ *and* $d \geqslant 1$. *Moreover,* $e(x,t;y) \equiv 0$ *for* $t \leqslant 1$.

Remark 3.5. In Proposition 3.4, and throughout the remainder of the paper, derivatives in $y \in \mathbb{R}^d$ refer to total derivatives, just as with the variable $x \in \mathbb{R}^d$.

Proof of Proposition 3.4. In the degenerate case (D3) that 0 is a semisimple eigenvalue of L_0, these estimates have been established in [21, Proposition 6.3] and [13, Proposition 2.4, Lemma 2.7., Corollary 3.1]. Without loss of generality, therefore, we hereafter assume that 0 is a non-semisimple eigenvalue of L_0, i.e. that (D3′) holds but (D3) does not, with the consequences described in Lemma 2.1. In particular, recalling that

$$\begin{aligned} G^I(x,t;y) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j=1}^{n+1} e^{\lambda_j(\xi)t} q_j(\xi,x_1)\tilde{q}_j(\xi,y_1)^*\,d\xi \\ &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j,k,l=1}^{n+1} e^{\lambda_j(\xi)t}\beta_{j,k} v_k(\xi,x_1)\tilde{\beta}_{j,l}\tilde{v}_l(\xi,y_1)^*\,d\xi, \end{aligned} \tag{3.14}$$

the fact that $\beta_{j,n} = O(|\xi|^{-1})$ suggests the $k=n$ terms (corresponding to translation) dominate the low-frequency Green kernel. With this motivation, we define

$$\tilde{e}(x,t;y) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j,l} e^{\lambda_j(\xi)t}\beta_{j,n}\tilde{\beta}_{j,l}\tilde{v}_l(\xi,y_1)^*\,d\xi \tag{3.15}$$

so that

$$\begin{aligned} &G^I(x,t;y) - \bar{u}'(x_1)\tilde{e}(x,t;y) \\ &\quad= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j,k\neq n,l} e^{\lambda_j(\xi)t}\beta_{j,k}\tilde{\beta}_{j,l} v_k(\xi,x_1)\tilde{v}_l(\xi,y_1)^*\,d\xi \\ &\qquad+ \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j,l} e^{\lambda_j(\xi)t}\beta_{j,n}\tilde{\beta}_{j,l}\big(v_n(\xi,x_1)-\bar{u}'(x_1)\big)\tilde{v}_l(\xi,y_1)^*\,d\xi, \end{aligned} \tag{3.16}$$

where, by analyticity of v_n, $v_n(\xi,x_1)-\bar{u}'(x_1) = O(|\xi|)$, and so, by Lemma 2.1,

$$\beta_{j,n}\tilde{\beta}_{j,l}\big(v_n(\xi,x_1)-\bar{u}'(x_1)\big)\tilde{v}_l(\xi,y_1)^* = O(1) \tag{3.17}$$

and

$$\beta_{j,k}\tilde{\beta}_{j,l} v_k(\xi,x_1)\tilde{v}_l(\xi,y_1)^* = O(1) \quad \text{for } k\neq n. \tag{3.18}$$

Further more, note that $\tilde{v}_l$ is identically constant unless $l=n$, in which case $\tilde{\beta}_{jl} = O(|\xi|)$ by Lemma 2.1; hence

$$\partial_{y_1}\big(\beta_{j,n}\tilde{\beta}_{j,l}\big(v_n(\xi,x_1)-\bar{u}'(x_1)\big)\tilde{v}_l(\xi,y_1)^*\big) = O\big(|\xi|\big) \tag{3.19}$$

and

$$\partial_{y_1}\big(\beta_{j,k}\tilde{\beta}_{j,l} v_k(\xi,x_1)\tilde{v}_l(\xi,y_1)^*\big) = O\big(|\xi|\big) \quad \text{for } k\neq n. \tag{3.20}$$

From representation (3.16), bounds (3.17)–(3.18), and assumption (D2), we obtain by the triangle inequality

$$\left\|\tilde{G}^1(\cdot,t;\cdot)\right\|_{L^\infty(x,y)} = \left\|G^I - \bar{u}'\tilde{e}\right\|_{L^\infty(x,y)} \leqslant C\left\|e^{-\theta|\xi|^2 t}\phi(\xi)\right\|_{L^1(\xi)} \leqslant C(1+t)^{-\frac{d}{2}}. \tag{3.21}$$

Derivative bounds follow similarly, since x_1-derivatives falling on v_{jk} are harmless, whereas, by (3.19)–(3.20), y_1- or t-derivatives falling on $\tilde{v}_{jl}$ or on $e^{i\xi\cdot(x-y)}$ bring down a factor of $|\xi|$ improving the decay rate by factor $(1+t)^{-1/2}$. (Note that $|\xi|$ is bounded because of the cutoff function ϕ, so there is no singularity at $t=0$.)

To obtain the corresponding bounds for $p=2$, we note that (3.14) may be viewed itself as a Bloch–Fourier decomposition with respect to variable $z := x-y$, with y appearing as a parameter. Recalling (1.14), we may thus estimate

