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The relational model is injective for multiplicative exponential linear logic (without weakenings)

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ABSTRACT

We show that for Multiplicative Exponential Linear Logic (without weakenings) the syntactical equivalence relation on proofs induced by cut-elimination coincides with the semantic equivalence relation on proofs induced by the multiset based relational model: one says that the interpretation in the model (or the semantics) is injective. We actually prove a stronger result: two cut-free proofs of the full multiplicative and exponential fragment of linear logic whose interpretations coincide in the multiset based relational model are the same "up to the connections between the doors of exponential boxes".

1. Introduction

Separation is an important mathematical property, and several theorems are often referred to as "separation theorems". In theoretical computer science, one of the most well-known examples of separation theorem is Böhm's theorem [2] for pure λ-calculus: if t, t' are two distinct closed $\beta\eta$-normal terms, then there exists a context $C[\ ]$ s.t. $C[t] \simeq_\beta 0$ and $C[t'] \simeq_\beta 1$. Such a result induces an order relation (i.e. a T_0 topology) on the $\beta\eta$-equivalence classes of (normalizable) λ-terms. Later on, this kind of question has been studied by Friedman and Statman for the simply typed λ-calculus [29], leading to what is often called "typed Böhm's theorem" (see also [17,11] for sharper formulations). We believe that if no other result of this kind has been produced for a long time, it is due to the absence of interesting logical systems where proofs could be represented in a nice "canonical" way.

The situation radically changed in the nineties, mainly due to Linear Logic (LL [13]), a refinement of intuitionistic (and classical) logic characterized by the introduction of new connectives (the exponentials) which give a *logical* status to the operations of erasing and copying (corresponding to the *structural rules* of logic): this change of viewpoint had striking consequences in proof-theory, like the introduction of proof-nets, a geometric way of representing computations. In the framework of proof-nets, the separation property can be studied: the first work on the subject is [20] where the authors

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deal with the translation in LL of the pure λ-calculus; it is a key property of ludics [14] and has been studied more recently for the intuitionistic multiplicative fragment of LL [21] and for differential nets [24]. For Parigot's $\lambda\mu$-calculus, see [5,28].

Still in LL's framework, a semantic approach to the question of separation is developed in [9,10], where the (very natural) question of "injectivity" of the semantics is addressed: do the equivalence relation on proofs defined by the cut-elimination procedure and the one defined by a given denotational model (sometimes/always) coincide? When the answer is positive one says that the model is *injective* (it separates syntactically different proofs). Indeed, two proofs are "syntactically" equivalent when (roughly speaking) they have the same cut-free form (in a confluent and weakly normalizing system), and they are "semantically" equivalent in a given denotational model (a semantics of proofs in logical terms) when they have the same *interpretation*. It is worth noticing that the study of both these equivalence relations is at the heart of the whole research area between proof-theory and theoretical computer science: on the one hand, cut-elimination is a crucial property of logical systems since Gentzen. In the second part of last century, there was a renewal of interest in this property after the discovery of the Curry–Howard correspondence: a proof is a program whose execution corresponds to applying the cut-elimination procedure to the proof. On the other hand, the general goal of denotational semantics is to give a "mathematical" counterpart to syntactical devices such as proofs and programs, bringing to the fore their essential properties: the basic pattern is to associate with every formula/type an object of some category and with every proof/program a morphism of this category (its interpretation). In the theoretical computer science tradition, once a notion of "value" is defined, one often wants to consider that two programs are equivalent when whatever context one chooses, the two programs are either both non-correct or they are both correct and yield the same value: this equivalence is then called "observational equivalence". When the semantic equivalence in a given model coincides with observational equivalence, one says that the model is "fully abstract". Full abstraction is among the most studied properties of theoretical computer science in the last decades. In LL, if one considers cut-free proof-nets as values, the syntactic equivalence relation is observational equivalence, and a model is injective precisely when it is fully abstract. To be more precise, in an untyped framework one should also have that two proof-nets with the same interpretation are either both normalizable[1] or both non-normalizable: in the relational model this is a consequence of the semantic characterization of normalizable proof-nets given in [6].

The works [9,10] give partial results and counterexamples to the question of injectivity, mainly for the (multiset based) coherent model: in particular the counterexamples show that this model is not injective for multiplicative and exponential LL (*MELL*). Also, it was conjectured that the (multiset based) relational model is injective for *MELL*, but despite many efforts [9], [10], [3], [25], [24], [26]... all the attempts to prove the conjecture failed up to now: no real progress has been done since [10], where a proof of injectivity of the relational model is given for a fragment of *MELL*.[2] Game semantics is much closer to syntax than relational and coherent semantics, and positive answers have been obtained for little fragments like the multiplicative fragment *MLL* or the fragment corresponding to the λ-calculus [1,16], but also for the polarized fragment of LL [19].

We prove here that for *MELL* without weakenings (and without the multiplicative unit $\perp$) relational semantics is injective (Corollary 55). This tremendous improvement with respect to the previous situation is an immediate consequence of a much stronger result: in the full *MELL* fragment (with units) two proof-nets R and R' with the same interpretation are the same "up to the connections between the doors of exponential boxes" (we say they have the same LPS: Theorem 50 and Corollary 52). This result can be expressed in terms of differential nets [12]: two cut-free proof-nets with different LPS have different Taylor expansions. We also believe this work is an essential step towards the proof of the full conjecture.

In the style of [8,6] we work in an untyped framework; we do not define proof-nets nor cut-elimination but only cut-free proof-structures (PS, Definition 17): we prove that two PS with the same interpretation have the same LPS (Corollary 52). A proof-net (as defined in [6]) is a particular case of PS so that the result holds for untyped (so as for typed) *MELL* proof-nets (Remark 56).

Since we want to prove that two PS are isomorphic in Theorem 50, it is mandatory to have a (simple and clear) notion of isomorphism between PS (Definition 19), and this is why in Section 2 we give a very sharp description of the syntax in the style of interaction nets [18,22]: we cannot only rely on a graphic intuition. The notion of Linear Proof-Structure (LPS), which comes from [10], is our main syntactical tool: with every proof-net R of (say) [6] is associated a LPS, which is obtained from R by forgetting some information about R's exponential boxes, namely which auxiliary doors correspond to which !-link (using standard LL's terminology); this is particularly clear in Definition 17 of PS: a PS is a LPS and a function allowing to recover boxes. Recovering this function from the interpretation of a PS is the only missing point in the proof of the full conjecture, but a simple remark shows that the function can be recovered from the LPS when the PS is a connected graph: this yields injectivity for *MELL* without weakenings and $\perp$ (Corollary 55).

In Section 3, we introduce a domain D to interpret PS which is essentially the one already defined in [6]. Like in [10], we use here experiments (introduced in [13]) which can be thought as objects in between syntax and semantics and are related to type derivations in the λ-calculus [7]. Experiments are functions defined on proof-nets allowing to compute the interpretation pointwise: the set of *results* of all the experiments of a given proof-net is its interpretation.[3] Usually

1 We mean that it is possible to apply the cut-elimination procedure to both the proof-nets and obtain a cut-free proof-net.

2 Precisely, for the $(?\wp)$LL fragment given by $A ::= X \mid ?A\wp A \mid A\wp ?A \mid A\wp A \mid A\otimes A \mid !A$.

3 The result of an experiment e is the image of the conclusions of the proof-net through the function e; so that contrary to an experiment its result is a truly semantic object.

an experiment e of a proof-net R is a labeling of R at depth 0 and a function associating with every !-link l of R a set of experiments of the content of the box associated with l. We noticed that a particular kind of experiment called *k-experiment* (Definition 35) can be defined directly on LPS (boxes are not needed).

In Section 4, we state our results and reduce the problem of injectivity to Proposition 40, which is proven in Section 5.

The paper ends with a technical appendix, containing some obvious definitions and the formal details of some constructions previously used.

In [10], a single (well-chosen!) point of the interpretation of a proof-net allowed to "rebuild" the entire proof-net (in some particular cases and for coherent semantics). Something similar happens in this paper, with a notable difference that makes everything much more complicated: in [10] the well-chosen point of the interpretation of a proof-net allowed not only to rebuild the proof-net but also the experiment having this point as result. This is not the case here, where the well-chosen points of the interpretation of a PS are atomic injective k-points (Definition 22): we show (see Example 28 and Fig. 2) that there exist different experiments having as result the same atomic injective k-point. Let us conclude by mentioning the main novelties in our proof.

- We use injective experiments in a completely different sense than in [10]: intuitively, our injective k-experiments associate with an axiom link with depth d, k^d different labels, while the injective k-obsessional experiments of [10] associate a unique label with such an axiom link (see Remark 23). A crucial aspect of our new injective k-experiments is that they can be recognized by their results (Definition 22), and this was not the case for *relational* injective k-obsessional experiments.
- We define some kind of "prototype" of injective atomic k-experiments: the notion of injective atomic k-experiment of LPS (Definitions 35 and 36). It is true that the two experiments of the PS of Fig. 2 previously mentioned (see again Example 28 for the details) are different, but we would like to consider them as "the same" experiment: any atomic injective k-experiment of a given LPS Φ, allows to generate the set of injective atomic k-points of the interpretation of every PS having Φ as LPS (Fact 48).
- We consider the results of experiments after forgetting the names of the atoms (see again Fact 48): two experiments having as results injective and atomic k-points that are the same "up to the names of the atoms" might not be the same experiment, but they are necessarily experiments of two PS having the same LPS (Proposition 40).

Summing up, we show that if the interpretation of the PS R contains an atomic injective k-point, then every R' with the same interpretation as R has the same LPS as R (Corollary 52); and contrary to [10] we do not know the experiment which produced this point.

Conventions. We use the notation [ ] for multisets while the notation { } is, as usual, for sets. For any set A, we denote by $\mathfrak{M}_{\mathrm{fin}}(A)$ the set of finite multisets a whose support, denoted by $Supp(a)$, is a subset of A. The pairwise union of multisets given by term-by-term addition of multiplicities is denoted by a $+$ sign and, following this notation, the generalized union is denoted by a $\sum$ sign. The neutral element for this operation, the empty multiset, is denoted by [ ]. For $k \in \mathbb{N}$ and a multiset, we denote by $k \cdot a$ the multiset defined by $Supp(k \cdot a) = Supp(a)$ and for every $\alpha \in Supp(a)$, $(k \cdot a)(\alpha) = ka(\alpha)$.

For any $k \in \mathbb{N}$, we set $\ulcorner k \urcorner = \{1, \ldots, k\}$. For any set A, we denote by $A^{<\omega}$ the set of finite sequences of elements of A, by $\mathfrak{P}(A)$ the powerset of A, by $\mathfrak{P}_{\mathrm{fin}}(A)$ the finite powerset of A and by $\mathfrak{P}_2(A)$ the set $\{\{a, b\} \in \mathfrak{P}(A) \mid a, b \in A \text{ and } a \neq b\}$. A function $f : A \to B$ has domain $A = \mathrm{dom}(f)$, codomain $B = \mathrm{codom}(f)$, image $\mathrm{im}(f) = \{f(a) \mid a \in A\}$; we denote by $f\big|_{A'}^{B'}$ the restriction of f to the domain A' and to the codomain B' and by $\mathfrak{P}(f)$ the function $\mathfrak{P}(A) \to \mathfrak{P}(B)$ which associates with every $X \subseteq A$ the set $\{f(x) \mid x \in X\}$. We denote by ε the unique element of $\ulcorner k \urcorner^0$ for any $k \in \mathbb{N}$ and by $A \uplus B$ the disjoint union of the sets A and B.

2. Syntax

This section is devoted to present in full details the syntactical objects for which we prove our main result: proof-structures (Definition 17). We adopt the interaction nets point of view and pass through intermediate objects: cells and ports (Section 2.1), Pre-Linear Proof-Structures (Section 2.2), Linear Proof-Structures (Section 2.3), Proof-Structures (Section 2.4). Cells come with a notion of isomorphism which is then adapted to its refinements; isomorphisms between Linear Proof-Structures and Proof-Structures will be crucial to prove the results presented in the paper (see for example Proposition 40 and Theorem 50).

2.1. Cells and ports

We introduce cells and ports, which intuitively correspond to "links with their premises and conclusions" in the theory of linear logic proof-nets ([13,4,9], ...). Our presentation is in the style of interaction nets [18,22], where principal (resp. auxiliary) ports correspond to the conclusions (resp. the premises) of the links and axiom links of the usual syntax become wires (see Definition 7). We deal with (the analogue of) unary !-links, while ?-links can have an arbitrary number of premises. More precisely, we set $\mathcal{T} = \{\otimes, \parr, 1, \perp, !, ?\}$ and we define **Cells** as follows.

Definition 1. A *cell base* is a 6-tuple $\mathbb{C} = (\mathsf{t}, \mathcal{P}, \mathsf{C}, \mathsf{P}^{\mathrm{pri}}, \mathsf{P}^{\mathrm{left}}, \#)$ such that

- t is a function such that $\mathrm{dom}(t)$ is finite and $\mathrm{codom}(\mathsf{t}) = \mathcal{T}$; the elements of $\mathrm{dom}(\mathsf{t})$ are *the cells of* $\mathbb{C}$;
- $\mathcal{P}$ is a finite set whose elements are *the ports of* $\mathbb{C}$;
- C is a surjection $\mathcal{P} \to \mathrm{dom}(\mathsf{t})$ such that for any $l \in \mathrm{dom}(\mathsf{t})$, we have
  - $\mathsf{t}(l) \in \{\otimes, \parr\} \Rightarrow \mathrm{Card}(\{p \in \mathcal{P} \mid \mathsf{C}(p) = l\}) = 3$;
  - $\mathsf{t}(l) = ! \Rightarrow \mathrm{Card}(\{p \in \mathcal{P} \mid \mathsf{C}(p) = l\}) = 2$;
  - and $\mathsf{t}(l) \in \{1, \bot\} \Rightarrow \mathrm{Card}(\{p \in \mathcal{P} \mid \mathsf{C}(p) = l\}) = 1$;

  the set $\{p \in \mathcal{P} \mid \mathsf{C}(p) = l\}$ is the set of *the ports of* l;
- $\mathsf{P}^{\mathrm{pri}}$ is a function $\mathrm{dom}(\mathsf{t}) \to \mathcal{P}$ such that $\mathsf{C} \circ \mathsf{P}^{\mathrm{pri}} = id_{\mathrm{dom}(\mathsf{t})}$; the port $\mathsf{P}^{\mathrm{pri}}(l)$ is *the principal port of* l. A port of l different from $\mathsf{P}^{\mathrm{pri}}(l)$ is an *auxiliary port of* l;
- $\mathsf{P}^{\mathrm{left}}$ is a function $\mathcal{C}^{\mathrm{m}} \to \mathcal{P}$ such that $\mathsf{P}^{\mathrm{left}}(l)$ is an auxiliary port of l, where $\mathcal{C}^{\mathrm{m}} = \{l \in \mathrm{dom}(\mathsf{t}) \mid \mathsf{t}(l) \in \{\otimes, \parr\}\}$;
- $\#$ is a function $\bigcup_{l \in \mathcal{C}^?} \{p \in \mathcal{P} \setminus \{\mathsf{P}^{\mathrm{pri}}(l)\} \mid \mathsf{C}(p) = l\} \to \mathbb{N}$, where $\mathcal{C}^?$ is the set $\{l \in \mathrm{dom}(t) \mid \mathsf{t}(l) = ?\}$.

We denote by **Cells** the set of cell bases.

Notations 2. Let $\mathbb{C} \in$ **Cells**. We set $\mathcal{C}(\mathbb{C}) = \mathrm{dom}(\mathsf{t})$, $\mathsf{t}_{\mathbb{C}} = \mathsf{t}$, $\mathcal{P}(\mathbb{C}) = \mathcal{P}$, $\mathsf{C}_{\mathbb{C}} = \mathsf{C}$, $\mathsf{P}^{\mathrm{pri}}_{\mathbb{C}} = \mathsf{P}^{\mathrm{pri}}$, $\mathsf{P}^{\mathrm{left}}_{\mathbb{C}} = \mathsf{P}^{\mathrm{left}}$ and $\#_{\mathbb{C}} = \#$. Moreover, for any $t \in \mathcal{T}$, we define the set $\mathcal{C}^t(\mathbb{C})$ by setting $\mathcal{C}^t(\mathbb{C}) = \{l \in \mathcal{C}(\mathbb{C}) | \mathsf{t}_{\mathbb{C}}(l) = t\}$. We set $\mathcal{C}^{\mathrm{m}}(\mathbb{C}) = \mathcal{C}^{\otimes}(\mathbb{C}) \cup \mathcal{C}^{\parr}(\mathbb{C})$.

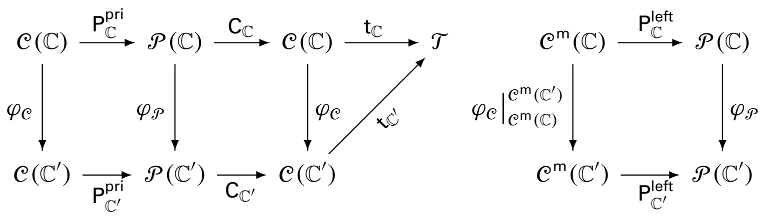
Remark 3. (i) Intuitively, $\mathbb{C} \in$ **Cells** corresponds to what is called "a set of links" in the usual syntax of [10]. Notice that the functions $\mathsf{P}^{\mathrm{pri}}$ and $\mathsf{P}^{\mathrm{left}}$ of Definition 1 induce the functions $\mathsf{P}^{\mathrm{aux}}_{\mathbb{C}} : \mathcal{C}(\mathbb{C}) \to \mathfrak{P}(\mathcal{P}(\mathbb{C}))$ and $\mathsf{P}^{\mathrm{right}}_{\mathbb{C}} : \mathcal{C}^{\mathrm{m}}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ defined by $\mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l) = \{p \in \mathcal{P}(\mathbb{C}) \setminus \{\mathsf{P}^{\mathrm{pri}}_{\mathbb{C}}(l)\} \mid \mathsf{C}_{\mathbb{C}}(p) = l\}$ and by $\{\mathsf{P}^{\mathrm{right}}_{\mathbb{C}}(l)\} = \mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l) \setminus \{\mathsf{P}^{\mathrm{left}}_{\mathbb{C}}(l)\}$: the functions $\mathsf{P}^{\mathrm{pri}}_{\mathbb{C}}$ and $\mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}$ allow to distinguish the principal ports (conclusions in [10]) from the auxiliary ports (premises in [10]), while for multiplicative cells the functions $\mathsf{P}^{\mathrm{left}}_{\mathbb{C}}$ and $\mathsf{P}^{\mathrm{right}}_{\mathbb{C}}$ allow to distinguish the left auxiliary port (left premise in [10]) from the right one. We denote by $\mathcal{P}^{\mathrm{pri}}(\mathbb{C})$ (resp. $\mathcal{P}^{\mathrm{aux}}(\mathbb{C})$) the set of principal (resp. auxiliary) ports of $\mathbb{C}$. Moreover, we denote by $\mathsf{a}_{\mathbb{C}}$ the function $\mathcal{C}(\mathbb{C}) \to \mathbb{N}$ defined by $\mathsf{a}_{\mathbb{C}}(l) = \mathrm{Card}(\mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l))$; the integer $\mathsf{a}_{\mathbb{C}}(l)$ is *the arity of* l.

(ii) There is however a notable difference w.r.t. [18] in the way we handle boxes in our **PS** (Definition 17): here the function $\#$ plays a crucial role. If $p \in \mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l)$ for some $l \in \mathcal{C}^?(\mathbb{C})$, then the integer $\#_{\mathbb{C}}(p)$ is in the syntax of [10] the number of auxiliary doors of boxes of the exponential branch corresponding to p. For instance, for the $\mathbb{C}$ in Fig. 2, we have $\#_{\mathbb{C}}(p_1) = 0$ and $\#_{\mathbb{C}}(p_2) = 1$. In the spirit of LL, we split the set $\mathcal{C}^?(\mathbb{C})$ into the four following disjoint sets:

- $\mathcal{C}^{?\mathrm{w}}(\mathbb{C}) = \{l \in \mathcal{C}^?(\mathbb{C}) \mid \mathsf{a}_{\mathbb{C}}(l) = 0\}$ which (in [10]) corresponds to the set of weakening links of $\mathbb{C}$
- $\mathcal{C}^{?\mathrm{d}}(\mathbb{C}) = \{l \in \mathcal{C}^?(\mathbb{C}) \mid \mathsf{a}_{\mathbb{C}}(l) = 1$ and $\#_{\mathbb{C}}(p) = 0$, where $\{p\} = \mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l)\}$, which (in [10]) corresponds to the set of dereliction links of $\mathbb{C}$
- $\mathcal{C}^{?\mathrm{c_b}}(\mathbb{C}) = \{l \in \mathcal{C}^?(\mathbb{C}) \mid \mathsf{a}_{\mathbb{C}}(l) > 1$ and $(\exists p \in \mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l))\, \#_{\mathbb{C}}(p) = 0\}$, which (in [10]) corresponds to the set of contraction links of $\mathbb{C}$ having at least the conclusion of one dereliction link among their premises
- $\mathcal{C}^{?\mathrm{c_{auxd}}}(\mathbb{C}) = \{l \in \mathcal{C}^?(\mathbb{C}) \mid \mathsf{a}_{\mathbb{C}}(l) \geq 1$ and $(\forall p \in \mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l))\, \#_{\mathbb{C}}(p) > 0\}$, which (in [10]) corresponds to the set of contraction links having only conclusions of auxiliary doors of boxes among their premises.

The *auxiliary ports of the* ?*-cells* of $\mathbb{C}$ are the ports belonging to the set $\mathrm{Aux}^?(\mathbb{C}) = \bigcup_{l \in \mathcal{C}^?(\mathbb{C})} \mathsf{P}^{\mathrm{aux}}_{\mathbb{C}}(l)$, while the *auxiliary doors* of $\mathbb{C}$ are the elements of $\mathrm{Auxdoors}(\mathbb{C}) = \{p \in \mathrm{Aux}^?(\mathbb{C}) \mid \#_{\mathbb{C}}(p) > 0\}$.

Definition 4. Let $\mathbb{C}, \mathbb{C}' \in$ **Cells** and let $\varphi = (\varphi_{\mathcal{C}}, \varphi_{\mathcal{P}})$ be a pair of bijections with $\varphi_{\mathcal{C}} : \mathcal{C}(\mathbb{C}) \to \mathcal{C}(\mathbb{C}')$ and $\varphi_{\mathcal{P}} : \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C}')$. For writing $\varphi : \mathbb{C} \simeq \mathbb{C}'$, we require that the following diagrams commute:



If these diagrams commute, then we have $\mathrm{im}(\varphi_{\mathcal{P}\,|\mathrm{Aux}^?(\mathbb{C})}) = \mathrm{Aux}^?(\mathbb{C}')$. Hence we can consider $\varphi' = \varphi_{\mathcal{P}}|^{\mathrm{Aux}^?(\mathbb{C}')}_{\mathrm{Aux}^?(\mathbb{C})}$. We then require moreover that $\#_{\mathbb{C}'} \circ \varphi' = \#_{\mathbb{C}}$.

2.2. Pre-Linear Proof-Structures (PLPS)

With **PPLPS** (Pre-Pre-Linear Proof-Structures) we shift from "sets of cells" (elements of **Cells**) to graphs, and this amounts to give the rules allowing to connect the ports of the different cells. We introduce a set $\mathcal{I}$ (intuitively, $p \in \mathcal{I}$ when p is a port of some axiom and a conclusion of a PPLPS) and we give conditions on the set of wires of our graphs: condition 1 implies that three ports cannot be connected by two wires, condition 2 implies that auxiliary ports can never be conclusions of PPLPS (see Definition 7), condition 3 implies that when the principal port of a cell is connected to another port this is necessarily a port of some cell, condition 4 corresponds to the fact that PPLPS are cut-free.

The reader acquainted with the theory of linear logic proof-nets might be interested in the reasons why our structures (PPLPS and later PLPS, LPS and PS) never contain cuts. There are essentially two reasons:

- (cut-free) PS are enough for our purpose, since the property we want to prove (injectivity) deals with cut-free proofs: once a precise notion of "identity" (or better said isomorphism) between cut-free PS is given (see Definition 19), if we prove that two different PS have different interpretations, then injectivity is proven (w.r.t. the chosen interpretation) whatever system of proofs one considers, provided the notion of cut-free proof of this system coincides with the one of PS.[4]
- We can thus avoid a technical problem related to the presence of cuts in untyped proof-structures: it might happen that applying a cut-elimination step to an untyped proof-structure which "contains a cycle" (meaning that it does not satisfy the proof-net correctness criterion) yields a graph without cuts but containing "vicious cycles" (a premise of some link is also its conclusion: see the discussion before Definition 13 of PLPS). It is precisely to avoid this problem that in [6] we decided to restrict to nets (proof-structures "without cycles" i.e. satisfying the correctness criterion).

Definition 5. Let **PPLPS** be the set of triples $\Phi = (\mathbb{C}, \mathcal{I}, \mathcal{W})$ with $\mathbb{C} \in$ **Cells**, $\mathcal{I}$ a finite set satisfying $\mathcal{I} \cap \mathcal{P}(\mathbb{C}) = \emptyset$ and $\mathcal{W} \subseteq \mathfrak{P}_2(\mathcal{P}(\mathbb{C}) \cup \mathcal{I})$ such that

1. for any $w, w' \in \mathcal{W}$ such that $w \cap w' \neq \emptyset$, we have $w = w'$;
2. we have $\mathcal{P}^{\mathsf{aux}}(\mathbb{C}) \cup \mathcal{I} \subseteq \bigcup \mathcal{W}$;
3. for any $w \in \mathcal{W}$ such that $w \cap \mathcal{I} \neq \emptyset$, we have $w \cap \mathcal{P}^{\mathsf{pri}}(\mathbb{C}) = \emptyset$;
4. for any $w \in \mathcal{W}$, there exists $p \in w$ such that $p \notin \mathcal{P}^{\mathsf{pri}}(\mathbb{C})$.

