Travelling wave solutions for time-delayed nonlinear evolution equations

Hyunsoo Kim\textsuperscript{a}, Rathinasamy Sakthivel\textsuperscript{b,∗}

\textsuperscript{a} Department of Applied Mathematics, College of Applied Science, Kyung Hee University, Yongin 446-701, South Korea
\textsuperscript{b} Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea

\textbf{ARTICLE INFO}

Article history:
Received 28 April 2009
Received in revised form 14 January 2010
Accepted 15 January 2010

Keywords:
\( (G'/G) \)-expansion method
Homogeneous balance
Hyperbolic function solutions
Trigonometric function solutions
Solitary wave solutions
Time-delayed equations

\textbf{ABSTRACT}

Time-delayed nonlinear evolution equations have a wide range of applications in science and engineering. In this paper, the \( (G'/G) \)-expansion method is implemented to establish travelling wave solutions for time-delayed Burgers and time-delayed Burgers–Fisher equations. The travelling wave solutions are expressed by hyperbolic functions and trigonometric functions. The results reveal that \( (G'/G) \)-expansion method is very effective and a powerful tool for solving nonlinear time-delayed evolution equations arising in mathematical physics.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

A number of nonlinear phenomena such as physical, biochemical and biological processes are described by the interplay of reaction and diffusion, or by the interaction between convection and diffusion. The well-known partial differential equation which governs a wide variety of these phenomena is the Burgers equation. The diffusion processes, however, get significantly modified when the memory effects are taken into account, i.e., when the dispersal of the particles is not mutually independent. This implies that the correlation between the successive movements of the diffusing particles may be understood as a delay in the flux for a given concentration gradient. Thus existence of time-delay is an important feature in convection–diffusion systems. It is interesting to point out that a well-known generalization of the Burgers–Fisher equation, namely the generalized time-delayed Burgers–Fisher equation is given by [1,2]

\[
\tau u_{tt} + [1 - \tau f_u]u_t = u_{xx} - pu_t u_x + f(u), \quad f(u) = qu(1 - u^s),
\]

where \( p, q, s \) are constants and \( \tau \) is a time-delayed constant. Eq. (1) reduces to the classical Burgers equation when \( q = \tau = 0 \) and \( p = s = 1 \).

On the other hand, directly seeking exact solutions of nonlinear partial differential equations has become one of the central themes of perpetual interest in mathematical physics. In order to understand better the nonlinear phenomena as well as further application in practical life, it is important to seek more of their exact travelling wave solutions. Many methods are used to obtain travelling solitary wave solutions to nonlinear PDEs, such as the Adomian decomposition method [1,3], the sub-ODE method [4,5], \( F \)-expansion method [6,7], the homotopy perturbation method [8,9], the exp-function method [10–12], Hirota’s bilinear method [13,14], Jacobi elliptic function method [15] and so on.

∗ Tel.: +82 31 299 4527; fax: +82 31 290 7033.
\E-mail address: krsakthivel@yahoo.com (R. Sakthivel).

0893-9659/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2010.01.005
However, practically there is no unified method that can be used to handle all types of nonlinear partial differential equations. Recently, Wang et al. [16] introduced a new direct method called the \( \left( \frac{G'}{G} \right) \)-expansion method to look for travelling wave solutions of nonlinear evolution equations. One of the most effective straightforward methods to construct exact solutions of PDEs is the \( \left( \frac{G'}{G} \right) \)-expansion method [17–19]. The main ideas of the \( \left( \frac{G'}{G} \right) \)-expansion method are that the travelling wave solutions of a nonlinear evolution equation can be expressed by a polynomial in \( \left( \frac{G'}{G} \right) \), where \( G = G(\xi) \) satisfies a second order linear ordinary differential equation (LODE) as follows: \( G'' + \lambda G' + \mu G = 0, G' = \frac{dG(\xi)}{d\xi}, \xi = x - \omega t \), the degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in a given nonlinear evolution equation, and the coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the \( \left( \frac{G'}{G} \right) \)-expansion method. Later Zhang et al. [20] proposed a generalized \( \left( \frac{G'}{G} \right) \)-expansion method for solving variable-coefficient equations and high-dimensional equations.