$$\begin{aligned}
&\sup_y \left\|G^I(\cdot,t;y) - \bar{u}'\tilde{e}(\cdot,t;y)\right\|_{L^2(x)} \\
&\quad\leqslant C\sum_{j,k\neq n,l}\sup_y \left\|\phi(\xi)e^{\lambda_j(\xi)t}v_k(\cdot,z_1)\tilde{v}_l^*(\cdot,y_1)\tilde{v}_l(\cdot,y_1)^*\right\|_{L^2(\xi;L^2(z_1\in[0,X]))} \\
&\qquad + C\sum_{j,l}\sup_y \left\|\phi(\xi)e^{\lambda_j(\xi)t}\left(\frac{v_n(\cdot,x_1)-\bar{u}'(x_1)}{|\cdot|}\right)\tilde{v}_l(\cdot,y_1)^*\right\|_{L^2(\xi;L^2(z_1\in[0,X]))} \\
&\quad\leqslant C\sum_{j,k\neq n,l}\sup_y \left\|\phi(\xi)e^{-\theta|\xi|^2 t}\right\|_{L^2(\xi)}\sup_\xi \left\|v_k(\cdot,z_1)\right\|_{L^2(0,X)}\left\|\tilde{v}_l(\cdot,y_1)^*\right\|_{L^\infty(0,X)} \\
&\qquad + C\sum_{j,l}\sup_y \left\|\phi(\xi)e^{-\theta|\xi|^2 t}\right\|_{L^2(\xi)}\sup_\xi \left\|\left(\frac{v_n(\xi,x_1)-\bar{u}'(x_1)}{|\xi|}\right)\right\|_{L^2(0,X)}\left\|\tilde{v}_l(\cdot,y_1)^*\right\|_{L^\infty(0,X)} \\
&\quad\leqslant C(1+t)^{-\frac{d}{4}},
\end{aligned} \tag{3.22}$$

where we have used in a crucial way the boundedness of $\tilde{v}_l$ in L^∞,[8] and also the boundedness of

$$\left(\frac{v_n(\xi,x_1)-\bar{u}'(x_1)}{|\xi|}\right) \sim \partial_{|\xi|}v_n(\omega;r)$$

in L^2, where $0<r<|\xi|$. Derivative bounds follow similarly as above, noting that y- or t-derivatives bring down a factor $|\xi|$, while x-derivatives are harmless, to obtain an additional factor of $(1+t)^{-1/2}$ decay. Finally, bounds for $2\leqslant p\leqslant\infty$ follow by L^p-interpolation.

Defining

$$e(x,t;y) := \chi(t)\tilde{e}(x,t;y), \tag{3.23}$$

where χ is a smooth cutoff function such that $\chi(t)\equiv 1$ for $t\geqslant 2$ and $\chi(t)\equiv 0$ for $t\leqslant 1$, and setting $\tilde{G}^I := G^I - \bar{u}'(x_1)e(x,t;y)$, we readily obtain the estimates (3.11).

[8] This is clear for $\xi=0$, since v_j are linear combinations of genuine and generalized eigenfunctions, which are solutions of the homogeneous or inhomogeneous eigenvalue ODE. More generally, note that resolvent of $L_\xi-\gamma$ gains one derivative, hence the total eigenprojection, as a contour integral of the resolvent, does too—now, use the one-dimensional Sobolev inequality for periodic boundary conditions to bound the L^∞ difference from the mean by the (bounded) H^1 norm, then bound the mean by the L^1 norm, which is controlled by the L^2 norm.

Finally, recalling, by Lemma 2.1, that $\tilde{v}_l \equiv \text{const}$ for $l \neq n$ while $\tilde{\beta}_{j,n} = O(|\xi|)$, we have

$$\partial_{y_1}\left(\beta_{j,n}\tilde{\beta}_{j,l}\tilde{v}_l(\xi, y_1)^*\right) = o\left(|\xi|\right).$$

Bounds (3.12) thus follow from (3.15) by the argument used to prove (3.11), together with the observation that x- or t-derivatives bring down factors of $|\xi|$. Bounds (3.13) follow similarly for $j + l \geqslant 1$, in which case the integrand on the right-hand side of (3.15) (now differentiated in x and/or t) is Lebesgue integrable. They follow likewise for $p = \infty$, $d \geqslant 2$ and $j + l \geqslant 0$.

In the critical case $d = 1$, $p = \infty$, and $j = l = 0$, (2.1) becomes a simpler one-parameter perturbation in ξ, and eigenvalues become analytic in ξ and not just $(|\xi|, \omega)$. Without loss of generality taking $t \geqslant 1$, expanding

$$\lambda_j(\xi) = -i\xi a_j - b_j\xi^2 + O\left(|\xi|^3\right)$$

and setting $\check{\lambda}_j(\xi) := -i\xi a_j - b_j\xi^2$, we may write $\tilde{e}(x, y; y)$ in (3.15) as

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \sum_{j=1,\ l\neq n}^{n+1} \check{\beta}_{j,n}(0)\tilde{\beta}_{j,l}(0)\tilde{v}_l(0, y)^*\xi^{-1}e^{\check{\lambda}_j(\xi)t}\,d\xi \\
&\quad = \left(\frac{1}{2\pi}\right)^d \text{P.V.}\int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \sum_{j=1,\ l\neq n}^{n+1} \check{\beta}_{j,n}(0)\tilde{\beta}_{j,l}(0)\tilde{v}_l(0, y)^*\xi^{-1}e^{\check{\lambda}_j(\xi)t}\,d\xi \\
&\quad = \sum_{j=1,\ l\neq n}^{n+1} \check{\beta}_{j,n}(0)\tilde{\beta}_{j,l}(0)\tilde{v}_l(0, y)^*\left(\frac{1}{2\pi}\right)^d \text{P.V.}\int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\xi^{-1}e^{\check{\lambda}_j(\xi)t}\,d\xi, \qquad (3.24)
\end{aligned}$$

where $\check{\beta}_{j,n}(0) := \lim_{|\xi|\to 0}(|\xi|\beta_{j,n}(\xi))$, plus a negligible error term

$$\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\,O\left(e^{-\theta|\xi|^2 t}\right)d\xi,$$

for which the integrand is Lebesgue integrable, hence, by the previous argument, obeys the bound for $j + l = 1$.

