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Supersymmetry does not imply mass degeneracy

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Abstract

It is commonly believed that unbroken supersymmetry (SUSY) implies that all members of a supermultiplet have the same mass. We demonstrate that this is not true, by exhibiting a simple counterexample. We employ the formalism of homeotic fermions, in a simple model where *CPT* conjugate fermions have different masses. This model can be supersymmetrized to a hypermultiplet of fields which form a representation of the conventional $N = 2$ SUSY algebra. Nevertheless, *CPT* conjugate states in this hypermultiplet have different masses. These surprising results do not violate either the *CPT* theorem or the Haag–Lopuszański–Sohnius theorem.

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1. Introduction

Supersymmetries are the only possible extensions of the four-dimensional Poincaré invariance observed ubiquitously in particle interactions [1,2]. Supersymmetry (SUSY) plays a fundamental role in string theory, and there are many strong phenomenological motivations for believing that supersymmetry is realized in nature, in spontaneously broken form. On the other hand, no superpartner particles have yet been observed, and spontaneously broken supersymmetry makes a prediction for the cosmological vacuum energy density which is too large by at least 60 orders of magnitude. Thus it is important to push the boundaries of our fundamental understanding of supersymmetry, and more especially to look for novel ways of expressing supersymmetry in physical systems.

It is widely believed that unbroken supersymmetry implies that all members of a supermultiplet have the same mass. For example, Sohnius' authoritative review article states explicitly that "supersymmetry must be broken in nature where elementary particles do not come in mass-degenerate multiplets" [3]. This belief is based upon the strong constraint that any conserved supercharge Q must commute with the 4-momentum operator P_μ, which in turn implies the O'Raifeartaigh theorem:

$$[Q, P_\mu P^\mu] = 0. \tag{1}$$

In spite of these facts, we demonstrate in this Letter that exact supersymmetry does *not* always imply mass-degenerate multiplets. We employ the formalism of homeotic fermions [4], developed previously by us as a toy model for *CPT* violation in the neutrino sector

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[5–9]. Our starting point is an extremely simple model where *CPT* conjugate fermions have different masses. This model can be supersymmetrized to a hypermultiplet of fields which form a representation of the conventional $N=2$ SUSY algebra. Nevertheless, *CPT* conjugate states in this hypermultiplet have different masses. Each fermion state is still mass-degenerate with a boson state, however, the hypermultiplet is *not* reducible to a pair of mass-degenerate supermultiplets. These surprising results do not violate either the *CPT* theorem [10] or the Haag–Lopuszański–Sohnius theorem [1].

2. Homeotic fermions

Homeotic fermions, like Dirac fermions, are denoted by 4-component complex spinor fields $\psi(x)$. In the free homeotic theory the Fourier-transformed fermion fields obey the equation of motion:

$$\big(\not p - m\epsilon(p_0)\big)\psi(p) = 0, \tag{2}$$

which differs from the Dirac equation by the presence of $\epsilon(p_0)$, the sign function of delta calculus. Solutions of (2) are solutions of the Klein–Gordon equation; the homeotic theory can be regarded as “the other square root” of the Klein–Gordon equation. The homeotic case is usually neglected because the equation of motion is non-local in position space:

$$i\not\partial\psi(t,\mathbf{x}) = -\frac{im}{\pi}\mathbf{P}\int dt'\frac{1}{t-t'}\psi(t',\mathbf{x}), \tag{3}$$

where $\mathbf{P}$ denotes the principal value, which we assume throughout. There is some dispute [4,11] as to whether causal interacting homeotic field theories exist, but there is no question that the homeotic equation (2) is Lorentz invariant on-shell. In this Letter we will only be concerned with the explicit on-shell properties of the homeotic formalism.

As noted in [4], the combination of a homeotic mass term with a Dirac mass term violates *CPT*. Consider the simple free theory defined by the Lagrangian

$$\int d^3x\,\bar\psi(i\not\partial - m_d)\psi + \frac{im_h}{\pi}\int\frac{d^3x\,dt'}{t-t'}\big(\bar\psi(t)\psi(t') - \bar\psi(t')\psi(t)\big). \tag{4}$$

If we define the *CPT* operator such that the Dirac mass term is *CPT* even, then the homeotic mass term is *CPT* odd, and vice-versa. *CPT* conjugate spinors in this theory have mass-squared eigenvalues $(m_d \pm m_h)^2$.