In the present work, \( \left( \frac{G'}{G} \right) \)-expansion method is implemented to obtain travelling wave solutions of the following time-delayed Burgers equation of the form

\[
\tau u_{tt} + u_t = -pu''u_x + u_{xx},
\]

for different values of \( \tau, p, s \) and time-delayed Burgers–Fisher equations. Eq. (2) have been applied to forest fire [21], population growth models, Neolithic transitions [22,23] and many other areas of applied sciences.

2. Summary of the \( \left( \frac{G'}{G} \right) \)-expansion method

A partial differential equation (PDE)

\[
P(u, u_t, u_x, u_{xx}, u_{tt}, \ldots ) = 0,
\]

can be converted to an ordinary differential equation (ODE)

\[
Q(u, -kau', ku', k^2u'', k^2\omega^2u'', \ldots ) = 0
\]

by using a wave variable \( \xi = k(x - \omega t) \).

By \( \left( \frac{G'}{G} \right) \)-expansion method the solution of ODE (4) can be expressed by a polynomial in \( \left( \frac{G'}{G} \right) \) as follows [16]:

\[
u(\xi) = \sum_{i=0}^{m} a_i \left( \frac{G'}{G} \right)^i,
\]

where \( G = G(\xi) \) satisfies the second order LODE in the form

\[
G'' + \lambda G' + \mu G = 0,
\]

here \( a_1, \ldots, a_m, \lambda \) and \( \mu \) are constants to be determined later, \( a_m \neq 0 \). The positive integer \( m \) can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (4).

By substituting Eq. (5) in Eq. (4) and using second order LODE (6), collecting all terms with the same order of \( \left( \frac{G'}{G} \right) \) together, the left-hand side of Eq. (4) is converted into another polynomial in \( \left( \frac{G'}{G} \right) \). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for \( a_1, \ldots, a_m, \lambda, k, \omega \) and \( \mu \). Solving this set of algebraic equations we can obtain the explicit expressions for \( a_1, \ldots, a_m, \lambda, k, \omega \) and \( \mu \). Substituting this values and general solutions of (6) in (5), we can obtain the exact solutions of (3).

In the remaining sections we will find the travelling wave solutions to nonlinear time-delayed evolution equations using \( \left( \frac{G'}{G} \right) \)-expansion method.

3. Solutions to the time-delayed Burgers equation

To obtain the solution for Eq. (2), we consider the transformation \( u = U(\xi), \xi = k(x - \omega t) \), where \( k \) and \( \omega \) are constants to be determined later. We can rewrite time-delayed Burgers equation (2) in the following nonlinear ordinary differential equation of the form

\[
\left( \tau \omega^2 - 1 \right) k^2 U'' - k\omega U' + pkU''U' = 0.
\]

Balancing \( U''U' \) with \( U'' \) in Eq. (7) gives \( m = 1/s \). To get a closed form solution we use the transformation \( U = v^{1/s} \) to change
Eq. (7) into
\[
(τω^2 − 1)k^2 \left[ v''v + \left( \frac{1}{s^2} - 1 \right) v^2 \right] - kv u'v' + p kv^2v' = 0.
\]