By (D2), we have a_j real and $\Re b_j > 0$. Moreover, the operator L, since real-valued, has spectrum with complex conjugate symmetry, hence b_j is real as well. Observing that the principal value integral

$$\left(\frac{1}{2\pi}\right)\text{P.V.}\int_{\mathbb{R}} e^{i\xi\cdot(x-y)}\xi^{-1}e^{\check{\lambda}_j(\xi)t}\,d\xi$$

(convergent, by the alternating series test) is an antiderivative in x of the inverse Fourier transform

$$\left(\frac{1}{2\pi}\right)\int_{\mathbb{R}} e^{i\xi\cdot(x-y)}e^{\check{\lambda}_j(\xi)t}\,d\xi = \frac{e^{-(x-y-a_jt)^2/4b_jt}}{\sqrt{4\pi b_j t}},$$

a Gaussian, we find that the principal part (3.24) is a sum of error-functions, hence bounded in L^∞ as claimed. This verifies bound (3.13) in the final case $d = 1$, $j = l = 0$, completing the proof. □

Remark 3.6. Underlying Proposition 3.4 are the estimates $\|u\|_{L^\infty(x)} \leqslant \|\hat{u}\|_{L^1(\xi;L^\infty(0,X))}$ and $\|u\|_{L^2(x)} \leqslant \|\hat{u}\|_{L^2(\xi;L^2(0,X))}$, both instances of the generalized Hausdorff–Young inequality[9]

$$\|u\|_{L^p(x)} \leqslant \|\hat{u}\|_{L^q(\xi,L^p(0,X))} \quad \text{for } q \leqslant 2 \leqslant p \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.25}$$

3.3. Final linearized estimates

Finally, we combine the high- and low-frequency estimates from the previous sections in order to obtain estimates on the full Green kernel $G(x,t;y)$. In particular, we obtain a decomposition of the Green kernel G analogous to that given in Proposition 3.4 on the low-frequency Green kernel G^I.

Corollary 3.7. *Under assumptions* (H1)–(H3), (D1)–(D3′), *the Green function* $G(x,t;y)$ *of* (1.9) *decomposes as* $G = E + \tilde{G}$,

$$E = \bar{u}'(x)e(x,t;y), \tag{3.26}$$

where, for some $C > 0$, *all* $t > 0$, $1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty$, $0 \leqslant j,k,l$, $j+l \leqslant K$, *and* $1 \leqslant r \leqslant 2$ *we have the estimates*

$$\begin{aligned}
\left\| \int_{-\infty}^{+\infty} \tilde{G}(\cdot,t;y) f(y)\,dy \right\|_{L^p(x)} &\leqslant C(1+t)^{-\frac{d}{2}(1/q-1/2)} t^{-\frac{d}{2}(1/2-1/p)} \|f\|_{L^q\cap L^2}, \\
\left\| \int_{-\infty}^{+\infty} \partial_y^r \tilde{G}(\cdot,t;y) f(y)\,dy \right\|_{L^p(x)} &\leqslant C(1+t)^{-\frac{d}{2}(1/q-1/2)-\frac{1}{2}+\frac{r}{2}} t^{-\frac{d}{2}(1/2-1/p)-\frac{r}{2}} \|f\|_{L^q\cap L^2}, \\
\left\| \int_{-\infty}^{+\infty} \partial_t^r \tilde{G}(\cdot,t;y) f(y)\,dy \right\|_{L^p(x)} &\leqslant C(1+t)^{-\frac{d}{2}(1/q-1/2)-\frac{1}{2}+r} t^{-\frac{d}{2}(1/2-1/p)-r} \|f\|_{L^q\cap L^2},
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\left\| \int_{-\infty}^{+\infty} \partial_x^j \partial_t^k e(\cdot,t;y) f(y)\,dy \right\|_{L^p} &\leqslant (1+t)^{-\frac{d}{2}(1/q-1/p)-\frac{(j+k)}{2}+\frac{1}{2}} \|f\|_{L^q}, \\
\left\| \int_{-\infty}^{+\infty} \partial_x^j \partial_t^k \partial_y^r e(\cdot,t;y) f(y)\,dy \right\|_{L^p} &\leqslant (1+t)^{-\frac{d}{2}(1/q-1/p)-\frac{(j+k)}{2}} \|f\|_{L^q}.
\end{aligned} \tag{3.28}$$

Moreover, $e(x,t;y) \equiv 0$ *for* $t \leqslant 1$.

Proof. We prove only the bounds (3.27) here, as the corresponding proofs of (3.28) follow analogously. To begin, we consider the special case $q = 1$. From (3.11) and the triangle inequality we obtain

$$\left\| \int_{\mathbb{R}^d} \tilde{G}^I(\cdot,t;y) f(y)\,dy \right\|_{L^p(x)} \leqslant \int_{\mathbb{R}^d} \sup_y \|\tilde{G}^I(\cdot,t;y)\|_{L^p} |f(y)|\,dy \leqslant C(1+t)^{-\frac{d}{2}(1-1/p)} \|f\|_{L^1}$$

[9] Estimate 3.25 follows from the extremal cases $p = 2$ and ∞ by Thorin's proof of the Riesz–Thorin Interpolation Theorem adapted to the mixed norm $\|\cdot\|_{L^q(\xi;L^p(0,X))}$; see Appendix A.

and similarly for y- and t-derivative estimates, which, together with (3.4), yield (3.27) by considering $0 \leqslant t \leqslant 1$ and $t \geqslant 1$ separately.

