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Electrohydrodynamic Kelvin–Helmholtz instability for a velocity stratified fluid

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Abstract

The electrohydrodynamic stability of a conducting incompressible stratified fluid topping a dielectric fluid layer is studied. The stability of the system is discussed theoretically and numerically. It is found that the normal electric field has a destabilising effect while the increase of the thickness of the layer has a stabilising influence. The special case of the Rayleigh–Taylor instability is also examined.

Keywords: Linear hydrodynamic stability; Electrohydrodynamics; Kelvin–Helmholtz and Rayleigh–Taylor instabilities; Dielectric and conducting fluids; Stratified fluids; Confluent hypergeometric function; Sturm's functions

1. Introduction

The stability of an interface separating two streaming fluids has received considerable attention of investigators due to its physical applications. A review of classical Kelvin–Helmholtz theory is provided in [5]. The theory is limited in that in most physical situations the two fluid components are not moving with constant velocities [1, 9, 29]. Thus one has to consider flows possessing velocity stratification and the results of the theory of stability of parallel flow [3, 6, 27] are brought into action due to the velocity stratifications. Recent works on the stability of superposed fluids that are initially streaming with variable velocities [15, 20–25] show different results than those of the classical Kelvin–Helmholtz instability.

On the other hand, increasing interest in the electrohydrodynamic stability is due to the important role played by electric fields in biophysics [32], chemical engineering [13, 14], and other domains of interest [4, 8, 10, 11, 23, 28, 30]. The presence of an electric field produces electric stresses on the interface separating two dielectric fluids. In the linear stability theory, the tangential field has a stabilising effect [13], while the normal field has a destabilising influence [13, 16, 19].

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However, recent studies of the nonlinear electrohydrodynamic stability showed that both the normal field and the tangential field play roles in stability criterion [17, 18].

The aim of this work is to study the effect of a normal electric field on a dielectric fluid layer topped by a streaming conducting fluid with sinusoidal boundary wave profile. The linear stability of the interface will be examined.

2. Formulation of the problem

The basic flow is assumed to be the steady flow of two inviscid incompressible fluids in a gravitational field. A rectangular coordinate system is used, the coordinate axes x–y as shown in Fig. 1, with origin at the interface.

The upper fluid is of density $\rho_2(y)$ and extends to infinity and it is assumed to be a conducting inviscid incompressible fluid whose density and velocity in the stationary state are given by

$$\rho_0^{(2)} = \rho_2(y) = \rho_0 e^{-\beta y}, \qquad u_0^{(2)} = U_2(y) = Ay \quad \text{and} \quad v_0^{(2)} = 0,$$

where u and v are the velocity components in the directions of increasing x and y and ρ_0, β and A are positive constants.

The lower fluid is of density ρ_1 and depth L, and it is assumed to be a dielectric inviscid incompressible fluid at rest. Thus

$$\rho_0^{(1)} = \rho_1 = \text{constant}, \qquad u_0^{(1)} = U_1 = 0 \quad \text{and} \quad v_0^{(1)} = 0.$$

The lower fluid is bounded from below by a rigid conducting plane having potential V_0 at $y = -L$.

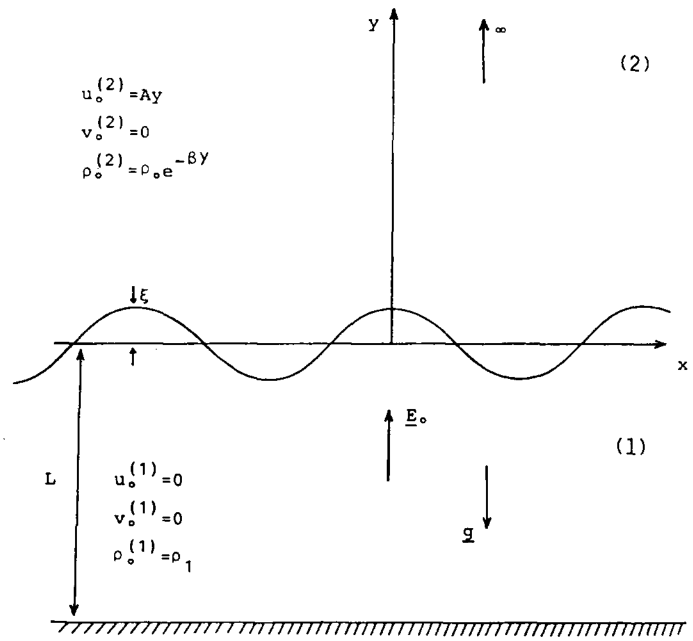

Fig. 1. Definition sketch of the problem.

In the stationary state, the interface is given by $y = 0$, and the lower fluid only is subjected to a constant electric field $\boldsymbol{E}_0$ and directed perpendicular to the interface, i.e. $\boldsymbol{E}_0 = E_0\boldsymbol{j}$, where $E_0 = V_0/L$. Also the gravitational force is taken into account in the two regions, and there exist surface charges on the interface. The equations governing the motion are the usual equations of fluid mechanics along with the equations governing the electric field.

The equations of motion are:

$$\rho^{(l)}\left[\frac{\partial u^{(l)}}{\partial t} + u^{(l)}\frac{\partial u^{(l)}}{\partial x} + v^{(l)}\frac{\partial u^{(l)}}{\partial y}\right] = -\frac{\partial \Pi^{(l)}}{\partial x}, \tag{2.1}$$

$$\rho^{(l)}\left[\frac{\partial v^{(l)}}{\partial t} + u^{(l)}\frac{\partial v^{(l)}}{\partial x} + v^{(l)}\frac{\partial v^{(l)}}{\partial y}\right] = -\frac{\partial \Pi^{(l)}}{\partial y} - \rho^{(l)}g, \tag{2.2}$$

where the superscripts $l = 1, 2$ refer to the lower and upper fluid, respectively, and the total pressure is defined by [12]

$$\Pi = p - \frac{1}{2}E^2\left.\frac{\partial \varepsilon}{\partial \rho}\right|_{\tau}\rho, \tag{2.3}$$

where t is the time, p is pressure and the subscript τ means that the derivation is evaluated at constant temperature.