3. The $N=2$ SUSY algebra

We would now like to extend this theory of 4-component complex spinors to a supermultiplet of fields which furnish a representation of the standard $N=2$ SUSY algebra. In the 4-component notation of Sohnius [3] the relevant parts of the $N=2$ algebra are:

$$\begin{aligned}&\{Q^i,\bar Q_j\} = 2\delta^i_j\gamma^\mu P_\mu + 2i\delta^i_j Z,\\&[Q_i,P_\mu] = 0,\end{aligned} \tag{5}$$

where the Q_i, $i=1,2$, are symplectic Majorana spinor supercharges, and Z is the (anti-Hermitian) central charge operator, which commutes with all of the other generators of the algebra. The supercharges form a doublet under the $SU(2)$ R symmetry of $N=2$ SUSY; the index i is raised and lowered with the two-dimensional Levi-Civita tensors $\epsilon_{ij}=\epsilon^{ij}$. In our derivation we will need a number of identities for bilinears of symplectic Majorana spinors:

$$\begin{aligned}&\bar\zeta^i\eta_i = -\big(\bar\zeta^i\eta_i\big)^\dagger = \bar\eta_i\zeta^i = -\bar\zeta_i\eta^i,\\&\bar\zeta^i\gamma^\mu\eta_i = -\big(\bar\zeta^i\gamma^\mu\eta_i\big)^\dagger = \bar\eta_i\gamma^\mu\zeta^i = -\bar\zeta_i\gamma^\mu\eta^i,\\&\bar\zeta_i\eta^j - \bar\eta_i\zeta^j = \delta^j_i\bar\zeta_k\eta^k,\\&\bar\zeta_i\gamma^\mu\eta^j - \bar\eta_i\gamma^\mu\zeta^j = \delta^j_i\bar\zeta_k\gamma^\mu\eta^k.\end{aligned} \tag{6}$$

Let $A_i(x)$ denote a doublet of complex scalar fields, and $\psi(x)$ a 4-component complex fermion field. We want these fields to form an $N=2$ hypermultiplet, i.e., to furnish a representation of the algebra (5). The fundamental relation between the fields is

$$\big[Q^i, A_j(x)\big] = -i\sqrt{2}\,\delta^i_j\psi(x) \tag{7}$$

one additional input is required:

$$\big[Z, A_i(x)\big] = F_i, \tag{8}$$

where $F_i(x)$ is a doublet of complex bosonic auxiliary fields; we will specify the precise form of $F_i(x)$ later.

Expressions (7) and (8), together with the algebra (5), the Jacobi identities, and the identities (6), now imply:

$$\{\bar{Q}_i, \psi\} = \sqrt{2}\, i\gamma^\mu [P_\mu, A_i] - \sqrt{2}\, F_i,$$
$$[Z, \psi] = -i\gamma^\mu [P_\mu, \psi] = \partial\!\!\!/ \psi,$$
$$\left[Q^i, F_j\right] = -i\sqrt{2}\,\delta^i_j [Z, \psi],$$
$$[Z, F_i] = -\left[P^\mu, [P_\mu, A_i]\right] = \partial_\mu \partial^\mu A_i. \tag{9}$$

These relations define a set of hypermultiplet fields $A_i(x)$, $\psi(x)$, and $F_i(x)$ which form a representation of $N = 2$ SUSY. These relations determine uniquely the equations of motion for the dynamical fields $A_i(x)$ and $\psi(x)$, once we specify the auxiliary fields $F_i(x)$. If, e.g., we write

$$F_i(x) = -im_d A_i, \tag{10}$$

then (8) combined with (9) implies that the $A_i(x)$ satisfy the Klein–Gordon equation, while (7) combined with (9) implies that $\psi(x)$ obeys the Dirac equation.