We set $\mathbb{C}(\Phi) = \mathbb{C}$, $\mathcal{I}(\Phi) = \mathcal{I}$, $\mathcal{W}(\Phi) = \mathcal{W}$ and $\mathcal{P}(\Phi) = \mathcal{P}(\mathbb{C}(\Phi)) \cup \mathcal{I}$. The elements of $\mathcal{P}(\Phi)$ are *the ports of* Φ, the elements of $\mathcal{C}(\mathbb{C}(\Phi))$ are the *cells of* Φ and those of $\mathcal{W}(\Phi)$ are *the wires* of Φ.

Notations 6. Let $\Phi \in$ **PPLPS**. We set $\mathcal{C}(\Phi) = \mathcal{C}(\mathbb{C}(\Phi))$ and $\mathcal{C}^{\alpha}(\Phi) = \mathcal{C}^{\alpha}(\mathbb{C}(\Phi))$ for any $\alpha \in \mathcal{T} \cup \{?\mathsf{w}, ?\mathsf{d}, ?\mathsf{c}_{\mathsf{b}}, ?\mathsf{c}_{\mathsf{auxd}}\}$, $\mathsf{t}_{\Phi} = \mathsf{t}_{\mathbb{C}(\Phi)}$, $\mathsf{C}_{\Phi} = \mathsf{C}_{\mathbb{C}(\Phi)}$, $\mathsf{P}^{\mathsf{pri}}_{\Phi} = \mathsf{P}^{\mathsf{pri}}_{\mathbb{C}(\Phi)}$, $\mathsf{P}^{\mathsf{left}}_{\Phi} = \mathsf{P}^{\mathsf{left}}_{\mathbb{C}(\Phi)}$ and $\mathsf{Auxdoors}(\Phi) = \mathsf{Auxdoors}(\mathbb{C}(\Phi))$.

We now introduce precisely axioms and conclusions of a PPLPS Φ; a consequence of our definition is that a conclusion p of Φ is either the principal port of some cell or an axiom port.

Definition 7. For any $\Phi \in$ **PPLPS**, we set:
- $\mathcal{P}^{\mathsf{f}}(\Phi) = \mathcal{I}(\Phi) \cup \{p \in \mathcal{P}(\mathbb{C}(\Phi)) \mid p \notin \bigcup \mathcal{W}(\Phi)\}$; the elements of $\mathcal{P}^{\mathsf{f}}(\Phi)$ are the *free ports* or the *conclusions* of Φ;
- $\mathcal{C}^{\mathsf{t}}(\Phi) = \{l \in \mathcal{C}(\Phi) \mid \mathsf{P}^{\mathsf{pri}}_{\Phi}(l) \in \mathcal{P}^{\mathsf{f}}(\Phi)\}$; the elements of $\mathcal{C}^{\mathsf{t}}(\Phi)$ are the *terminal cells* of Φ;
- $\mathsf{Ax}(\Phi) = \{\{p, q\} \in \mathcal{W}(\Phi) \mid p, q \notin \mathcal{P}^{\mathsf{pri}}(\mathbb{C}(\Phi))\}$; the wire $\{p, q\} \in \mathsf{Ax}(\Phi)$ is *an axiom* of ϕ and the ports p and q are *axiom ports*;
- $\mathsf{Ax}^{\mathsf{t}}(\Phi)$ is the set $\{w \in \mathsf{Ax}(\Phi) \mid (\exists p \in w) p \in \mathcal{P}^{\mathsf{f}}(\Phi)\}$ and $\mathsf{Ax}^{\mathsf{i}}(\Phi)$ is the set $\{w \in \mathsf{Ax}(\Phi) \mid (\forall p \in w)\ p \in \mathcal{P}^{\mathsf{f}}(\Phi)\}$[5]; the wires of $\mathsf{Ax}^{\mathsf{t}}(\Phi)$ (resp. $\mathsf{Ax}^{\mathsf{i}}(\Phi)$) are the *terminal axioms* (resp. the *isolated axioms*) of Φ.

Definition 8. Let $\Phi, \Phi' \in$ **PPLPS**. We write $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) : \Phi \simeq \Phi'$ if, and only if,
- $\varphi_{\mathcal{P}}$ is a bijection $\mathcal{P}(\Phi) \to \mathcal{P}(\Phi')$ such that $\mathsf{im}(\varphi_{\mathcal{P}\,|\mathcal{I}(\Phi)}) = \mathcal{I}(\Phi')$;
- $(\varphi_{\mathcal{P}}|^{\mathcal{P}(\mathbb{C}(\Phi'))}_{\mathcal{P}(\mathbb{C}(\Phi))}, \varphi_{\mathcal{C}}) : \mathbb{C}(\Phi) \simeq \mathbb{C}(\Phi')$;
- and for every $\{p, q\} \in \mathfrak{P}_2(\mathcal{P}(\Phi))$, we have $\{p, q\} \in \mathcal{W}(\Phi)$ if, and only, if $\{\varphi_{\mathcal{P}}(p), \varphi_{\mathcal{P}}(q)\} \in \mathcal{W}(\Phi')$.

For any $\Phi, \Phi' \in$ **PPLPS**, for any $\varphi = (\varphi_{\mathcal{C}}, \varphi_{\mathcal{P}}) : \Phi \simeq \Phi'$, we set $\mathcal{P}(\varphi) = \varphi_{\mathcal{P}}$ and $\mathcal{C}(\varphi) = \varphi_{\mathcal{C}}$.

Intuitively, an axiom port is "above" a unique conclusion. But for general PPLPS this is wrong and we can only say that an axiom port cannot be "above" two different conclusions (Lemma 10). We thus consider the reflexive and transitive closure $\leq_{\Phi}$ of the relation $<^1_{\Phi}$ "p is immediately below p' in Φ" (see Definition 9) and show that our statement holds provided $\leq_{\Phi}$ is antisymmetric (Lemma 14), that is for PLPS (Definition 13).

Definition 9. For any $\Phi \in$ **PPLPS**, we define the binary relation $<^1_{\Phi}$ on $\mathcal{P}(\Phi)$ as follows: $p <^1_{\Phi} p'$ if, and only if, one of the following conditions holds:
- there exists a cell l of Φ such that p is the principal port of l and p' is an auxiliary port of l
- p' is the principal port of some cell l' of Φ, p is an auxiliary port of some cell l of Φ and $\{p, p'\}$ is a wire of Φ.

The binary relation $\leq_{\Phi}$ (or simply $\leq$) on $\mathcal{P}(\Phi)$ is the transitive reflexive closure of $<^1_{\Phi}$.

Lemma 10. *Let $\Phi \in$ **PPLPS**. We have $(\forall w \in Ax(\Phi))\ (\forall p \in w)\ (\forall c, c' \in \mathcal{P}^f(\Phi))\ ((c \leq_{\Phi} p \text{ and } c' \leq_{\Phi} p) \Rightarrow c = c')$.*

The proof of Lemma 10 is just an application of Facts 11 and 12:

Fact 11. *Let $\Phi \in$ **PPLPS** and $p, q_1, q_2 \in \mathcal{P}(\Phi)$. If $q_1 \leq_{\Phi} p$ and $q_2 \leq_{\Phi} p$, then $q_1 \leq_{\Phi} q_2$ or $q_2 \leq_{\Phi} q_1$.*

Proof. If $q_1 <^1_{\Phi} p$ and $q_2 <^1_{\Phi} p$, then $q_1 = q_2$. □

Fact 12. *Let $\Phi \in$ **PPLPS**. If $c \in \mathcal{P}^f(\Phi)$ and $p \leq_{\Phi} c$, then $p = c$.*

Proof. If $c \in \mathcal{P}^{\mathsf{f}}(\Phi)$ then $\neg p <^1_{\Phi} c$ for every $p \in \mathcal{P}(\Phi)$. □

[4] We already mentioned in the introduction that a standard cut-free proof-net (as defined for example in [10] or in [6]) is a particular case of PS.

[5] Notice that $\mathsf{Ax}^{\mathsf{i}}(\Phi) = \{w \in \mathcal{W}(\Phi) \mid w \in \mathfrak{P}_2(\mathcal{I}(\Phi))\}$.

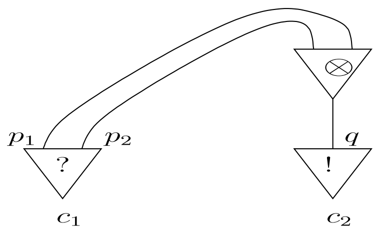


Fig. 1. Example of LPS. Let $\Psi_2 \in$ **PPLPS** be as in the figure and such that $\#_{\Psi_2}(p_1) = 1 = \#_{\Psi_2}(p_2)$. Then we have $\Psi_2 \in$ **LPS**. Actually $\Psi_2 \in$ **?-box-PLPS** $\cap$ **LPS** (see Definition 43).

A PPLPS Φ can have "vicious cycles" like for example a cell l such that p (resp. p') is the principal (resp. an auxiliary) port of l and $\{p, p'\}$ is a wire of Φ: in [10] this corresponds to a link having a premise which is also the conclusion of the link. Let us stress that such a cycle is called "vicious" to distinguish it from the cycles in the so-called correctness graphs, which are related to the issue of sequentialization (see the discussion before Corollary 55). A PLPS is a PPLPS without vicious cycles:

Definition 13. We set **PLPS** $= \{\Phi \in$ **PPLPS** $\mid$ the relation $\leq_\Phi$ is antisymmetric$\}$.

The fact that an axiom port is above a conclusion follows from the antisymmetry of $\leq$ and from the fact that minimal elements are conclusions. Indeed:

Lemma 14. *Let $\Phi \in$ **PLPS**. We have $(\forall w \in \mathsf{Ax}(\Phi))\ (\forall p \in w)\ (\exists! c \in \mathcal{P}^{\mathsf{f}}(\Phi))\ c \leq_\Phi p$.*

Proof. For the unicity, apply Lemma 10. For the existence, use the antisymmetry of $\leq_\Phi$ and the following property: we have

$$(\forall q \in \mathcal{P}(\Phi))\ ((\forall p \in \mathcal{P}(\Phi))(p \leq_\Phi q \Rightarrow p = q) \Rightarrow q \in \mathcal{P}^{\mathsf{f}}(\Phi)). \quad \square$$

The depth of a cell l is (in the usual syntax see [10]) the number of exponential boxes containing l. We have not yet defined our notion of box (Definition 17), but since we are cut-free, l's depth can also be defined as the number of doors of boxes below l; this makes sense in our framework too. We thus obtain the following definition (where the function # plays a crucial role, as mentioned in Remark 3):

Definition 15. Let $\Phi \in$ **PLPS**. For any $p \in \mathcal{P}(\Phi)$:
- we denote by $\mathsf{c}_\Phi(p)$ the unique $c \in \mathcal{P}^{\mathsf{f}}(\Phi)$ such that $c \leq_\Phi p$
- $depth_\Phi(p) = Card(\{l \in \mathcal{C}^{!}(\Phi) \mid \mathsf{P}^{\mathsf{pri}}_\Phi(l) <_\Phi p\}) + \sum_{q \in \mathsf{Auxdoors}(\Phi), q \leq p} \#_\Phi(q)$.

The depth of a PLPS Φ is the maximal depth of its ports and it is denoted by $depth(\Phi)$.

2.3. Linear Proof-Structures (LPS)

In a (cut-free) Proof-Structure of [10], the depth of an axiom link is easily defined as the number of boxes in which the link is contained. In our framework this notion makes sense only when the two ports of an axiom have the same depth (Definition 15). This condition is not fulfilled by every PLPS: when this is the case we have a LPS.

Definition 16. A LPS is a PLPS Φ such that $(\forall \{p_1, p_2\} \in \mathsf{Ax}(\Phi))\ depth_\Phi(p_1) = depth_\Phi(p_2)$. We denote by **LPS** the set of LPS.[6]

2.4. Proof-Structures (PS)

Intuitively, what is still missing in $\Phi \in$ **LPS** to be a (cut-free) Proof-Structure in the standard sense [10] is the connection between the doors of exponential boxes (once this information has been correctly produced, it automatically yields boxes). We then introduce a function b associating with every $v \in \mathcal{C}^{!}(\Phi)$ a set of auxiliary doors of Φ: this is precisely what was missing, provided certain conditions are satisfied (Definition 17). In particular, one asks that with every $v \in \mathcal{C}^{!}(\Phi)$ is associated a Proof-Structure: this is the usual notion of exponential box (see for example [4]). In our framework, in order to define the Proof-Structure associated with v,[7] we first build a PLPS Φ_v by taking "everything what is above v and the doors associated by b with v" and add a ?-cell under every "auxiliary conclusion"; doing this we take care to change the value of # on the auxiliary doors. We then remove v (using Definition 85); finally we define from b the new function b_v:

Definition 17. A *Proof-Structure (PS)* is a pair $R = (\Phi, b)$ where $\Phi \in$ **LPS** and b is a function $\mathcal{C}^{!}(\Phi) \to \mathfrak{P}(\mathsf{Auxdoors}(\Phi))$ such that for any $p \in \mathsf{Auxdoors}(\Phi)$, $\#_\Phi(p) = Card\{l \in \mathcal{C}^{!}(\Phi) \mid p \in b(l)\}$. Proof-Structures are defined by induction on the number of !-cells: we ask that with every $v \in \mathcal{C}^{!}(\Phi)$ is associated a PS called *the box* of v (denoted by $\overline{\mathsf{B}}(R)(v)$),[8] and defined from the following subset B_v of $\mathcal{P}(\Phi)$:

$$B_v = \{q \in \mathcal{P}(\Phi) \mid (\exists p \in \mathsf{P}^{\mathsf{aux}}_\Phi(v) \cup b(v))\ p \leq_\Phi q\}.$$

6 Our notion of *LPS* has not to be confused with what is sometimes called "the linearization of a proof-net": the "linearization" forgets the auxiliary doors, and obviously there are some PS that have the same "linearization" but different LPS.

7 We use the fact v's box is itself a Proof-Structure in Definition 24.

8 Two examples of boxes are in Figs. 3 and 4.

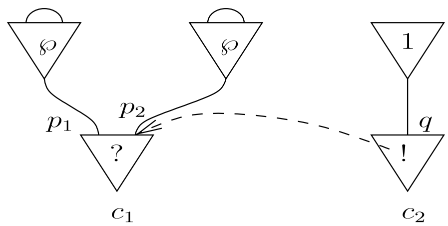


Fig. 2. Example of PS. In the standard syntax of [6] we have a box with a unique auxiliary door represented by the port p_2 (the dashed arrow allows to determine the doors of the box) and a dereliction link (the port p_1); the conclusions of the auxiliary door and the dereliction are then contracted.

We ask that for $v, v' \in \mathcal{C}^!(\Phi)$ either $B_v \cap B_{v'} = \emptyset$ or $B_v \subseteq B_{v'}$ or $B_{v'} \subseteq B_v$.[9]

In order to define $\overline{\mathsf{B}}(R)(v)$ one first defines $\Psi \in$ **PLPS**, starting from two sets $\mathcal{L}_0$ and $\mathcal{P}_0$ and from two bijections $p_1 : \mathcal{L}_0 \to b(v)$ and $p_0 : \mathcal{L}_0 \to \mathcal{P}_0$, by setting:

- $\mathcal{C}(\Psi) = \mathcal{L}_0 \uplus (\mathfrak{P}(\mathsf{C}_\Phi)(B_v) \setminus \mathfrak{P}(\mathsf{C}_\Phi)(b(v)))$;
  $\mathsf{t}_\Psi\restriction_{\mathfrak{P}(\mathsf{C}_\Phi)(B_v)\setminus\mathfrak{P}(\mathsf{C}_\Phi)(b(v))} = \mathsf{t}_\Phi\restriction_{\mathfrak{P}(\mathsf{C}_\Phi)(B_v)\setminus\mathfrak{P}(\mathsf{C}_\Phi)(b(v))}$ and $\mathsf{t}_\Psi(l) = ?$ for every $l \in \mathcal{L}_0$;
- $\mathcal{P}(\mathbb{C}(\Psi)) = (B_v \cup \{\mathsf{P}^{\mathsf{pri}}_\Phi(v)\}) \uplus \mathcal{P}_0$;
- $\mathsf{C}_\Psi(p) = \begin{cases} \mathsf{C}_\Phi(p) & \text{if } p \in B_v \backslash b(v); \\ l & \text{if } p = p_1(l) \text{ for } p \in b(v); \\ l & \text{if } p = p_0(l) \text{ for } p \in \mathcal{P}_0; \\ v & \text{if } p = \mathsf{P}^{\mathsf{pri}}_\Phi(v); \end{cases}$ [10]
- $\mathsf{P}^{\mathsf{pri}}_\Psi(l) = \begin{cases} \mathsf{P}^{\mathsf{pri}}_\Phi(l) & \text{if } l \notin \mathcal{L}_0; \\ p_0(l) & \text{if } l \in \mathcal{L}_0; \end{cases}$
- $\mathsf{P}^{\mathsf{left}}_\Psi = \mathsf{P}^{\mathsf{left}}_\Phi\restriction_{\mathcal{C}^{\mathsf{m}}(\Phi)\cap\mathfrak{P}(\mathsf{C}_\Phi)(B_v)}$;
- $\#_\Psi(p) = \mathsf{Card}\{w \in \mathcal{C}^!(\Phi) \cap \mathfrak{P}(\mathsf{C}_\Phi)(B_v) \mid w \neq v \text{ and } p \in b(w)\}$;
- $\mathcal{I}(\Psi) = \emptyset$[11];
- $\mathcal{W}(\Psi) = \{\{p, q\} \in \mathcal{W}(\Phi) \mid p, q \in B_v\}$.

The box of v, denoted by $\overline{\mathsf{B}}(R)(v)$, is the pair (Φ_v, b_v), where Φ_v is obtained from Ψ by eliminating the terminal link v (Definition 85) and $b_v = b\restriction^{\mathfrak{P}(\mathsf{Auxdoors}(\Phi_v))}_{\mathcal{C}^!(\Phi_v)}$.

We set $\mathsf{LPS}(R) = \Phi$, $\mathsf{b}(R) = b$ and we will write *the ports of R* (resp. *the cells of R*) meaning the ports of Φ (resp. the cells of Φ).

In order to establish the equality (or better said an isomorphism) between two graphs representing (some kind of) proof we need to say how the conclusions of the two graphs correspond one another: we thus introduce the notion of indexed PPLPS (resp. PLPS, LPS, PS).

Definition 18. We denote by $\mathbf{PPLPS}_{\mathsf{ind}}$ the set of pairs (Φ, ind) such that $\Phi \in$ **PPLPS** and ind is a bijection $\mathcal{P}^{\mathsf{f}}(\Phi) \to \ulcorner Card(\mathcal{P}^{\mathsf{f}}(\Phi))\urcorner$.

We set $\mathbf{PS}_{\mathsf{ind}} = \{(R, \mathsf{ind}) \mid R \in \mathbf{PS} \text{ and } (\mathsf{LPS}(R), \mathsf{ind}) \in \mathbf{PPLPS}_{\mathsf{ind}}\}$.

Definition 19. Let $(\Phi, \mathsf{ind}) \in \mathbf{PLPS}_{\mathsf{ind}}$ and let $(\Phi', \mathsf{ind'}) \in \mathbf{PPLPS}_{\mathsf{ind}}$. We write $\varphi : (\Phi, \mathsf{ind}) \simeq (\Phi', \mathsf{ind'})$ if, and only if, $\varphi : \Phi \simeq \Phi'$ and, for every $c \in \mathcal{P}^{\mathsf{f}}(\Phi)$, we have $\mathsf{ind'}(\mathcal{P}(\varphi)(c)) = \mathsf{ind}(c)$.

Definition 20. Let $(R, \mathsf{ind}), (R', \mathsf{ind'}) \in \mathbf{PS}_{\mathsf{ind}}$. We write $\varphi : (R, \mathsf{ind}) \simeq (R', \mathsf{ind'})$ if, and only if, $\varphi : (\mathsf{LPS}(R), \mathsf{ind}) \simeq (\mathsf{LPS}(R'), \mathsf{ind'})$ and the following diagram commutes[12]:

$$\begin{array}{ccc} \mathcal{C}^!(\mathsf{LPS}(R)) & \xrightarrow{\mathsf{b}(R)} & \mathfrak{P}(\mathsf{Auxdoors}(\mathsf{LPS}(R))) \\ {\scriptstyle \mathcal{C}(\varphi)\restriction^{\mathcal{C}^!(\mathsf{LPS}(R'))}_{\mathcal{C}^!(\mathsf{LPS}(R))}}\downarrow & & \downarrow{\scriptstyle \mathfrak{P}(\mathcal{P}(\varphi)\restriction^{\mathsf{Auxdoors}(\mathsf{LPS}(R'))}_{\mathsf{Auxdoors}(\mathsf{LPS}(R))})} \\ \mathcal{C}^!(\mathsf{LPS}(R')) & \xrightarrow[\mathsf{b}(R')]{} & \mathfrak{P}(\mathsf{Auxdoors}(\mathsf{LPS}(R'))) \end{array}$$

---

9 This is the usual nesting condition of the definition of proof-net: two boxes are either disjoint or contained one in the other.

10 This implies that every $l \in \mathcal{L}_0$ is a ?-cell with two ports : $p_0(l)$ and $p_1(l)$, where (see next item) $p_0(l)$ is the principal port and $p_1(l)$ is the unique auxiliary port of l.

11 As mentioned at the beginning of Section 2.2, $p \in \mathcal{I}(\Psi)$ when p is an axiom port and a conclusion of Ψ. Following our construction, none of the ports of Ψ can be in such a position. Notice, in particular, that in case the unique auxiliary port of v is an axiom port of Φ, then it is not a conclusion of Ψ (and thus it is not an element of $\mathcal{I}(\Psi)$) but it is a conclusion of $\mathsf{LPS}(\overline{\mathsf{B}}(R)(v))$ and the unique element of $\mathcal{I}(\mathsf{LPS}(\overline{\mathsf{B}}(R)(v)))$, following Definition 85 of the Appendix.

12 Recall that the notation $\mathcal{C}(\varphi)$ refers to Definition 8 and that for a function f the notation $\mathfrak{P}(f)$ is among the ones introduced in the conventions at the beginning of the paper.

3. Experiments

We introduce in Section 3.1 experiments for Proof-Structures (a well-known notion coming from [13]), adapted to our framework (Definition 24), and in Section 3.2 a new notion, the one of k-experiment of PLPS (Definition 35), that will be crucial in the sequel of the paper.

3.1. Experiments of PS

In [8,6] experiments are defined in an untyped framework; we follow here the same approach in our Definition 24. Experiments allow to compute the semantics of proof-nets (more generally of proof-structures): the *interpretation* $[\![\pi]\!]$ of a proof-net π is the set of the results of π's experiments, and the same happens in our framework for PS (Definition 27). Like in [6], in the following definition the set $\{+, -\}$ is used in order to "semantically distinguish" cells of type $\otimes$ from cells of type $\invamp$, which is mandatory in an untyped framework (as already discussed and used in [6]). The function $(\ )^{\perp}$ (which is the semantic version of linear negation) flips polarities (see Definition 86 of the appendix for the details).

Definition 21. We fix a set A which does not contain any couple nor any 3-tuple and such that $* \notin A$; we call *atoms* the elements of A. By induction on n we define D_n:

- $D_0 = A \cup (\{+, -\} \times \{*\})$
- and $D_{n+1} = D_0 \cup (\{+, -\} \times D_n \times D_n) \cup (\{+, -\} \times \mathfrak{M}_{\text{fin}}(D_n))$.

We set $D = \bigcup_{n\in\mathbb{N}} D_n$.

We need in the sequel the notion of injective k-point of $D^{<\omega}$, and for $E \in \mathfrak{P}(D^{<\omega})$ the notion of E-atomic element. In a typed framework, we would not have to define the latter notion, but in our untyped framework we need to restrict the set E of all results of all experiments of a PS to the set of the results of the *atomic* experiments (see footnote 26) of this PS. Of course, a given point of D can be the result of an atomic experiment of a PS and the result of a non-atomic experiment of another PS. However, once the subset E of $D^{<\omega}$ is fixed, it makes sense for $r \in E$ to say that it is E-atomic: this means that no other element of E is "more atomic" than r.

Definition 22. Given $k \in \mathbb{N}$, we say that $r \in D^{<\omega}$ is a *k-point* if, for any $m \in \mathbb{N}$, for any $\alpha_1, \dots, \alpha_m \in D$ such that $(+, [\alpha_1, \dots, \alpha_m])$ occurs in r,[13] we have $m = k$.

We say that $r \in D^{<\omega}$ is *injective* when for every $\gamma \in A$, either γ does not occur in r (see footnote 13) or there are exactly two occurrences of γ in r (see footnote 13).

Given $E \in \mathfrak{P}(D^{<\omega})$, we say that $r \in E$ is *E-atomic* when for every $r' \in E$ and every substitution[14] σ such that $\sigma(r') = r$ one has $\sigma(\gamma) \in A$ for every $\gamma \in A$ that occurs in r'. For $E \in \mathfrak{P}(D^{<\omega})$, we denote by E_{At} the subset of E consisting of the E-atomic elements.

Remark 23. The notion of k-point is reminiscent of the notion of "result of a k-obsessional experiment" [10], and it is also used in [8]. Notice however that the notion of injective point *is not* related to what is called in [10] a result of an injective k-obsessional experiment: we keep the idea that all positive multisets have the same size, but we are very far from obsessionality. In some sense we do here exactly the opposite than obsessional experiments do: a k-obsessional experiment takes k copies of the same (k-obsessional) experiment every time it crosses a box, while the intuition here is that injective k-points are results of experiments obtained by taking k *pairwise different* (k-)experiments every time a box is crossed.