The equation of continuity is

$$\frac{\partial u^{(l)}}{\partial x} + \frac{\partial v^{(l)}}{\partial y} = 0. \tag{2.4}$$

Both the upper and lower fluids are assumed to be incompressible, so that

$$\frac{\partial \rho^{(l)}}{\partial t} + u^{(l)}\frac{\partial \rho^{(l)}}{\partial x} + v^{(l)}\frac{\partial \rho^{(l)}}{\partial y} = 0. \tag{2.5}$$

It is assumed that the quasi-static approximation is valid for the problem on hand, and therefore Maxwell's equations reduce to

$$\nabla \times \boldsymbol{E} = 0, \qquad \nabla \cdot [\varepsilon \boldsymbol{E}] = 0, \tag{2.6}$$

where ε is the dielectric constant in the lower fluid.

3. Perturbation equations

As usual, the motion is resolved into primary motion and perturbation. Here the suffix 0 is used to denote values in the undisturbed flow, while the subscript 1 will refer to the perturbations in various quantities, we may assume:

$$u^{(l)} = u_0^{(l)} + u_1^{(l)}, \qquad v^{(l)} = v_0^{(l)} + v_1^{(l)}, \qquad \Pi^{(l)} = \Pi_0^{(l)} + \Pi_1^{(l)},$$

$$\rho^{(2)} = \rho_0^{(2)} + \rho_1^{(2)}, \qquad \boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1$$

in which $\Pi_0^{(l)}$ is the total pressure for the primary flow.

The linearized equation of continuity allows the use of a stream function $\Psi^{(l)}$, in terms of which $u_1^{(l)}$ and $v_1^{(l)}$ solving (2.4) can be expressed as follows:

$$u_1^{(l)} = \Psi_y^{(l)}, \qquad v_1^{(l)} = -\Psi_x^{(l)}, \tag{3.1}$$

with subscripts indicating partial differentiation.

We assume an exponential time factor for all perturbation quantities, such that

$$[\Psi^{(l)}, \Pi_1^{(l)}, \rho_1^{(2)}] = [\psi_l(y), f_l(y), h_2(y)]\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{3.2}$$

in which $\alpha = 2\pi/\lambda$ is the wave number, λ being the wavelength, and $c = c_r + \mathrm{i}c_i$ is the wave velocity. The stability or instability is then decided by the sign of c_i [19].

Following the usual procedure [5], from Eqs. (3.1), (3.2) and the linearized form of Eqs. (2.1), (2.2) and (2.5) we obtain the following two equations for the upper and lower fluid, respectively:

$$\frac{\mathrm{d}^2\psi_2(y)}{\mathrm{d}y^2} - \beta\frac{\mathrm{d}\psi_2(y)}{\mathrm{d}y} + \left[-\alpha^2 + \frac{A\beta}{(Ay-c)} + \frac{g\beta}{(Ay-c)^2}\right]\psi_2(y) = 0, \tag{3.3}$$

$$\frac{\mathrm{d}^2\psi_1(y)}{\mathrm{d}y^2} - \alpha^2\psi_1(y) = 0. \tag{3.4}$$

We can now imagine that the equilibrium interface separating the two fluids is perturbed and the surface of the deformed interface is given by

$$y = \xi = \delta\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)} \tag{3.5}$$

where δ is a smallness parameter having the dimensions of a length.

If $\boldsymbol{n}$ is the unit normal vector to the disturbed interface $y = \xi$, then to the first order terms, $\boldsymbol{n}$ is given by

$$\boldsymbol{n} = -\mathrm{i}\alpha\xi\boldsymbol{i} + \boldsymbol{j}. \tag{3.6}$$

From the linearized form of Eq. (2.6), $\boldsymbol{E}_1$ is an irrotational vector, and therefore there exists an electric potential ϕ_1 such that $\boldsymbol{E}_1 = -\nabla\phi_1$ and

$$\nabla^2\phi_1 = 0, \tag{3.7}$$

with the solution

$$\phi_1 = E_0\xi\frac{\sinh\alpha(y+L)}{\sinh(\alpha L)}. \tag{3.8}$$

Therefore, the total electric field is given by

$$\boldsymbol{E} = E_0\left\{-\mathrm{i}\alpha\xi\boldsymbol{i} + \left[1 - \alpha\xi\frac{\cosh\alpha(y+L)}{\sinh(\alpha L)}\right]\boldsymbol{j}\right\}, \tag{3.9}$$

where the potential ϕ_1 vanishes at the rigid boundary $y = -L$.

Using the interfacial conditions:

(i) The normal component of the velocity is continuous at the interface $y = \xi$, then it is required that

$$\psi_1(0) = \psi_2(0). \tag{3.10}$$

From the linearized kinematic conditions at the interface $y = \xi$,

$$v_1^{(l)} = \frac{\partial \xi}{\partial t} + U_l \frac{\partial \xi}{\partial x} \quad \text{at } y = 0, \tag{3.11}$$

it follows that

$$\xi = \frac{\psi_1(0)}{c} \mathrm{e}^{\mathrm{i}\alpha(x-ct)}. \tag{3.12}$$

(ii) The normal component of the stress tensor is discontinuous across the surface of separation $y = \xi$ by the surface tension T. Then, to the first-order terms, we obtain

$$c\rho_1\psi_1'(0) - \rho_0[c\psi_2'(0) + A\psi_2(0)] + [(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2 \coth(\alpha L)]\frac{\psi_1(0)}{c} = 0. \tag{3.13}$$

The vanishing of the normal component of the velocity $v_1^{(1)}$ at the rigid boundary $y = -L$ implies that

$$\psi_1(-L) = 0. \tag{3.14}$$

4. The dispersion equation

The solution of Eqs. (3.4) and (3.14) can be written in the form

$$\psi_1(y) = G \sinh \alpha(y + L), \tag{4.1}$$

where G is an arbitrary constant.