Suppose that the solution of ODE (8) can be expressed by a polynomial in \( \left( \frac{G'}{G} \right) \) as follows:
\[
v(ξ) = \sum_{i=0}^{m} a_i \left( \frac{G'}{G} \right)^i,
\]
where \( G = G(ξ) \) satisfies the second order LODE in the form
\[
G'' + λG' + μG = 0.
\]
By using Eqs. (9) and (10) it is derived that
\[
v^2 = a_m^2 \left( \frac{G'}{G} \right)^{2m} + \cdots ,
\]
\[
v' = -ma_m \left( \frac{G'}{G} \right)^{m+1} + \cdots ,
\]
\[
v'' = m(m + 1)a_m \left( \frac{G'}{G} \right)^{m+2} + \cdots .
\]
To determine the parameter \( m \), we balance the linear terms of highest order in Eq. (8) with the highest order nonlinear terms. Considering the homogeneous balance between \( v'' \) and \( v^2 \) in (8), based on (11) and (13) we obtain that \( 2m + 2 = 2m + m + 1 \Rightarrow m = 1 \), so we can write (9) as
\[
v(ξ) = a_1 \left( \frac{G'}{G} \right) + a_0, \ a_1 \neq 0.
\]
By using Eqs. (9) and (10) it is derived that
\[
v^2 = a_1^2 \left( \frac{G'}{G} \right)^2 + 2a_1a_0 \left( \frac{G'}{G} \right) + a_0^2,
\]
\[
v' = -a_1μ - a_1λ \left( \frac{G'}{G} \right) - a_1 \left( \frac{G'}{G} \right)^2,
\]
\[
v'' = 2a_1 \left( \frac{G'}{G} \right)^3 + 3a_1λ \left( \frac{G'}{G} \right)^2 + (2a_1μ + a_1λ^2) \left( \frac{G'}{G} \right) + a_1λμ.
\]
By substituting (14)-(17) in (8) and collecting all terms with the same power of \( \left( \frac{G'}{G} \right) \) together, the left-hand side of (8) is converted into another polynomial in \( \left( \frac{G'}{G} \right) \). Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for \( a_1, a_0, μ, ν, k \) and \( κ \) as follows:
\[
\begin{align*}
\left( \frac{G'}{G} \right)^0 & : k^2a_1^2μ^2 + kωa_0a_1μ - pka_0^2a_1μ + \frac{k^3τω^2a_1^2κ^2}{s} - k^2a_1λμa_0 - \frac{k^2a_1^2κ^2}{s} + k^2τω^2a_1λμa_0 - k^2τω^2a_1^2κ^2 = 0 \\
\left( \frac{G'}{G} \right)^1 & : kωa_1^2μ + kωa_0a_1λ - 2pka_0a_1^2μ - pka_0^2a_1λ - \frac{2k^2a_1^2λκ^2}{s} + \frac{2k^2τω^2a_1^2κ^2}{s} + 2k^2τω^2a_1^2λμ + k^2a_1^2λμ - k^2a_1κ^2a_0 + k^2τω^2a_1^2κ^2 - 2k^2a_1^2μa_0 = 0 \\
\left( \frac{G'}{G} \right)^2 & : kωa_1^2λ + kωa_0a_1 - pka_1^2μ - pka_0^2a_1λ - \frac{2k^2τω^2a_1^2κ^2}{s} + \frac{2k^2τω^2a_1^2κ^2}{s} + 3k^2τω^2a_1^2λμ - 3k^2a_1^2μa_0 - 3k^2a_1^2λ - \frac{2k^2a_1^2κ^2}{s} - 2k^2a_1^2μ \frac{κ^2}{s} = 0 \ \\
\left( \frac{G'}{G} \right)^3 & : -pka_1^2λ - 2pka_0a_1^2 + 2k^2τω^2a_1a_0 + \frac{2k^2τω^2a_1^2κ^2}{s} - \frac{2k^2a_1^2κ^2}{s} + k^2τω^2a_1^2λ - 2k^2a_1^2a_0 - k^2a_1^2λ + kωa_1 = 0 \ \\
\left( \frac{G'}{G} \right)^4 & : -pka_1^3 + \frac{k^2τω^2a_1^2κ^2}{s} - k^2a_1^2 - k^2τω^2a_1^2κ^2 = 0.
\end{align*}
\]
Solving the above algebraic equations using Maple, we obtain the following set of nontrivial solutions
\[
a_1 = \pm \frac{(s + 1)ω}{p\sqrt{κ^2 - 4μ}}, \quad a_0 = \pm \frac{(s + 1)κω}{2p\sqrt{κ^2 - 4μ}} + \frac{(s + 1)ω}{2p}, \quad κ = \pm \frac{sω}{(τω^2 - 1)\sqrt{κ^2 - 4μ}},
\]
where \( λ, μ, s, p, τ \) and \( ω \) are constants.
By using (18), expression (14) can be written as

\[ v(\xi) = \pm \frac{(s + 1)\omega}{p\sqrt{\lambda^2 - 4\mu}} \left( \frac{C'}{C} \right) \pm \frac{(s + 1)\omega}{2p\sqrt{\lambda^2 - 4\mu}} + \frac{(s + 1)\omega}{2p}, \]  

where \( \xi = \pm \frac{s\omega}{(\tau^2 - 1)\sqrt{\lambda^2 - 4\mu}} (x - \omega t) \).