We now adapt to our framework the definition of experiment (given in [13]; see also [9,10,6] for alternative definitions), the key tool to define the interpretation of a PS. Intuitively, an experiment of a PS Φ is a labeling of its ports by elements of D: this works perfectly well in the multiplicative fragment of LL (see for example [27]), but of course for PS with depth greater than zero things become a bit more complicated. One can either say that an experiment is defined only on ports p such that $depth_{\Phi}(p) = 0$ and that with every !-cell with depth zero is associated a multiset of experiments of its box (allowing to define the labels of the ports with depth zero): this is the choice made in [8] and [6]. Or one can follow (as we are going to do here in the spirit of [9] and [10]) the intuition that even with ports p such that $depth_{\Phi}(p) > 0$, an experiment associates labels, but not necessarily a unique label for every port (they might be several or none): formally it will associates with p a multiset of elements of D (and thus with every !-cell a multiset of multisets of experiments). Of course the two definitions associate the same interpretation with a given PS (Definition 27).

Definition 24. An experiment e of a PS $R = (\Phi, b)$ is given by a function $\mathcal{P}(\Phi) \to \mathcal{M}_{\text{fin}}(D)$[15] and for every $v \in \mathcal{C}^{!}(\Phi)$ a finite multiset of finite multisets of experiments of v's box (i.e. $\overline{\mathsf{B}}(R)(v)$) $e(v) = [[e_1^1, \dots, e_{n_1}^1], \dots, [e_1^{l_v}, \dots, e_{n_{l_v}}^{l_v}]]$, where $l_v \geq 0$ and $n_i \geq 0$ for every $1 \leq i \leq l_v$. Experiments are defined by induction on $\mathsf{depth}(\Phi)$ and we ask that $\mathsf{Card}(e(v)) = 1$ for $v \in \mathcal{C}^{!}(\Phi)$ such that $depth_{\Phi}(\mathsf{P}_{\Phi}^{\text{pri}}(v)) = 0$ and that $\mathsf{Card}(e(p)) = 1$ for $p \in \mathcal{P}(\Phi)$ such that $depth_{\Phi}(p) = 0$. For ports at depth 0 the following conditions hold:

13 See Definition 87 of the appendix for a formal definition of this expression.

14 A substitution is a function $\sigma : D \to D$ induced by a function $\sigma^A : A \to D$ (see Definition 88 of the appendix for the details).

15 The elements of $e(p)$ are often called *the labels* of p. Notice that $e(p) \notin D$.

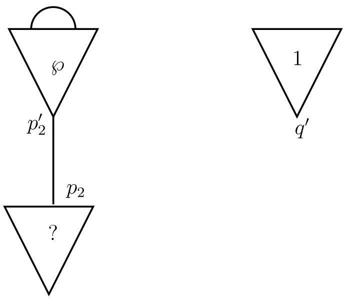


Fig. 3. The box $\overline{\mathsf{B}}(R)(v)$ of the unique !-cell v of the PS R of Fig. 2.

- for any $\{p, q\} \in \mathsf{Ax}(\Phi)$, we have $\alpha = \beta^{\perp}$, where $e(p) = [\alpha]$ and $e(q) = [\beta]$;
- for any $l \in \mathcal{C}^{\otimes}(\Phi)$, we have $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = [(+, \alpha, \beta)]$, where $e(\mathsf{P}^{\mathsf{left}}_{\Phi}(l)) = [\alpha]$ and $e(\mathsf{P}^{\mathsf{right}}_{\Phi}(l)) = [\beta]$;
- for any $l \in \mathcal{C}^{\wp}(\Phi)$, we have $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = [(-, \alpha, \beta)]$, where $e(\mathsf{P}^{\mathsf{left}}_{\Phi}(l)) = [\alpha]$ and $e(\mathsf{P}^{\mathsf{right}}_{\Phi}(l)) = [\beta]$;
- for any $l \in \mathcal{C}^{1}(\Phi)$, we have $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = [(+, *)]$;
- for any $l \in \mathcal{C}^{\perp}(\Phi)$, we have $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = [(-, *)]$;
- for any $l \in \mathcal{C}^{?}(\Phi)$, we have $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = [(-, \sum_{p \in \mathsf{P}^{\mathsf{aux}}_{\Phi}(l)} e(p))]$;
- for any $\{p, q\} \in \mathcal{W}(\Phi) \setminus \mathsf{Ax}(\Phi)$, we have $e(p) = e(q)$.

If $\mathsf{depth}(\Phi) = 0$, the definition is already complete. Otherwise for every $v \in \mathcal{C}^{!}(\Phi)$ such that $depth_{\Phi}(\mathsf{P}^{\mathsf{pri}}_{\Phi}(v)) = 0$ we know the multiset $[e_1, \ldots, e_{n_v}]$ of experiments of v's box such that $e(v) = [[e_1, \ldots, e_{n_v}]]$ and we know for every port p of Φ which is also a port of $\overline{B}(R)(v)$ the multiset $e_i(p)$ (for $i \in \{1, \ldots, n_v\}$). Then we set

- $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(v)) = [(+, \sum_{i=1}^{n_v} e_i(p))]$, where p is the unique free port of $\overline{B}(R)(v)$ such that $\mathsf{P}^{\mathsf{pri}}_{\Phi}(v) \leq_{\Phi} p$;[16]
- $e(p) = \sum_{i=1}^{n_v} e_i(p)$ for every port p of Φ which is also a port of $\overline{B}(R)(v)$;[17]
- $e(w) = \sum_{i=1}^{n_v} e_i(w)$ for every !-cell w of Φ which is also a cell of $\overline{B}(R)(v)$ (see footnote 17).

Example 25. Consider the PS R of Fig. 2 and the box $\overline{B}(R)(v)$ of its unique !-cell v represented in Fig. 3. We can define two experiments e_1 and e_2 of $\overline{\mathsf{B}}(R)(v)$ by choosing $\gamma_1, \gamma_2 \in D$: we obtain $e_i(p_2) = e_i(p_2') = [(-, \gamma_i, \gamma_i^{\perp})]$ and $e_i(q') = [(+, *)]$ where $\{q, q'\}, \{p_2, p_2'\} \in \mathcal{W}(\mathsf{LPS}(R))$. By choosing $\alpha \in D$, we have an experiment e of R such that $e(p_1) = [(-, \alpha, \alpha^{\perp})]$, $e(p_2') = e(p_2) = [(-, \gamma_1, \gamma_1^{\perp}), (-, \gamma_2, \gamma_2^{\perp})]$, $e(c_1) = [(-, [(-, \gamma_1, \gamma_1^{\perp}), (-, \gamma_2, \gamma_2^{\perp}), (-, \alpha, \alpha^{\perp})])]$, $e(q') = e(q) = [(+, *), (+, *)]$, $e(c_2) = [(+, [(+, *), (+, *)])]$, and $e(v) = [[e_1, e_2]]$.

Definition 26. Let $(R, \mathsf{ind}) \in \mathbf{PS}_{\mathsf{ind}}$. We set $n = Card(\mathcal{P}^{\mathsf{f}}(\mathsf{LPS}(R)))$. For any experiment e of R, for any $r \in D^n$, we say that (e, r) is an experiment of (R, ind) and that r is *the result* of (e, r) if, and only if, $r = (x_1, \ldots, x_n)$, where x_i is the unique element of the multiset $e \circ \mathsf{ind}^{-1}(i)$.

Definition 27. If $(R, \mathsf{ind}) \in \mathbf{PS}_{\mathsf{ind}}$, we define *the interpretation of* (R, ind) as the set $[\![(R, \mathsf{ind})]\!] = \{r \in D^{Card(\mathcal{P}^{\mathsf{f}}(R))} \mid r$ is the result of an experiment of $(R, \mathsf{ind})\}$.

The crucial result proven in [13] is that if π' is a proof-net obtained by applying to π some steps of cut-elimination, then $[\![\pi]\!] = [\![\pi']\!]$. Since any cut-free untyped net of [6] (and thus any cut-free proof-net of, for example, [10]) is a PS, in order to prove injectivity for the nets of [6] (and thus for the usual proof-nets of, for example, [10]) it is enough to prove that two PS with the same interpretation are the same (Corollary 54 and Corollary 55).

Example 28. We can define two experiments e_1 and e_2 of the PS R represented in Fig. 2 in such a way that $e_1(p_1) = [\zeta_1]$, $e_2(p_1) = [\zeta_2]$, $e_1(p_2) = [\zeta_2, \zeta_3, \zeta_4]$ and $e_2(p_2) = [\zeta_1, \zeta_3, \zeta_4]$, where $\zeta_j = (-, \gamma_j, \gamma_j)$ and the γ_j are distinct atoms. The two (different) experiments have the same result. More precisely: we define ind by setting $\mathsf{ind}(c_1) = 1$ and $\mathsf{ind}(c_2) = 2$, and we set $r = ((-, a), (+, b))$ with $a = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ and $b = [(+, *), (+, *), (+, *)]$. Then (e_1, r) and (e_2, r) are experiments of (R, ind), and r is an $[\![(R, \mathsf{ind})]\!]$-atomic injective 3-point.

3.2. Experiments of PLPS

In general, if we want to know whether a point is the result of any experiment, it is not enough to know the LPS of the (proof-)net: we have to know "the connection between the doors of the boxes". But if one takes k copies every time

[16] Let $\{q_v\} = \mathsf{P}^{\mathsf{aux}}_{\Phi}(v)$; then for some port q_v' of Φ we have $\{q_v, q_v'\} \in \mathcal{W}(\Phi)$. If $\{q_v, q_v'\} \in \mathsf{Ax}(\Phi)$ (resp. $\{q_v, q_v'\} \notin \mathsf{Ax}(\Phi)$), then q_v (resp. q_v') is the unique free port p of $\overline{B}(R)(v)$ such that $\mathsf{P}^{\mathsf{pri}}_{\Phi}(v) \leq_{\Phi} p$.

[17] We are using here the nesting condition of Definition 17 : see footnote 9.

one crosses a box, then it is enough: results of k-experiments can be defined directly on LPS. This yields the notion of k-experiment of a LPS (Definition 35). Actually k-experiments are defined "up to the names of the atoms" and we thus introduce sequences of indexes: the intuition is that for $\gamma \in A$ and $s \in \mathbb{N}^n$, (γ, s) is one of the k^n copies of γ.

We set $A' = A \times \mathbb{N}^{<\omega}$, and we denote by $|\ |$ (resp. loc) the first (resp. second) projection with domain A' and codomain A (resp. $\mathbb{N}^{<\omega}$) : the function $|\ |$ associates with $\delta \in A'$ its "support" $|\delta| \in A$, while loc associates with $\delta \in A'$ its "location" $\mathrm{loc}(\delta) \in \mathbb{N}^{<\omega}$.
The embedding that associates $(a, \varepsilon) \in A'$ with every $a \in A$ allows to consider A as a proper subset of A'.

Definition 29. For any $s \in \mathbb{N}^{<\omega}$, we denote by $\mathrm{dig}(s)$ the function $A' \to A'$ defined by $\mathrm{dig}(s)(\delta) = (|\delta|, conc(\mathrm{loc}(\delta), s))$, where $conc$ is the function $\mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega} \to \mathbb{N}^{<\omega}$ defined by $conc((d_1, \ldots, d_m), (d'_1, \ldots, d'_{m'})) = (d_1, \ldots, d_m, d'_1, \ldots, d'_{m'})$.

A construction similar to the one used to define D from A allows to define D' from A': intuitively, an element of D' is an element of D where every atom is followed by a sequence of integers. Notice that since $A \subseteq A'$ one has $D \subseteq D'$, and this will be used in Definition 35 (last item) of experiment of a PLPS.

Definition 30. By induction on n we define D'_n: $D'_0 = A' \cup (\{+,-\} \times \{*\})$ and $D'_{n+1} = D'_0 \cup (\{+,-\} \times D'_n \times D'_n) \cup (\{+,-\} \times \mathfrak{M}_{\mathrm{fin}}(D'_n))$. We set $D' = \bigcup_{n\in\mathbb{N}} D'_n$.

Definition 31. We define $At' : D' \to \mathfrak{P}_{\mathrm{fin}}(A')$ the function which associates with $\alpha \in D'$ its atoms, by induction on $\min\{n \in \mathbb{N} \mid \alpha \in D'_n\}$:

- $At'(\delta) = \{\delta\}$ if $\delta \in A'$;
- $At'(\iota, *) = \emptyset$;
- $At'(\iota, \alpha_1, \alpha_2) = At'(\alpha_1) \cup At'(\alpha_2)$;
- $At'(\iota, [\alpha_1, \ldots, \alpha_m]) = \cup_{j=1}^{m} At'(\alpha_j)$.

We also denote by At' the two following functions:

$$\begin{array}{rcl} \mathfrak{P}_{\mathrm{fin}}(D') & \longrightarrow & \mathfrak{P}_{\mathrm{fin}}(A') \\ \mathfrak{a} & \longmapsto & \bigcup_{\alpha\in\mathfrak{a}} At'(\alpha) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathfrak{M}_{\mathrm{fin}}(D')^{<\omega} & \longrightarrow & \mathfrak{P}_{\mathrm{fin}}(A') \\ (a_1, \ldots, a_n) & \longmapsto & \bigcup_{i=1}^{n} At'(Supp(a_i)) \end{array}$$

Definition 32. The set of partial injections from A' to A' is denoted by **pInj**.
Let $\tau \in$ **pInj**. For any $\alpha \in D'$ such that $At'(\alpha) \subseteq \mathrm{dom}(\tau)$, we define $\tau \cdot \alpha \in D'$ by induction on $\min\{n \in \mathbb{N} \mid \alpha \in D'_n\}$:

$$\tau \cdot \alpha = \begin{cases} \tau(\delta) & \text{if } \alpha = \delta \in A'; \\ (\iota, *) & \text{if } \alpha = (\iota, *); \\ (\iota, \tau\cdot\alpha_1, \tau\cdot\alpha_2) & \text{if } \alpha = (\iota, \alpha_1, \alpha_2); \\ (\iota, [\tau\cdot\alpha_1, \ldots, \tau\cdot\alpha_m]) & \text{if } \alpha = (\iota, [\alpha_1, \ldots, \alpha_m]). \end{cases}$$

For any $a = [\alpha_1, \ldots, \alpha_m] \in \mathfrak{M}_{\mathrm{fin}}(D')$ such that $At'(a) \subseteq \mathrm{dom}(\tau)$, we set $\tau \cdot a = [\tau\cdot\alpha_1, \ldots, \tau\cdot\alpha_m] \in \mathfrak{M}_{\mathrm{fin}}(D')$. For any $r = (\alpha_1, \ldots, \alpha_n) \in D'^{<\omega}$ such that $At'([\alpha_1, \ldots, \alpha_n]) \subseteq \mathrm{dom}(\tau)$, we set $\tau \cdot r = (\tau\cdot\alpha_1, \ldots, \tau\cdot\alpha_n) \in D'^{<\omega}$. For any $r = (a_1, \ldots, a_n) \in \mathfrak{M}_{\mathrm{fin}}(D')^{<\omega}$ such that $At'(r) \subseteq \mathrm{dom}(\tau)$, we set $\tau \cdot r = (\tau\cdot a_1, \ldots, \tau\cdot a_n) \in \mathfrak{M}_{\mathrm{fin}}(D')^{<\omega}$.

Definition 33. For any $\tau \in$ **pInj**, for any function h such that $\mathrm{im}(h) \subseteq D'$ and $At'(\mathrm{im}(h)) \subseteq \mathrm{dom}(\tau)$, we define $\tau \cdot h : \mathrm{dom}(h) \to D'$ as follows: $(\tau \cdot h)(x) = \tau \cdot h(x)$.

The function dig_d^k associates with $a \in \mathfrak{M}_{\mathrm{fin}}(D')$ the multiset of the k^d copies of a: if for example $a = [\alpha, \beta, \beta]$ for some $\alpha, \beta \in A$, then one has $\mathrm{dig}_1^2(a) = [(\alpha, 1), (\alpha, 2), (\beta, 1), (\beta, 2), (\beta, 1), (\beta, 2)]$. An immediate consequence of the following definition is that for every $a \in \mathfrak{M}_{\mathrm{fin}}(D')$ and for every $d \in \mathbb{N}$ one has $\mathrm{dig}_{d+1}^k(a) = \mathrm{dig}_1^k(\mathrm{dig}_d^k(a))$.

Definition 34. For any $k, d \in \mathbb{N}$, let dig_d^k be the function $\mathfrak{M}_{\mathrm{fin}}(D') \to \mathfrak{M}_{\mathrm{fin}}(D')$ defined by $\mathrm{dig}_d^k(a) = \sum_{s\in\ulcorner k\urcorner^d} \sum_{\alpha\in\mathrm{Supp}(a)} a(\alpha) \cdot [\mathrm{dig}(s) \cdot \alpha]$.

We now have all the tools to define (a particular kind of) experiments directly on LPS and not on PS as in the usual setting (Definition 24 in our framework). It clearly appears in Section 4.2 (and precisely in Fact 48) how (injective atomic) k-experiments of LPS are used in our proof. It is worth noticing that we recover in the framework of LPS the simplicity of the definition of experiment in the multiplicative fragment of linear logic proof-nets (see for example [27] and [25]): despite the presence of exponentials (here ?-cells and !-cells) a k-experiment of a PLPS is just a labeling of its ports by elements of D' satisfying some conditions.

Definition 35. Let $k \in \mathbb{N}$. For any $\Phi \in$ **PLPS**, a k-experiment e of Φ is a function $\mathcal{P}(\Phi) \to D'$ such that

- for any $l \in \mathcal{C}^{\otimes}(\Phi)$, we have $e(\mathrm{P}_\Phi^{\mathrm{pri}}(l)) = (+, e(\mathrm{P}_\Phi^{\mathrm{left}}(l)), e(\mathrm{P}_\Phi^{\mathrm{right}}(l)))$;
- for any $l \in \mathcal{C}^{⅋}(\Phi)$, we have $e(\mathrm{P}_\Phi^{\mathrm{pri}}(l)) = (-, e(\mathrm{P}_\Phi^{\mathrm{left}}(l)), e(\mathrm{P}_\Phi^{\mathrm{right}}(l)))$;
- for any $l \in \mathcal{C}^{1}(\Phi)$ (resp. $l \in \mathcal{C}^{\perp}(\Phi)$), we have $e(\mathrm{P}_\Phi^{\mathrm{pri}}(l)) = (+, *)$ (resp. $e(\mathrm{P}_\Phi^{\mathrm{pri}}(l)) = (-, *)$);
- for any $l \in \mathcal{C}^{!}(\Phi)$, we have $e(\mathrm{P}_\Phi^{\mathrm{pri}}(l)) = (+, \sum_{p\in\mathrm{P}_\Phi^{\mathrm{aux}}(l)} \mathrm{dig}_1^k([e(p)]))$[18];

[18] Notice that $\sum_{p\in\mathrm{P}_\Phi^{\mathrm{aux}}(l)} \mathrm{dig}_1^k([e(p)]) = \mathrm{dig}_1^k([e(p)])$ where $\{p\} = \mathrm{P}_\Phi^{\mathrm{aux}}(l)$.

- for any $l \in \mathcal{C}^?(\Phi)$, we have $e(\mathsf{P}^{\mathrm{pri}}_{\Phi}(l) = (-, \sum_{p \in \mathsf{P}^{\mathrm{aux}}_{\Phi}(l)} \mathrm{dig}^k_{\#_\Phi(p)}([e(p)]))$;
- for any $\{p, q\} \in \mathsf{Ax}(\Phi)$, we have $e(p) = e(q)^{\perp}$[19] and $e(p) \in D$;
- for any $\{p, q\} \in \mathcal{W}(\Phi) \setminus \mathsf{Ax}(\Phi)$, we have $e(p) = e(q)$.

Definition 36. Let $k \in \mathbb{N}$, let $\Phi \in$ **PLPS**. Let e be any k-experiment of Φ.

We say that e is *atomic* if for any $w \in \mathsf{Ax}(\Phi)$, for any $p \in w$, we have $e(p) \in A$.

We say that e is *injective* if for any $w, w' \in \mathsf{Ax}(\Phi)$, for any $p \in w, p' \in w'$, we have $At'(e(p)) \cap At'(e(p')) \neq \emptyset \Rightarrow w = w'$.

Definition 37. Let $k \in \mathbb{N}$. Let $(\Phi, \mathsf{ind}) \in \mathbf{PLPS}_{\mathsf{ind}}$. Let e be a k-experiment of Φ and let $r \in (D')^{Card(\mathcal{P}^{\mathsf{f}}(\Phi))}$. We say that (e, r) is a k-experiment of (Φ, ind) and that r is the result of (e, r) iff $r = e \circ \mathsf{ind}^{-1}$.

Example 38. Let Ψ_2 be as in Fig. 1, $\mathsf{ind}_2(c_1) = 1$ and $\mathsf{ind}_2(c_2) = 2$. Let $\gamma_1 \in A$, $\gamma_2 \in A$ such that $\gamma_1 \neq \gamma_2$. Let $a_1 = [(\gamma_1, 1), (\gamma_1, 2), (\gamma_1, 3), (\gamma_2, 1), (\gamma_2, 2), (\gamma_2, 3)]$ and $a_2 = [(+, (\gamma_1, 1), (\gamma_2, 1)), (+, (\gamma_1, 2), (\gamma_2, 2)), (+, (\gamma_1, 3), (\gamma_2, 3))]$.

Then $r_2 = ((-, a_1), (+, a_2))$ is the result of (e_2, r_2), where e_2 is the injective atomic 3-experiment of (Ψ_2, ind_2) such that $e_2(p_1) = \gamma_2$ and $e_2(p_2) = \gamma_1$. Notice that once we have chosen the labels of p_1 and p_2 and the integer k (here $k = 3$), the k-experiment of Ψ_2 is entirely determined.

Remark 39. As mentioned in Example 38, once an integer $k \geq 1$ and the labels of the axiom ports of the LPS Φ are chosen, the k-experiment of Φ is entirely determined. In particular, given a 1-experiment e_1 of Φ, for every $k \geq 1$ there exists a unique k-experiment e_k associating with the axiom ports of Φ the same labels as e_1. Clearly, e_1 is atomic (resp. injective) iff e_k is atomic (resp. injective).

4. Main result

In Section 4.2, we prove the main result on PS (Theorem 50), based on a crucial proposition (Proposition 40) concerning only LPS (and not PS anymore). In Section 4.1, we introduce the main syntactical tools to prove this crucial proposition, and we explain the technique we adopt in Section 5 to fully prove it.

4.1. Main result on LPS

When there exist two injective atomic experiments of two LPS with the same result (up to the name of the atoms), then the two LPS are the same:

Proposition 40. *Let $(\Phi, ind), (\Phi', ind') \in \boldsymbol{LPS}_{ind}$. Let $k > cosize(\Phi), cosize(\Phi')$.*[20] *For any k-experiment (e, r) of (Φ, ind), for any k-experiment (e', r') of (Φ', ind'), e and e' atomic and injective, if there exist $\rho, \rho' \in$ **pInj** such that $\rho \cdot r = \rho' \cdot r'$, then $(\Phi, ind) \simeq (\Phi', ind')$.*

Remark 41. As already noticed (Example 38, Remark 39), for every integer k there exists a unique atomic injective k-experiment of $\Phi \in$ **LPS** (up to the name of the atoms). This entails that by giving the suitable definition of isomorphism between experiments, one could easily substitute the conclusion of Proposition 40 by a(n apparently) stronger statement, namely $(e, r) \simeq (e', r')$.

Our strategy is to define a "measure" ($mes(\Phi)$, see Definition 42) of the size of an LPS Φ and to prove Proposition 40 by induction on this measure. More precisely, relying on the fact that LPS (and actually PLPS) can be inductively built, our idea can be roughly summed up as follows:

1. we start with the data contained in the hypothesis of Proposition 40, namely with $(\Phi, \mathsf{ind}), (\Phi', \mathsf{ind'}) \in \mathbf{LPS}_{\mathsf{ind}}$ and two k-experiments ((e, r) of (Φ, ind) and (e', r') of $(\Phi', \mathsf{ind'})$), both atomic and injective, and such that $\rho \cdot r = \rho' \cdot r'$, for some $\rho, \rho' \in$ **pInj**
2. from $\rho \cdot r = \rho' \cdot r'$ we can deduce that[21]:
   (a) (Φ, ind) and $(\Phi', \mathsf{ind'})$ can be obtained from some suitable LPS (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind'}_1)$ by "adding the same cell(s)", where $mes(\Phi_1) < mes(\Phi)$ and $mes(\Phi'_1) < mes(\Phi')$
   (b) (e, r) and (e', r') can be obtained from some suitable injective atomic k-experiments (e_1, r_1) of (Φ_1, ind_1) and (e'_1, r'_1) of $(\Phi'_1, \mathsf{ind'}_1)$ such that $\rho_1 \cdot r_1 = \rho'_1 \cdot r'_1$, for some $\rho_1, \rho'_1 \in$ **pInj**
3. we can thus apply the induction hypothesis on the measure to (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind'}_1)$ (with their injective atomic k-experiments (e_1, r_1) and (e'_1, r'_1)): we obtain $(\Phi_1, \mathsf{ind}_1) \simeq (\Phi'_1, \mathsf{ind'}_1)$
4. since by "adding the same cell(s)" to (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind'}_1)$ one obtains (Φ, ind) and $(\Phi', \mathsf{ind'})$, from $(\Phi_1, \mathsf{ind}_1) \simeq (\Phi'_1, \mathsf{ind'}_1)$ one easily deduces that $(\Phi, \mathsf{ind}) \simeq (\Phi', \mathsf{ind'})$.