4. Simple example of homeotic supersymmetry

We obtain the simplest example of homeotic supersymmetry by specifying the auxiliary fields as follows:

$$F_i(x) = -im_d A_i(x) - \frac{m_h}{\pi} \int \frac{dt'}{t - t'} A_i(t', \mathbf{x}). \tag{11}$$

Applying the relations (7)–(9), we determine the equations of motion to be:

$$\partial_\mu \partial^\mu A_i + \left(m_d^2 + m_h^2\right) A_i - \frac{2im_d m_h}{\pi} \int \frac{dt'}{t - t'} A_i(t', \mathbf{x}) = 0,$$
$$i\partial\!\!\!/ \psi - m_d \psi + \frac{im}{\pi} \int \frac{dt'}{t - t'} \psi(t', \mathbf{x}) = 0. \tag{12}$$

The action which reproduces (11), (12) is given by

$$S = \int dt \left(L_{\text{kin}} + L_d + L_h + L_h^\dagger\right), \tag{13}$$

where

$$L_{\text{kin}} = \int d^3x \left[\partial_\mu A^{\dagger i} \partial^\mu A_i + F^{\dagger i} F_i + i\bar{\psi}\partial\!\!\!/ \psi\right],$$
$$L_d = -m_d \int d^3x \left[iA^{\dagger i} F_i - iF^{\dagger i} A_i + \bar{\psi}\psi\right],$$
$$L_h = \frac{im_h}{\pi} \int \frac{d^3x\, dt'}{t - t'} \big[iA^{\dagger i}(t) F_i(t') - iF^{\dagger i}(t) A_i(t') + \bar{\psi}(t)\psi(t')\big]. \tag{14}$$

It is easy to check that all four pieces of the action are separately invariant under the standard $N = 2$ SUSY transformations of the fields:

$$\delta A_i = \sqrt{2}\, \bar{\zeta}_i \psi,$$
$$\delta \psi = -i\sqrt{2}\, F_i - i\sqrt{2}\, \gamma^\mu \zeta^i \partial_\mu A_i,$$
$$\delta F_i = \sqrt{2}\, \bar{\zeta}_i \partial\!\!\!/ \psi. \tag{15}$$

We can proceed further to construct the conserved supercurrent in terms of the component fields. However, the usual Noether procedure does not yield a conserved current; this is a generic feature of nonlocal field theories, and was noted in our previous Letter with regard to the fermion number current of the homeotic fermion theory. Let us first review that case, in which we employed a trick of Pauli's construct the conserved fermion number current:

$$J^\mu = \bar{\psi}\gamma^\mu \psi + \delta^{\mu 0} \frac{m_h}{\pi} \int^t dt' \int \frac{dt''}{t' - t''} \times \left[\bar{\psi}(t')\psi(t'') + \bar{\psi}(t'')\psi(t')\right].$$

It is easily seen using the fermion equation of motion (12) that $J^\mu(x)$ is conserved on-shell. Despite its ugly form in position space, $J^\mu(x)$ reduces to the usual number current in the creation/annihilation Fock basis.

In a supersymmetric theory the conserved bosonic current $J^\mu(x)$ must belong to a supermultiplet of conserved currents. In particular, we can immediately obtain an expression for the supercurrent $j^{\mu i}$, by applying the supersymmetry transformations (15) to the component fields in (16). The result is:

$$\frac{j^{\mu i}(x)}{\sqrt{2}} = \gamma^\mu \psi F^{\dagger i} + \gamma^\nu \gamma^\mu \psi \partial_\nu A^{\dagger i} + \delta^{\mu 0} \frac{m_h}{\pi} \int^t dt' \int \frac{dt''}{t' - t''} \times \left[\psi(t'') F^{\dagger i}(t') + \gamma^\nu \psi(t'') \partial_\nu A^{\dagger i}(t') + (t' \leftrightarrow t'')\right]. \tag{16}$$

Using the equations of motion (11), (12), one finds that $j^{\mu i}(x)$ is conserved on-shell. It thus represents the supercurrent modulo possible "improvement" terms [3].