Substituting the general solutions of (10) in (19), we obtain three types of travelling wave solutions of the time-delayed Burgers equation (Eq. (2)). We get the following hyperbolic functions solution when \( \lambda^2 - 4\mu > 0 \)

\[ u_1(x, t) = \pm \frac{(s + 1)\omega}{2p} \left( \frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi} \right) + \frac{(s + 1)\omega}{2p} \right]^{\frac{1}{4}}, \]  

where \( \xi = \pm \frac{s\omega}{(\tau^2 - 1)\sqrt{\lambda^2 - 4\mu}} (x - \omega t) \), \( C_1 \) and \( C_2 \) are arbitrary constants.

In particular, if \( C_1 \neq 0, C_2 = 0 \), then \( u_1 = u_1(\xi) \) can be written as the following kink-like solution

\[ u_1(x, t) = \pm \frac{(s + 1)\omega}{2p} \tanh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \frac{(s + 1)\omega}{2p} \right]^{\frac{1}{4}}, \]

where \( \xi = \pm \frac{s\omega}{(\tau^2 - 1)\sqrt{\lambda^2 - 4\mu}} (x - \omega t) \).

In particular, if \( C_1 = 0, C_2 \neq 0 \), then \( u_1 = u_1(\xi) \) can be written as the following soliton solution

\[ \hat{u}(x, t) = \pm \frac{(s + 1)\omega}{2p} \coth \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \frac{(s + 1)\omega}{2p} \right]^{\frac{1}{4}}, \]

where \( \xi = \pm \frac{s\omega}{(\tau^2 - 1)\sqrt{\lambda^2 - 4\mu}} (x - \omega t) \).

The solution (22) is identical to those obtained in [1]. Next we obtain the following triangular periodic solution when \( \lambda^2 - 4\mu < 0 \)

\[ u_2(x, t) = \pm \frac{(s + 1)\omega}{2p} \left( \frac{-C_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi + C_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi}{C_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi + C_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi} \right) + \frac{(s + 1)\omega}{2p} \right]^{\frac{1}{4}}, \]

where \( \xi = \pm \frac{s\omega}{(\tau^2 - 1)\sqrt{\lambda^2 - 4\mu}} (x - \omega t) \), \( C_1 \) and \( C_2 \) are arbitrary constants.

It is clear that when \( \tau = 0 \), Eq. (2) reduces to the classical Burgers equation [1]. Taking the limit when \( \tau \to 0 \) in (21) and (22), we can recover exact particular solutions for the classical Burgers equation as in [1]. The effect of the time-delay of solution (21) is shown in Fig. 1.
4. Solutions to the time-delayed Burgers–Fisher equation

In this section, we obtain more travelling solutions of the time-delayed Burgers–Fisher equation. When \( p = s = q = 1 \) Eq. (1) reduces to the form

\[
\tau u_{tt} + (1 - \tau)u_t + 2\tau u_w_t - u_{xx} + uu_x - u + u^2 = 0,
\]

where \( \tau \) is a time-delayed constant. This equation may be called as time-delayed Burgers–Fisher equation. This equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect and diffusion transport [2,24]. It is clear that when \( \tau = 0 \), Eq. (24) reduces to the classical Burgers–Fisher equation discussed in [2,24].