19 $\delta^{\perp}$ is obtained from $\delta \in D'$ by substituting every occurrence of $+$ (resp. $-$) by $-$ (resp. $+$): see Definition 86 of the appendix for the details.

20 The integer $cosize(\Phi)$ is the maximal arity of the ?-cells of Φ (see Definition 42).

21 This is the difficult part of the proof.

The first thing we do is to define the measure, by introducing two sizes on elements of **PPLPS**: an integer and an ordered pair (pairs are lexicographically ordered).

Definition 42. Let $\Phi \in$ **PPLPS**. We set $cosize(\Phi) = \max\{\mathsf{a}_\Phi(l) \mid l \in \mathcal{C}^?(\Phi)\}$ and $mes(\Phi) = (\sum_{l \in \mathcal{C}^?(\Phi)} \mathsf{a}_\Phi(l), Card(\mathcal{P}(\Phi)) + \sum_{p \in \mathsf{Auxdoors}(\Phi)} \#_\Phi(p))$.

The aim of the rest of the section is to give a precise meaning to the expression "(Φ, ind) and (Φ', ind') can be obtained from some suitable LPS (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind}'_1)$ by adding the same cell(s)". The intuition is that Φ_1 (resp. Φ'_1) is obtained from Φ (resp. Φ') by "eliminating some terminal cell" (thus decreasing the measure). So the general problem is to define a procedure to "eliminate a terminal cell" from $\Phi \in$ **PLPS**,[22] which of course depends on the available terminal cells. We thus first classify PLPS depending on their terminal cells: $\Phi \in$ **PLPS** can have different terminal cells, but notice that in case $\Phi \in$ **?-box-PLPS** defined below, every terminal cell of Φ belongs to the set $\mathcal{C}^!(\Phi) \cup \mathcal{C}^{?\mathsf{c_{auxd}}}(\Phi)$.

Definition 43. We set:
- $\emptyset$-**PLPS** $= \{\Phi \in$ **PLPS** $\mid \mathcal{W}(\Phi) = \emptyset\}$.
- ax-**PLPS** $= \{\Phi \in$ **PLPS** $\mid \mathsf{Ax}^{\mathsf{i}}(\Phi) \neq \emptyset\}$.
- mult-**PLPS** $= \{\Phi \in$ **PLPS** $\mid (\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi))\ \mathsf{t}_\Phi(l) \in \{\otimes, \parr\}\}$.
- unit-**PLPS** $= \{\Phi \in$ **PLPS** $\mid (\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi))\ \mathsf{t}_\Phi(l) \in \{1, \bot\}\}$.
- $?_{\mathsf{w}}$-**PLPS** $= \{\Phi \in$ **PLPS** $\mid (\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi))\ l \in \mathcal{C}^{?\mathsf{w}}(\Phi)\}$.
- $?_{\mathsf{d}}$-**PLPS** $= \{\Phi \in$ **PLPS** $\mid (\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi))\ l \in \mathcal{C}^{?\mathsf{d}}(\Phi)\}$.
- $?_{\mathsf{c_b}}$-**PLPS** $= \{\Phi \in$ **PLPS** $\mid (\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi))\ l \in \mathcal{C}^{?\mathsf{c_b}}(\Phi)\}$.
- **?unit-PLPS** $= \{\Phi \in$ **PLPS** $|(\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi)) l \in \mathcal{C}^{?\mathsf{unit}}(\Phi)\}$, where $\mathcal{C}^{?\mathsf{unit}}(\Phi) = \{l \in \mathcal{C}^?(\Phi) \setminus \mathcal{C}^{?\mathsf{c_b}}(\Phi) | (\exists p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l))(\#_\Phi(p) \geq 1 \text{ and } (\forall q \geq_\Phi p) q \notin \bigcup \mathsf{Ax}(\Phi))\}$;
- **!unit-PLPS** $= \{\Phi \in$ **PLPS** $\mid (\exists l \in \mathcal{C}^{\mathsf{t}}(\Phi) \cap \mathcal{C}^!(\Phi))(\exists p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l))(\forall q \geq_\Phi p) q \notin \bigcup \mathsf{Ax}(\Phi))\}$;
- **?-box-PLPS** $=$ **PLPS** $\setminus$ ($\emptyset$-**PLPS** $\cup$ ax-**PLPS** $\cup$ mult-**PLPS** $\cup$ unit-**PLPS** $\cup ?_{\mathsf{w}}$-**PLPS** $\cup ?_{\mathsf{d}}$-**PLPS** $\cup ?_{\mathsf{c_b}}$-**PLPS** $\cup$ **?unit-PLPS** $\cup$ **!unit-PLPS**).

If $\Phi \in$ ax-**PLPS** it is obvious how to remove an isolated axiom. And to "eliminate a terminal cell l" from a particular PLPS is immediate when $l \in \mathcal{C}^{?\mathsf{w}}(\Phi)$ or $\mathsf{t}_\Phi(l) \in \{1, \bot\}$ since there is nothing "above" l. In case $\mathsf{t}_\Phi(l) \in \{\otimes, \parr, !\}$ or $l \in \mathcal{C}^{?\mathsf{d}}(\Phi)$, "to eliminate l" is intuitively clear, that is why we do not give the formal definition.[23] But of course a non-empty $\Phi \in$ **PLPS** does not always have an isolated axiom or contain the previously mentioned terminal cells: in that case we are in one of the last four cases of Definition 43. When $\Phi \in ?_{\mathsf{c_b}}$-**PLPS**, there exists $l \in \mathcal{C}^{?\mathsf{c_b}}(\Phi) \cap \mathcal{C}^{\mathsf{t}}(\Phi)$ and $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ such that $\#_\Phi(p) = 0$; one can obtain $\Phi_1 \in$ **PLPS** from Φ by removing p from the auxiliary ports of l: this operation (which is precisely described in the proof of Proposition 40 in Section 5 and in the Appendix) is also intuitively clear, and yields a PLPS Φ_1 with one more conclusion and with a strictly smaller measure, since the number of premises of l has strictly decreased. It then remains to describe operations allowing to shrink the measure when $\Phi \in$ **?unit-PLPS** $\cup$ **!unit-PLPS** $\cup$ **?-box-PLPS**.

The peculiarity of the PLPS elements of **?unit-PLPS** $\cup$ **!unit-PLPS** is that they contain "isolated subgraphs": if "above" an auxiliary port p of $l \in \mathcal{C}^!(\Phi) \cup \mathcal{C}^?(\Phi)$ there are no axioms, then the subgraph "above" p is isolated. In presence of "isolated subgraphs", we can apply to the PLPS Φ the following transformations *without damage* (Fact 44) and shrinking the measure of Φ. For any $\Phi \in$ **PLPS**, for any $l \in \mathcal{C}^{\mathsf{t}}(\Phi) \cap (\mathcal{C}^!(\Phi) \cup \mathcal{C}^?(\Phi))$, we denote by $\Phi_{[l]}$ the PLPS obtained as follows:

- if $l \in \mathcal{C}^!(\Phi)$, then we distinguish between two cases:
  - if $\{p \in \bigcup \mathsf{Ax}(\Phi) \mid p \geq_\Phi \mathsf{P}^{\mathsf{pri}}_\Phi(l)\} \neq \emptyset$, then $\Phi_{[l]} = \Phi$;
  - otherwise, we remove l;
- if $l \in \mathcal{C}^?(\Phi)$, $\Phi_{[l]}$ is Φ, except when there exists $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ such that $\#_\Phi(q) \geq 1$ and $\{p \in \bigcup \mathsf{Ax}(\Phi) \mid p \geq_\Phi q\} = \emptyset$: in that case $\Phi_{[l]}$ is Φ where for every such q one has $\#_{\Phi_{[l]}}(q) = \#_\Phi(q) - 1$.

The reader can easily check that when $\Phi \in$ **?unit-PLPS** $\cup$ **!unit-PLPS**, it is always possible to select a suitable cell l such that $mes(\Phi_{[l]}) < mes(\Phi)$. And we now show that whatever l we choose, **LPS** is stable with respect to the transformation previously defined.

Fact 44. *For any $\Phi \in$ **LPS**, for any $l \in \mathcal{C}^t(\Phi) \cap (\mathcal{C}^!(\Phi) \cap \mathcal{C}^?(\Phi))$, we have $\Phi_{[l]} \in$ **LPS**.*

Proof. We have $\mathsf{Ax}(\Phi_{[l]}) = \mathsf{Ax}(\Phi)$ and for any $\{p, q\} \in \mathsf{Ax}(\Phi)$, $depth_\Phi(p) = depth_{\Phi_{[l]}}(p)$. □

We turn to the last case $\Phi \in$ **?-box-PLPS**: here the intuition is that we eliminate one layer, the most external one. In order to do so, we must be sure that there is no terminal axiom port in such a Φ.

Fact 45. *For any $\Phi \in$ **?-box-PLPS** $\cap$ **LPS**, we have $Ax^t(\Phi) = \emptyset$.*

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[22] In the proof of Proposition 40, we actually "eliminate terminal cells" from $\Phi \in$ **LPS**. However, the definition makes sense for general PLPS, and it seems more natural to define it on PLPS. We then have to take care that when applying this operation to a PLPS which is also a LPS we still get a LPS.

[23] See Definition 85 in the appendix for such a definition.

Proof. Let $\{p, q\} \in \mathsf{Ax}(\Phi)$, suppose $p \in \mathcal{P}^{\mathsf{f}}(\Phi)$ and let c_q be the unique conclusion below q: by Definition 15 $depth_\Phi(p) = 0$. Since $\Phi \notin$ **ax-PLPS** we have $q \neq c_q$ and thus c_q is not an axiom port: in this case c_q is the principal port of some cell l of Φ. By Definition 43 this means that $l \in \mathcal{C}^{!}(\Phi) \cup \mathcal{C}^{?\mathsf{c}_{\mathsf{auxd}}}(\Phi)$, which entails that $depth_\Phi(q) > 0$, thus contradicting Definition 16 of LPS. □

A consequence of Fact 45 is that in case $\Phi \in$ **?-box-PLPS** $\cap$ **LPS** all Φ's conclusions are principal ports of some cells of the set $\mathcal{C}^{!}(\Phi) \cup \mathcal{C}^{?\mathsf{c}_{\mathsf{auxd}}}(\Phi)$; in the syntax of [10] this corresponds to a proof-structure Φ with no links at depth 0 except boxes and contraction links. We call $\overline{\Phi}$ the PLPS obtained from such a Φ by decreasing Φ's depth by 1, which can be easily done since $\Phi \in$ **?-box-PLPS** $\cap$ **LPS**[24]; the reader will notice that **LPS** is stable with respect to this operation, hence we actually have $\overline{\Phi} \in$ **LPS**. Furthermore, from Definition 42 it clearly follows that $mes(\overline{\Phi}) < mes(\Phi)$.

Since in the proof of Proposition 40 we deal with indexed LPS, we conclude the section by defining the indexing function on $\Phi_{[l]}$ and $\overline{\Phi}$, based on the indexing function of Φ.

Definition 46. Let $R = (\Phi, \mathsf{ind}) \in \mathbf{PLPS}_{\mathsf{ind}}$ and let $l \in \mathcal{C}^{\mathsf{t}}(\Phi) \cap (\mathcal{C}^{!}(\Phi) \cup \mathcal{C}^{?}(\Phi))$. We set $R_{[l]} = (\Phi_{[l]}, \mathsf{ind}_{[l]})$, where $\mathsf{ind}_{[l]}(p) = \mathsf{ind}(\mathsf{c}_\Phi(p))$ for $p \in \mathcal{P}^{\mathsf{f}}(\Phi_{[l]})$.

Definition 47. Let $(\Phi, \mathsf{ind}) \in \mathbf{LPS}_{\mathsf{ind}}$ such that $\Phi \in$ **?-box-PLPS**. We set $\overline{(\Phi, \mathsf{ind})} = (\overline{\Phi}, \overline{\mathsf{ind}})$, where $\overline{\Phi}$ has been defined above[25] and $\overline{\mathsf{ind}}(p) = \mathsf{ind}(\mathsf{c}_\Phi(p))$.

Coming back to the last four cases of Definition 43, we want to mention that the cases $\Phi \in$ **?unit-PLPS** and $\Phi \in$ **!unit-PLPS** have to be distinguished because our graphs (PLPS, LPS, PS) are disconnected (as already mentioned they can contain "isolated subgraphs"); if we decided from the beginning to restrict to connected graphs these cases would not occur (and Corollary 54 would hold, but our main result Theorem 50 would be much weaker). On the other hand, even in the connected case, the two most delicate cases in the proof of Proposition 40 would still be $\Phi \in ?_{\mathsf{c_b}}$**-PLPS** and $\Phi \in$ **?-box-PLPS**.

4.2. Main result on PS

An injective atomic k-experiment of an LPS Φ can be considered as a "prototype" of (atomic) k-experiment of *any* PS (Φ, b).[26] Indeed, every k-point of $[\![(\Phi, b)]\!]_{At}$ can be obtained from the result of an injective atomic k-experiment of Φ: to be precise, if $(R, \mathsf{ind}) \in \mathbf{PS}_{\mathsf{ind}}$ and (e, r) is any injective atomic k-experiment of $(\mathsf{LPS}(R), \mathsf{ind})$, we have

$$\{r_0 \in [\![(R, \mathsf{ind})]\!]_{At} \mid r_0 \text{ is a } k\text{-point}\} = \{\rho \cdot r \mid \rho \text{ is a partial map from } A' \text{ to } A\}$$

where $\rho \cdot r$ is defined by a straightforward generalization of Definition 32. In our proof we will only use Fact 48, namely that for a PS $R = (\Phi, b)$, the restriction of $[\![R]\!]$ to the injective k-points which are $[\![R]\!]$-atomic is precisely the set of the results of the atomic injective k-experiments of Φ (up to the name of the atoms):

Fact 48. *Let $k \in \mathbb{N}$, let $(R, ind) \in \mathbf{PS}_{ind}$ and let (e, r) be an injective atomic k-experiment of $(LPS(R), ind)$. We have:*

$$\{r_0 \in [\![(R, ind)]\!]_{At} \mid r_0 \text{ is an injective } k\text{-point}\} = \{\rho \cdot r \mid \rho \in \mathbf{pInj} \text{ and } codom(\rho) = A\}$$

Proof. One of the two inclusions is easy to prove: given an injective atomic k-experiment (e, r) of $(\mathsf{LPS}(R), \mathsf{ind})$ and given $\rho \in \mathbf{pInj}$ such that $\mathsf{codom}(\rho) = A$, there is an experiment (e_ρ, r_0) of (R, ind) such that $r_0 = \rho \cdot r$. The experiment (e_ρ, r_0) of (R, ind) can be defined by induction on $mes(\mathsf{LPS}(R))$ (see Definition 42 and see also Example 49).

Conversely, let $r_0 \in [\![(R, \mathsf{ind})]\!]_{At}$ be an injective k-point and let (e_0, r_0) be an experiment of (R, ind). We prove that for every atomic injective k-experiment (e, r) of $(\mathsf{LPS}(R), \mathsf{ind})$, there exists $\rho \in \mathbf{pInj}$ such that $\mathsf{im}(\rho) \subseteq At'(r_0)$ and $\rho \cdot r = r_0$. The proof is by induction on $mes(\mathsf{LPS}(R))$, the unique case deserving some details being the one where there is a unique terminal !-cell v of R and every other terminal cell is a ?-cell having a unique auxiliary port which is an element of $\mathsf{b}(R)(v)$.[27] The situation is represented in Fig. 4. We set $\{p_1, \dots, p_{\underline{l}}\} = \mathsf{b}(R)(v)$, we call $\overline{B}(R)(v)$ the box of v (we still denote by ind the obvious bijection $\mathcal{P}^{\mathsf{f}}(\mathsf{LPS}(\overline{B}(R)(v))) \simeq \ulcorner Card(\mathcal{P}^{\mathsf{f}}(\mathsf{LPS}(\overline{B}(R)(v)))\urcorner)$ and we call p the unique free port of $\overline{B}(R)(v)$ such that $\mathsf{P}^{\mathsf{pri}}_{\mathsf{LPS}(R)}(v) \leq_{\mathsf{LPS}(R)} p$.

In the sequel of the proof, it is important to distinguish between experiments of PS (Definition 24) and k-experiments of LPS (Definition 35): the experiments of PS have 0 as index (e_0 and f_0^i), while all the others are k-experiments of LPS.

Let $e_0(v) = [[f_0^1, \dots, f_0^1]]$, where (f_0^i, r_0^i) is an experiment of $(\overline{B}(R)(v), \mathsf{ind})$. Clearly, $r_0^i \in [\![(\overline{B}(R)(v), \mathsf{ind})]\!]_{At}$ is an injective k-point. The restriction (f, s) of (e, r) to $\mathsf{LPS}(\overline{B}(R)(v))$ is an atomic injective k-experiment of $(\mathsf{LPS}(\overline{B}(R)(v)), \mathsf{ind})$. We can then apply the induction hypothesis: for every $i \in \ulcorner k \urcorner$ there exists $\rho_i \in \mathbf{pInj}$ such that $\mathsf{im}(\rho_i) \subseteq At'(r_0^i)$ and $\rho_i \cdot s = r_0^i$.[28]

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[24] See Definition 89 in the appendix for a formal definition.

[25] and, more formally, in Definition 89 of the appendix.

[26] Notice that we did not define k-experiments of PS but only of LPS: k-experiments of nets have been defined in [8] and by *(injective) k-experiment of a PS* we mean here an experiment having a(n injective) k-point as result. A k-experiment of a PS R is said to be *atomic* if for any $p \in \bigcup \mathsf{Ax}(\mathsf{LPS}(R))$, we have $Supp(e(p)) \subseteq A$.

[27] In the standard terminology of linear logic proof-nets one would say that R is an exponential box.

[28] Notice that for every $i \in \ulcorner k \urcorner$ one has $At'(s) \subseteq \mathsf{dom}(\rho_i)$.

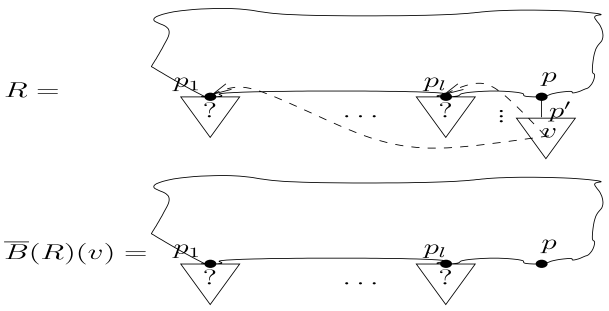

Fig. 4. The critical case of Fact 48. We have $p = p'$ if, and only if, $p' \in \bigcup \mathsf{Ax}(\mathsf{LPS}(R))$.

Since $\mathsf{im}(\rho_i) \subseteq At'(r_0^i)$ and since r_0 is injective, one has $At'(r_0^i) \cap At'(r_0^j) = \emptyset$ when $i \neq j$ and thus $\mathsf{im}(\rho_i) \cap \mathsf{im}(\rho_j) = \emptyset$ when $i \neq j$. We can then define $\rho \in \mathbf{pInj}$ on the elements $\gamma \in At'(r)$: since for every such γ there exist a unique $i \in \ulcorner k \urcorner$ and a unique $\beta \in At'(s)$ such that $\gamma = \mathsf{dig}(i)(\beta)$, we can set $\rho(\gamma) = \rho_i(\beta)$.

We now check that ρ is indeed the function we look for. With the notations introduced we have:

- $r_0 = ((-, \sum_{i=1}^k f_0^i(p_1)), \ldots, (-, \sum_{i=1}^k f_0^i(p_l)), (+, \sum_{i=1}^k f_0^i(p)))$;
- $r_0^i = ((-, f_0^i(p_1)), \ldots, (-, f_0^i(p_l)), \beta_i)$, where $f_0^i(p) = [\beta_i]$, for every $i \in \ulcorner k \urcorner$;
- $s = ((-, \mathsf{dig}_{d_1}^k([f(p_1)])), \ldots, (-, \mathsf{dig}_{d_l}^k([f(p_l)])), f(p))$, with, for any $j \in \ulcorner l \urcorner$, $d_j = \#_{\mathsf{LPS}(\overline{B}(R)(v))}(p_j)$;
- $r = ((-, a_1), \ldots, (-, a_l), (+, \mathsf{dig}_1^k([f(p)])))$, where, for any $j \in \ulcorner l \urcorner$, $a_j = \mathsf{dig}_1^k(\mathsf{dig}_{d_j}^k([f(p_j)]))$.

Now notice that for every $j \in \ulcorner l \urcorner$ we have $\mathsf{dig}_1^k(\mathsf{dig}_{d_j}^k([f(p_j)])) = \sum_{i=1}^k \mathsf{dig}(i) \cdot \mathsf{dig}_{d_j}^k([f(p_j)])$; and, since we have $At'(\mathsf{dig}_{d_j}^k([f(p_j)])) \subseteq At'(s)$, we can deduce for every $\beta \in At'(\mathsf{dig}_{d_j}^k([f(p_j)]))$ and for every $i \in \ulcorner k \urcorner$ that $\mathsf{dig}(i)(\beta) \in \mathsf{dom}(\rho)$ and $\rho(\mathsf{dig}(i)(\beta)) = \rho_i(\beta)$. This entails that, for every $j \in \ulcorner l \urcorner$, one has:

$$\rho \cdot \mathsf{dig}_1^k(\mathsf{dig}_{d_j}^k([f(p_j)])) = \sum_{i=1}^k \rho \cdot (\mathsf{dig}(i) \cdot \mathsf{dig}_{d_j}^k([f(p_j)])) = \sum_{i=1}^k \rho_i \cdot \mathsf{dig}_{d_j}^k([f(p_j)])$$

In the same way, we have $\rho \cdot \mathsf{dig}_1^k([f(p)]) = \sum_{i=1}^k \rho \cdot (\mathsf{dig}(i) \cdot [f(p)]) = \sum_{i=1}^k \rho_i \cdot [f(p)]$. Then the following equalities hold: $\rho \cdot r = ((-, \sum_{i=1}^k \rho_i \cdot \mathsf{dig}_{d_1}^k([f(p_1)])), \ldots, (-, \sum_{i=1}^k \rho_i \cdot \mathsf{dig}_{d_l}^k([f(p_l)])), (+, \sum_{i=1}^k \rho_i \cdot [f(p)])) = ((-, \sum_{i=1}^k f_0^i(p_1)), \ldots, (-, \sum_{i=1}^k f_0^i(p_l)), (+, \sum_{i=1}^k f_0^i(p))) = r_0$. □

Example 49. Consider the LPS Ψ_2 of Fig. 1. The experiment (e_2, r_2) considered in Example 38 is an injective atomic 3-experiment of (Ψ_2, ind_2). Let $\rho \in \mathbf{pInj}$ be such that for $j \in \ulcorner 2 \urcorner$ and $i \in \ulcorner 3 \urcorner$ one has $\rho(\gamma_j, i) = \gamma_{j,i}$, where $\gamma_{j,i} \in A$ (since $\rho \in \mathbf{pInj}$ the $\gamma_{j,i}$s are pairwise different). Then for any[29] PS R such that $\mathsf{LPS}(R) = \Psi_2$, there exists an experiment $e_0 = (e_2)_\rho$ of R with result $r_0 = \rho \cdot r_2 = ((-, \sum_{j=1}^2 \sum_{i=1}^3 [\gamma_{j,i}]), (+, [(+, \gamma_{1,1}, \gamma_{2,1}), (+, \gamma_{1,2}, \gamma_{2,2}), (+, \gamma_{1,3}, \gamma_{2,3})]))$. Indeed, if we call v the unique !-cell of R, we can set $e_0(v) = [[f_1, f_2, f_3]]$, where f_i is the experiment of v's box obtained by setting $f_i(p_1) = [\gamma_{2,i}]$ and $f_i(p_2) = [\gamma_{1,i}]$ (which entirely determines f_i). One can easily check that r_0 is indeed e_0's result.

Theorem 50. *Let $(R, ind), (R', ind') \in \mathbf{PS}_{ind}$, $k \in \mathbb{N}$ such that $k > cosize(LPS(R))$ and $k > cosize(LPS(R'))$. We set $E = \{r_0 \in [\![(R, ind)]\!]_{At} | r_0$ is an injective k-point$\}$ and $E' = \{r_0 \in [\![(R', ind')]\!]_{At} | r_0$ is an injective k-point$\}$. If $E \cap E' \neq \emptyset$, then $(LPS(R), ind) \simeq (LPS(R'), ind')$.*

Proof. Let r_0 be an injective $[\![(R, \mathsf{ind})]\!]$-atomic k-point of $[\![(R, \mathsf{ind})]\!]$ which is also an injective $[\![(R', \mathsf{ind'})]\!]$-atomic k-point of $[\![(R', \mathsf{ind'})]\!]$. Let (e, r) (resp. (e', r')) be an injective atomic k-experiment of $(\mathsf{LPS}(R), \mathsf{ind})$ (resp. $(\mathsf{LPS}(R'), \mathsf{ind'})$). By Fact 48, there exists $\rho \in \mathbf{pInj}$ (resp. $\rho' \in \mathbf{pInj}$) such that $\rho \cdot r = r_0 = \rho' \cdot r'$. By Proposition 40 we thus have $(\mathsf{LPS}(R), \mathsf{ind}) \simeq (\mathsf{LPS}(R'), \mathsf{ind'})$. □

Remark 51. Of course, as illustrated by Fig. 5, there are different PS with the same LPS. The k-experiments of two PS[30] have the same results if, and only if, the PS have the same LPS, but we do not say anything about the results of the other experiments.