In the case $q = 2$, we see from (3.17)–(3.18) and the analyticity of v_j, $\tilde{v}_j$ on the variable ξ, we have boundedness from $L^2([0, X]) \to L^2([0, X])$ of the projection-type operators

$$f \to \beta_{j,n}\tilde{\beta}_{j,l}\big(v_n(\xi, x_1) - \bar{u}'(x_1)\big)\langle \tilde{v}_l, f\rangle \tag{3.29}$$

and

$$f \to \beta_{j,k}\tilde{\beta}_{j,l}v_k(\xi, x_1)\langle \tilde{v}_l, f\rangle \quad \text{for } k \neq n, \tag{3.30}$$

uniformly with respect to ξ, from which we obtain by (3.16), (3.23), and (1.14) the bound

$$\left\| \int_{-\infty}^{+\infty} \tilde{G}^I(x, t; y) f(y)\, dy \right\|_{L^2(x)} \leqslant C\|f\|_{L^2(x)} \tag{3.31}$$

for all $t \geqslant 0$, yielding together with (3.4) the result (3.27) for $p = 2$, $r = 1$. Similarly, by boundedness of $\tilde{v}_j$, v_j, $\bar{u}'$ in all $L^p[0, X]$, we have

$$\begin{aligned}
&\left\| e^{\lambda_j(\xi)t}\beta_{j,n}\tilde{\beta}_{j,l}\big(v_n(\xi, x_1) - \bar{u}'(x_1)\big)\langle \tilde{v}_l, \hat{f}\rangle \right\|_{L^\infty(x_1)} \leqslant Ce^{-\theta|\xi|^2 t}\left\| \hat{f}(\xi, \cdot)\right\|_{L^2(x_1)},\\
&\left\| e^{\lambda_j(\xi)t}\beta_{j,k}\tilde{\beta}_{j,l}v_k(\xi, x_1)\langle \tilde{v}_l, \hat{f}\rangle \right\|_{L^\infty(x_1)} \leqslant Ce^{-\theta|\xi|^2 t}\left\| \hat{f}(\xi, \cdot)\right\|_{L^2(x_1)} \quad \text{for } k \neq n,
\end{aligned}$$

$C, \theta > 0$, yielding by definitions (3.16), (3.23) the bound

$$\begin{aligned}
\left\| \int_{-\infty}^{+\infty} \tilde{G}^I(x, t; y) f(y)\, dy \right\|_{L^\infty(x)} &\leqslant \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} C\phi(\xi)e^{-\theta|\xi|^2 t}\left\| \hat{f}(\xi, \cdot)\right\|_{L^2(x_1)} d\xi_1\, d\tilde{\xi}\\
&\leqslant C\left\|\phi(\xi)e^{-\theta|\xi|^2 t}\right\|_{L^2(\xi)}\|\hat{f}\|_{L^2(\xi, x_1)}\\
&= C(1+t)^{-\frac{d}{4}}\|f\|_{L^2([0,X])},
\end{aligned} \tag{3.32}$$

hence giving the result for $p = \infty$, $r = 0$. The result for $r = 0$ and general $2 \leqslant p \leqslant \infty$ then follows by L^p-interpolation between $p = 2$ and $p = \infty$. Derivative bounds $1 \leqslant r \leqslant 2$ follow by similar arguments, using (3.19)–(3.20). Bounds (3.28) follow similarly.

Finally, using Riesz–Thorin interpolation between the cases $q = 1$ and $q = 2$ yields the bounds asserted in the general case $1 \leqslant q \leqslant 2$, $2 \leqslant p \leqslant \infty$. □

Remark 3.8. The bounds on $\tilde{G}$, e_t, e_x may be recognized as the standard diffusive bounds satisfied for the heat equation [30]. For dimension $d = 1$, it may be shown using pointwise techniques as in [23] that the bounds of Corollary 3.7 extend to all $1 \leqslant q \leqslant p \leqslant \infty$.

We note a striking analogy between the Green function decomposition of Corollary 3.7 and that of [17,32] in the viscous shock case; compare [30, Proposition 3.3].

4. Nonlinear stability in dimension one

With the bounds of Corollary 3.7, nonlinear stability follows by exactly the same argument as in [13], included here for completeness. We carry out the nonlinear stability analysis only in the most difficult, one-dimensional, case. The extension to the multi-dimensional case is straightforward [13, 21]. (Recall that the nonlinear iteration is easier to close in multi-dimensions, since the linearized behavior is faster decaying [21,13,26,25,27].) Hereafter, take $x \in \mathbb{R}^1$, dropping the indices on f^j and x_j and writing $u_t + f(u)_x = u_{xx}$.

4.1. Nonlinear perturbation equations

Given a solution $\tilde{u}(x,t)$ of (1.5), define the nonlinear perturbation variable

$$v = u - \bar{u} = \tilde{u}\big(x + \psi(x,t)\big) - \bar{u}(x), \tag{4.1}$$

where

$$u(x,t) := \tilde{u}\big(x + \psi(x,t)\big) \tag{4.2}$$

and $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is to be chosen later.

Lemma 4.1. *For v, u as in* (4.1), (4.2),

$$u_t + f(u)_x - u_{xx} = (\partial_t - L)\bar{u}'(x_1)\psi(x,t) + \partial_x R + \big(\partial_t + \partial_x^2\big)S, \tag{4.3}$$

where

$$R := v\psi_t + v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x} = O\left(|v|\big(|\psi_t| + |\psi_{xx}|\big) + \left(\frac{|\bar{u}_x| + |v_x|}{1 - |\psi_x|}\right)|\psi_x|^2\right)$$

and

$$S := -v\psi_x = O\big(|v||\psi_x|\big).$$

Proof. To begin, notice from the definition of u in (4.2) we have by a straightforward computation

$$\begin{aligned} u_t(x,t) &= \tilde{u}_x\big(x + \psi(x,t), t\big)\psi_t(x,t) + \tilde{u}_t(x + \psi, t), \\ f\big(u(x,t)\big)_x &= df\big(\tilde{u}\big(x + \psi(x,t), t\big)\big)\tilde{u}_x(x + \psi, t) \cdot \big(1 + \psi_x(x,t)\big) \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= \big(\tilde{u}_x\big(x + \psi(x,t), t\big) \cdot \big(1 + \psi_x(x,t)\big)\big)_x \\ &= \tilde{u}_{xx}\big(x + \psi(x,t), t\big) \cdot \big(1 + \psi_x(x,t)\big) + \big(\tilde{u}_x\big(x + \psi(x,t), t\big) \cdot \psi_x(x,t)\big)_x. \end{aligned}$$

