Some eigenvalue properties in graphs (conjectures of Graffiti – II)

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Abstract

In this paper we improve some classical bounds on the greatest eigenvalue of the adjacency matrix of a graph. We also give inequalities between the eigenvalues and some other parameters. These results allow us to prove some conjectures of the program Graffiti written by Fajtlowicz. Moreover, the study of the spectrum of graphs obtained by some simple constructions yields infinite families of counterexamples for other conjectures of this program.

1. Introduction

1.1. Definitions

The computer program Graffiti written by Fajtlowicz [6] gives a lot of conjectures involving graph parameters, some of which are classical, others needing new definitions.

Let $G = (V, E)$ be a simple graph of order $n$, size $m$, minimum degree $\delta$ and maximum degree $\Delta$.

An induced subgraph of $G$ is a graph $G' = (V', E')$, where $V'$ is a subset of $V$ and $E'$ is the subset of all the edges of $E$ which are incident to two vertices of $V'$. We also call $G'$ the subgraph induced by $V'$. A partial subgraph of $G$ is a graph $G' = (V', E')$, where $E'$ is a subset of $E$ and $V'$ is the subset of the vertices of $V$ which are incident to at least one edge of $E'$. Note that $V'$ may be equal to $V$.

A matching of $G$ is a set of mutually nonincident edges. The matching number $\nu$ is the largest size of a matching. A perfect matching of $G$ is a matching spanning $V$.

An independent set of $G$ is a set of mutually nonadjacent vertices. The independence number $\alpha$ is the maximum order of an independent set.

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The chromatic number $\chi$ of $G$ is the smallest number of classes into a partition of $V$ in independent sets.

We denote by $\omega$ the maximum order of a clique of $G$.

The dual degree of the vertex $x$ is the mean of the degrees of the neighbours of $x$. The Dual Degree of $G$ is the vector the components of which are the dual degrees of the vertices.

The residue $R$ of a graph $G$ of degree sequence $S$: $d_1 \geq d_2 \geq \cdots \geq d_n$ is the number of zeros obtained by the iterative process consisting of deleting the first term $d_1$ of $S$, subtracting 1 from the $d_1$ following ones and re-sorting the new sequence in decreasing order. The depth is the number $n - R$ of steps in this algorithm.

The Randic of $G$ is the number given by the formula $Rc = \sum_{x,y \in E} \frac{1}{\sqrt{d(x) \cdot d(y)}}$, where $d(x)$ is the degree of the vertex $x$.

The Harmonic of $G$ is the number given by the formula $Hc = \sum_{x,y \in E} \frac{2}{d(x) + d(y)}$.

The girth of a graph is the length of a smallest cycle.

The distance $d(x, y)$ between two vertices of a connected graph $G$ is the length of a shortest path joining $x$ and $y$. The diameter $D$ of $G$ is the maximum of the distances between any two vertices. The mean distance $\mu$ of $G$ is the average value $\frac{1}{n(n-1)} \sum_{x,y \in V} d(x, y)$ taken over all the ordered pairs of vertices. The eccentricity of a vertex $x$ is $e(x) = \max_{y \in V} d(x, y)$. The radius of $G$ is $r = \min_{x \in V} e(x)$.

Given an arbitrary enumeration of the vertices of a connected graph $G$, let $\text{even}(i)$ be the number of vertices $j$ such that $d(i, j)$ is an even (including 0) integer. The resulting vector is called $\text{Even}$. Similarly, one can define $\text{Odd}$ of $G$.

The range of a vector is the number of distinct components of this vector. The scope of a vector is the difference between the largest and the smallest component. The mode of a vector is the component which occurs most often. The derivative $V'$ of a nonincreasing vector $V$ is the vector of components $V'(i) = V(i) - V(i+1)$.