To obtain the solution of Eq. (3.3), we may use the transformation [5]

$$\psi_2(y) = W(y)\mathrm{e}^{(1/2)\beta y}. \tag{4.2}$$

Then Eq. (3.3) can be written in the form

$$\frac{\mathrm{d}^2 W(\zeta)}{\mathrm{d}\zeta^2} + \left(-\frac{1}{4} + \frac{k}{\zeta} + \frac{\frac{1}{4} - m^2}{\zeta^2}\right) W(\zeta) = 0, \tag{4.3}$$

where

$$k = \frac{\beta}{(4\alpha^2 + \beta^2)^{1/2}}, \qquad m = \left(\frac{1}{4} - Q\right)^{1/2}, \qquad \zeta = \left(y - \frac{c}{A}\right)(4\alpha^2 + \beta^2)^{1/2}$$

and the Richardson number Q is $Q = g\beta/A^2$.

We recognize in Eq. (4.3) Whittaker's standard form of the equation for the confluent hypergeometric function. The condition at infinity requires that the solution of Eq. (4.3) appropriate to the problem on hand be Whittaker's function.

$$W = H W_{k,m}(\zeta), \tag{4.4}$$

where H is an arbitrary constant. This solution has the asymptotic behaviour [21, 31]:

$$W_{k,m}(\zeta) \sim \zeta^k e^{-(1/2)\zeta}, \quad (\zeta \to \infty).$$

From Eqs. (4.2) and (4.4), we get

$$\psi_2(y) = H e^{(1/2)\beta y} W_{k,m}(\zeta). \tag{4.5}$$

Using Eqs. (4.1) and (4.5), then the boundary condition (3.10) gives

$$G \sinh(\alpha L) - H W_{k,m}\left[-\frac{c}{A}(4\alpha^2 + \beta^2)^{1/2}\right] = 0 \tag{4.6}$$

and the boundary condition (3.13) gives

$$\begin{aligned} &G\left\{c\rho_1 \alpha \cosh(\alpha L) + [(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2 \coth(\alpha L)]\frac{\sinh(\alpha L)}{c}\right\} \\ &\quad - H\left\{(A + \tfrac{1}{2}\beta c)\rho_0 W_{k,m}\left[-\frac{c}{A}(4\alpha^2 + \beta^2)^{1/2}\right]\right. \\ &\quad \left. + \rho_0 c(4\alpha^2 + \beta^2)^{1/2} W'_{k,m}\left[-\frac{c}{A}(4\alpha^2 + \beta^2)^{1/2}\right]\right\} = 0, \end{aligned} \tag{4.7}$$

with respect to the unknowns G and H, then the system of Eqs. (4.6) and (4.7) has a solution different from zero, if the determinant of coefficients is equal to zero, and it then follows that

$$\begin{aligned} &-\rho_0 c(4\alpha^2 + \beta^2)^{1/2} \sinh(\alpha L) W'_{k,m}\left[-\frac{c}{A}(4\alpha^2 + \beta^2)^{1/2}\right] \\ &\quad + \left\{c\rho_1 \alpha \cosh(\alpha L) + [(\rho_0 - \rho_1)g - \alpha^2 T - (A + \tfrac{1}{2}\beta c)\rho_0 c\right. \\ &\quad \left. + \alpha\varepsilon E_0^2 \coth(\alpha L)]\frac{\sinh(\alpha L)}{c}\right\} W_{k,m}\left[-\frac{c}{A}(4\alpha^2 + \beta^2)^{1/2}\right] = 0. \end{aligned} \tag{4.8}$$

Eq. (4.8) is the dispersion equation characterising c. Accordingly the stability or instability is determined through the solutions for c resulting from the dispersion equation. Unfortunately, the above relation is rather complicated implicit transcendental equation. The relation can be simplified considerably for large wave numbers. It is well known that the behaviour for the Whittaker's function for large arguments has the property [25]:

$$W'_{k,m}(Z) \sim -\tfrac{1}{2} W_{k,m}(Z), \quad |\arg Z| < \tfrac{3}{2}\pi \tag{4.9}$$

and therefore, the dispersion equation (4.8) reduces to the simplified form

$$\begin{aligned} &c^2[\tfrac{1}{2}\rho_0(4\alpha^2 + \beta^2)^{1/2} + \rho_1 \alpha \coth(\alpha L) - \tfrac{1}{2}\beta\rho_0] - cA\rho_0 \\ &\quad + [(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2 \coth(\alpha L)] = 0. \end{aligned} \tag{4.10}$$

Eq. (4.10) is quadratic in c. From the nature of the roots, we find that the principal of overstability is valid. The marginal state is given by

$$\lambda^* = [2\rho_0(4\alpha^2 + \beta^2)^{1/2} + 4\rho_1\alpha\coth(\alpha L) - 2\beta\rho_0][(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2\coth(\alpha L)] - A^2\rho_0^2 = 0. \tag{4.11}$$

The system is stable if $\lambda^* \leqslant 0$. The equality of the relation (4.11) can be expressed in the form

$$E_0^{*^2} = E_0^2 = \frac{\tanh(\alpha L)}{\alpha\varepsilon}\left\{\frac{A^2\rho_0^2}{[2\rho_0\sqrt{4\alpha^2+\beta^2} + 4\rho_1\alpha\coth(\alpha L) - 2\beta\rho_0]} - [(\rho_0 - \rho_1)g - \alpha^2 T]\right\}. \tag{4.12}$$