Using the fact that $\tilde{u}_t + df(\tilde{u})\tilde{u}_x - \tilde{u}_{xx} = 0$, it follows that

$$\begin{aligned} u_t + f(u)_x - u_{xx} &= \tilde{u}_x\psi_t + df(\tilde{u})\tilde{u}_x\psi_x - \tilde{u}_{xx}\psi_x - (\tilde{u}_x\psi_x)_x \\ &= \tilde{u}_x\psi_t - \tilde{u}_t\psi_x - (\tilde{u}_x\psi_x)_x, \end{aligned} \tag{4.4}$$

where it is understood that derivatives of $\tilde{u}$ appearing on the right-hand side are evaluated at $(x+\psi(x,t),t)$. Moreover, by another direct calculation, using the fact that $L(\bar{u}'(x))=0$ by translation invariance, we have

$$(\partial_t - L)\bar{u}'(x)\psi = \bar{u}_x\psi_t - \bar{u}_t\psi_x - (\bar{u}_x\psi_x)_x.$$

Subtracting, and using the facts that, by differentiation of $(\bar{u}+v)(x,t)=\tilde{u}(x+\psi,t)$,

$$\begin{aligned} \bar{u}_x + v_x &= \tilde{u}_x(1+\psi_x), \\ \bar{u}_t + v_t &= \tilde{u}_t + \tilde{u}_x\psi_t, \end{aligned} \tag{4.5}$$

so that

$$\begin{aligned} \tilde{u}_x - \bar{u}_x - v_x &= -(\bar{u}_x + v_x)\frac{\psi_x}{1+\psi_x}, \\ \tilde{u}_t - \bar{u}_t - v_t &= -(\bar{u}_x + v_x)\frac{\psi_t}{1+\psi_x}, \end{aligned} \tag{4.6}$$

we obtain

$$u_t + f(u)_x - u_{xx} = (\partial_t - L)\bar{u}'(x)\psi + v_x\psi_t - v_t\psi_x - (v_x\psi_x)_x + \left((\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x}\right)_x,$$

yielding (4.3) by $v_x\psi_t - v_t\psi_x = (v\psi_t)_x - (v\psi_x)_t$ and $(v_x\psi_x)_x = (v\psi_x)_{xx} - (v\psi_{xx})_x$. □

Corollary 4.2. *The nonlinear residual v defined in* (4.1) *satisfies*

$$(\partial_t - L)v = (\partial_t - L)\bar{u}'(x_1)\psi - Q_x + R_x + \left(\partial_t + \partial_x^2\right)S, \tag{4.7}$$

where

$$Q := f\left(\tilde{u}\left(x+\psi(x,t),t\right)\right) - f\left(\bar{u}(x)\right) - df\left(\bar{u}(x)\right)v = \mathcal{O}\left(|v|^2\right), \tag{4.8}$$

$$R := v\psi_t + v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x}, \tag{4.9}$$

and

$$S := -v\psi_x = O\left(|v||\psi_x|\right). \tag{4.10}$$

Proof. Straightforward Taylor expansion comparing (4.3) and $\bar{u}_t + f(\bar{u})_x - \bar{u}_{xx} = 0$. □

Remark 4.3. In the case $\psi(x,t)=\psi(t)$, the term $(\partial_t - L)\bar{u}'(x_1)\psi(t)$ reduces to the term $\dot{\psi}(t)\bar{u}'(x_1)$ appearing in the shock wave case [33,32,30,19,18]. The fact that the shift function ψ must be chosen to be x-dependent is a reflection of the periodic boundary conditions imposed by the problem. Indeed, in the shock wave case the exponential decay of $\bar{u}'(x_1)$ allows one to consider the simplified case of an x-independent phase shift. In the periodic case considered here, however, one must proceed with more care.

4.2. Cancellation estimate

Our strategy in writing (4.7) is motivated by the following basic cancellation principle.

Proposition 4.4. *(See [9, Lemma 5.5].) For any $f(y,s) \in L^p \cap C^2$ with $f(y,0) \equiv 0$, there holds*

$$\int_0^t \int G(x,t-s;y)(\partial_s - L_y) f(y,s)\,dy\,ds = f(x,t). \tag{4.11}$$

Proof. Integrating the left-hand side by parts, we obtain

$$\int G(x,0;y) f(y,t)\,dy - \int G(x,t;y) f(y,0)\,dy + \int_0^t \int (\partial_t - L_y)^* G(x,t-s;y) f(y,s)\,dy\,ds. \tag{4.12}$$

Noting that, by duality,

$$(\partial_t - L_y)^* G(x,t-s;y) = \delta(x-y)\delta(t-s),$$

$\delta(\cdot)$ here denoting the Dirac delta-distribution, we find that the third term on the right-hand side vanishes in (4.12), while, because $G(x,0;y) = \delta(x-y)$, the first term is simply $f(x,t)$. The second term vanishes by $f(y,0) \equiv 0$. $\square$

Remark 4.5. Proposition 4.4 amounts, by the principle of linear superposition, to the evident statement that the solution of $(\partial_t - L)v = (\partial_t - L)f$ with $v|_{t=0} = 0$ is $v \equiv f$.