Corollary 52. *Assume A is infinite. Let $(R, ind), (R', ind') \in \mathbf{PS}_{ind}$. If $[\![(R, ind)]\!] = [\![(R', ind')]\!]$, then $(LPS(R), ind) \simeq (LPS(R'), ind')$.*

Proof. For any $k \in \mathbb{N}$, we set $E_k = \{r_0 \in [\![(R, \mathsf{ind})]\!]_{At} | r_0$ is an injective k-point$\}$ and $E'_k = \{r_0 \in [\![(R', \mathsf{ind'})]\!]_{At} | r_0$ is an injective k-point$\}$. Since A is infinite, for every $k \in \mathbb{N}$, one has $E_k \cap E'_k \neq \emptyset$. Apply Theorem 50. □

29 Corollary 54 shows that in this particular case (Ψ_2 is a connected graph) there is actually a unique PS R such that $\mathsf{LPS}(R) = \Psi_2$.

30 See footnote 26.

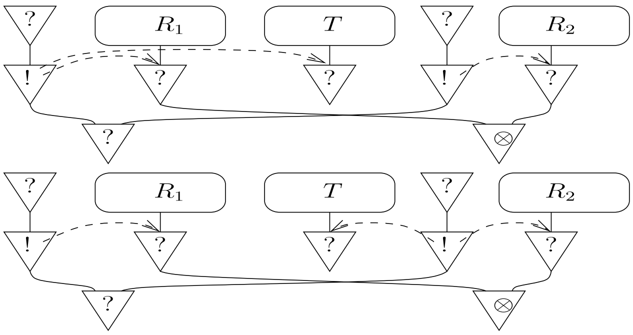

Fig. 5. Two different PS with the same LPS. The PS R_1, R_2 and T are PS of depth 0.

Remark 53. In the proof of Corollary 52, we use the fact that there always exists an $[\![R]\!]$-atomic injective k-point in the interpretation of any PS R and thus there always exists an atomic injective k-experiment of R (see footnote 30) (and we already noticed in Remark 41 that such an atomic injective k-experiment (see footnote 30) is unique "up to the names of the atoms").

The reader acquainted with *injective k-obsessional experiments* (defined in [9,10]) knows that, in the coherent model, not every PS has an injective k-obsessional experiment: this is precisely the reason why the proof of injectivity of the coherent model given in [9,10] for the $(?\wp)$LL fragment (already mentioned in the introduction) cannot be extended to *MELL*; and still for that reason injectivity of the coherent model fails for *MELL* as shown in [9,10].

The following corollary is based on a simple and crucial remark, already used in [10] (for the same purpose): since in LPS the depth of every port is known, given two !-cells v and w with the same depth in a PS (Φ, b) and given an auxiliary port p of some ?-cell of Φ, there might be an ambiguity on whether $p \in b(v)$ or $p \in b(w)$ (we would say in the standard terminology of linear logic proof-nets whether p is an auxiliary door of v or w's box) only in case Φ is not a connected graph[31]. Indeed (using again the standard terminology of linear logic proof-nets), in case Φ is connected, p and v are two "doors of the same box" iff there exists a path of Φ connecting p and v and crossing only cells with depth greater than the depth of v. More precisely:

Corollary 54. *Assume A is infinite. Let $(R, ind), (R', ind') \in$ **PS**$_{ind}$ such that LPS(R) is a connected graph. If $[\![(R, ind)]\!] = [\![(R', ind')]\!]$, then $(R, ind) \simeq (R', ind')$.*

Proof. By Corollary 52 $(\mathsf{LPS}(R), \mathsf{ind}) \simeq (\mathsf{LPS}(R'), \mathsf{ind'})$. Now notice that when LPS(R) is connected, there is a unique function b such that $(\mathsf{LPS}(R), b) \in$ **PS**. Indeed, given $v \in \mathcal{C}^!(\mathsf{LPS}(R))$, we have $p \in b(v)$ iff the two following conditions hold:

- $depth_{\mathsf{LPS}(R)}(\mathsf{P}^{\mathsf{pri}}_{\mathsf{LPS}(R)}(w)) \leq depth_{\mathsf{LPS}(R)}(\mathsf{P}^{\mathsf{pri}}_{\mathsf{LPS}(R)}(v))$, where $p \in \mathsf{P}^{\mathsf{aux}}_{\mathsf{LPS}(R)}(w)$
- there exists a path $d_{p_0 p}$ starting from the unique auxiliary port p_0 of v and ending in p such that for every port q crossed by $d_{p_0 p}$ we have that $depth_{\mathsf{LPS}(R)}(q) > depth_{\mathsf{LPS}(R)}(\mathsf{P}^{\mathsf{pri}}_{\mathsf{LPS}(R)}(v))$. □

As already pointed out in the introduction, the theory of proof-nets is among the striking novelties introduced with Linear Logic. Right from the start [13], it appeared very natural to first introduce graphs (called like in this paper "proof-structures") not necessarily representing sequent calculus proofs, and then look for "intrinsic" (usually graph-theoretical) properties allowing to characterize, among proof-structures, precisely those corresponding to sequent calculus proofs (in this case the proof-structure is called *proof-net*). Such a property is called *correctness criterion*; the most used one is the Danos–Regnier criterion: a proof-structure π of Multiplicative Linear Logic is a proof-net iff every correctness graph (every graph obtained from π by erasing one of the two premises of every ⅋ link) is acyclic and connected.
As soon as one leaves the purely multiplicative fragment of Linear Logic, things become less simple; for Multiplicative and Exponential Linear Logic *MELL*, one often considers (like for example in [6]) a weaker correctness criterion: a proof-structure is a proof-net when every correctness graph is acyclic (and not necessarily connected); such a criterion corresponds to a particular version of Linear Logic sequent calculus (see for example [9]). But it is also well-known (see again for example [9]) that in the absence of weakening and $\perp$ links, the situation is much better, in the sense that one can strengthen the criterion so as to capture the standard Linear Logic sequent calculus (very much in the style of the purely multiplicative case): in this framework, an *MELL* proof-structure is a proof-net iff every correctness graph is not only acyclic, but also connected. By *MELL* net we mean in the following corollary the (indexed) untyped version (in the style of [6]) of this strong notion of proof-net:

Corollary 55. *Assume A is infinite. Let R and R' be two MELL nets without weakening nor $\perp$ links. If $[\![R]\!] = [\![R']\!]$, then R and R' have the same (cut-free) normal form.*

[31] Here, we consider $\Phi \in$ **LPS** as the following graph: cells and terminal axiom ports of Φ are the nodes and two nodes ν and ν' are connected by an edge iff $\{p, p'\}$ is a wire, where p (resp. p') is a port of ν (resp. ν') if ν (resp. ν') is a cell, and $p = \nu$ (resp. $p' = \nu'$) if ν (resp. ν') is a terminal axiom port.

Proof. Let R_0 (resp. R'_0) be a cut-free normal form of R (resp. R'). Then $[\![R]\!] = [\![R_0]\!] = [\![R'_0]\!] = [\![R']\!]$. Since we are in *MELL* without weakening nor $\perp$, $\mathsf{LPS}(R_0)$ (so as $\mathsf{LPS}(R'_0)$) is a connected graph. Apply Corollary 54. □

Remark 56. Theorem 50, Corollaries 52, 54 and 55 hold for the standard typed *MELL* proof-nets of [10]: in particular if every propositional variable of the logical language is interpreted by the infinite set A and if π and π' are two cut-free typed proof-nets with atomic axioms, without weakenings nor $\perp$[32], and such that $[\![\pi]\!] = [\![\pi']\!]$, then $\pi = \pi'$[33].

5. Proof of Proposition 40

In this last section, we use the tools previously introduced in order to prove the key-proposition (Proposition 40) concerning only LPS (and not PS anymore). Since we need to consider isomorphisms between several kinds of objects (elements of D', t-uples of elements of D', finite multisets of D', t-uples of finite multisets of D',...) we use the notion of groupoid (Section 5.1).

Sections 5.2–5.4 establish the main results that will be used in the different cases of the proof by induction of Proposition 40, given in Section 5.5. More precisely, let us come back to the general strategy described in Section 4.1: following the classification of Definition 43, we already explained in that subsection how from (Φ, ind) and (Φ', ind') one can obtain (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind}'_1)$ by "eliminating the same cell(s)", in such a way that $mes(\Phi_1) < mes(\Phi)$ and $mes(\Phi'_1) < mes(\Phi')$ (this is item 2a of the description given in Section 4.1). We now turn to item 2b of the description given in Section 4.1: starting from (e, r) and (e', r') of Proposition 40, we want to define some suitable injective atomic k-experiments (e_1, r_1) of (Φ_1, ind_1) and (e'_1, r'_1) of $(\Phi'_1, \mathsf{ind}'_1)$ such that $\rho_1 \cdot r_1 = \rho'_1 \cdot r'_1$, for some $\rho_1, \rho'_1 \in$ **pInj**. This is more or less obvious except in four of the cases of Definition 43, namely for $\Phi, \Phi' \in ?_{c_b}$**-PLPS**∪**?unit-PLPS**∪**!unit-PLPS**∪**?-box-PLPS**.

When $\Phi, \Phi' \in ?_{c_b}$**-PLPS**, the LPS (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind}'_1)$, so as the experiments (e_1, r_1) and (e'_1, r'_1), are defined directly in the proof of Proposition 40; and Section 5.2 is mainly devoted to define an equivalence relation allowing to split the multiset associated with the principal port of $l \in \mathfrak{C}^{?c_b}(\Phi) \cap \mathfrak{C}^{t}(\Phi)$ in such a way that all the "possible" labels of a given auxiliary port p of l such that $\#_\Phi(p) = 0$ are in the same equivalence class.

On the other hand, Sections 5.3 and 5.4 have a similar structure[34]: we first define the injective atomic k-experiments (e_1, r_1) and (e'_1, r'_1) and we then have a "purely semantic" part (dealing only with points of D' and not with experiments anymore), allowing to prove (in the corresponding case of the proof of Proposition 40) that from $\rho \cdot r = \rho' \cdot r'$ it follows that $\rho_1 \cdot r_1 = \rho'_1 \cdot r'_1$, for some $\rho_1, \rho'_1 \in$ **pInj**.

Finally, in Section 5.5 we prove Proposition 40 by induction on the measure introduced in Definition 42.

Let e be an atomic k-experiment of $\Phi \in$ **PLPS** and suppose $e(p) = \alpha$ for $p \in \mathcal{P}^{f}(\Phi)$. If $\alpha = (+, \alpha_1, \alpha_2)$, then since e is atomic we can say that p is not an axiom port, so that p is necessarily the principal port of a cell of type $\otimes$. When $\alpha = (-, a)$ for some $a \in \mathfrak{M}_{\mathrm{fin}}(D')$, even if we know that p is not an axiom port, there are several possibilities for the ?-cell having p as principal port. The following fact will be used several times in Section 5.5: it allows (in particular) to distinguish between ?-cells having only auxiliary doors (remember Remark 3) among their premises from the others.

Fact 57. *Let $\Phi \in$ **PLPS**. Let $l \in \mathfrak{C}^{?}(\Phi)$. Let $k > \mathsf{a}_\Phi(l)$. Let $\mathcal{P}_0 \subseteq P^{aux}_\Phi(l)$. Let e be a k-experiment of Φ. We set $a = \sum_{p \in \mathcal{P}_0} dig^{k}_{\#_\Phi(p)}([e(p)])$. Then k divides $Card(a)$ if, and only if, $(\forall p \in \mathcal{P}_0)\ \#_\Phi(p) \neq 0$.*

Proof. We have

$$\begin{aligned} Card(a) &= \sum_{p \in \mathcal{P}_0} k^{\#_\Phi(p)} \\ &= Card(\{p \in \mathcal{P}_0 \mid \#_\Phi(p) = 0\}) + k \sum_{\substack{p \in \mathcal{P}_0 \\ \#_\Phi(p) \neq 0}} k^{\#_\Phi(p)-1} \end{aligned}$$

Hence k divides $Card(a)$ if, and only if, k divides $Card(\{p \in \mathcal{P}_0 \mid \#_\Phi(p) = 0\})$. Now

$$\begin{aligned} Card(\{p \in \mathcal{P}_0 \mid \#_\Phi(p) = 0\}) &\leq \mathsf{a}_\Phi(l) \\ &< k. \end{aligned}$$

So k divides $Card(\{p \in \mathcal{P}_0 \mid \#_\Phi(p) = 0\})$ iff $Card(\{p \in \mathcal{P}_0 \mid \#_\Phi(p) = 0\}) = 0$ i.e. $(\forall p \in \mathcal{P}_0)\ \#_\Phi(p) \neq 0$. □

5.1. Groupoids

We recall that a groupoid is a category such that any morphism is an iso and that a morphism of groupoids is a functor between two groupoids. For any groupoid **G**, we will denote by $\mathbf{G}_0$ the class of objects of the groupoid **G**. In the following, we sometimes think of a set as a groupoid such that the morphisms are identities on the elements of the set. We now define

32 We still refer here to the strong notion of proof-net corresponding to *MELL* sequent calculus.

33 More formally, one should write that if $[\![(\pi, \mathsf{ind})]\!] = [\![(\pi', \mathsf{ind}')]\!]$, then $(\pi, \mathsf{ind}) \simeq (\pi', \mathsf{ind}')$.

34 Notice by the way that since the cases $\Phi, \Phi' \in$ **?unit-PLPS** and $\Phi, \Phi' \in$ **!unit-PLPS** are very similar we treat them in the same Section 5.3.

some useful groupoids; some of them rely on the definition of the subset D'^{At} of D', consisting of those points of D' containing at least one atom (see Definition 64):

- The groupoid **D**: let $\mathbf{D}_0 = D'$ and $\rho : \alpha \to \alpha'$ in **D** if, and only if, we have $\rho \in \mathbf{pInj}$ and $\rho \cdot \alpha = \alpha'$.
- The groupoid **sD**: let $\mathbf{sD}_0 = D'^{<\omega}$ and $\rho : (\alpha_1, \dots, \alpha_n) \to (\alpha'_1, \dots, \alpha'_{n'})$ in **sD** if, and only if, we have $n = n'$ and $(\forall i \in \ulcorner n \urcorner)\ \rho : \alpha_i \to \alpha'_i$ in **D**.
- The groupoid **M**: let $\mathbf{M}_0 = \mathfrak{M}_{\mathrm{fin}}(D')$ and $\rho : a \to a'$ in **M** if, and only if, $\rho \cdot a = a'$.
- The groupoid **sDM**: let $\mathbf{sDM}_0 = (D'^{<\omega} \times \mathfrak{M}_{\mathrm{fin}}(D'))$ and $\rho : (r, a) \to (r', a')$ in **sDM** if, and only if, $\rho : r \to r'$ in **sD** and $\rho : a \to a'$ in **M**.
- the groupoid **pM**: let $\mathbf{pM}_0 = \mathfrak{P}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(D'))$ and $\rho : \mathfrak{a} \to \mathfrak{a}'$ in **pM** if, and only if, for any $a' \in \mathfrak{M}_{\mathrm{fin}}(D')$, we have $a' \in \mathfrak{a}' \Leftrightarrow (\exists a \in \mathfrak{a})\ \rho : a \to a'$ in **M**.
- The groupoid **sM**: let $\mathbf{sM}_0 = \mathfrak{M}_{\mathrm{fin}}(D'^{At})^{<\omega}$ and $\rho : (a_1, \dots, a_n) \to (a'_1, \dots, a'_n)$ in **sM** if, and only if, for any $i \in \ulcorner n \urcorner$, we have $\rho : a_i \to a'_i$ in **M**.
- the groupoid **psM**: let $\mathbf{psM}_0 = \mathfrak{P}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(D'^{At})^{<\omega})$ and $\rho : \mathfrak{r} \to \mathfrak{r}'$ in **psM** if, and only if, for any $r' \in \mathfrak{M}_{\mathrm{fin}}(D'^{At})^{<\omega}$, we have $r' \in \mathfrak{r}' \Leftrightarrow (\exists r \in \mathfrak{r})\ \rho : r \to r'$ in **sM**.
- the groupoid **ppsM**: let $\mathbf{ppsM}_0 = \mathfrak{P}_{\mathrm{fin}}(\mathfrak{P}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(D'^{At})^{<\omega}))$ and $\rho : \mathcal{A} \to \mathcal{A}'$ in **ppsM** if, and only if, for any $\mathfrak{a}' \in \mathfrak{P}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(D'^{At})^{<\omega})$, the following holds: $\mathfrak{a}' \in \mathcal{A}' \Leftrightarrow (\exists \mathfrak{a} \in \mathcal{A})\ \rho : \mathfrak{a} \to \mathfrak{a}'$ in **psM**.
- the groupoid **Bij**: objects are sets and morphisms are bijections.

In the sequel, we will write $\rho : r \to r'$ (referring to a given groupoid) in order to indicate that ρ is an iso between r and r', while we will write $r \simeq r'$ meaning that there exists some iso $\rho : r \to r'$.

Definition 58. We denote by *Card* the morphism of groupoids $\mathbf{M} \to \mathbb{N}$ defined by: $Card(a) = \sum_{\alpha \in Supp(a)} a(\alpha)$; and $Card(\rho) = id_{Card(a)}$ for any $\rho : a \to a'$.

5.2. The case of $?_{c_b}$-PLPS

The main result of this subsection is Lemma 63, where we establish a precise correspondence between equivalence classes of a multiset which is the label given by an experiment to the principal port of a ?-cell and the auxiliary ports of this same ?-cell. So we start by defining, for every multiset a, an equivalence relation on $Supp(a)$ allowing to split a:

Definition 59. Let $a \in \mathfrak{M}_{\mathrm{fin}}(\mathcal{E})$ such that $Supp(a) = \mathcal{E}$. Let $\mathcal{R}$ be an equivalence relation on $\mathcal{E}$. We set

$$a/\mathcal{R} = \{a_0 \in \mathfrak{M}_{\mathrm{fin}}(\mathcal{E}) \mid Supp(a_0) \in \mathcal{E}/\mathcal{R} \text{ and } (\forall \alpha \in Supp(a_0))\ a_0(\alpha) = a(\alpha)\}.$$

Consider again the LPS Ψ_2 of Fig. 1 and the 3-experiment (e_2, r_2) of (Ψ_2, ind_2) already defined in Example 38. We have that $(r_2, (\gamma_1, 1)), (r_2, (\gamma_1, 2)) \in \mathbf{sD}_0$ and if we define $\rho \in \mathbf{pInj}$ by setting $\rho(\gamma_1, 1) = (\gamma_1, 2)$, $\rho(\gamma_1, 2) = (\gamma_1, 1)$, $\rho(\gamma_2, 1) = (\gamma_2, 2)$, $\rho(\gamma_2, 2) = (\gamma_2, 1)$, $\rho(\gamma_3, 1) = (\gamma_3, 1)$ and $\rho(\gamma_3, 2) = (\gamma_3, 2)$, we have that $\rho : (r_2, (\gamma_1, 1)) \to (r_2, (\gamma_1, 2))$ in **sD**[35]: the effect of the morphism ρ of **sD** is to exchange two elements of $a_1 = [(\gamma_1, 1), (\gamma_2, 1), (\gamma_1, 2), (\gamma_2, 2), (\gamma_1, 3), (\gamma_2, 3)]$, without changing r_2. This suggests the definition of an equivalence relation on any $a \in \mathfrak{M}_{\mathrm{fin}}(D')$ (w.r.t. a given $r \in \mathbf{sD}_0$):

Definition 60. For any $(r, a) \in \mathbf{sDM}_0$, we set $Q(r, a) = a/\simeq$, where for $\alpha_1, \alpha_2 \in Supp(a)$ one has $\alpha_1 \simeq \alpha_2$ if, and only if, $(r, \alpha_1) \simeq (r, \alpha_2)$ in **sD**.

Fact 61. *By extending the definition of Q to the morphisms of **sDM** in setting $Q(\rho) = \rho$, we obtain a morphism of groupoids **sDM** $\to$ **pM**.*

Proof. For any $(r, \alpha_1), (r, \alpha_2) \in \mathbf{sD}_0$, for any $\rho \in \mathbf{pInj}$ such that $At'(r, \alpha_1, \alpha_2) \subseteq \mathrm{dom}(\rho)$, we have $(r, \alpha_1) \simeq (r, \alpha_2)$ in **sD** if, and only if, we have $(\rho \cdot r, \rho \cdot \alpha_1) \simeq (\rho \cdot r, \rho \cdot \alpha_2)$ in **sD**. □

We now prove a fact concerning experiments and their results, that allows to "exchange" two indexes (elements of $\ulcorner k \urcorner$) without changing the result of a given experiment: thanks to this property we will be able (in Lemma 63) to exchange two "copies" of $\alpha \in a$ for some multiset a of D'.

Fact 62. *Let $k \in \mathbb{N}$. Let $(\Phi, ind) \in \mathbf{PLPS}_{ind}$. Let (e, r) be a k-experiment of (Φ, ind). Let $d \in \mathbb{N}$. Let $j_1, j_2 \in \ulcorner k \urcorner$. Let $\rho \in \mathbf{pInj}$ defined by setting*

$$\rho(\delta) = \begin{cases} dig(s)(dig(j_2)(\delta_0)) \text{ if } \delta = dig(s)(dig(j_1)(\delta_0)) \text{ with } s \in \ulcorner k \urcorner^d \text{ and } \delta_0 \in A'; \\ dig(s)(dig(j_1)(\delta_0)) \text{ if } \delta = dig(s)(dig(j_2)(\delta_0)) \text{ with } s \in \ulcorner k \urcorner^d \text{ and } \delta_0 \in A'; \\ \delta \text{ otherwise.} \end{cases}$$

Then we have $\rho \cdot r = r$.

Proof. We first try to explain the intuition behind this fact: whenever, for $j \in \ulcorner k \urcorner$, an atom $(\gamma, conc(\sigma, conc(j, \sigma')))$ (where $\sigma, \sigma' \in \mathbb{N}^{<\omega}$) occurs[36] in the label $\alpha \in D'$ associated by an experiment with a port of some cell, the atom

35 Notice that we do not have, for example, $(r_2, (\gamma_1, 1)) \simeq (r_2, (\gamma_2, 2))$ in **sD**.

36 Recall that Definition 87 of the appendix gives a precise meaning to this notion.

$(\gamma, conc(\sigma, conc(i, \sigma')))$ occurs in α too, for every $i \in \ulcorner k \urcorner$. And (most important) there always exists a multiset a occurring in α such that $\beta_j, \beta_i \in a$ and $(\gamma, conc(\sigma, conc(j, \sigma')))$ (resp. $(\gamma, conc(\sigma, conc(i, \sigma')))$) occurs in β_j (resp. in β_i). This means that one can always "exchange" $(\gamma, conc(\sigma, conc(j, \sigma')))$ and $(\gamma, conc(\sigma, conc(i, \sigma')))$, without changing α (and thus without changing the result r of the experiment of Φ). This is essentially due to the fact that following Definition 35 of k-experiment, indexes are introduced precisely when (following the top-down propagation of labels) multisets appear.

More precisely, one can proceed by induction on $mes(\Phi)$.

If $\mathcal{C}^!(\Phi) \cap \mathcal{C}^t(\Phi) \neq \emptyset$, we choose some $l_0 \in \mathcal{C}^!(\Phi) \cap \mathcal{C}^t(\Phi)$, we set $i_0 = \mathsf{ind}(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l_0))$ and we consider the PLPS Ψ obtained from Φ by removing l_0 and the bijection ind' : $\mathcal{P}^{\mathsf{f}}(\Psi) \to \mathsf{Card}(\mathcal{P}^{\mathsf{f}}(\Psi))$ defined by $\mathsf{ind'} = \mathsf{ind} \circ \mathsf{c}_{\Phi}$. We have $r(i_0) = (+, \mathsf{dig}^k_1([\beta]))$ with $r' = (r(1), \ldots, r(i_0 - 1), \beta, r(i_0 + 1), \ldots, r(n))$ a result of a k-experiment of $(\Psi, \mathsf{ind'})$. By induction hypothesis, we have $\rho \cdot r' = r'$.

For $d = 0$, we clearly have $\rho \cdot \mathsf{dig}^k_1([\beta]) = \mathsf{dig}^k_1([\beta])$, hence $\rho \cdot r = r$.

For $d > 0$, we consider $\rho' \in \mathbf{pInj}$ defined by

$$\rho'(\delta) = \begin{cases} \mathsf{dig}(s')(\mathsf{dig}(j_2)(\delta_0)) \text{ if } \delta = \mathsf{dig}(s')(\mathsf{dig}(j_1)(\delta_0)) \text{ with } s' \in \ulcorner k \urcorner^{d-1} \text{ and } \delta_0 \in A'; \\ \mathsf{dig}(s')(\mathsf{dig}(j_1)(\delta_0)) \text{ if } \delta = \mathsf{dig}(s')(\mathsf{dig}(j_2)(\delta_0)) \text{ with } s' \in \ulcorner k \urcorner^{d-1} \text{ and } \delta_0 \in A'; \\ \delta \text{ otherwise.} \end{cases}$$

Again by induction hypothesis, we have $\rho' \cdot \beta = \beta$, hence, for any $j \in \ulcorner k \urcorner$, we have $\rho \cdot (\mathsf{dig}(j) \cdot \beta) = \mathsf{dig}(j) \cdot (\rho' \cdot \beta) = \mathsf{dig}(j) \cdot \beta$, so $\rho \cdot \mathsf{dig}^k_1([\beta]) = \mathsf{dig}^k_1([\beta])$.

If there exists $l_0 \in \mathcal{C}^?(\Phi) \cap \mathcal{C}^t(\Phi)$ such that for any auxiliary port p of l_0, we have $\#_{\Phi}(p) > 0$, we proceed in the same way as before, except that instead of applying the induction hypothesis on the PLPS obtained by removing l_0, we apply the induction hypothesis on the PLPS obtained by decreasing the function # on the auxiliary ports of l_0.