5. Mass spectrum

Both the bosonic and fermionic parts of the action (13) violate *CPT*. Let us focus first on the bosonic sector. We can expand the fields $A_i(x)$ in positive and negative frequency plane wave solutions of the equations of motion (12):

$$A_i = \int \frac{d^3p}{(2\pi)^3}\left(\frac{a_{\mathbf{p}i}}{\sqrt{2\omega_+}}\mathrm{e}^{-i\omega_+ t + i\mathbf{p}\cdot\mathbf{x}} + \frac{b^\dagger_{\mathbf{p}i}}{\sqrt{2\omega_-}}\mathrm{e}^{i\omega_- t - i\mathbf{p}\cdot\mathbf{x}}\right), \tag{17}$$

where

$$\omega_\pm \equiv \sqrt{\mathbf{p}^2 + (m_d \pm m_h)^2}. \tag{18}$$

We quantize the theory by assuming that $a_{\mathbf{p}i}$, $b_{\mathbf{p}i}$ satisfy the commutation relations of creation/annihilation operators. It follows that the general commutator of $A_i(x)$ with its conjugate $\Pi^j(x)$ is given by

$$\begin{aligned}&\left[A_i(x), \Pi^j(x')\right]\\ &\quad = \frac{i}{2}\delta_i^j \int \frac{d^3p}{(2\pi)^3}\\ &\qquad \times \left(\mathrm{e}^{-i\omega_+(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} + \mathrm{e}^{i\omega_-(t-t')-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\right).\end{aligned}$$

Thus the equal-time commutator is canonical, but the general commutator is not. Using (19) we can now verify that the supercharge extracted from (16) satisfies (7).

Another novel feature appears when we construct the bosonic part of the Hamiltonian in terms of $a_{\mathbf{p}i}$, $b_{\mathbf{p}i}$. The canonical Hamiltonian is not diagonalized in the basis defined by (18); it is instead diagonalized in the basis defined by

$$A_i = \int \frac{d^3p}{(2\pi)^3}\left(\frac{a_{\mathbf{p}i}}{\sqrt{2\omega_+}}\mathrm{e}^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}} + \frac{b^\dagger_{\mathbf{p}i}}{\sqrt{2\omega_-}}\mathrm{e}^{i\omega t - i\mathbf{p}\cdot\mathbf{x}}\right), \tag{19}$$

where

$$\omega = \sqrt{\mathbf{p}^2 + m_d^2}. \tag{20}$$

In this basis the bosonic Hamiltonian is

$$H_b = \int d^3x\left[\omega_+ a^{\dagger i}_{\mathbf{p}} a_{\mathbf{p}i} + \omega_- b^{\dagger i}_{\mathbf{p}} b_{\mathbf{p}i}\right], \tag{21}$$

showing that the *CPT* conjugate single particle states have a mass-squared splitting equal to $|4m_d m_h|$.

A similar analysis for the fermions diagonalizes the fermionic part of the Hamiltonian in terms of the anticommuting Fock operators $\tilde{a}_{\mathbf{p}s}$, $\tilde{b}_{\mathbf{p}s}$, where s is the spin label:

$$H_f = \int d^3x\left[\omega_+ \tilde{a}^\dagger_{\mathbf{p}s}\tilde{a}_{\mathbf{p}s} + \omega_- \tilde{b}^\dagger_{\mathbf{p}s}\tilde{b}_{\mathbf{p}s}\right]. \tag{22}$$

Again the *CPT* conjugate states have a mass-squared splitting equal to $|4m_d m_h|$.

6. Comments

It would be interesting to extend the above construction to produce an interacting theory. A conventional $N = 2$ hypermultiplet can interact with an $N = 2$ vector multiplet, or can have self-interactions describing a non-linear sigma model [12]. For the homeotic case neither extension appears entirely straight–forward.

The homeotic $N = 2$ hypermultiplet has 8 real on-shell degrees of freedom. We have just seen that half of these describe a boson–fermion pair with mass $|m_d + m_h|$, while the other half describe a boson–fermion pair with mass $|m_d - m_h|$. The homeotic $N = 2$ hypermultiplet is thus a *CPT* violating BPS saturated multiplet. The BPS shortening of the multiplet is of course essential to our construction.