4.3. Nonlinear damping estimate

The following technical result is a key ingredient in our forthcoming nonlinear stability analysis. Recalling that $\psi(x,t) = 0$ for $0 \leqslant t \leqslant 1$, we may apply Duhamel's principle to (4.7) and use Proposition 4.4 to obtain the (implicit) integral representation

$$\begin{aligned} v(x,t) = {} & \int_{-\infty}^{\infty} G(x,t;y) v_0(y)\,dy \\ & + \int_0^t \int_{-\infty}^{\infty} G(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds + \psi(x,t)\bar{u}'(x) \end{aligned} \tag{4.13}$$

of the nonlinear residual v defined in (4.1). In particular, the loss of derivatives in (4.13) presents a formidable problem and requires delicate analysis in order to close any type of nonlinear iteration scheme. The next proposition is an adaptation of the methods of [18,31], which allows derivatives lost in the linearized level to be regained at the nonlinear level. While not strictly necessary in the present, strictly parabolic case, this is a convenience which allows the analysis to go through in straightforward fashion (for partially parabolic systems, it appears to be essential [19,18,31,32]).

Proposition 4.6. *Let $v(\cdot,0)\in H^K$ (K as in (H1)), and suppose that for $0\leqslant t\leqslant T$, the H^K norm of $v(\cdot,t)$ and the H^K norms of $\psi_t(\cdot,t)$ and $\psi_x(\cdot,t)$ remain bounded by a sufficiently small constant. There are then constants $\theta_{1,2}>0$ so that, for all $0\leqslant t\leqslant T$,*

$$\|v(\cdot,t)\|_{H^K}^2\leqslant Ce^{-\theta_1 t}\|v(\cdot,0)\|_{H^K}^2+C\int_0^t e^{-\theta_2(t-s)}\big(\|v(\cdot,s)\|_{L^2}^2+\|(\psi_t,\psi_x)(\cdot,s)\|_{H^K}^2\big)\,ds. \tag{4.14}$$

Proof. Subtracting from Eq. (4.4) for u the equation for $\bar u$, we may write the nonlinear perturbation equation as

$$v_t+\big(df(\bar u)v\big)_x-v_{xx}=Q(v)_x+\tilde u_x\psi_t-\tilde u_t\psi_x-(\tilde u_x\psi_x)_x, \tag{4.15}$$

where it is understood that derivatives of $\tilde u$ appearing on the right-hand side are evaluated at $(x+\psi(x,t),t)$. Using (4.6) to replace $\tilde u_x$ and $\tilde u_t$ respectively by $\bar u_x+v_x-(\bar u_x+v_x)\frac{\psi_x}{1+\psi_x}$ and $\bar u_t+v_t-(\bar u_x+v_x)\frac{\psi_t}{1+\psi_x}$, and moving the resulting $v_t\psi_x$ term to the left-hand side of (4.15), we obtain

$$\begin{aligned}(1+\psi_x)v_t-v_{xx}=&-\big(df(\bar u)v\big)_x+Q(v)_x+\bar u_x\psi_t\\&-\big((\bar u_x+v_x)\psi_x\big)_x+\left((\bar u_x+v_x)\frac{\psi_x^2}{1+\psi_x}\right)_x.\end{aligned} \tag{4.16}$$

Taking the L^2 inner product in x of $\sum_{j=0}^K\frac{\partial_x^{2j}v}{1+\psi_x}$ against (4.16), integrating by parts, and rearranging the resulting terms, we arrive at the inequality

$$\partial_t\|v(\cdot,t)\|_{H^K}^2\leqslant-\theta\|\partial_x^{K+1}v(\cdot,t)\|_{L^2}^2+C\big(\|v(\cdot,t)\|_{H^K}^2+\|(\psi_t,\psi_x)(\cdot,t)\|_{H^K}^2\big),$$

for some $\theta>0$, $C>0$, so long as $\|\tilde u(\cdot,t)\|_{H^K}$ remains bounded, and $\|v(\cdot,t)\|_{H^K}$ and $\|(\psi_t,\psi_x)(\cdot,t)\|_{H^K}$ remain sufficiently small. Using the Sobolev interpolation $\|v(\cdot,t)\|_{H^K}^2\leqslant\|\partial_x^{K+1}v(\cdot,t)\|_{L^2}^2+\tilde C\|v(\cdot,t)\|_{L^2}^2$ for $\tilde C>0$ sufficiently large, we obtain

$$\partial_t\|v(\cdot,t)\|_{H^K}^2\leqslant-\tilde\theta\|v(\cdot,t)\|_{H^K}^2+C\big(\|v(\cdot,t)\|_{L^2}^2+\|(\psi_t,\psi_x)(\cdot,t)\|_{H^K}^2\big)$$

from which (4.14) follows by Gronwall's inequality. □

4.4. Integral representation/ψ-evolution scheme

Recalling the Duhamel representation (4.13) of the perturbation v we follow [33,32,19,17] and defining ψ implicitly as

$$\begin{aligned}\psi(x,t)=&-\int_{-\infty}^{\infty}e(x,t;y)u_0(y)\,dy\\&-\int_0^t\int_{-\infty}^{+\infty}e(x,t-s;y)(-Q_y+R_y+S_t+S_{yy})(y,s)\,dy\,ds,\end{aligned} \tag{4.17}$$

where e is defined as in (3.26). Substituting in (4.13) the decomposition $G=\bar u'(x)e+\tilde G$ of Corollary 3.7, we obtain the (implicit) *integral representation*

$$v(x,t) = \int_{-\infty}^{\infty} \tilde{G}(x,t;y)v_0(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds, \tag{4.18}$$

for the nonlinear residual v and, differentiating (4.17) with respect to x and t, and recalling that $e(x,s;y) \equiv 0$ for $s \leqslant 1$,

$$\partial_t^j \partial_x^k \psi(x,t) = -\int_{-\infty}^{\infty} \partial_t^j \partial_x^k e(x,t;y)u_0(y)\,dy - \int_0^t \int_{-\infty}^{+\infty} \partial_t^j \partial_x^k e(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds. \tag{4.19}$$

Eqs. (4.18), (4.19) together form a complete system in the variables $(v, \partial_t^j \psi, \partial_x^k \psi)$, $0 \leqslant j \leqslant 1$, $0 \leqslant k \leqslant K$, from the solution of which we may afterward recover the shift ψ via (4.17). From the original differential equation (4.7) together with (4.19), we readily obtain short-time existence and continuity with respect to t of solutions $(v, \psi_t, \psi_x) \in H^K$ by a standard contraction-mapping argument based on (4.14), (4.17), and (3.28).