For values of the field such that $E_0 \leqslant E_0^*$, the system is stable. The values of the critical field E_0^* depends on the ratio of the densities, the wave number and the thickness of the layer L. The surface tension plays a stabilising role. If $\rho_0 > \rho_1$, then stability is possible if $E_0 < E_0^*$ up to a critical value of the difference $(\rho_0 - \rho_1)$. If the difference $(\rho_0 - \rho_1)$ is very large, then the term $-(\rho_0 - \rho_1)$ in Eq. (4.12) dominates and stability is not possible for any value of the electric field. Thus if

$$(\rho_0 - \rho_1) > \left[\frac{A^2\rho_0^2}{g\{2\rho_0\sqrt{4\alpha^2+\beta^2} + 4\rho_1\alpha\coth(\alpha L) - 2\beta\rho_0\}} + \frac{\alpha^2 T}{g}\right],$$

the system is unstable regardless of the value of E_0. If the lower fluid is denser, then $(\rho_0 - \rho_1)$ is negative and one can always find an electric potential such that $E_0 < E_0^*$ and stability is achieved.

5. Numerical discussion

From Eq. (4.12) we calculate the values of the electric field square E_0^2 corresponding to some given values of the wave number α from $\alpha = 10^{-2}$ to $\alpha = 3000$ for various constant values of the thickness L of the lower fluid. The other parameters are taken to be: $g = 9.8\,\mathrm{m/s^2}$, $\rho_0 = 1.2\,\mathrm{kg/m^3}$, $\rho_1 = 998.2\,\mathrm{kg/m^3}$, $T = 0.0728\,\mathrm{N/m}$, $\varepsilon = 78.54$, $A = 1.0$ and $\beta = 1.1 \times 10^{-6}$.

The relation (4.12) is drawn between the electric field square E_0^2 and the wave number α for given values of the thickness L of the lower fluid. The resulting curves in Figs. 2–4 represent the neutral curves or the marginal state separating the stable and unstable regions. For a given curve, we observe that the stable region is decreased by the increase of α till a critical point (α_c, E_{0c}^2) after which the stable region is increased by the increase of α, e.g. for $L \geqslant 0.03$; $\alpha_c \cong 366.14$ and $E_{0c}^2 \cong 0.6714958$, and as L decreases the values α_c and E_{0c}^2 also decrease. We observe from the figures that there are stable regions under the curves. Thus there are small values of E_0^2 for which instability is not possible: as E_0^2 increases an unstable region is reached; and the system can be brought into the unstable state for values of $E_0^2 > E_{0c}^2$. We also observe that the stable areas

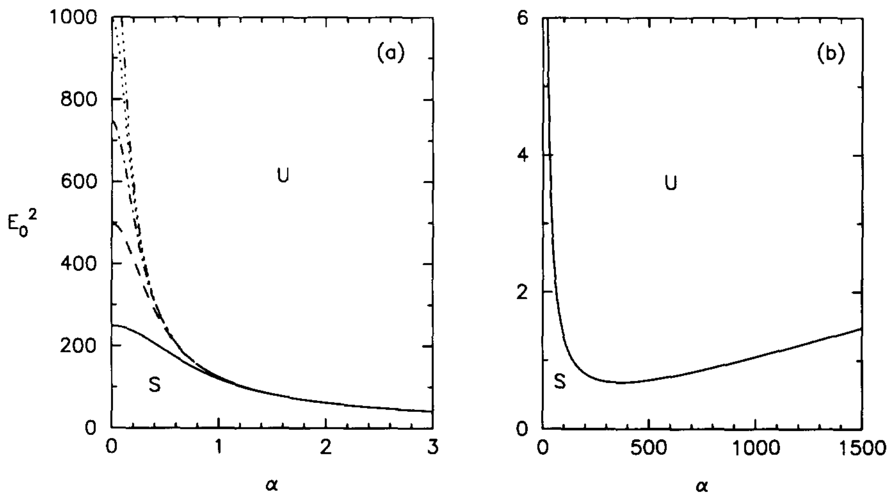

Fig. 2. Variation of E_0^2 with the wave number α for various values of the liquid depth L for a system having $g = 9.8\,\mathrm{m/s^2}$, $\rho_0 = 1.2\,\mathrm{kg/m^3}$, $\rho_1 = 998.2\,\mathrm{kg/m^3}$, $T = 0.0728\,\mathrm{N/m}$, $\varepsilon = 78.54$, $A = 1.0$ and $\beta = 1.1 \times 10^{-6}$. U and S denote unstable and stable regions above and under each curve, respectively, where (a) the solid, dashed, dotted-dash, dotted, 3 dotted-dash curves correspond to the values $L = 2, 4, 6, 8, 10$, respectively, (b) all the above curves coincide.

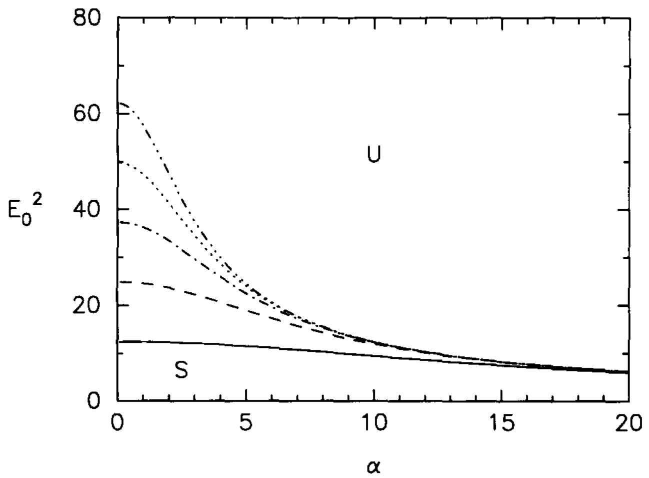

Fig. 3. For the system considered in Fig. 2, but with the solid, dashed, dotted-dash, dotted, 3 dotted-dash curves corresponding to the values $L = 0.1, 0.2, 0.3, 0.4, 0.5$, respectively, and by increasing α, we can get a curve similar to that in Fig. 2(b).