The other cases are left to the reader. □

Suppose (e, r) is an experiment of $(\Phi, \mathsf{ind}) \in \mathbf{PLPS}_{\mathsf{ind}}$, suppose $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = (-, a)$ for some $l \in \mathcal{C}^?(\Phi) \cap \mathcal{C}^t(\Phi)$ and suppose that $e(p) = \alpha$ for $p \in \mathsf{P}^{\mathsf{aux}}_{\Phi}(l)$ such that $\#_{\Phi}(p) = d$. Then the idea is that (like we did in the example before Definition 60) one can exchange two "copies" of α in a without changing r: the intuition is that for every $\alpha_1, \alpha_2 \in Supp(\mathsf{dig}^k_d([\alpha]))$ one has $(r, \alpha_1) \simeq (r, \alpha_2)$ in $\mathbf{sD}$. More precisely, the following lemma holds:

Lemma 63. *Let $k \in \mathbb{N}$. Let $(\Phi, ind) \in \mathbf{PLPS}_{ind}$. Let $l \in \mathcal{C}^?(\Phi)$. Let (e, r) be a k-experiment of (Φ, ind). Let $a \in \mathfrak{M}_{fin}(D')$ such that $e(P^{pri}_{\Phi}(l)) = (-, a)$. Let $a_0 \in \mathcal{Q}(r, a)$. Then there exists $\mathcal{P}_0 \subseteq P^{aux}_{\Phi}(l)$ such that $a_0 = \sum_{q \in \mathcal{P}_0} dig^k_{\#_{\Phi}(q)}(e(q))$.*

Proof. We prove, by induction on d and using Fact 62, that for any $d \in \mathbb{N}$, for any $\alpha \in D'$, for any $\alpha_1, \alpha_2 \in Supp(\mathsf{dig}^k_d([\alpha]))$, we have $(r, \alpha_1) \simeq (r, \alpha_2)$ in $\mathbf{sD}$. □

5.3. *The case of **?unit-PLPS** and **!unit-PLPS***

In the first part of this subsection (and similarly in the first part of the following Section 5.4), we first define some suitable injective atomic k-experiments (e_1, r_1) of (Φ_1, ind_1) and (e'_1, r'_1) of $(\Phi'_1, \mathsf{ind'}_1)$[37], and we then establish some *purely semantic* statements, that will allow in the final Section 5.5 to show that $r_1 \simeq r'_1$ (and thus apply the induction hypothesis). Notice that in the second part of Sections 5.3 and 5.4 we often refer to k-experiments and LPS, but only in discussions and examples: the intuition is that the points of D' we consider in mathematical statements are results of k-experiments of LPS, but the statements themselves hold without any reference to experiments.

For every $\rho \in \mathbf{pInj}$ (Definition 32) and for every $\alpha \in D'$, when $At'(\alpha) = \emptyset$, one has $\rho \cdot \alpha = \alpha$. We will use in the sequel the remark that any multiset $b \in \mathfrak{M}_{\mathsf{fin}}(D')$ can be decomposed into a (possibly empty) multiset b^{At} in which atoms occur and a (possibly empty) multiset b^* in which no atom occurs: $b = b^{At} + b^*$, where b^{At} and b^* are precisely defined as follows.

Definition 64. For any $D_0 \subseteq D'$, we set ${D_0}^{At} = \{\alpha \in D_0 \mid At'(\alpha) \neq \emptyset\}$ and ${D_0}^* = \{\alpha \in D_0 \mid At'(\alpha) = \emptyset\}$.

For any $a \in \mathfrak{M}_{\mathsf{fin}}(D')$, we set $a^{At} = a_{|Supp(a)^{At}}$ and $a^* = a_{|Supp(a)^*}$.

When (for some $\Phi \in \mathbf{PLPS}$) "above" an auxiliary port p of $l \in \mathcal{C}^?(\Phi) \cup \mathcal{C}^!(\Phi)$[38] there are no axiom ports, it is obvious that whatever k-experiment e of Φ one considers, the label $\alpha = e(p)$ of p contains no atom. And the converse holds too when e is atomic: if $At'(e(p)) = \emptyset$, there are no axiom ports "above" p. This implies that $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = (\iota, b)$ for some $b \in \mathfrak{M}_{\mathsf{fin}}(D')$ such that $b^* \neq []$ iff "above" one of the auxiliary ports of l there are no axiom ports, as the following fact shows.

Fact 65. *Let $k \geq 1$, let $\Phi \in \mathbf{PLPS}$ and let e be an atomic k-experiment of Φ. Suppose that $l \in \mathcal{C}(\Phi)$ and $e(P^{pri}_{\Phi}(l)) = (\iota, b)$ for some $b \in \mathfrak{M}_{fin}(D')$.*

We have that $b^* \neq []$ iff there exists $p \in P^{aux}_{\Phi}(l)$ such that for every $q \geq_{\Phi} p$ one has $q \notin \bigcup Ax(\Phi)$.

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37 Notations still refer to the general strategy described in Section 4.1.

38 In case $l \in \mathcal{C}^!(\Phi)$ such a premise is the unique premise of l.

Proof. Since e is atomic[39] and $k \geq 1$, we have $At'(e(q)) \neq \emptyset$ for any $q \in \bigcup \mathsf{Ax}(\Phi)$, hence one can easily prove, by induction on the number of ports "above" the port p of Φ (that is on $Card(\{q \in \mathcal{P}(\Phi) \ / \ q \geq_\Phi p\})$), that there exists $q \geq_\Phi p$ such that $q \in \bigcup \mathsf{Ax}(\Phi)$ iff $At'(e(p)) \neq \emptyset$. This immediately yields the conclusion: for every $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ there exists $q \geq_\Phi p$ such that $q \in \bigcup \mathsf{Ax}(\Phi)$ iff $At'(\alpha) \neq \emptyset$ for every $\alpha \in b$ iff $b^* = []$. □

The following Fact 66 and Fact 68 are similar in spirit to Fact 71 of the following Section 5.4: they allow to obtain a k-experiment $e_{[l_0]}$ of $\Phi_{[l_0]}$ from a k-experiment e of $\Phi \in$ **LPS**, and they will be used in the cases $\Phi \in$ **!unit-PLPS** and $\Phi \in$ **?unit-PLPS** of the proof of Proposition 40. In both the facts the hypothesis $a^* \neq []$ (for $a \in \mathfrak{M}_{\mathrm{fin}}(D')$ such that $e(p) = (\iota, a)$ with p port of Φ) is crucial: it implies that "above" p there is an "isolated subgraph", which allows to apply the transformations defined in Section 4.1, thus shrinking the measure of Φ.

Fact 66. *Let $k \geq 1$. Let $R = (\Phi, ind) \in \boldsymbol{LPS}_{ind}$ and let (e, r) be an atomic k-experiment of (Φ, ind). Let $l_0 \in \mathcal{C}^!(\Phi) \cap \mathcal{C}^t(\Phi)$ and $\beta \in D'$ such that $e(P^{pri}_\Phi(l_0)) = (+, dig^k_1([\beta]))$ and $(dig^k_1([\beta]))^* \neq []$. Then $mes(\Phi_{[l_0]}) < mes(\Phi)$ and there exists a unique atomic k-experiment $(e_{[l_0]}, r_{[l_0]})$ of $R_{[l_0]}$ such that*

- *for any $p \in (\mathcal{P}(\Phi) \setminus \mathcal{P}^f(\Phi)) \cap \mathcal{P}(\Phi_{[l_0]})$, we have $e_{[l_0]}(p) = e(p)$;*
- $$r_{[l_0]}(i) = \begin{cases} r(i) & \text{if } i \neq ind(P^{pri}_\Phi(l_0)); \\ \beta & \text{if } i = ind(P^{pri}_\Phi(l_0)). \end{cases}$$

Moreover, if e is injective, then $e_{[l_0]}$ is injective.

Proof. By Fact 65, if we call p the unique auxiliary port of l_0, we have that for every $q \geq_\Phi p$ one has $q \notin \bigcup \mathsf{Ax}(\Phi)$, that is $\{p \in \bigcup \mathsf{Ax}(\Phi) \mid p \geq_\Phi \mathsf{P}^{\mathsf{pri}}_\Phi(l)\} = \emptyset$: this implies that $\Phi \in$ **!unit-PLPS**, thus $mes(\Phi_{[l_0]}) < mes(\Phi)$.

We then set $e_{[l_0]}(p) = e(p)$ for any $p \in \mathcal{P}(\Phi_{[l_0]})$. □

Remark 67. If e is a k-experiment of $\Phi \in$ **PLPS** and $l \in \mathcal{C}^?(\Phi)$, we know by Definition 35 that $e(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = (-, a)$, where $a = \sum_{p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)} \mathsf{dig}^k_{\#_\Phi(p)}([e(p)])$. When $l \in \mathcal{C}^{?\mathsf{c_{auxd}}}(\Phi)$ we have $\#_\Phi(p) \geq 1$ for every $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$, which implies that $a = \mathsf{dig}^k_1(b)$ for $b = \sum_{p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)} \mathsf{dig}^k_{\#_\Phi(p)-1}([e(p)])$. It then follows that when $\Phi \in$ **?unit-PLPS** there always exists $l \in \mathcal{C}^{\mathsf{t}}(\Phi)$ such that $e(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = (-, \mathsf{dig}^k_1(b))$ for some $b \in \mathfrak{M}_{\mathrm{fin}}(D')$.

Fact 68. *Let $k > 1$. Let $R = (\Phi, ind) \in \boldsymbol{LPS}_{ind}$ and let (e, r) be an atomic k-experiment of R. Let $l_0 \in (\mathcal{C}^?(\Phi) \setminus \mathcal{C}^{?c_b}(\Phi)) \cap \mathcal{C}^t(\Phi)$ and $b \in \mathfrak{M}_{fin}(D')$ such that $e(P^{pri}_\Phi(l_0)) = (-, dig^k_1(b))$ and $(dig^k_1(b))^* \neq []$. Then $mes(\Phi_{[l_0]}) < mes(\Phi)$ and there exists a unique atomic k-experiment $(e_{[l_0]}, r_{[l_0]})$ of $R_{[l_0]}$ such that*

- *for any $p \in (\mathcal{P}(\Phi) \setminus \mathcal{P}^f(\Phi)) \cap \mathcal{P}(\Phi_{[l_0]})$, we have $e_{[l_0]}(p) = e(p)$;*
- $$r_{[l_0]}(i) = \begin{cases} r(i) & \text{if } i \neq ind(P^{pri}_\Phi(l_0)); \\ (-, (dig^k_1(b))^{At} + b^*) & \text{if } i = ind(P^{pri}_\Phi(l_0)). \end{cases}$$

Moreover, if e is injective then $e_{[l_0]}$ is injective.

Proof. By Fact 65 there exists $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l_0)$ such that for every $q \geq_\Phi p$ one has $q \notin \bigcup \mathsf{Ax}(\Phi)$. From $k > 1$, $e(\mathsf{P}^{\mathsf{pri}}_\Phi(l_0)) = (-, \mathsf{dig}^k_1(b))$ and $(\mathsf{dig}^k_1(b))^* \neq []$ we deduce that $l_0 \notin \mathcal{C}^{?\mathsf{w}}(\Phi) \cup \mathcal{C}^{?\mathsf{d}}(\Phi)$, and since $l_0 \notin \mathcal{C}^{?\mathsf{c_b}}(\Phi)$, we have $l_0 \in \mathcal{C}^{?\mathsf{c_{auxd}}}(\Phi)$ and thus $\#_\Phi(p) \geq 1$. Summing up, we have the existence of $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l_0)$ such that $\#_\Phi(p) \geq 1$ and $\{q \in \bigcup \mathsf{Ax}(\Phi) \mid q \geq_\Phi p\} = \emptyset$: this implies that $l_0 \in \mathcal{C}^{?\mathsf{unit}}(\Phi)$ and $\Phi \in$ **?unit-PLPS**, thus $mes(\Phi_{[l_0]}) < mes(\Phi)$.

We then set $e_{[l_0]}(p) = \begin{cases} e(p) & \text{if } p \neq \mathsf{P}^{\mathsf{pri}}_\Phi(l_0); \\ (-, (\mathsf{dig}^k_1(b))^{At} + b^*) & \text{if } p = \mathsf{P}^{\mathsf{pri}}_\Phi(l_0). \end{cases}$ □

We now prove two "purely semantic" facts, that will be used in the following cases of the proof of Proposition 40: the case **?unit-PLPS** and the case **!unit-PLPS**. The first one intuitively states that given an (injective atomic) experiment e (resp. e') of Φ (resp. Φ') such that $e(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) \simeq e'(\mathsf{P}^{\mathsf{pri}}_{\Phi'}(l'))$ for some suitable terminal link l (resp. l'), there exists $p \in \mathcal{P}^{\mathsf{f}}(\Phi_{[l]})$ such that for the "corresponding" $p' \in \mathcal{P}^{\mathsf{f}}(\Phi'_{[l']})$ one has $e_{[l]}(p) \simeq e'_{[l']}(p')$.

Fact 69. *Let $k \geq 1$. Let $b, b' \in \mathfrak{M}_{fin}(D')$. Let $\rho : dig^k_1(b) \to dig^k_1(b')$ in $\boldsymbol{M}$. Then we have $\rho : b^* + (dig^k_1(b))^{At} \to b'^* + (dig^k_1(b'))^{At}$ in $\boldsymbol{M}$.*

Proof. We have $\mathsf{dig}^k_1(b^*) = (\mathsf{dig}^k_1(b))^* = (\mathsf{dig}^k_1(b'))^* = \mathsf{dig}^k_1(b'^*)$, hence (since $k \neq 0$) $b^* = b'^*$. From $\rho : \mathsf{dig}^k_1(b) \to \mathsf{dig}^k_1(b')$ one deduces that $\rho : (\mathsf{dig}^k_1(b))^{At} \to (\mathsf{dig}^k_1(b'))^{At}$, and since for $\rho \in$ **pInj** we already noticed that $\rho(b^*) = b^*$, we can conclude that $\rho : b^* + (\mathsf{dig}^k_1(b))^{At} \to b^* + (\mathsf{dig}^k_1(b'))^{At} = b'^* + (\mathsf{dig}^k_1(b'))^{At}$. □

Fact 70. *Let $k \in \mathbb{N}$. Let $\beta \in D'$ such that $(dig^k_1([\beta]))^* \neq []$. Then $([\beta])^* = [\beta]$.*

Proof. From $(\mathsf{dig}^k_1([\beta]))^* \neq []$, we deduce that $At'(\beta) = \emptyset$. □

[39] In case e is not atomic, one might have for example $e(q) = (+, *)$ for some $q \in \bigcup \mathsf{Ax}(\Phi)$.

5.4. The case of **?-box-PLPS** $\cap$ **LPS**

The last case to analyze ($\Phi \in$ **?-box-PLPS** $\cap$ **LPS**) is the most complicated one. The first part of this subsection allows to define a k-experiment $\overline{e}$ of $\overline{\Phi}$ from a k-experiment e of the LPS Φ (where $\overline{\Phi}$ has been defined in Section 4.1, where we already noticed that $mes(\overline{\Phi}) < mes(\Phi)$) and consists only of Fact 71. All the rest of the subsection is "purely semantic".

Fact 71. *Let $k \in \mathbb{N}$. Let $(\Phi, ind) \in$ **LPS**$_{ind}$ such that $\Phi \in$ **?-box-PLPS** and let (e, r) be a k-experiment of (Φ, ind). Then there exists a unique k-experiment $(\overline{e}, \overline{r})$ of $\overline{(\Phi, ind)} = (\overline{\Phi}, \overline{ind})$ such that*
- *for any $p \in (\mathcal{P}(\Phi) \setminus \mathcal{P}^f(\Phi)) \cap \mathcal{P}(\overline{\Phi})$, we have $\overline{e}(p) = e(p)$;*
- *let $i \in \ulcorner \mathcal{P}^f(\Phi) \urcorner$; if $r(i) = (+, a)$, then there exists $\alpha \in D'$ such that $\overline{r}(i) = \alpha$ and $a = \sum_{j=1}^{k} dig(j) \cdot [\alpha]$; if $r(i) = (-, a)$, then there exists $b \in \mathfrak{M}_{fin}(D')$ such that $\overline{r}(i) = (-, b)$ and $a = \sum_{j=1}^{k} dig(j) \cdot b$.*

Moreover, if e is atomic (resp. injective), then $\overline{e}$ is atomic (resp. injective).

Proof. For any $l \in \mathcal{C}^{?c_{auxd}}(\Phi) \cap \mathcal{C}^{t}(\Phi)$, we have

$$e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = \sum_{p \in \mathsf{P}^{\mathsf{aux}}_{\Phi}(l)} \mathsf{dig}^{k}_{\#_{\Phi}(p)}([e(p)]) = \sum_{j=1}^{k} \mathsf{dig}(j) \cdot \sum_{p \in \mathsf{P}^{\mathsf{aux}}_{\Phi}(l)} \mathsf{dig}^{k}_{\#_{\Phi}(p)-1}([e(p)]).$$

For any $l \in \mathcal{C}^{!}(\Phi) \cap \mathcal{C}^{t}(\Phi)$, we have

$$e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l)) = \sum_{p \in \mathsf{P}^{\mathsf{aux}}_{\Phi}(l)} \mathsf{dig}^{k}_{1}([e(p)]) = \sum_{j=1}^{k} \mathsf{dig}(j) \cdot [e(q)], \quad \text{where } \{q\} = \mathsf{P}^{\mathsf{aux}}_{\Phi}(l). \quad \square$$

In the following informal discussion, we fix an LPS Φ and an atomic k-experiment (e, r) of (Φ, ind). Suppose Φ consists of 2 cells: a !-cell and a ?-cell with a unique auxiliary port p such that $\#_{\Phi}(p) = 1$, and suppose that the two auxiliary ports of the two cells are connected by an axiom (in the language of the usual theory of linear logic proof-nets, Φ would correspond to an axiom link inside an exponential box). In this case $r = ((-, \mathsf{dig}^{k}_{1}([\delta])), (+, \mathsf{dig}^{k}_{1}([\delta]))) \in D'^{At} \times D'^{At}$ for some $\delta \in A$. If $\alpha, \alpha' \in Supp(\mathsf{dig}^{k}_{1}([\delta]))$ such that $\alpha \neq \alpha'$, then $At'(\alpha) \cap At'(\alpha') = \emptyset$: two elements of the multiset associated with the principal port of the ?-cell have no atom in common, since they "come from" two different copies of the content of the box.

Suppose now that, more generally, $\Phi \in$ **?-box-PLPS** $\cap$ **LPS** has two conclusions, one is the principal port of a !-cell and the other one is the principal port of a ?-cell, but now this last cell has several auxiliary ports and for every such port p one has $\#_{\Phi}(p) \geq 1$; suppose also that the graph obtained by removing this ?-cell is connected (in the language of the usual theory of linear logic proof-nets, Φ would now correspond to a connected proof-net inside an exponential box, where the ?-conclusions of the box are contracted): an example of such an **LPS** is in Fig. 1 (see also the following Example 75). The previous remark can be generalized to such an LPS: let a (resp. b) be the multiset associated by e with the principal port of the ?-cell (resp. !-cell) conclusion of Φ; we have that $\alpha, \alpha' \in Supp(a)$ "come from" the same copy of the content of the box if and only if there is a "bridge" between α and α'[40], meaning that there is a sequence $\alpha_0, \ldots, \alpha_n$ such that $\alpha_i \in Supp(a + b)$ and $\alpha_0 = \alpha$, $\alpha_n = \alpha'$ and for any $i \in \ulcorner n \urcorner$, we have $At'(\alpha_{i-1}) \cap At'(\alpha_i) \neq \emptyset$. This means that one can split the multiset a into equivalence classes given by the relation "being connected by a bridge", and every equivalence class will identify a copy of the box.

For general $\Phi \in$ **?-box-PLPS** $\cap$ **LPS**, the situation is more complex: it might be the case that the elements α and α' above come from the same copy of a box even though they are not connected by a bridge. On the other hand, the converse still holds: when there is a bridge between α and α' they do come from the same copy of the box. We thus define a function sB, that splits the result r of the experiment e into equivalence classes of this relation.

Definition 72. For any $D_0 \subseteq D'^{At}$, we define the equivalence relation $\simeq_{D_0}$ on D_0 as follows: $\alpha \simeq_{D_0} \alpha'$ if, and only if, there exist $\alpha_0, \ldots, \alpha_n \in D_0$ such that $\alpha_0 = \alpha$, $\alpha_n = \alpha'$ and for any $i \in \ulcorner n \urcorner$, we have $At'(\alpha_{i-1}) \cap At'(\alpha_i) \neq \emptyset$.

Definition 73. We denote by B the function $\mathfrak{P}_{\mathsf{fin}}(D'^{At}) \to \mathfrak{P}_{\mathsf{fin}}(\mathfrak{P}_{\mathsf{fin}}(D'^{At}))$ defined by $B(D_0) = D_0/\simeq_{D_0}$.

The function sB that we are going to define "splits" a t-uple of multisets, following the equivalence classes of the "bridge" equivalence relation:

Definition 74. We denote by sB the morphism of groupoids **sM** $\to$ **psM** defined by: $sB(a_1, \ldots, a_n) = \{(a_{1\,|Supp(a_1)\cap\mathfrak{a}}, \ldots, a_{n\,|Supp(a_n)\cap\mathfrak{a}}) \mid \mathfrak{a} \in B(Supp(\sum_{i=1}^{n} a_i))\}$; and $sB(\rho) = \rho$.

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40 Notice that by Definition 43 $\Phi \notin$ **?unit-PLPS** $\cup$ **!unit-PLPS**, so that $\alpha, \alpha' \in D'^{At}$.

Example 75. Let a_1 and a_2 be as in Example 38. Then we have $B(Supp(a_1 + a_2)) = \{c_1, c_2, c_3\}$ with $c_z = \{(\gamma_1, z), (\gamma_2, z), (+, (\gamma_1, z), (\gamma_2, z))\}$ and we have $sB(a_1, a_2) = \{r_1, r_2, r_3\}$ with $r_z = ([(\gamma_1, z), (\gamma_2, z)], [(+, (\gamma_1, z), (\gamma_2, z))])$. Notice that every element of $sB(a_1, a_2)$ corresponds to a copy of the box.

Given $r = (a_1, \dots, a_n) \in \mathbf{sM}_0$ and two different equivalence classes $\mathfrak{a}, \mathfrak{b} \in B(Supp(\sum_{i=1}^n a_i))$, we clearly have that $At'(\mathfrak{a}) \cap At'(\mathfrak{b}) = \emptyset$. This implies that any element of the restriction of r to the elements of $\mathfrak{a}$ has no atom in common with any element of the restriction of r to the elements of $\mathfrak{b}$, as the following fact precisely states. A consequence that will be used in Lemma 81 is that if for some $r, r' \in \mathbf{sM}_0$ one has $\rho : sB(r) \to sB(r')$ in **psM**, then $\rho : r \to r'$ in **sM**.

Fact 76. *Let $r \in \mathbf{sM}_0$. For any $r_1, r_2 \in sB(r)$, we have $At'(r_1) \cap At'(r_2) \neq \emptyset \Rightarrow r_1 = r_2$.*

Proof. Suppose $r = (a_1, \dots, a_n)$, $r_1 = (c_1, \dots, c_n)$ and $r_2 = (d_1, \dots, d_n)$. By Definition 74, for every $i \in \{1, \dots, n\}$ we have that $c_i = a_{i\,|Supp(a_i)\cap\mathfrak{a}}$ and $d_i = a_{i\,|Supp(a_i)\cap\mathfrak{b}}$ for some $\mathfrak{a}, \mathfrak{b} \in B(Supp(\sum_{i=1}^n a_i))$.

If $At'(r_1) \cap At'(r_2) \neq \emptyset$, then since $At'(r_1) \subseteq At'(\mathfrak{a})$ and $At'(r_2) \subseteq At'(\mathfrak{b})$, we have $At'(\mathfrak{a}) \cap At'(\mathfrak{b}) \neq \emptyset$, which means that $At'(\xi) \cap At'(\eta) \neq \emptyset$ for some $\xi \in \mathfrak{a}$ and $\eta \in \mathfrak{b}$: this implies by Definition 72 that $\xi \simeq_{Supp(\sum_{i=1}^n a_i)} \eta$ and thus $\mathfrak{a} = \mathfrak{b}$ and $r_1 = r_2$. □

In the language of the usual theory of linear logic proof-nets, given a proof-net one can "box it"; we have generalized this boxing operation in the framework of *LPS*: for $\Phi \in$ **?-box-PLPS** ∩ **LPS** this corresponds to the passage from $\overline{\Phi}$ to Φ. From an experiment (e_1, r_1) of $\overline{(\Phi, \mathrm{ind})}$, one can naturally obtain an experiment (e, r) of (Φ, ind). The following lemma (intuitively) relates the effect of applying the splitting function sB after boxing to the effect of applying the splitting function sB before boxing.