After the customary transformation \( u = \psi(\xi), \eta = k(x - \omega t) \), we obtain

\[
k^2(\tau \omega^2 - 1)v'' - k\omega(1 - \tau)v' - (2k\omega\tau - k)v' - v + v^2 = 0.
\]

To look for the travelling wave solutions of Eq. (24), we balance \( v'' \) and \( vv' \) in (25), which gives \( m + 2 = 2m + 1 \) so that \( m = 1 \). Now we can write (5) as

\[
v(\xi) = a_1 \left( \frac{G'}{G} \right) + a_0, \quad a_1 \neq 0.
\]

By substituting (15)–(17) and (26) in (25) and collecting all terms with the same power of \( \left( \frac{G'}{G} \right) \) together, the left-hand side of (25) is converted into another polynomial in \( \left( \frac{G'}{G} \right) \). Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for \( a_2, a_1, a_0, \omega \) and \( k \). Solving the system of algebraic equations using Maple, gives the following set of nontrivial solutions

\[
a_1 = \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}}, \quad a_0 = \pm \frac{\lambda}{2\sqrt{\lambda^2 - 4\mu}} + \frac{1}{2}, \quad k = \pm \frac{1 + \tau}{\tau - 4} \frac{2}{\sqrt{\lambda^2 - 4\mu}}, \quad \omega = \frac{5}{2(1 + \tau)}
\]

where \( \lambda, \mu, \delta \) and \( a_0 \) are arbitrary constants.

By using (27), expression (26) can be written as

\[
v(\xi) = \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}} \left( \frac{G'}{G} \right) \pm \frac{\lambda}{2\sqrt{\lambda^2 - 4\mu}} + \frac{1}{2},
\]

where \( \xi = \pm \frac{1 + \tau}{\tau - 4} \frac{2}{\sqrt{\lambda^2 - 4\mu}}(x - \frac{5}{2(1 + \tau)}t) \).

Substituting the general solutions of (6) in (28), we obtain three types of travelling wave solutions of the time-delayed Burgers–Fisher equation (Eq. (25)). We get the following hyperbolic functions solution when \( \lambda^2 - 4\mu > 0 \)

\[
u_1(x, t) = \pm \frac{1}{2} \left( \frac{C_1 \sinh \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \sqrt{\lambda^2 - 4\mu} \xi} \right) + \frac{1}{2}.
\]

where \( \xi = \pm \frac{1 + \tau}{\tau - 4} \frac{2}{\sqrt{\lambda^2 - 4\mu}}(x - \frac{5}{2(1 + \tau)}t) \).

In particular, if \( C_1 \neq 0, C_2 = 0 \), then \( u_1 = u_1(\xi) \) can be written as the following soliton solution

\[
u_1(x, t) = \pm \frac{1}{2} \tanh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \frac{1}{2}
\]

where \( \xi = \pm \frac{1 + \tau}{\tau - 4} \frac{2}{\sqrt{\lambda^2 - 4\mu}}(x - \frac{5}{2(1 + \tau)}t) \).

Next we obtain the following triangular periodic solution when \( \lambda^2 - 4\mu < 0 \)

\[
u_2(x, t) = \mp \frac{1}{2} \left( -\frac{C_1 \sin \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \sqrt{4\mu - \lambda^2} \xi} \right) + \frac{1}{2},
\]

where \( \xi = \pm \frac{1 + \tau}{\tau - 4} \frac{2}{\sqrt{\lambda^2 - 4\mu}}(x - \frac{5}{2(1 + \tau)}t) \). The effect of the time-delay of solution (30) is shown in Fig. 2. And we found that the time-delay is effective in smoothing out the shock-wave nature of the travelling wave. When \( \tau = 0 \), Eq. (24) reduces to the classical Burgers–Fisher equation [24]. Taking the limit when \( \tau \to 0 \) in (30), we can recover an exact particular solution for the classical Burgers–Fisher equation as in [25].
5. Conclusion

In this paper, the \((G'/G)\)-expansion method is implemented to obtain more exact solutions for time-delayed Burgers and time-delayed Burgers–Fisher equations. All solutions presented in this paper have been verified by using Mathematica by putting them back into the original equation. Finally, it is worth mentioning that the implementation of these proposed methods is very simple and straightforward, and it can also be applied to many other nonlinear time-delayed evolution equations.

References