S increase with the increase of the thickness L. Thus the increase of the thickness yields a stabilising influence. The upper bounds of the electric field required for stability are increased by the increase of L for fixed α. The upper values of the electric field and the corresponding values of L are given in Table 1 for $\alpha = 10^{-2}$.

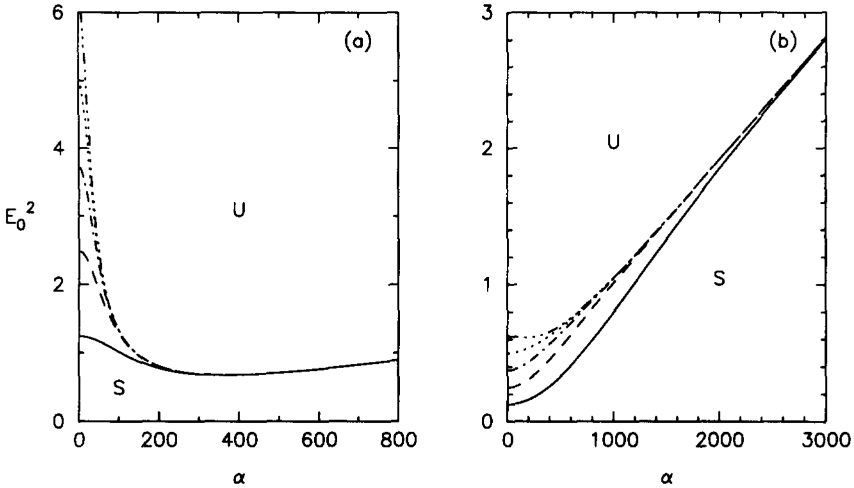

Fig. 4. For the system considered in Fig. 2, but with the solid, dashed, dotted-dash, dotted, 3 dotted-dash curves corresponding to the values $L = 0.01, 0.02, 0.03, 0.04, 0.05$, respectively, in (a), and the values $L = 0.001, 0.002, 0.003, 0.004, 0.005$, respectively, in (b), and it shows the destabilising influence of the electric field and the stabilising effect of the liquid depth.

Table 1
The table shows the upper values of a stable field E_0^2 corresponding to the thickness of the fluid layer L for a given $\alpha = 10^{-2}$

L	$E_0^2\|_{\alpha=10^{-2}}$	L	$E_0^2\|_{\alpha=10^{-2}}$
0.00001	0.0012440	0.05	6.2201420
0.0005	0.0622014	0.1	12.4402809
0.001	0.1244029	0.2	24.8805389
0.002	0.2488057	0.3	37.3207428
0.003	0.3732086	0.4	49.7608734
0.004	0.4976114	0.5	62.2009094
0.005	0.6220142	2.0	248.7725830
0.01	1.2440285	4.0	497.3462402
0.02	2.4880569	6.0	745.5229492
0.03	3.7320854	8.0	993.1054688
0.04	4.9761135	10.0	1239.8986328

The analysis becomes more obvious when we discuss the system in the L–α plane for fixed E_0^2. The curve is multibranched and the determination of the asymptote to the curve is necessary. The procedure is to solve the equality of (4.12) for $\coth(\alpha L)$ and hence for L. The asymptotes are obtained by letting $\coth(\alpha L) = 1$ which results in the sixth-order algebraic equation:

$$A_1\alpha^6 + A_2\alpha^5 + A_3\alpha^4 + A_4\alpha^3 + A_5\alpha^2 + A_6\alpha + A_7 = 0, \tag{5.1}$$

where

$$A_1 = 4T^2(\rho_0^2 - \rho_1^2),$$
$$A_2 = 4T\{\beta T\rho_0\rho_1 - 2\varepsilon E_0^2(\rho_0^2 - \rho_1^2)\},$$
$$A_3 = -8T\beta\rho_0\rho_1\varepsilon E_0^2 - 4(\rho_0^2 - \rho_1^2)\{2T(\rho_0 - \rho_1)g - \varepsilon^2 E_0^4\},$$
$$A_4 = 8g\varepsilon E_0^2(\rho_0 - \rho_1)^2(\rho_0 + \rho_1) - 2TA^2\rho_0^2\rho_1 - 4\beta\rho_0\rho_1\{2T(\rho_0 - \rho_1)g - \varepsilon^2 E_0^4\},$$
$$A_5 = 4g^2(\rho_0 - \rho_1)^3(\rho_0 + \rho_1) + TA^2\rho_0^3\beta + 2\rho_0\rho_1\varepsilon E_0^2\{A^2\rho_0 + 4\beta(\rho_0 - \rho_1)g\},$$
$$A_6 = 2\rho_0\rho_1 g(\rho_0 - \rho_1)\{A^2\rho_0 + 2\beta(\rho_0 - \rho_1)g\} - \beta A^2\rho_0^3\varepsilon E_0^2,$$
$$A_7 = -\tfrac{1}{4}A^2\rho_0^3\{A^2\rho_0 + 4\beta(\rho_0 - \rho_1)g\}.$$

The above equation admits two positive real roots, and Table 2 shows the computed values of the roots $\alpha = \alpha_{A_1}, \alpha_{A_2}$ (corresponding to the asymptotes) for a given value of E_0^2. The other parameters of the system are as tabulated before. Figs. 5 and 6 show the system in the L–α plane for different values of E_0^2, and from which, we observe that for a given value of E_0^2, the stable region S is reduced (or increased) by the increase of $\alpha < \alpha_{A_1}$ (or $\alpha > \alpha_{A_2}$). For $\alpha > \alpha_{A_1}$ and $\alpha < \alpha_{A_2}$ stability is not possible for this value of the electric field, and the value of α_{A_1} (or α_{A_2}) decreases (or increases) with the increase of E_0^2 allowing more unstable regions to appear as shown in Figs. 5 and 6. This emphasises the destabilising influence of the electric field.