If $G$ is a simple graph of order $n$ with vertices labelled $1, \ldots, n$, the adjacency matrix $A$ of $G$ is the symmetric matrix with entries $a_{ij} = 1$ or 0, according as $ij$ is or is not an edge of $G$. The eigenvalues of $G$ are those of $A$, and their set is independent of the labelling of the vertices. Since $A$ is real and symmetric, these eigenvalues are real numbers and we label them in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_n$. The separator of a graph is the difference between the largest and the second largest eigenvalues of its adjacency matrix.

If the name of an invariant $M$ is preceded by the prefix $\text{mis}$ then it denotes the value $\tilde{M}$ of $M$ computed for the complement $\tilde{G}$ of $G$. The notation $bi-M$ denotes $M + \text{mis-M}$. The only exception is the notation $\tilde{d}$ for the average degree $2m/n$ of $G$.

We are interested, in this paper, in proving or disproving some conjectures related to the eigenvalues of a graph. These conjectures are stated as inequalities involving eigenvalues and other parameters. Most often, we establish general properties, from which we deduce the conjecture as a particular case. When possible, we study the graphs realizing equality.
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First of all, let us recall some definitions and results about eigenvalues, in general, and about the spectrum of a graph in particular. Most parts of these statements can be found in [3], the relevant pages of which we quote for convenience.

1.2. Known results about eigenvalues of a graph

Note that the spectrum of a disconnected graph is simply the disjoint union of the spectra of its components; so, in many cases one may restrict one’s study to connected graphs. Recall that the spectrum of an empty graph of order $n$ is $\lambda_1 = \cdots = \lambda_n = 0$, the spectrum of a clique $K_n$ is $\lambda_1 = n - 1$, $\lambda_2 = \cdots = \lambda_n = -1$, and that of a complete bipartite $K_{r,s}$ is $\lambda_1 = \sqrt{pq}$, $\lambda_2 = \cdots = \lambda_{n-1} = 0$, $\lambda_n = -\sqrt{pq}$ [3, p. 72].

The values of the traces of $A$ and of $A^2$ yield the following properties:

1. For any graph of order $n$, $\lambda_1 + \cdots + \lambda_n = 0$.
2. If $G$ is nonempty, $\lambda_1 > 0 > \lambda_n$.
3. For any graph of order $n$ and size $m$, $\lambda_1^2 + \cdots + \lambda_n^2 = 2m$.

We may consider $A$ as the matrix of a quadratic form $Q$ on $\mathbb{R}^n$ and, thus, the eigenvalues $\lambda_1$ and $\lambda_n$ are the maximum and minimum values of $Q(X)$ on the compact $S^{n-1} = \{X \mid \|X\| = 1\} \subset \mathbb{R}^n$. The other eigenvalues are critical values of $Q_{|S^{n-1}}$ taken on unit associated eigenvectors. Recall also that eigenspaces corresponding to different eigenvalues are orthogonal in the euclidean space $\mathbb{R}^n$. Consequently, we have the following proposition.

Proposition 1.1. For any vector of $\mathbb{R}^n$, $\lambda_n \|X\|^2 \leq Q(X) \leq \lambda_1 \|X\|^2$.

Proposition 1.2. For any graph, $|\lambda_i| \leq \lambda_1$ for $i > 2$.

Generally, this inequality is strict according to the two following results.

Proposition 1.3. $\lambda_1$ has an eigenvector with nonnegative coordinates. If an eigenvector $X$ has all its coordinates nonnull and of the same sign, it is associated with $\lambda_1$. The converse is true iff $G$ is connected and, in this case, $\lambda_1$ is simple.

Proposition 1.4. If $G$ is connected, the equality $\lambda_n = -\lambda_1$ holds if and only if $G$ is bipartite. Moreover, the spectrum of a bipartite graph is symmetric.

If $G$ is not connected, one needs the stricter hypothesis that the spectrum is symmetric to conclude that $G$ is bipartite. Nevertheless, $\lambda_n = -\lambda_1$ implies at least that $G$ has a bipartite component.