Lemma 77. *Let $k, n \in \mathbb{N}$ such that $k > 0$. Let $b_1, \dots, b_n \in \mathfrak{M}_{fin}(D'^{At})$. We have:*

$$sB(dig_1^k(b_1), \dots, dig_1^k(b_n)) = \{(dig(j_0) \cdot f_1, \dots, dig(j_0) \cdot f_n) \mid j_0 \in \ulcorner k \urcorner \text{ and } (f_1, \dots, f_n) \in sB(b_1, \dots, b_n)\}$$

Proof. For any $b \in \mathfrak{M}_{\mathrm{fin}}(D'^{At})$, we have $B(Supp(\mathrm{dig}_1^k(b))) = B(Supp(\sum_{j=1}^k \mathrm{dig}(j) \cdot b)) = \{\{\mathrm{dig}(j_0) \cdot \beta \mid \beta \in \mathfrak{b}\} \mid j_0 \in \ulcorner k \urcorner \text{ and } \mathfrak{b} \in B(Supp(b))\}$. Now notice that $\mathrm{dig}_1^k(\sum_{i=1}^n b_i) = \sum_{i=1}^n \mathrm{dig}_1^k(b_i)$; hence

$$\begin{aligned} B\left(Supp\left(\sum_{i=1}^n \mathrm{dig}_1^k(b_i)\right)\right) &= B\left(Supp\left(\mathrm{dig}_1^k\left(\sum_{i=1}^n b_i\right)\right)\right) \\ &= \left\{\{\mathrm{dig}(j_0) \cdot \beta \mid \beta \in \mathfrak{b}\} \mid j_0 \in \ulcorner k \urcorner \text{ and } \mathfrak{b} \in B\left(Supp\left(\sum_{i=1}^n b_i\right)\right)\right\}. \end{aligned}$$

Thus

$$\begin{aligned} sB(\mathrm{dig}_1^k(b_1), \dots, \mathrm{dig}_1^k(b_n)) &= \left\{(\mathrm{dig}_1^k(b_1)\big|_{Supp(\mathrm{dig}_1^k(b_1))\cap\mathfrak{a}}, \dots, \mathrm{dig}_1^k(b_n)\big|_{Supp(\mathrm{dig}_1^k(b_n))\cap\mathfrak{a}}) \mid \mathfrak{a} \in B\left(Supp\left(\sum_{i=1}^n \mathrm{dig}_1^k(b_i)\right)\right)\right\} \\ &= \left\{(\mathrm{dig}_1^k(b_1)\big|_{Supp(\mathrm{dig}_1^k(b_1))\cap\{\mathrm{dig}(j_0)\cdot\beta \mid \beta\in\mathfrak{b}\}}, \dots, \mathrm{dig}_1^k(b_n)\big|_{Supp(\mathrm{dig}_1^k(b_n))\cap\{\mathrm{dig}(j_0)\cdot\beta \mid \beta\in\mathfrak{b}\}}) \mid \right. \\ &\qquad \left. j_0 \in \ulcorner k \urcorner \text{ and } \mathfrak{b} \in B\left(Supp\left(\sum_{i=1}^n b_i\right)\right)\right\} \\ &= \left\{(\mathrm{dig}_1^k(b_1)\big|_{\{\mathrm{dig}(j_0)\cdot\beta \mid \beta\in Supp(b_1)\cap\mathfrak{b}\}}, \dots, \mathrm{dig}_1^k(b_n)\big|_{\{\mathrm{dig}(j_0)\cdot\beta \mid \beta\in Supp(b_n)\cap\mathfrak{b}\}}) \mid \right. \\ &\qquad \left. j_0 \in \ulcorner k \urcorner \text{ and } \mathfrak{b} \in B\left(Supp\left(\sum_{i=1}^n b_i\right)\right)\right\} \\ &= \left\{((\mathrm{dig}(j_0) \cdot b_1)\big|_{\{\mathrm{dig}(j_0)\cdot\beta \mid \beta\in Supp(b_1)\cap\mathfrak{b}\}}, \dots, (\mathrm{dig}(j_0) \cdot b_n)\big|_{\{\mathrm{dig}(j_0)\cdot\beta \mid \beta\in Supp(b_n)\cap\mathfrak{b}\}}) \mid \right. \\ &\qquad \left. j_0 \in \ulcorner k \urcorner \text{ and } \mathfrak{b} \in B\left(Supp\left(\sum_{i=1}^n b_i\right)\right)\right\} \\ &= \left\{(\mathrm{dig}(j_0) \cdot b_{1\,|Supp(b_1)\cap\mathfrak{b}}, \dots, \mathrm{dig}(j_0) \cdot b_{n\,|Supp(b_n)\cap\mathfrak{b}}) \;\middle|\; \begin{array}{c} j_0 \in \ulcorner k \urcorner \text{ and} \\ \mathfrak{b} \in B\left(Supp\left(\sum_{i=1}^n b_i\right)\right) \end{array}\right\} \\ &= \{(\mathrm{dig}(j_0) \cdot f_1, \dots, \mathrm{dig}(j_0) \cdot f_n) \mid j_0 \in \ulcorner k \urcorner \text{ and } (f_1, \dots, f_n) \in sB(b_1, \dots, b_n)\}. \quad \square \end{aligned}$$

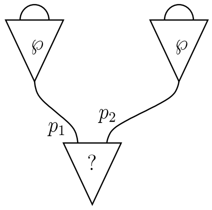

Fig. 6. Example described in Example 80 illustrating the case ?-box-PLPS ∩ LPS. Let $\Phi \in$ **PLPS** be as in the figure and such that $\#_\Phi(p_1) = 1 = \#_\Phi(p_2)$. We have $\Phi \in$ **?-box-PLPS ∩ LPS**.

Our aim is now to prove Lemma 81: both the following Definition 78 and Fact 79 are just tools to prove this result (in order to get some intuition, see Example 80).

Definition 78. We denote by R the morphism of groupoids **psM** → **ppsM** defined by: $R(\mathfrak{a}) = \mathfrak{a}/\simeq_{\mathbf{sM}}$, where $r \simeq_{\mathbf{sM}} r'$ if, and only if, $r \simeq r'$ in **sM**; and $R(\rho) = \rho$.

Fact 79. *Let $k \in \mathbb{N} \setminus \{0\}$. Let $r, r' \in \mathbf{sM}_0$. Let $\mathfrak{b} \in R(sB(r))$, $\mathfrak{b}' \in R(sB(r'))$ such that $\{dig(j_0) \cdot r_0 \mid j_0 \in \ulcorner k \urcorner \text{ and } r_0 \in \mathfrak{b}\} \simeq \{dig(j_0) \cdot r'_0 \mid j_0 \in \ulcorner k \urcorner \text{ and } r'_0 \in \mathfrak{b}'\}$ in **psM**. Then we have $\mathfrak{b} \simeq \mathfrak{b}'$ in **psM**.*

Proof. Let ρ be the morphism

$$\{\mathsf{dig}(j_0) \cdot r_0 \mid j_0 \in \ulcorner k \urcorner \text{ and } r_0 \in \mathfrak{b}\} \to \{\mathsf{dig}(j_0) \cdot r'_0 \mid j_0 \in \ulcorner k \urcorner \text{ and } r'_0 \in \mathfrak{b}'\}$$

in **psM**. Let $r_0 \in \mathfrak{b}$. Let $r'_0 \in \mathfrak{b}'$ and $j_0 \in \ulcorner k \urcorner$ such that $\rho : \mathsf{dig}(1) \cdot r_0 \to \mathsf{dig}(j_0) \cdot r'_0$ in **sM**; then we have $r_0 \simeq r'_0$ in **sM**. Thus the following holds:

- there exists $r_0 \in \mathfrak{b}$, $r'_0 \in \mathfrak{b}'$ such that $r_0 \simeq r'_0$ in **sM**;
- for any $r_1, r_2 \in \mathfrak{b}$, we have $r_1 \simeq r_2$ in **sM** and for any $r'_1, r'_2 \in \mathfrak{b}'$, $r'_1 \simeq r'_2$ in **sM**;
- for any $r_1, r_2 \in \mathfrak{b}$, we have $At'(r_1) \cap At'(r_2) \neq \emptyset \Rightarrow r_1 = r_2$ and for any $r'_1, r'_2 \in \mathfrak{b}'$, we have $At'(r'_1) \cap At'(r'_2) \neq \emptyset \Rightarrow r'_1 = r'_2$ (by Fact 76);
- $Card(\mathfrak{b}) = Card(\mathfrak{b}')$.

Hence $\mathfrak{b} \simeq \mathfrak{b}'$ in **psM**. Indeed: let $\tau : r_0 \to r'_0$ in **sM** and let $\varphi : \mathfrak{b} \to \mathfrak{b}'$ in **Bij**; for any $r_1 \in \mathfrak{b}$, let $\tau_{r_1} : r_1 \to r_0$ in **sM**; for any $r'_1 \in \mathfrak{b}'$, let $\tau'_{r'_1} : r'_0 \to r'_1$ in **sM**; for any $r_1 \in \mathfrak{b}$, we set $\rho_{r_1} = \tau'_{\varphi(r_1)} \circ \tau \circ \tau_{r_1}$; we define $\rho' : \mathfrak{b} \to \mathfrak{b}'$ in **psM** by setting $\rho'(\delta) = \rho_{r_1}(\delta)$ if $\delta \in At'(r_1)$. □

Example 80. In order to help the reader to get some intuition of what we want to do here, let us consider the LPS Φ represented in Fig. 6: the contraction of two auxiliary doors p_1 and p_2 such that $\#_\Phi(p_1) = \#_\Phi(p_2) = 1$; above each auxiliary door, a ⅋; above each ⅋, an axiom. Let $e = e'$ be the injective atomic k-experiment of Φ such that the label associated by e with every auxiliary port of the ?-cell is $(-, \gamma_z, \gamma_z)$, where $\gamma_z \in A$, $z \in \ulcorner 2 \urcorner$ and $\gamma_1 \neq \gamma_2$. The result $r = r'$ is $(-, \sum_{1 \leq j \leq k, 1 \leq z \leq 2}[(-, (\gamma_z, j), (\gamma_z, j))])$. We have $\rho : \mathfrak{a} \to \mathfrak{a}'$ in **psM**, where $\mathfrak{a} = \bigcup_{1 \leq j \leq k, 1 \leq z \leq 2}\{([(-, (\gamma_z, j), (\gamma_z, j))])\} = \mathfrak{a}'$, with ρ that can send any (γ_z, j) to any $(\gamma_{z'}, j')$. Fact 79 will be useful to deduce very generally that in situations of this kind, we have $\mathfrak{b} \simeq \mathfrak{b}'$ in **psM**, where here $\mathfrak{b} = \{([(-, \gamma_1, \gamma_1)]), ([(-, \gamma_2, \gamma_2)])\} = \mathfrak{b}'$.

The following lemma is the crucial step allowing to apply the induction hypothesis in the proof of the key-Proposition 40 in the **?-box-PLPS** case: it intuitively states that if there is an isomorphism between the results of two experiments of $\Phi_1, \Phi_2 \in$ **?-box-PLPS ∩ LPS**, then there exists also an isomorphism between the results of two experiments of $\overline{\Phi_1}$ and $\overline{\Phi_2}$. In the proof, we denote by U the forgetful functor **ppsM** → **Bij**.

Lemma 81. *Let $k, n \in \mathbb{N}$ such that $k > 0$. Let $b_1, \ldots, b_n, b'_1, \ldots, b'_n \in \mathfrak{M}_{fin}(D'^{At})$ such that $(dig_1^k(b_1), \ldots, dig_1^k(b_n)) \simeq (dig_1^k(b'_1), \ldots, dig_1^k(b'_n))$ in **sM**. Then we have $(b_1, \ldots, b_n) \simeq (b'_1, \ldots, b'_n)$ in **sM**.*

Proof. We set

$$\mathfrak{a} = \{\mathsf{dig}(j_0) \cdot (f_1, \ldots, f_n) \mid j_0 \in \ulcorner k \urcorner \text{ and } (f_1, \ldots, f_n) \in sB(b_1, \ldots, b_n)\}$$

and

$$\mathfrak{a}' = \{\mathsf{dig}(j_0) \cdot (f'_1, \ldots, f'_n) \mid j_0 \in \ulcorner k \urcorner \text{ and } (f'_1, \ldots, f'_n) \in sB(b'_1, \ldots, b'_n)\}.$$

Since sB is a morphism of groupoids, by Lemma 77, there exists $\rho : \mathfrak{a} \to \mathfrak{a}'$ in **psM**.

Since for any $r, r' \in \mathbf{sM}_0$, for any $j_1, j_2 \in \ulcorner k \urcorner$, we have $\mathsf{dig}(j_1) \cdot r \simeq \mathsf{dig}(j_2) \cdot r'$ in **sM** if, and only if, $r \simeq r'$ in **sM**, we can define $\varphi : U(R(sB(b_1, \ldots, b_n))) \to U(R(\mathfrak{a}))$ in **Bij** by setting $\varphi(\{(f_1^1, \ldots, f_n^1), \ldots, (f_1^q, \ldots, f_n^q)\}) = \{\mathsf{dig}(j) \cdot (f_1^z, \ldots, f_n^z) \mid j \in \ulcorner k \urcorner \text{ and } z \in \ulcorner q \urcorner\}$ and $\varphi' : U(R(sB(b'_1, \ldots, b'_n))) \to U(R(\mathfrak{a}'))$ in **Bij** by setting $\varphi'(\{(f'^1_1, \ldots, f'^1_n), \ldots, (f'^q_1, \ldots, f'^q_n)\}) = \{\mathsf{dig}(j) \cdot (f'^z_1, \ldots, f'^z_n) \mid j \in \ulcorner k \urcorner \text{ and } z \in \ulcorner q \urcorner\}$. We have $\varphi'^{-1} \circ U(R(\rho)) \circ \varphi : U(R(sB(b_1, \ldots, b_n))) \to U(R(sB(b'_1, \ldots, b'_n)))$ in **Bij**.

For any $\mathfrak{b} \in U(R(sB(b_1, \ldots, b_n)))$, we have $\rho : \varphi(\mathfrak{b}) = \{\mathsf{dig}(j_0) \cdot r_0 \mid j_0 \in \ulcorner k \urcorner \text{ and } r_0 \in \mathfrak{b}\} \to \{\mathsf{dig}(j_0) \cdot r'_0 \mid j_0 \in \ulcorner k \urcorner \text{ and } r'_0 \in (\varphi'^{-1} \circ U(R(\rho)) \circ \varphi)(\mathfrak{b})\} = (U(R(\rho)) \circ \varphi)(\mathfrak{b})$ in **psM**. Hence by Fact 79, for any $\mathfrak{b} \in U(R(sB(b_1, \ldots, b_n)))$ there exists $\tau_{\mathfrak{b}} : \mathfrak{b} \to (\varphi'^{-1} \circ U(R(\rho)) \circ \varphi)(\mathfrak{b})$ in **psM**.

Now, by applying a first time Fact 76, we can define an application

$$\tau : \bigcup_{r \in sB(b_1, \ldots, b_n)} At'(r) \to \bigcup_{r' \in sB(b'_1, \ldots, b'_n)} At'(r')$$

by setting $\tau(\delta) = \tau_{\mathfrak{b}}(\delta)$ for $\delta \in At'(r)$, $r \in \mathfrak{b}$ and $\mathfrak{b} \in R(sB(b_1, \ldots, b_n))$.

We thus obtain $\tau : R(sB(b_1, \ldots, b_n)) \to R(sB(b'_1, \ldots, b'_n))$ in **ppsM**. By applying a second time Fact 76, we obtain that τ is a morphism

$$\bigcup R(sB(b_1, \ldots, b_n)) = sB(b_1, \ldots, b_n) \to sB(b'_1, \ldots, b'_n) = \bigcup R(sB(b'_1, \ldots, b'_n))$$

in **psM**. Lastly, by applying a third time Fact 76, we obtain $\tau : (b_1, \ldots, b_n) \to (b'_1, \ldots, b'_n)$ in **sM**. □

5.5. *Key-Proposition*

We can now conclude the paper by giving the complete proof of the missing result:

Proposition 40. *Let $(\Phi, ind), (\Phi', ind') \in \boldsymbol{LPS}_{ind}$, let $k > cosize(\Phi), cosize(\Phi')$, let (e, r) (resp. (e', r')) be an atomic injective k-experiment of (Φ, ind) (resp. (Φ', ind')). If $r \simeq r'$ in* **sD**, *then $(\Phi, ind) \simeq (\Phi', ind')$.*

Proof. The proof is by induction on $mes(\Phi)$. We have $mes(\Phi) = (0, 0)$ if, and only if, $\Phi \in \emptyset$-**PLPS**; in this case, it is obvious that we have $(\Phi, \mathsf{ind}) \simeq (\Phi', \mathsf{ind'})$. If $mes(\Phi) > (0, 0)$, then let $\rho : r \to r'$ in **sD**, we set $n = Card(\mathcal{P}^{\mathsf{f}}(\Phi))$ and we distinguish between the several cases.

- In the case where $\Phi \in \mathsf{ax}$-**PLPS**, let $w = \{p_0, q_0\} \in \mathsf{Ax}^{\mathsf{i}}(\Phi)$ and let $i_0, j_0 \in \ulcorner n \urcorner$ such that $\mathsf{ind}(p_0) = i_0$ and $\mathsf{ind}(q_0) = j_0$. Let $p'_0, q'_0 \in \mathcal{P}^{\mathsf{f}}(\Phi')$ such that $\mathsf{ind'}(p'_0) = i_0$ and $\mathsf{ind'}(q'_0) = j_0$. As e is atomic and e' is injective, we have $w' = \{p'_0, q'_0\} \in \mathsf{Ax}^{\mathsf{i}}(\Phi')$.
  Let $(\Phi_1, \mathsf{ind}_1) \in \mathbf{PLPS}_{\mathsf{ind}}$ (resp. $(\Phi'_1, \mathsf{ind'}_1) \in \mathbf{PLPS}_{\mathsf{ind}}$) obtained from (Φ, ind) (resp. $(\Phi', \mathsf{ind'})$) by removing w (resp. w').[41] Since $\Phi, \Phi' \in$ **LPS**, we have $\Phi_1, \Phi'_1 \in$ **LPS**. We set $e_1 = e|_{\mathcal{P}(\Phi_1)}$ and $e'_1 = e'|_{\mathcal{P}(\Phi'_1)}$. We set $r_1 = e \circ \mathsf{ind}_1{}^{-1}$ and $r'_1 = e' \circ \mathsf{ind'}_1{}^{-1}$: it is immediate that (e_1, r_1) is an injective atomic experiment of (Φ_1, ind_1) and that (e'_1, r'_1) is an injective atomic experiment of $(\Phi'_1, \mathsf{ind'}_1)$; and from $\rho : r \to r'$ in **sD** one deduces $\rho : r_1 \to r'_1$ in **sD**. Notice that $mes(\Phi_1) < mes(\Phi)$: by induction hypothesis we have $(\Phi_1, \mathsf{ind}_1) \simeq (\Phi'_1, \mathsf{ind'}_1)$, which obviously implies $(\Phi, \mathsf{ind}) \simeq (\Phi', \mathsf{ind'})$.
- In the case where $\Phi \in ?_{c_b}$-**PLPS**, let $l_0 \in \mathcal{C}^{?c_b}(\Phi) \cap \mathcal{C}^{\mathsf{t}}(\Phi)$ and let $i_0 \in \ulcorner n \urcorner$ such that $\mathsf{ind}(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l_0)) = i_0$. As e' is atomic, there exists $l'_0 \in \mathcal{C}^{?}(\Phi') \cap \mathcal{C}^{\mathsf{t}}(\Phi')$ such that $\mathsf{P}^{\mathsf{pri}}_{\Phi'}(l'_0) = \mathsf{ind'}^{-1}(i_0)$. Let $a \in \mathfrak{M}_{\mathsf{fin}}(D')$ such that $e(\mathsf{P}^{\mathsf{pri}}_{\Phi}(l_0)) = (-, a)$. Let $a' \in \mathfrak{M}_{\mathsf{fin}}(D')$ such that $\rho \cdot (-, a) = (-, a')$. Let $p \in \mathsf{P}^{\mathsf{aux}}_{\Phi}(l_0)$ such that $\#_{\Phi}(p) = 0$. We set $\beta = e(p)$. We have $\beta \in Supp(a)$, hence there exists $a_0 \in Q(r, a)$ such that $\beta \in Supp(a_0)$. By Lemma 63, there exists $\mathcal{P}_0 \subseteq \mathsf{P}^{\mathsf{aux}}_{\Phi}(l_0)$ such that $a_0 = \sum_{q \in \mathcal{P}_0} \mathsf{dig}^k_{\#_{\Phi}(q)}(e(q))$. We have $p \in \mathcal{P}_0$ (otherwise, we would have $a(\beta) > a_0(\beta)$). Hence, by Fact 57, k does not divide $Card(a_0) = Card(\rho \cdot a_0)$. As we have $\rho : (r, a) \to (r', a')$ in **sDM** and by Fact 61 Q is a morphism of groupoids, we have $\rho \cdot a_0 \in Q(r', a')$. Hence, by Lemma 63, there exists $\mathcal{P}'_0 \subseteq \mathsf{P}^{\mathsf{aux}}_{\Phi'}(l'_0)$ such that $\rho \cdot a_0 = \sum_{q \in \mathcal{P}'_0} \mathsf{dig}^k_{\#_{\Phi'}(q)}(e'(q))$. By Fact 57, there exists $p' \in \mathcal{P}'_0$ such that $\#_{\Phi'}(p') = 0$. Let $\beta' = e'(p')$; we have $(r', \rho \cdot \beta) \simeq (r', \beta')$ and $(r, \beta) \simeq (r', \rho \cdot \beta)$ in **sD**, hence $(r, \beta) \simeq (r', \beta')$ in **sD**.
  Let $\Phi_1 \in$ **PLPS** (resp. $\Phi'_1 \in$ **PLPS**) obtained from Φ (resp. Φ') by removing p (resp. p') from the auxiliary ports of l_0 (resp. l'_0).[42] Notice that $mes(\Phi_1) < mes(\Phi)$. Both Φ_1 and Φ'_1 have $n + 1$ free ports: for Φ_1, those of Φ and a new free port p_0; for Φ'_1, those of Φ' and a new free port p'_0. We set
  $$\mathsf{ind}_1(q) = \begin{cases} \mathsf{ind}(q) & \text{if } q \neq p_0; \\ n+1 & \text{if } q = p_0; \end{cases} \quad \text{and} \quad \mathsf{ind'}_1(q) = \begin{cases} \mathsf{ind'}(q) & \text{if } q \neq p'_0; \\ n+1 & \text{if } q = p'_0. \end{cases}$$
  We have $(\Phi_1, \mathsf{ind}_1), (\Phi'_1, \mathsf{ind'}_1) \in \mathbf{LPS}_{\mathsf{ind}}$. For any $q \in \mathcal{P}(\Phi_1) \setminus \{\mathsf{P}^{\mathsf{pri}}_{\Phi_1}(l_0)\}$, we set $e_1(q) = e(q)$. Let $b \in \mathfrak{M}_{\mathsf{fin}}(D')$ such that $a = b + [\beta]$; we set $e_1(\mathsf{P}^{\mathsf{pri}}_{\Phi_1}(l_0)) = (-, b)$. For any $q \in \mathcal{P}(\Phi'_1) \setminus \{\mathsf{P}^{\mathsf{pri}}_{\Phi'_1}(l'_0)\}$, we set $e'_1(q) = e'(q)$. Let $b' \in \mathfrak{M}_{\mathsf{fin}}(D')$ such

41 See the appendix for a formal definition of (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind'}_1)$.
42 See the appendix for a formal definition of (Φ_1, ind_1) and $(\Phi'_1, \mathsf{ind'}_1)$.

that $a' = b' + [\beta']$; we set $e'_1(\mathsf{P}^{\mathrm{pri}}_{\Phi'_1}(l'_0)) = (-, b')$.

$$\text{We set } r_1(i) = \begin{cases} r(i) & \text{if } i \notin \{i_0, n+1\}; \\ (-, b) & \text{if } i = i_0; \\ \beta & \text{if } i = n+1. \end{cases}$$

$$\text{We set } r'_1(i) = \begin{cases} r'(i) & \text{if } i \notin \{i_0, n+1\}; \\ (-, b') & \text{if } i = i_0; \\ \beta' & \text{if } i = n+1. \end{cases}$$

Since (e, r) (resp. (e', r')) is an atomic injective k-experiment of (Φ, ind) (resp. $(\Phi', \mathrm{ind'})$), (e_1, r_1) (resp. (e'_1, r'_1)) is an atomic injective k-experiment of (Φ_1, ind_1) (resp. $(\Phi'_1, \mathrm{ind'}_1)$) and since $(r, \beta) \simeq (r', \beta')$ in **sD** we have $r_1 \simeq r'_1$ in **sD**. By induction hypothesis we deduce that $(\Phi_1, \mathrm{ind}_1) \simeq (\Phi'_1, \mathrm{ind'}_1)$, from which the conclusion $(\Phi, \mathrm{ind}) \simeq (\Phi', \mathrm{ind'})$ immediately follows.