We observe from the figures that there are unstable regions under the curves, and these regions are not the same for the same value of E_0^2, but it depends on whether $\alpha < \alpha_{A_1}$ or $\alpha > \alpha_{A_2}$, where the curves corresponding to a given value of E_0^2 with $\alpha > \alpha_{A_2}$ are too close to the α-axis more than the

Table 2
The table gives the positive real roots of Eq. (5.1) for some given values of E_0^2

E_0^2	α_{A_1}	α_{A_2}
1.0	143.4865	935.3597
2.0	64.0885	2093.5862
3.0	42.0298	3194.5251
4.0	31.3328	4284.0565
5.0	25.0065	5366.9495
10.0	12.4580	10776.0078
20.0	6.2244	21570.7016
30.0	4.1473	32371.7926
40.0	3.1103	43150.7385
50.0	2.4877	53934.6911
60.0	2.0725	64723.2651
70.0	1.7761	75512.7952
80.0	1.5539	86292.3223
90.0	1.3808	97068.8351
100.0	1.2427	107868.3414

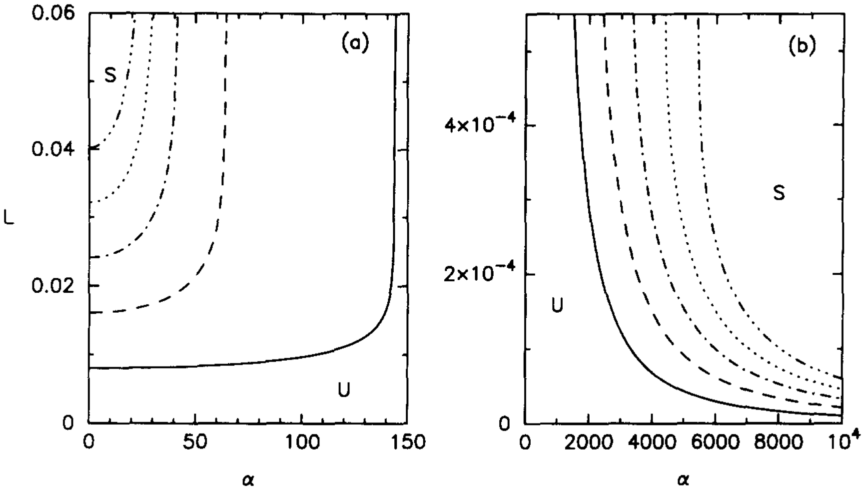

Fig. 5. Stability diagram in the (L–α) plane for various values of the electric field square E_0^2 for a system having $g = 9.8\,\mathrm{m/s^2}$, $\rho_0 = 1.2\,\mathrm{kg/m^3}$, $\rho_1 = 998.2\,\mathrm{kg/m^3}$, $T = 0.0728\,\mathrm{N/m}$, $\varepsilon = 78.54$, $A = 1.0$ and $\beta = 1.1 \times 10^{-6}$. The solid, dashed, dotted-dash, dotted, 3 dotted-dash curves correspond to the values $E_0^2 = 1, 2, 3, 4, 5$, respectively, where (a) represents the curves in the region $\alpha < \alpha_{A_1}$ (before the first asymptote to each curve), and (b) represents the same curves in the region $\alpha > \alpha_{A_2}$ (after the second asymptotes), where the values α_{A_1} and α_{A_2} are given in Table 2.

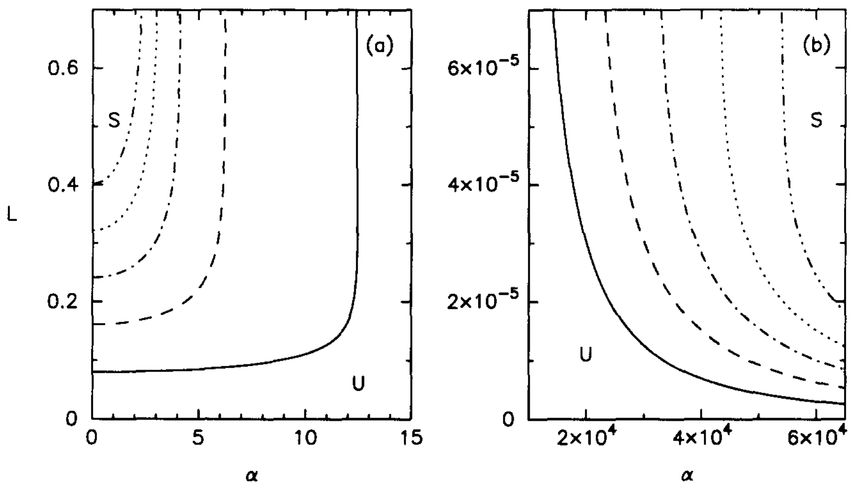

Fig. 6. Stability diagram for the same system considered in Fig. 5, but with the solid, dashed, dotted-dash, dotted, 3 dotted-dash curves corresponding to $E_0^2 = 10, 20, 30, 40, 50$, and figures (a) and (b) are drawn for the values of $\alpha < \alpha_{A_1}$ and $\alpha > \alpha_{A_2}$, respectively, and they show also that the electric field has a destabilising effect and the liquid depth has a stabilising effect.

same curves corresponding to the same value of E_0^2 with $\alpha < \alpha_{A_1}$. Thus there are small values of L for which stability is not possible, and as L increases a stable region is reached (for $\alpha > \alpha_{A_2}$ more quickly than for $\alpha < \alpha_{A_1}$ with the same value of E_0^2); and the system can be brought into a stable state for a given $\alpha < \alpha_{A_1}$ or $\alpha > \alpha_{A_2}$.