Some results on eigenvalues are peculiar to regular graphs.

Proposition 1.5. If $G$ is $d$-regular, $\lambda_1 = d$ with eigenvector $1 = (1, \ldots, 1)$, and the $(n-1-d)$-regular graph $\overline{G}$ has spectrum $\lambda'_1 = n-1-d$, $\lambda'_2 = -\lambda_n - 1$, $\ldots$, $\lambda'_n = -\lambda_1 - 1$. 

Some inequalities on $\lambda_1$ proceed from these simple considerations.

**Proposition 1.6.** For any graph, $\lambda_1 \leq \Delta$. Equality holds if and only if $G$ has a $\Delta$-regular component.

**Proposition 1.7.** For any graph, $\lambda_1 \geq \bar{d} - 2m/n$. Equality holds iff $G$ is $\bar{d}$-regular.

Combining Propositions 1.1 and 1.3, we obtain the following proposition.

**Proposition 1.8.** If $H$ is a partial subgraph $G$, $\lambda_1(H) \leq \lambda_1(G)$.

Since $G$ has a $K_{1, \Delta}$ as a partial subgraph, we get the following corollary.

**Corollary 1.9.** For any graph, $\lambda_1 \geq \sqrt{\Delta}$.

**Theorem 1.10** (cf. Edwards and Elphick [4]). For any graph, $\lambda_1 \leq \sqrt{2m(\chi - 1)/\chi}$.

Other inequalities use more elaborate theorems, the basis of which is the principle of interlacing. This principle derives from the fact that subspaces of $\mathbb{R}^n$, with dimensions $p$ and $q$ such that $p + q \geq n + 1$, have necessarily a unit vector in common; thus, by application of Proposition 1.1, we obtain the following theorem.

**Theorem 1.11.** Let $F$ be a subspace of dimension $p$ of $\mathbb{R}^n$ and $Q$ a quadratic form on $\mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_n$ ($\lambda_1', \ldots, \lambda_p'$) be the eigenvalues of a representative matrix of $Q (Q/F)$ in an orthonormal basis of $\mathbb{R}^n (F)$. Then $\lambda_i \geq \lambda_i' \geq \lambda_{n+i-p}$ for $1 \leq i \leq p$.

An immediate interpretation in terms of graphs gives the following corollary.

**Corollary 1.12.** Let $G$ be a graph of order $n$ and $H$ an induced subgraph of order $p$. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of $G$ together with the eigenvalues $\lambda_1', \ldots, \lambda_p'$ of $H$ satisfy the inequalities

$$
\lambda_i \geq \lambda_i' \geq \lambda_{n+i-p} \quad \text{for} \ 1 \leq i \leq p.
$$

Corollary 1.12 has two applications important for our purpose. By a consideration of the subgraph induced by an independent set of $G$ of maximum order, one obtains [3, p. 88] the following proposition.

**Proposition 1.13.** Let $p_0$ be the multiplicity of 0 as an eigenvalue of $G$, and $p_-$ ($p_+$) the number of negative (positive) eigenvalues of $G$. Then $\alpha \leq p_0 + \min(p_-, p_+)$.

This property allows us to define new terms, which were first introduced by Fajtlowicz [6]. We say that a graph $G$ is a heliotropic (geotropic) plant if we have $\alpha = p_0 + p_+ (\alpha = p_0 + p_-)$. Fortunately, trees are both heliotropic and geotropic plants.
Since $G$ necessarily has an induced subgraph with minimum degree $\chi - 1$, Corollary 1.12 combined with Proposition 1.7 gives a theorem of Wilf [17] (cf. also [3, p. 90]).

**Proposition 1.14.** For any graph, $\chi \leq \lambda_1 + 1$.

Another interpretation of this principle leads to a theorem of Courant–Weyl [16] (cf. also [3, p. 51]).