4.5. Nonlinear iteration

Associated with the solution (u, ψ_t, ψ_x) of integral system (4.18)–(4.19), define

$$\zeta(t) := \sup_{0 \leqslant s \leqslant t} \big\|(v, \psi_t, \psi_x)(\cdot, s)\big\|_{H^K} (1+s)^{1/4}. \tag{4.20}$$

By short-time H^K existence theory, $\|(v, \psi_t, \psi_x)\|_{H^K}$ is continuous so long as it remains small, hence ζ is a continuous function of t so long as it remains small. We now use the linearized estimates of Section 3 to prove that if ζ is initially small then it must remain so.

Lemma 4.7. *For all $t \geqslant 0$ for which $\zeta(t)$ is finite, some $C > 0$, and $E_0 := \|v(\cdot,0)\|_{L^1 \cap H^K}$,*

$$\zeta(t) \leqslant C\big(E_0 + \zeta(t)^2\big). \tag{4.21}$$

Proof. By (4.9)–(4.10) and definition (4.20),

$$\begin{aligned} \big\|(Q,R,S)(\cdot,t)\big\|_{L^1 \cap L^2} &\leqslant \big\|(v, v_x, \psi_t, \psi_x)(\cdot,t)\big\|_{L^1}^2 + \big\|(v, v_x, \psi_t, \psi_x)(\cdot,t)\big\|_{L^2}^2 \\ &\leqslant C\zeta(t)^2(1+t)^{-\frac{1}{2}}, \end{aligned} \tag{4.22}$$

so long as $|\psi_x| \leqslant \|\psi_x\|_{H^K} \leqslant \zeta(t)$ remains small, and likewise (using the equation to bound t-derivatives in terms of x-derivatives of up to two orders)

$$\big\|\big(\partial_t + \partial_x^2\big)S(\cdot,t)\big\|_{L^1 \cap L^2} \leqslant \big\|(v,\psi_x)(\cdot,t)\big\|_{W^{1,1}}^2 + \big\|(v,\psi_x)(\cdot,t)\big\|_{H^2}^2 \leqslant C\zeta(t)^2(1+t)^{-\frac{1}{2}}. \tag{4.23}$$

Applying Corollary 3.7 with $q=1$, $d=1$ to representations (4.18)–(4.19), we obtain for any $2 \leqslant p < \infty$[10]

$$\begin{aligned}\|v(\cdot,t)\|_{L^p(x)} &\leqslant C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 \\ &\quad + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac{1}{4}}(t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}}(1+s)^{-\frac{1}{2}}\,ds \\ &\leqslant C\big(E_0+\zeta(t)^2\big)(1+t)^{-\frac{1}{2}(1-1/p)} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\|(\psi_t,\psi_x)(\cdot,t)\|_{W^{K,p}} &\leqslant C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)-1/2}(1+s)^{-\frac{1}{2}}\,ds \\ &\leqslant C\big(E_0+\zeta(t)^2\big)(1+t)^{-\frac{1}{2}(1-1/p)}. \end{aligned} \tag{4.25}$$

Now, by the nonlinear damping estimate given in Proposition 4.6 the size of v in $H^K(\mathbb{R}^d)$ can be controlled by its size in L^2 together with H^K estimates on the shift function ψ. In particular, we have that for some positive constants θ_1 and θ_2

$$\begin{aligned}\|v(\cdot,t)\|^2_{H^K(\mathbb{R}^d)} &\leqslant Ce^{-\theta_1 t}E_0^2 + C\big(E_0+\zeta(t)^2\big)^2\int_0^t e^{-\theta_2(t-s)}(1+s)^{-1/2} \\ &\leqslant Ce^{-\theta_1 t}E_0^2 + C\big(E_0+\zeta(t)^2\big)^2(1+t)^{-1/2} \\ &\leqslant C\big(E_0+\zeta(t)\big)^2(1+t)^{-1/2}. \end{aligned}$$

Combining with (4.25) in the case $p=2$ and recalling the definition of $\zeta(t)$ completes the proof. □

Proof of Theorem 1.3. Recalling that $\zeta(t)$ is continuous so long as it remains small, it follows by (4.7) and continuous induction that $\eta(t) \leqslant 2C\eta_0$ for $t \geqslant 0$, if $\eta_0 < 1/4C$, yielding by (4.20) the result (1.17) for $p=2$. Similarly, using (4.24)–(4.25), we obtain (1.17) for $2 \leqslant p \leqslant p_*$ for any $p_* < \infty$, with uniform constant C. Taking $p_* > 4$ and estimating

$$\|Q\|_{L^2}, \|R\|_{L^2}, \|S\|_{L^2}(t) \leqslant \|(v,\psi_t,\psi_x)\|^2_{L^4} \leqslant CE_0(1+t)^{-\frac{3}{4}}$$

in place of the weaker (4.22), then applying Corollary 3.7 with $q=2$, $d=1$, we obtain finally (1.17) for $2 \leqslant p \leqslant \infty$, by a computation similar to (4.24)–(4.25); we omit the details of this final bootstrap argument.