- In the case where $\Phi \in$ **!unit-PLPS**, by Fact 65, there exists $l_0 \in \mathcal{C}^!(\Phi) \cap \mathcal{C}^{\mathrm{t}}(\Phi)$ and $\beta \in D'$ such that $e(\mathsf{P}^{\mathrm{pri}}_{\Phi}(l_0)) = (+, \mathrm{dig}^k_1([\beta]))$ and $(\mathrm{dig}^k_1([\beta]))^* \neq []$. As e' is atomic, there exists $l'_0 \in \mathcal{C}^!(\Phi') \cap \mathcal{C}^{\mathrm{t}}(\Phi')$ such that $\mathsf{P}^{\mathrm{pri}}_{\Phi'}(l'_0) = \mathrm{ind'}^{-1}(i_0)$. Since $\rho : r \to r'$ in **sD** one has $\rho : e(\mathsf{P}^{\mathrm{pri}}_{\Phi}(l_0)) \to e'(\mathsf{P}^{\mathrm{pri}}_{\Phi'}(l'_0))$ in **D**, so that there exists $\beta' \in D'$ such that $e'(\mathsf{P}^{\mathrm{pri}}_{\Phi'}(l'_0)) = (+, \mathrm{dig}^k_1([\beta']))$ and $\rho : \mathrm{dig}^k_1([\beta]) \to \mathrm{dig}^k_1([\beta'])$ in **M**. Hence $(\mathrm{dig}^k_1([\beta']))^* \neq []$ and, by Fact 69, $\rho : ([\beta])^* + (\mathrm{dig}^k_1([\beta]))^{At} \to ([\beta'])^* + (\mathrm{dig}^k_1(([\beta']))^{At}$ in **M**: by Fact 70, we obtain $\rho : \beta \to \beta'$ in **D** and thus $\rho : r_{[l_0]} \to r'_{[l'_0]}$ in **sD**, where $r_{[l_0]}$ and $r'_{[l'_0]}$ have been defined in Fact 66. By this fact and by Fact 44, we can apply the induction hypothesis and deduce that $(\Phi_{[l_0]}, \mathrm{ind}_{[l_0]}) \simeq (\Phi'_{[l'_0]}, \mathrm{ind'}_{[l'_0]})$. Since $\Phi_{[l_0]}$ (resp. $\Phi'_{[l'_0]}$) has been obtained from Φ (resp. Φ') by removing the !-cell l_0 (resp. l'_0), the fact that $(\Phi_{[l_0]}, \mathrm{ind}_{[l_0]}) \simeq (\Phi'_{[l'_0]}, \mathrm{ind'}_{[l'_0]})$ entails that $(\Phi, \mathrm{ind}) \simeq (\Phi', \mathrm{ind'})$.
- In the case where $\Phi \in$ **?unit-PLPS**, by Remark 67 and Fact 65, there exists $l_0 \in (\mathcal{C}^?(\Phi) \setminus \mathcal{C}^{?c_b}(\Phi)) \cap \mathcal{C}^{\mathrm{t}}(\Phi)$ and $b \in \mathfrak{M}_{\mathrm{fin}}(D')$ such that $e(\mathsf{P}^{\mathrm{pri}}_{\Phi}(l_0)) = (-, \mathrm{dig}^k_1(b))$ and $(\mathrm{dig}^k_1(b))^* \neq []$. As e' is atomic, there exists $l'_0 \in \mathcal{C}^?(\Phi') \cap \mathcal{C}^{\mathrm{t}}(\Phi)$ such that $\mathsf{P}^{\mathrm{pri}}_{\Phi'}(l'_0) = \mathrm{ind'}^{-1}(i_0)$. We have $l_0 \notin \mathcal{C}^{?c_b}(\Phi)$, so that by Fact 57, k divides $Card(\mathrm{dig}^k_1(b))$. Still by Fact 57, we obtain that $l'_0 \notin \mathcal{C}^{?c_b}(\Phi')$. From $\rho : r \to r'$ in **sD**, we can deduce (using again Remark 67) that $\rho : \mathrm{dig}^k_1(b) \to \mathrm{dig}^k_1(b')$ in **M**, hence, by Fact 69, we get $\rho : b^* + (\mathrm{dig}^k_1(b))^{At} \to b'^* + \mathrm{dig}^k_1(b')^{At}$ in **M** and thus $\rho : r_{[l_0]} \to r'_{[l'_0]}$ in **sD**, where $r_{[l_0]}$ and $r'_{[l'_0]}$ have been defined in Fact 68. By this fact and by Fact 44, we can apply the induction hypothesis and deduce that $(\Phi_{[l_0]}, \mathrm{ind}_{[l_0]}) \simeq (\Phi'_{[l'_0]}, \mathrm{ind'}_{[l'_0]})$. Now notice that for $\varphi = (\varphi_{\mathcal{C}}, \varphi_{\mathcal{P}}) : (\Phi_{[l_0]}, \mathrm{ind}_{[l_0]}) \simeq (\Phi'_{[l'_0]}, \mathrm{ind'}_{[l'_0]})$, we also have $\varphi : (\Phi, \mathrm{ind}) \simeq (\Phi', \mathrm{ind'})$. Indeed: let $b_0 = \sum_{p \in \mathsf{P}^{\mathrm{aux}}_{\Phi}(l_0)} [e(p)]$ and $b'_0 = \sum_{p' \in \mathsf{P}^{\mathrm{aux}}_{\Phi'}(l'_0)} [e'(p')]$; then for any $p \in \mathsf{P}^{\mathrm{aux}}_{\Phi}(l_0)$, we have $e(p) \in Supp({b_0}^*)$ if, and only if, $e'(\varphi_{\mathcal{P}}(p)) \in Supp({b'_0}^*)$, hence $\#_{\Phi}(p) = \#_{\Phi'}(\varphi_{\mathcal{P}}(p))$.
- In the case where $\Phi \in$ **?-box-PLPS**, for every $i \in \ulcorner n \urcorner$ we have that $r(i) = (\iota_i, b_i)$ for some $b_i \in \mathfrak{M}_{\mathrm{fin}}(D')$ and from the existence of $\rho : r \to r'$ in **sD**, we deduce that $r'(i) = (\iota_i, b'_i)$ where $b'_i = \rho \cdot b_i$. Since $\Phi \notin$ **?unit-PLPS**, by Fact 65 we deduce $b^*_i = []$ for every $i \in \ulcorner n \urcorner$, thus $b'^*_i = []$ which implies $\Phi' \notin$ **?unit-PLPS**. By Fact 57, k divides $Card(b_i)$. Since $Card(b_i) = Card(b'_i)$ and e' is atomic, by applying again Fact 57, we can conclude that $\Phi' \in$ **?-box-PLPS**. We can thus now apply Fact 71 twice:

  1. there exists a unique atomic and injective k-experiment $(\overline{e}, \overline{r})$ of $\overline{(\Phi, \mathrm{ind})} = (\overline{\Phi}, \overline{\mathrm{ind}}) \in \mathbf{LPS}_{\mathrm{ind}}$ such that
     - for any $p \in (\mathcal{P}(\Phi) \setminus \mathcal{P}^{\mathrm{f}}(\Phi)) \cap \mathcal{P}(\overline{\Phi})$, we have $\overline{e}(p) = e(p)$;
     - if $r(i) = (+, \mathrm{dig}^k_1([\alpha_i]))$ for some $\alpha_i \in D'$, then $\overline{r}(i) = \alpha_i$ and if $r(i) = (-, \mathrm{dig}^k_1(c_i))$ then $\overline{r}(i) = (-, c_i)$.
  2. there exists a unique atomic and injective k-experiment $(\overline{e'}, \overline{r'})$ of $\overline{(\Phi', \mathrm{ind'})} = (\overline{\Phi'}, \overline{\mathrm{ind'}}) \in \mathbf{LPS}_{\mathrm{ind}}$ such that
     - for any $p \in (\mathcal{P}(\Phi') \setminus \mathcal{P}^{\mathrm{f}}(\Phi')) \cap \mathcal{P}(\overline{\Phi'})$, we have $\overline{e'}(p) = e'(p)$;
     - if $r'(i) = (+, \mathrm{dig}^k_1([\alpha'_i]))$ for some $\alpha'_i \in D'$, then $\overline{r'}(i) = \alpha'_i$ and if $r'(i) = (-, \mathrm{dig}^k_1(c'_i))$ then $\overline{r'}(i) = (-, c'_i)$.

  If we set $b_i = c_i$ (resp. $b_i = [\alpha_i]$) if $\overline{r}(i) = c_i$ (resp. $\overline{r}(i) = \alpha_i$), and $b'_i = c'_i$ (resp. $b'_i = [\alpha'_i]$) if $\overline{r'}(i) = c'_i$ (resp. $\overline{r'}(i) = \alpha'_i$), then $r \simeq r'$ in **sD** is equivalent to $(\mathrm{dig}^k_1(b_1), \ldots, \mathrm{dig}^k_1(b_n)) \simeq (\mathrm{dig}^k_1(b'_1), \ldots, \mathrm{dig}^k_1(b'_n))$ in **sM**. By Lemma 81 we can then conclude that $(b_1, \ldots, b_n) \simeq (b'_1, \ldots, b'_n)$ in **sM**, which immediately yields $\overline{r} \simeq \overline{r'}$ in **sD**. Since $mes(\overline{\Phi}) < mes(\Phi)$, by induction hypothesis we deduce that $(\overline{\Phi}, \overline{\mathrm{ind}}) \simeq (\overline{\Phi'}, \overline{\mathrm{ind'}})$. To conclude, notice that (since $r \simeq r'$ in **sD**) for $p \in \mathcal{P}^{\mathrm{f}}(\Phi)$, $p = \mathsf{P}^{\mathrm{pri}}_{\Phi}(l)$ (resp. $p' \in \mathcal{P}^{\mathrm{f}}(\Phi')$, $p' = \mathsf{P}^{\mathrm{pri}}_{\Phi'}(l')$) such that $\mathrm{ind}(p) = \mathrm{ind'}(p')$, we have $l \in \mathcal{C}^!(\Phi)$ iff $l' \in \mathcal{C}^!(\Phi')$. Thus from $(\overline{\Phi}, \overline{\mathrm{ind}}) \simeq (\overline{\Phi'}, \overline{\mathrm{ind'}})$ it follows that $(\Phi, \mathrm{ind}) \simeq (\Phi', \mathrm{ind'})$.
- the other cases are easier and left to the reader. □

Remark 82. A crucial point in the case $\Phi \in ?_{c_b}$**-PLPS** of the proof is that we have $\rho \cdot \beta \simeq \beta'$, but we do not necessarily have $\rho \cdot \beta = \beta'$ and this corresponds to the fact that, as illustrated in the introduction by an example using the PS of Fig. 2, there are different atomic k-experiments of PS[43] having the same injective result. Consider again this figure and let Φ be the LPS of this PS. Let $e = e'$ be a 3-experiment of Φ such that $e(p_z) = (-, \lambda_z, \lambda_z)$ with $\lambda_z \in A$ and $z \in \ulcorner 2 \urcorner$. We have $e(c_1) = (-, a)$

43 See footnote 26.

with $a = [(-, \lambda_1, \lambda_1)] + \sum_{j=1}^{3}[(-, (\lambda_2, j), (\lambda_2, j))]$. Let $r = r'$ be the result of $e = e'$. We have $Q(r, a) = \{a\}$, hence we can consider, for example, $\rho : (-, a) \to (-, a)$ in **sD** such that $\rho(\lambda_1) = (\lambda_2, 1)$. We have $\beta = (-, \lambda_1, \lambda_1) = \beta' \neq \rho \cdot \beta$.

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Technical appendix

A. Syntax

A.1. Pre-Linear Proof-Structures (PLPS)

We introduce a weaker notion than the one of PPLPS: *ωPPLPS*. An ωPPLPS[44] is a PPLPS, except that Condition 3 of Definition 5 is not required.

Definition 83. Let ω**PPLPS** be the set of triples $\Phi = (\mathbb{C}, \mathcal{I}, \mathcal{W})$ with $\mathbb{C} \in$ **Cells**, $\mathcal{I}$ a finite set satisfying $\mathcal{I} \cap \mathcal{P}(\mathbb{C}) = \emptyset$ and $\mathcal{W} \subseteq \mathfrak{P}_2(\mathcal{P}(\mathbb{C}) \cup \mathcal{I})$ such that

1. for any $w, w' \in \mathcal{W}$ such that $w \cap w' \neq \emptyset$, we have $w = w'$;
2. we have $\mathcal{P}^{\mathsf{aux}}(\mathbb{C}) \cup \mathcal{I} \subseteq \bigcup \mathcal{W}$;
3. for any $w \in \mathcal{W}$, there exists $p \in w$ such that $p \notin \mathcal{P}^{\mathsf{pri}}(\mathbb{C})$.

We set $\mathbb{C}(\Phi) = \mathbb{C}$, $\mathcal{I}(\Phi) = \mathcal{I}$, $\mathcal{W}(\Phi) = \mathcal{W}$ and $\mathcal{P}(\Phi) = \mathcal{P}(\mathbb{C}(\Phi)) \cup \mathcal{I}$. We use for ωPPLPS the notations introduced for PPLPS (see Notations 6).

With every ωPPLPS Φ, we associate a unique PPLPS $\omega(\Phi)$:

Definition 84. Let ω be the function ω**PPLPS** $\to$ **PPLPS** such that $\omega(\Phi) = \Phi'$ is defined as follows:

- $\mathbb{C}(\Phi') = \mathbb{C}(\Phi)$;
- $\mathcal{I}(\Phi') = \mathcal{I}(\Phi) \setminus \{p \in \mathcal{I}(\Phi) \mid (\exists q \in \mathcal{P}^{\mathsf{pri}}(\mathbb{C}(\Phi)))\ \{p, q\} \in \mathcal{W}(\Phi)\}$;
- $\mathcal{W}(\Phi') = \{w \in \mathcal{W}(\Phi) \mid w \subseteq \mathcal{P}(\Phi')\}$.

We give here the formal definition of *the PLPS Ψ obtained from Φ by removing* $\mathcal{C}_0$, where $\mathcal{C}_0 \subseteq \mathcal{C}^{\mathsf{t}}(\Phi)$ is such that ($\mathcal{C}_0 = \{l\}$ and $l \in \mathcal{C}^{\mathsf{m}}(\Phi) \cup \mathcal{C}^{?\mathsf{d}}(\Phi)$) or $\mathcal{C}_0 \subseteq \mathcal{C}^{!}(\Phi)$:

Definition 85. Let $\Phi \in$ **PLPS** and let $\mathcal{C}_0 \subseteq \mathcal{C}^{\mathsf{t}}(\Phi)$ such that ($\mathcal{C}_0 = \{l\}$ and $l \in \mathcal{C}^{\mathsf{m}}(\Phi) \cup \mathcal{C}^{?\mathsf{d}}(\Phi)$) or $\mathcal{C}_0 \subseteq \mathcal{C}^{!}(\Phi)$. *The PLPS Ψ obtained from Φ by removing* $\mathcal{C}_0$ is $\omega(\Phi')$, where Φ' is the ωPPLPS defined as follows[45]:

- $\mathcal{C}(\Phi') = \mathcal{C}(\Phi) \setminus \mathcal{C}_0$;
- $\mathcal{P}(\Phi') = \mathcal{P}(\Phi) \setminus \bigcup_{l \in \mathcal{C}_0}\{p \in \mathcal{P}(\Phi) \mid \mathsf{C}_\Phi(p) = l\}$;
- $\mathsf{t}_{\Phi'} = \mathsf{t}_{\Phi}{}_{|\mathcal{C}(\Phi')}$ and $\mathsf{C}_{\Phi'} = \mathsf{C}_\Phi\big|_{\mathcal{P}(\Phi')}^{\mathcal{C}(\Phi')}$;
- $\mathsf{P}^{\mathsf{pri}}_{\Phi'} = \mathsf{P}^{\mathsf{pri}}_{\Phi}\big|_{\mathcal{C}(\Phi')}^{\mathcal{P}(\Phi')}$ (resp. $\mathsf{P}^{\mathsf{left}}_{\Phi'} = \mathsf{P}^{\mathsf{left}}_{\Phi}\big|_{\mathcal{C}(\Phi')}^{\mathcal{P}(\Phi')}$);
- $\#_{\Phi'} = \#_\Phi$;
- $\mathcal{I}(\Phi') = \mathcal{I}(\Phi) \cup \bigcup_{l \in \mathcal{C}_0} \mathsf{P}^{\mathsf{aux}}_\Phi(l)$;
- $\mathcal{W}(\Phi') = \{w \in \mathcal{W}(\Phi) \mid w \subseteq (\mathcal{P}(\Phi') \cup \mathcal{I}(\Phi'))\}$.

A.2. Proof-Structures (PS)

In the same way that we introduced indexed PPLPS, indexed PLPS, indexed LPS and indexed PS, we introduce the notion of indexed ωPPLPS. Now, to every $(\Phi, \mathsf{ind}) \in \omega\mathbf{PPLPS}_{\mathsf{in}}$, we associate the indexed PPLPS $\omega(\Phi) = (\omega(\Phi), \mathsf{ind}_1)$ defined as follows: for $p \in \mathcal{P}^{\mathsf{f}}(\omega(\Phi))$ we set $\mathsf{ind}_1(p) = \mathsf{ind}(\mathsf{c}_\Phi(p))$.

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[44] ω is reminiscent of the definition of *ω-reduction* in [23].

[45] Concretely, Φ' is obtained from Φ by erasing the cells of $\mathcal{C}_0$ and their principal ports, and by "changing the status" of the auxiliary ports of the cells of $\mathcal{C}_0$, which become elements of $\mathcal{I}(\Phi')$.

B. Experiments

Definition 86. We call *depth of an element* $\alpha \in D$ the least number $n \in \mathbb{N}$ such that $\alpha \in D_n$.[46]

Let $+^\perp = -$ and $-^\perp = +$. We define $\alpha^\perp$ for any $\alpha \in D$, by induction on the depth of α:

- for $\gamma \in A$, $\gamma^\perp = \gamma$; and $(\iota, *)^\perp = (\iota^\perp, *)$;
- else, $(\iota, \alpha, \beta)^\perp = (\iota^\perp, \alpha^\perp, \beta^\perp)$ and $(\iota, [\alpha_1, \ldots, \alpha_n])^\perp = (\iota^\perp, [\alpha_1^\perp, \ldots, \alpha_n^\perp])$.

Definition 87. For any $\alpha \in D$, we define, by induction on the depth of α, $Sub(\alpha) \in \mathfrak{M}_{\text{fin}}(D)$ as follows:

- $Sub(\delta) = [\delta]$ if $\delta \in A \cup (\{+, -\} \times \{*\})$;
- $Sub(\iota, \alpha, \beta) = [(\iota, \alpha, \beta)] + Sub(\alpha) + Sub(\beta)$;
- $Sub(\iota, [\alpha_1, \ldots, \alpha_m]) = [(\iota, [\alpha_1, \ldots, \alpha_m])] + \sum_{j=1}^m Sub(\alpha_j)$.

For any $(\alpha_1, \ldots, \alpha_n) \in D^{<\omega}$, we set $Sub(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n Sub(\alpha_i)$.

For any $\beta \in D$, for any $r \in D^{<\omega}$, we say that *β occurs in r* if $\beta \in Supp(Sub(r))$.

For any $\gamma \in A$, for any $r \in D^{<\omega}$, for any $m \in \mathbb{N}$, we say that *there are exactly m occurrences of γ in r* if $Sub(r)(\gamma) = m$.

The following precise definition of substitution clearly entails that for every $\alpha \in D$ and for every substitution $\sigma : D \to D$, one has $\sigma(\alpha^\perp) = \sigma(\alpha)^\perp$:

Definition 88. A *substitution* is a function $\sigma : D \to D$ induced by a function $\sigma^A : A \to D$ and defined by induction on the depth of elements of D, as follows (as usual $\iota \in \{+, -\}$ and $\gamma \in A$):

- $\sigma(\gamma) = \sigma^A(\gamma)$ and $\sigma(\iota, *) = (\iota, *)$;
- $\sigma(\iota, \alpha, \beta) := (\iota, \sigma(\alpha), \sigma(\beta))$
- $\sigma(\iota, [\alpha_1, \ldots, \alpha_n]) = (\iota, [\sigma(\alpha_1), \ldots, \sigma(\alpha_n)])$.

C. Main result

We give the formal definition of $\overline{\Phi}$ for $\Phi \in$ **?-box-PLPS** $\cap$ **LPS**:

Definition 89. With $\Phi \in$ **?-box-PLPS** $\cap$ **LPS** one can associate the PLPS Φ_{-1} obtained from Φ by modifying the function # (all the rest is unchanged): $\mathcal{C}^?(\Phi_{-1}) \cap \mathcal{C}^t(\Phi_{-1}) = \mathcal{C}^{?c_{\text{auxd}}}(\Phi) \cap \mathcal{C}^t(\Phi)$ and for every cell $l \in \mathcal{C}^{?c_{\text{auxd}}}(\Phi) \cap \mathcal{C}^t(\Phi)$, the auxiliary ports of l in Φ are exactly those of l in Φ_{-1}; we can thus set $\#_{\Phi_{-1}}(p) = \#_\Phi(p) - 1$ for such an auxiliary port p.[47] For every $l \in \mathcal{C}^?(\Phi_{-1}) \setminus (\mathcal{C}^?(\Phi_{-1}) \cap \mathcal{C}^t(\Phi_{-1}))$ and for every auxiliary port p of l, we set $\#_{\Phi_{-1}}(p) = \#_\Phi(p)$.

The PLPS $\overline{\Phi}$ is then obtained from Φ_{-1} by removing $\mathcal{C}^!(\Phi_{-1}) \cap \mathcal{C}^t(\Phi_{-1})$.[48]

D. Proof of Proposition 40

D.1. The case of ax-PLPS

We give here the formal definition of (Φ_1, ind_1) and $(\Phi'_1, \text{ind'}_1)$ of the proof of Proposition 40 (case $\Phi \in$ ax-**PLPS**).

We set $m_0 = \min\{i_0, j_0\}$ and $M_0 = \max\{i_0, j_0\}$. We define $(\Phi_1, \text{ind}_1) \in \textbf{PLPS}_{\text{ind}}$ and $(\Phi'_1, \text{ind'}_1) \in \textbf{PLPS}_{\text{ind}}$ as follows:

- $\mathbb{C}(\Phi_1) = \mathbb{C}(\Phi)$ and $\mathbb{C}(\Phi'_1) = \mathbb{C}(\Phi')$;
- $\mathcal{I}(\Phi_1) = \mathcal{I}(\Phi) \setminus \{p_0, q_0\}$ and $\mathcal{I}(\Phi'_1) = \mathcal{I}(\Phi') \setminus \{p'_0, q'_0\}$;
- $\mathcal{W}(\Phi_1) = \mathcal{W}(\Phi) \setminus \{\{p_0, q_0\}\}$ and $\mathcal{W}(\Phi'_1) = \mathcal{W}(\Phi') \setminus \{\{p'_0, q'_0\}\}$;
- we define the value of $\text{ind}_1(p)$ as follows:

$$\begin{cases} \text{ind}(p) & \text{if } \text{ind}(p) < m_0; \\ \text{ind}(p) - 1 & \text{if } m_0 < \text{ind}(p) < M_0; \\ \text{ind}(p) - 2 & \text{if } M_0 < \text{ind}(p); \end{cases}$$

and the value of $\text{ind'}_1(p)$ as follows:

$$\begin{cases} \text{ind'}(p) & \text{if } \text{ind'}(p) < m_0; \\ \text{ind'}(p) - 1 & \text{if } m_0 < \text{ind'}(p) < M_0; \\ \text{ind'}(p) - 2 & \text{if } M_0 < \text{ind'}(p). \end{cases}$$

D.2. The case of $?_{c_b}$-PLPS

We give here the definition of $(\Phi_1, \text{ind}_1), (\Phi'_1, \text{ind'}_1) \in \textbf{PLPS}_{\text{ind}}$ of the proof of Proposition 40 (case: $\Phi \in ?_{c_b}$-**PLPS**): $(\Phi_1, \text{ind}_1) = \omega(\Psi_1, \text{ind}_2)$ and $(\Phi'_1, \text{ind'}_1) = \omega(\Psi'_1, \text{ind'}_2)$, where $(\Psi_1, \text{ind}_2), (\Psi'_1, \text{ind'}_2) \in \omega\textbf{PPLPS}_{\text{in}}$ are defined as follows:

46 The definition of D_n has been given in Definition 21.

47 We use here the crucial hypothesis that $l \in \mathcal{C}^{?c_{\text{auxd}}}(\Phi)$ which means that $\#_\Phi(p) > 0$.

48 Following Definition 85.

- $\mathcal{C}(\Psi_1) = \mathcal{C}(\Phi)$ and $\mathcal{C}(\Psi'_1) = \mathcal{C}(\Phi')$;
- $\mathsf{t}_{\Psi_1} = \mathsf{t}_\Phi$ and $\mathsf{t}_{\Psi'_1} = \mathsf{t}_{\Phi'}$;
- $\mathcal{P}(\Psi_1) = \mathcal{P}(\Phi) \setminus \{p\}$ and $\mathcal{P}(\Psi'_1) = \mathcal{P}(\Phi') \setminus \{p'\}$;
- $\mathsf{C}_{\Psi_1} = \mathsf{C}_\Phi\,|_{\mathcal{P}(\Psi_1)}$ and $\mathsf{C}_{\Psi'_1} = \mathsf{C}_{\Phi'}\,|_{\mathcal{P}(\Psi'_1)}$;
- $\mathsf{P}^{\mathrm{pri}}_{\Psi_1} = \mathsf{P}^{\mathrm{pri}}_\Phi$ and $\mathsf{P}^{\mathrm{pri}}_{\Psi'_1} = \mathsf{P}^{\mathrm{pri}}_{\Phi'}$; $\mathsf{P}^{\mathrm{left}}_{\Psi_1} = \mathsf{P}^{\mathrm{left}}_\Phi$ and $\mathsf{P}^{\mathrm{left}}_{\Psi'_1} = \mathsf{P}^{\mathrm{left}}_{\Phi'}$;
- $\#_{\Psi_1} = \#_\Phi\,|_{\mathrm{dom}(\#_\Phi)\setminus\{p\}}$ and $\#_{\Psi'_1} = \#_{\Phi'}\,|_{\mathrm{dom}(\#_{\Phi'})\setminus\{p'\}}$;
- $\mathcal{I}(\Psi_1) = \mathcal{I}(\Phi) \cup \{p\}$ and $\mathcal{I}(\Psi'_1) = \mathcal{I}(\Phi') \cup \{p'\}$;
- $\mathcal{W}(\Psi_1) = \mathcal{W}(\Phi)$ and $\mathcal{W}(\Psi'_1) = \mathcal{W}(\Phi')$;
- $\mathsf{ind}_2(q) = \begin{cases} \mathsf{ind}(q) & \text{if } q \neq p; \\ Card(\mathcal{P}^{\mathrm{f}}(\Phi)) + 1 & \text{if } q = p; \end{cases}$

  and $\mathsf{ind'}_2(q) = \begin{cases} \mathsf{ind'}(q) & \text{if } q \neq p'; \\ Card(\mathcal{P}^{\mathrm{f}}(\Phi')) + 1 & \text{if } q = p'. \end{cases}$

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