Since an ordinary $N = 2$ hypermultiplet can be split into two $N = 1$ chiral multiplets, it is important to ask whether our homeotic $N = 2$ hypermultiplet is reducible into two $N = 1$ multiplets. The answer is no. This can be seen by imposing the Majorana condition on the fermions in (4), and observing that the homeotic mass term then vanishes identically. Alternatively, one notes that the usual decomposition of the hypermultiplet into chiral multiplets can be written in the Fock basis as

$$a_{\mathbf{p}i}, b_{\mathbf{p}i}, \tilde{a}_{\mathbf{p}s}, \tilde{b}_{\mathbf{p}s} \to (a_{\mathbf{p}\pm}, b_{\mathbf{p}\mp}, \tilde{a}_{\mathbf{p}\pm}, \tilde{b}_{\mathbf{p}\mp}), \tag{23}$$

obviously in the homeotic case this would mix operators with different dispersion relations.

The irreducibility of the homeotic $N = 2$ hypermultiplet is in fact very analogous to the irreducibility of the ordinary $N = 1$ chiral multiplet. In this case one finds the 4 on-shell degrees of freedom consist of two

CPT conjugate pairs; each pair has one boson and one fermion state, related by supersymmetry. However, it is not possible to reduce the multiplet, due to the non-existence of Majorana–Weyl spinors in four dimensions [13]. This is the analog of the non-existence of homeotic Majorana spinors in four dimensions.

As a parting remark, let us inquire how one might attempt to construct a supermultiplet in which Bose–Fermi degeneracy is violated. At the level of the on-shell Hamiltonian, this does not appear to be particularly difficult. Let $H_0(m)$ be a mass term with mass parameter m for a free supersymmetric Hamiltonian containing two species of fermions, and construct a new Hamiltonian defined by

$$H_1 = H_0(m) + (-1)^{F_1+F_2} H_0(m'), \tag{24}$$

where F_1 and F_2 are the fermion number operators for the two species of fermions. Clearly the single particle eigenstates of H_1 have different masses for bosons and fermions: the bosons have mass $m + m'$ while the fermions have mass $m - m'$. It is easy to see that, acting on single particle states:

$$\left\{Q, (-1)^{F_1}\right\} = \left\{Q, (-1)^{F_2}\right\} = 0, \tag{25}$$

from which it follows that, acting on single particle states:

$$[Q, H_0] = 0 \quad \Rightarrow \quad [Q, H_1] = 0, \tag{26}$$

and thus supersymmetry is unbroken. The challenge, of course, is to realize such a scheme in field theory.

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References

[1] R. Haag, J.T. Lopuszański, M. Sohnius, Nucl. Phys. B 88 (1975) 257.
[2] S.R. Coleman, J. Mandula, Phys. Rev. 159 (1967) 1251.
[3] M.F. Sohnius, Phys. Rep. 128 (1985) 39.
[4] G. Barenboim, J. Lykken, Phys. Lett. B 554 (2003) 73, hep-ph/0210411.
[5] G. Barenboim, L. Borissov, J. Lykken, A.Y. Smirnov, JHEP 10 (2002) 001, hep-ph/0108199.
[6] G. Barenboim, L. Borissov, J. Lykken, Phys. Lett. B 534 (2002) 106, hep-ph/0201080.
[7] G. Barenboim, J.F. Beacom, L. Borissov, B. Kayser, Phys. Lett. B 537 (2002) 227, hep-ph/0203261.
[8] G. Barenboim, L. Borissov, J. Lykken, hep-ph/0212116.
[9] G. Barenboim, C. Quigg, hep-ph/0301220.
[10] R.F. Streater, A.S. Wightman, PCT, Spin and Statistics, and all that, Addison–Wesley Advanced Book Classics, Redwood City, CA, 1989.
[11] O.W. Greenberg, Phys. Lett. B 567 (2003) 179, hep-ph/0305276.
[12] S.J. Gates, M.T. Grisaru, M. Rocek, W. Siegel, Front. Phys. 58 (1983) 1, hep-th/0108200.
[13] J.D. Lykken, hep-th/9612114.