6. The Rayleigh–Taylor instability

It is interesting to examine the above system when the upper fluid is at rest (i.e. the Rayleigh–Taylor instability), and this can be done by letting $A \to 0$. Unfortunately, the transformation for ζ breaks down when $A \to 0$ and consequently the Whittaker equation as given by Eq. (4.3) is not appropriate for the problem. Thus the dispersion equation (4.8) and its consequences are not valid for the Rayleigh–Taylor instability.

However, the analysis can be restarted at a stage earlier to Eq. (4.2). If we let $A \to 0$ in Eq. (3.3), we get

$$\frac{d^2\psi_2(y)}{dy^2} - \beta\frac{d\psi_2(y)}{dy} - \alpha^2\left\{1 - \frac{g\beta}{(\alpha c)^2}\right\}\psi_2(y) = 0. \tag{6.1}$$

The general solution of Eq. (6.1) is [26]

$$\psi_2(y) = H_1 e^{m_+ y} + H_2 e^{m_- y}, \quad y > 0, \tag{6.2}$$

where H_1 and H_2 are arbitrary constants, and

$$m_\pm = \frac{\beta}{2} \pm \left[\left(\frac{\beta}{2}\right)^2 + \alpha^2\left\{1 - \frac{g\beta}{(\alpha c)^2}\right\}\right]^{1/2}.$$

Boundary conditions disallow disturbances which increase exponentially as the outer bound of the fluid is approached. Thus,

$$\psi_2(y) = H_2 e^{m_- y}, \quad y > 0 \tag{6.3}$$

with the requirement that

$$\Re\left[\left(\frac{\beta}{2}\right)^2 + \alpha^2\left\{1 - \frac{g\beta}{(\alpha c)^2}\right\}\right]^{1/2} \geqslant \frac{\beta}{2} \tag{6.4}$$

using Eqs. (4.1) and (6.3), then the boundary condition (3.10) gives

$$G\sinh(\alpha L) = H_2 \tag{6.5}$$

and the boundary condition (3.13) gives

$$G\left[\rho_1\alpha\cosh(\alpha L) + \{(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2\coth(\alpha L)\}\frac{\sinh(\alpha L)}{c}\right]$$
$$= H_2\rho_0 c m_-, \tag{6.6}$$

with respect to the unknowns G and H_2. The system of Eqs. (6.5) and (6.6) has a solution different from zero if the determinant of coefficients is equal to zero, and it then follows that

$$\lambda_1 c^5 - \lambda_2 c^3 - \lambda_3 c^2 + \lambda_4 c + \lambda_5 = 0, \tag{6.7}$$

where

$$\lambda_1 = \{\rho_0\beta - 2\rho_1\alpha\coth(\alpha L)\}^2,$$

$$\lambda_2 = 4\{\rho_0\beta - 2\rho_1\alpha\coth(\alpha L)\}\{(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2\coth(\alpha L)\},$$

$$\lambda_3 = \rho_0^2(4\alpha^2 + \beta^2),$$

$$\lambda_4 = 4\{(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2\coth(\alpha L)\}^2,$$

$$\lambda_5 = 4\rho_0^2 g\beta.$$

The stability of the system depends on the properties of the roots of Eq. (6.7). For stability it is necessary and sufficient that all the roots of Eq. (6.7) should be real and distinct [5]. In order to examine the nature of the roots we construct the Sturm functions [2] $f(c), f_1(c), f_2(c), f_3(c), f_4(c)$ and $f_5(c)$ (see the Appendix).

The roots of an equation of order n are real and distinct, if and only if two conditions are satisfied:

(i) The number of Sturm's functions must be $(n + 1)$.

(ii) The leading coefficients of all these functions must be positive.

Now, for the Eq. (6.7), the first condition (i) is satisfied, and the second condition (ii) leads to a set of inequalities. The details of these inequalities are very lengthy [7] and will not be included here (and they are available from the author on request). The first inequality is trivially satisfied, and the second one gives

$$\{\rho_0\beta - 2\rho_1\alpha\coth(\alpha L)\}\{(\rho_0 - \rho_1)g - \alpha^2 T + \alpha\varepsilon E_0^2\coth(\alpha L)\} > 0. \tag{6.8}$$

Either

$$E_0^2 < \frac{\tanh(\alpha L)}{\alpha\varepsilon}\{(\rho_1 - \rho_0)g + \alpha^2 T\} \tag{6.9}$$

and

$$\beta < 2(\rho_1\alpha/\rho_0)\coth(\alpha L) \tag{6.10}$$

or

$$E_0^2 < \frac{\tanh(\alpha L)}{\alpha\varepsilon}\{(\rho_1 - \rho_0)g + \alpha^2 T\} \tag{6.11}$$

and

$$\beta > 2(\rho_1\alpha/\rho_0)\coth(\alpha L). \tag{6.12}$$

The latter is inconsistant with condition (6.4), and therefore the stability is only governed by Eq. (6.9) or Eq. (6.10). For the classical Rayleigh–Taylor instability where the upper density is constant, i.e. $\beta = 0$, the stability is governed by Eq. (6.9) only.

7. Summary and conclusions

From the preceding, it is quite clear that we have studied the effect of a normal electric field on a dielectric fluid layer topped by a streaming conducting fluid with sinusoidal boundary wave profile. The linear stability of the interface between the two fluids is examined. In Section 2, we formulated the problem and wrote down the equations of motion. In Section 3, we put these equations in the perturbed form and by introducing the stream function Ψ, we obtained two differential equations (one of them is Whittaker's standard form of the equation for the confluent hypergeometric function) for the upper and lower fluids. We then expressed the form of the normal electric field $\boldsymbol{E}$ and wrote down the perturbed form of the relevant boundary conditions to our model. In Section 4, we derived the dispersion equation in a simple form by using the properties of the Whittaker function, and from the nature of the roots of this equation, we studied the stability of the system theoretically and numerically. The results obtained from the present study can be summarized as follows:

(i) The increase of the normal electric field has a destabilising effect, while the increase of the thickness L of the lower layer yields a stabilising influence. There are small values of L and E_0^2 for which instability is not possible. The value of the critical field E_0^* (above which the system is unstable) depends on the ratio of the densities, the wave number, and the thickness of the lower layer.