**Theorem 1.15.** Let $A, A_1, A_2$ be three $n \times n$ real symmetric matrices such that $A = A_1 + A_2$. The eigenvalues of these matrices satisfy the following inequalities: for $1 \leq i \leq n$ and $0 \leq j \leq \min\{i-1, n-i\}$, $\lambda_i - j(A_1) + \lambda_1 + j(A_2) \geq \lambda_i(A) \geq \lambda_i + j(A_1) + \lambda_{n-j}(A_2)$.

Applying this theorem with $A$ the adjacency matrix of $K_n$, one obtains the following corollary.

**Corollary 1.16.** Let $G$ be a graph of order $n$, with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\bar{G}$ the complementary graph, with eigenvalues $\lambda'_1, \ldots, \lambda'_n$. We have the inequality $\lambda_1 + \lambda'_1 \geq n - 1$ and, for $n \geq k \geq 2$, the inequalities $-\lambda_{n-k+2} - 1 \geq \lambda'_{k-1} \geq -\lambda_{n-k+1} - 1$ as well as the corresponding inequalities obtained by interchanging the roles of $G$ and $\bar{G}$.

A more intricate application of the interlacing principle leads to a result due to Hoffmann [11].

**Theorem 1.17.** Let $A = (A_{ij})_{i,j=1}^{p}$ be a partition of a real symmetric matrix of order $n$ into blocks, such that the diagonal blocks $A_{ii}$ are square (symmetric) matrices of orders $n_i$ (with $n = n_1 + \cdots + n_p$). The eigenvalues of $A$ and of its diagonal blocks satisfy the following inequalities:

$$\quad \text{for } j_k \in [1 \cdots n_k] \text{ and } 1 \leq k \leq p, \quad \lambda_{j_1} + \cdots + j_p(A) + \sum_{i=1}^{p-1} \lambda_i(A) \geq \sum_{k=1}^{p} \lambda_{j_k}(A_{kk}).$$

The Laplacian of a graph $G$ is the matrix $D - A$, where $D$ is the diagonal matrix with entries equal to the degrees of the vertices. Like the adjacency matrix, it is a real symmetric matrix. According to [12], we sort in increasing order the (real) eigenvalues of the Laplacian $A_1 \leq \cdots \leq A_n$. The quadratic form associated to this matrix is $Q_L = \sum_{i,j \in E} (x_i - x_j)^2$, a positive form. Therefore, the eigenvalues are nonnegative and the kernel of $D - A$ is the set of isotropic vectors of $Q_L$ and $A = 0$ is an eigenvalue with multiplicity equal to the number of connected components of $G$.

As for the ordinary spectrum, we note that $\text{Tr}(D - A) = 2m$ is the sum of the eigenvalues and $\text{Tr}((D - A)^2) = \sum_{i \in V} d_i^2 + 2m$ is the sum of their squares. Note that Proposition 1.1 implies inequalities for the maximum eigenvalue.
2. Proofs of conjectures and other results

The first result provides bounds for \( \lambda_1 \) which improve the inequality of Proposition 1.6.

**Theorem 2.1.** Let \( B=(a_{ij}) \) be a real symmetric matrix of order \( n \) and suppose \( \lambda \) is an eigenvalue of \( B \) having an eigenvector \( X=(x_1,\ldots,x_n) \) with nonnegative coordinates. Let \( Y=(y_1,\ldots,y_n) \) be a vector with arbitrary positive coordinates and, for \( 1 \leq i \leq n \), put \( y_i^*=(1/y_i) \sum_{j} a_{ij} y_j \). Then \( \min_{1 \leq i \leq n} y_i^* \leq \lambda \leq \max_{1 \leq i \leq n} y_i^* \).