Finally, notice that by (4.1) we have

$$\tilde{u}(x,t) - \bar{u}(x) = v\big(x-\psi(x,t),t\big) + \big(\bar{u}\big(x-\psi(x,t)\big) - \bar{u}(x)\big)$$

and hence the size of $\tilde{u}(x,t)-\bar{u}(x)$ in L^p or H^K is controlled by the corresponding size of the function $(v+\psi)(x,t)$ in the respective norm. Therefore, using (4.24) along with the estimate

[10] Notice the following bounds in the case $p=\infty$ do not necessarily hold due to a term of size $\log(1+t)$ appearing from integrating over $[\frac{t}{2},t]$.

$$\begin{aligned}\|\psi(t)\|_{L^p} &\leqslant CE_0(1+t)^{\frac{1}{2p}} + C\zeta(t)^2 \int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)}(1+s)^{-\frac{1}{2}}\,ds \\ &\leqslant C(1+t)^{\frac{1}{2p}}\big(E_0+\zeta(t)^2\big), \end{aligned} \tag{4.26}$$

which follows by (3.28) with $q=d=1$ for $1\leqslant p\leqslant\infty$, we obtain the estimate (1.18). This yields stability for $\|u-\bar{u}\|_{L^1\cap H^K}|_{t=0}$ sufficiently small, as described in the final line of the theorem. □

Acknowledgment

Thanks to Fedja Nazarov for suggesting the proof of Proposition A.1.

Appendix A. Riesz–Thorin interpolation in Bloch norms

For functions $h(\xi,x_1)$, periodic in x_1 with period X, define the Bloch norm $\|\cdot\|_{p,q}$ by

$$\|h(\xi,x)\|_{p,q} := \|h\|_{L^q(\xi;L^p(0,X))}, \tag{A.1}$$

and $B^{p,q}$ to be the space of x_1-periodic functions with finite Bloch norm. Then, we have the following generalization of the Riesz–Thorin Interpolation Theorem.

Proposition A.1. *Let T be a linear operator that is bounded from B^{p_0,q_0} to L^{r_0} with norm N_0 and from B^{p_1,q_1} to L^{r_1} with norm N_1, where p_0,q_0,r_0 are between 1 and ∞, and let*

$$\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right) = \theta\left(\frac{1}{p_1},\frac{1}{q_1},\frac{1}{r_1}\right) + (1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0},\frac{1}{r_0}\right) \tag{A.2}$$

for $0\leqslant\theta\leqslant 1$. Then, T is bounded from $B^{p,q}$ to L^r with norm $N\leqslant N_0^\theta N_1^{1-\theta}$.

Proof. For simplicity, take $r<\infty$; the case $r=\infty$ may be treated by a slight modification. Without loss of generality, take $h(\xi,x_1)=\phi(\xi)f(\xi,x_1)$, where $\|f(\xi,\cdot)\|_{L^p(0,X)}\equiv 1$, and $\|\phi\|_{L^q}=1$. By duality, it is sufficient to show that $\int |Th|(x)|g|(x)\,dx \leqslant N_0^\theta N_1^{1-\theta}$ for all g such that $\|g\|_{L^{r'}}=1$, where $\frac{1}{r}+\frac{1}{r'}=1$. Accordingly, define the analytic function

$$F(z) := \int (Th_z)(x)g_z(x)\,dx, \tag{A.3}$$

where

$$h_z := |f|^{p(z)/p}\big(f/|f|\big)|\phi|^{q(z)/q}\big(\phi/|\phi|\big), \qquad g_z := |g|^{r'(z)/r'}\big(g/|g|\big), \tag{A.4}$$

with

$$\left(\frac{1}{p},\frac{1}{q},\frac{1}{r},\frac{1}{r'}\right)(z) := z\left(\frac{1}{p_1},\frac{1}{q_1},\frac{1}{r_1},\frac{1}{r_1'}\right) + (1-z)\left(\frac{1}{p_0},\frac{1}{q_0},\frac{1}{r_0},\frac{1}{r_0'}\right). \tag{A.5}$$

Evidently, $\|h_z\|_{p(z),q(z)} = \|h\|_{p,q}^{p(z)} \equiv 1$, $\|g_z\|_{r'(z)} = \|g\|_{r'}^{r'(z)} \equiv 1$, by assumption. Thus, for $\Re z=0$,

$$\big|F(z)\big| \leqslant \|Th\|_{r_0}\|g\|_{r_0'} \leqslant N_0\|h\|_{p_0,q_0}\|g\|_{r_0'} = N_0$$

and, for $\Re z = 1$,

$$\left|F(z)\right| \leqslant \|Th\|_{r_1}\|g\|_{r_1'} \leqslant N_1\|h\|_{p_1,q_1}\|g\|_{r_1'} = N_1.$$

By the Three Lines Theorem, therefore, $|F(z)| \leqslant N_0^\theta N_1^{1-\theta}$ for $\Re z = \theta$ with $0 \leqslant \theta \leqslant 1$. Observing that $h_\theta = h$, $g_\theta = g$ by $(p,q,r)(\theta) = (p,q,r)$, we obtain the result. □

Corollary A.2 *(Generalized Hausdorff–Young inequality).*

$$\|u\|_{L^p(x)} \leqslant \|\hat{u}\|_{L^q(\xi,L^p(0,X))} \quad \textit{for } q \leqslant 2 \leqslant p \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{A.6}$$

Proof. The extremal cases $p = 2$ and $p = \infty$ follow respectively by Parseval's identity and the triangle inequality, whence the result follows by Proposition A.1, with T defined as the Bloch–Fourier transform, $r = p$, and without loss of generality $2 < r < \infty$. □

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