(ii) In the α–E_0^2 plane, the stable region (for a given curve) is decreased by the increase of α till a critical point (α_c, E_{0c}^2) after which the stable region is increased by the increase of α.

(iii) For a given value of E_0^2 in the α–L plane, the values $\alpha_{A_1}, \alpha_{A_2}$ correspond to the asymptotes, and the stable region is reduced or increased by the increase of $\alpha < \alpha_{A_1}$ or $\alpha > \alpha_{A_2}$. Outside these regions stability is not possible for this value of the electric field, and the unstable regions under the curves are not the same for the same value of E_0^2; that depends on whether $\alpha < \alpha_{A_1}$ or $\alpha > \alpha_{A_2}$.

Finally, the case of Rayleigh–Taylor instability is also investigated by using the Sturm functions to examine the nature of the roots of the resulting equation and to obtain the conditions for stability in the case of stratified fluids with the influence of a normal electric field.

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Appendix

The Sturmian functions for Eq. (6.7) are:

$$f(c) = \lambda_1 c^5 - \lambda_2 c^3 - \lambda_3 c^2 + \lambda_4 c + \lambda_5, \tag{A.1}$$

$$f_1(c) = 5\lambda_1 c^4 - 3\lambda_2 c^2 - 2\lambda_3 c + \lambda_4, \tag{A.2}$$

$$f_2(c) = \tfrac{2}{5}\lambda_2 c^3 + \tfrac{3}{5}\lambda_3 c^2 - \tfrac{4}{5}\lambda_4 c - \lambda_5, \tag{A.3}$$

$$\begin{aligned} f_3(c) = {} & \frac{1}{4\lambda_2^2}\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\}c^2 + \frac{1}{2\lambda_2^2}\{4\lambda_2^2\lambda_3 + 30\lambda_1\lambda_3\lambda_4 - 25\lambda_1\lambda_2\lambda_5\}c \\ & + \frac{1}{4\lambda_2^2}\{75\lambda_1\lambda_2\lambda_5 - 4\lambda_2^2\lambda_4\}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} f_4(c) = {} & [\{\{48\lambda_2^4 - 180\lambda_1\lambda_2\lambda_3^2 - 160\lambda_1\lambda_2^2\lambda_4 + 150\lambda_1\lambda_2^2\lambda_5 - 8\lambda_2^3\lambda_4\}\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\} \\ & - \{20\lambda_2^3\lambda_3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\}\{8\lambda_2^2\lambda_3 + 60\lambda_1\lambda_3\lambda_4 - 50\lambda_1\lambda_2\lambda_5\}\}c \\ & + \{5\lambda_5\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\}^2 - \{20\lambda_2^3\lambda_3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\} \\ & \times\{75\lambda_1\lambda_2\lambda_5 - 4\lambda_2^2\lambda_4\}\}]/5\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\}^2, \end{aligned} \tag{A.5}$$

$$\begin{aligned} f_5(c) = {} & [\{\{8\lambda_2^2\lambda_3 + 60\lambda_1\lambda_3\lambda_4 - 50\lambda_1\lambda_2\lambda_5\}(\{48\lambda_2^4 - 180\lambda_1\lambda_2\lambda_3^2 - 160\lambda_1\lambda_2^2\lambda_4 + 150\lambda_1\lambda_2^2\lambda_5 \\ & - 8\lambda_2^3\lambda_4\}\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\} - \{20\lambda_2^3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\} \\ & \times\{8\lambda_2^2\lambda_3 + 60\lambda_1\lambda_3\lambda_4 - 50\lambda_1\lambda_2\lambda_5\})(5\lambda_5\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\}^2 \\ & - \{20\lambda_2^3\lambda_3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\}\{75\lambda_1\lambda_2\lambda_5 - 4\lambda_2^2\lambda_4\}) \\ & - \{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\}(5\lambda_5\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\}^2 \\ & - \{20\lambda_2^3\lambda_3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\}\{75\lambda_1\lambda_2\lambda_5 - 4\lambda_2^2\lambda_4\})^2\} \\ & - \{\{75\lambda_1\lambda_2\lambda_5 - 4\lambda_2^2\lambda_4\}(\{48\lambda_2^4 - 180\lambda_1\lambda_2\lambda_3^2 - 160\lambda_1\lambda_2^2\lambda_4 + 150\lambda_1\lambda_2^2\lambda_5 - 8\lambda_2^3\lambda_4\} \\ & \times\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\} - \{20\lambda_2^3\lambda_3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\} \\ & \times\{8\lambda_2^2\lambda_3 + 60\lambda_1\lambda_3\lambda_4 - 50\lambda_1\lambda_2\lambda_5\})^2\}]/4\lambda_2^2[\{48\lambda_2^4 - 180\lambda_1\lambda_2\lambda_3^2 - 160\lambda_1\lambda_2^2\lambda_4 \\ & + 150\lambda_1\lambda_2^2\lambda_5 - 8\lambda_2^3\lambda_4\}\{12\lambda_2^3 - 45\lambda_1\lambda_3^2 - 40\lambda_1\lambda_2\lambda_4\} - \{20\lambda_2^3\lambda_3 - 240\lambda_1\lambda_2\lambda_3\lambda_4 \\ & - 135\lambda_1\lambda_3^3 + 100\lambda_1\lambda_2^2\lambda_5\}\{8\lambda_2^2\lambda_3 + 60\lambda_1\lambda_3\lambda_4 - 50\lambda_1\lambda_2\lambda_5\}]^2. \end{aligned} \tag{A.6}$$

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