**Proof.** We have the equalities \( \lambda \sum_{1 \leq i \leq n} x_i y_i = Y'BX = X'BY = \sum_{1 \leq i \leq n} x_i^* y_i^* \), and the last quantity belongs to the interval \( [ak, aK] \), where \( k=\min_{1 \leq i \leq n} y_i^* \), \( K=\max_{1 \leq i \leq n} y_i^* \) and \( a \) is the positive number \( \sum_{1 \leq i \leq n} x_i y_i \). It remains to divide by this quantity to obtain the required inequalities. \( \square \)

Equality holds (in both lower and upper bounds) if and only if \( Y \) is an eigenvector associated to \( \lambda \). Note that \( Y \) is not orthogonal to \( X \) and, thus, could not be associated to another eigenvalue.

Let us first take for \( B \) the adjacency matrix \( A \) of a graph \( G \). From Section 1.2 we know that \( \lambda_1 \) satisfies the hypothesis of Theorem 2.1. If we choose \( y_i = 1 \) we obtain the result of Proposition 1.6 as a corollary. By choosing \( y_i = d_i \), we obtain the next result, which was independently proved by Shearer.

**Corollary 2.2 (Conjectures 256 and 347).** For any graph without isolated vertices, minimum (Dual Degree) \( \leq \lambda_1 \leq \) maximum (Dual Degree).

Equality holds in this corollary if and only if every vertex has the same dual degree; in other words, if and only if \( G \) is dual-degree-regular. Note that since the characteristic polynomial can have only integers as rational roots, a dual-degree-regular graph has integer dual degree. We have no characterization of such graphs. As an example of a dual-degree-regular graph, consider a disjoint union of cycles of total order \( p \) and denote the vertices as \( x_1,\ldots,x_p \). Next, partition in an arbitrary way the vertices of \( p \) copies of \( K_2 \) into \( p \) subsets \( X_1,\ldots,X_p \) of two vertices, then join every \( x_i \) to the two vertices of \( X_i \). The graph obtained in this way is of dual degree 3 in each vertex without being regular. A generalization of this construction may be described in the following manner: let \( d_i = a_i d'_i \), \( i = 1, 2 \), be factorizations of two integers such that \( a_1 \neq a_2 \). Let \( k_1 = a_1 d'_2 \) and \( k_2 = a_2 d'_1 \) and take, for \( i = 1, 2 \), a \( d_i \)-regular graph \( G_i \) of order \( n_i \) with \( k_2 n_1 = k_1 n_2 \). By adding edges between the two graphs in such a way that every vertex of \( G_i \) is joined to exactly \( k_i \) vertices of the other graph, we obtain a graph of dual degree \( d_1 + d_2 \) which is not regular.

Note that if we proceed further in the process defining the dual degree, that is, if we calculate for every vertex the average dual degree of its neighbours and so on, we never achieve a constant average if the graph is not dual-degree-regular. In fact, if we
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denote by $D$ the diagonal matrix with entries equal to the degrees and by $d$ the vector $(d_1, \ldots, d_n)$, the process consists in calculating the vector $d_k = (D^{-1} A)^k d$ until we obtain a constant vector $\delta_k 1$. But this is equivalent to the property that $d - \delta_k 1$ is in the kernel of the matrix $(D^{-1} A)^{k+1}$. Since the matrix $D^{-1} A$ is similar to the symmetric matrix $D^{-1/2} A D^{-1/2}$, it is a diagonalizable matrix and the kernel of $(D^{-1} A)^k$ is the same as that of $D^{-1} A$. Therefore, if $d_k$ is constant, the vector Dual Degree is already constant.

Let us now take $B = A^2$ and $\lambda = \lambda^2$. Choosing $y_1 = 1$, we obtain $y_i^* = d_i d_i^*$ and, thus, the following corollary.

**Corollary 2.3.** For any graph, $\lambda_1 \leq \max_{i \in V} \sqrt{d_i d_i^*}$.

Equality holds if and only if $Y$ is an eigenvector and thus, if and only if $d_i d_i^*$ is constant. The graphs having such a property are characterized in the following lemma.

**Lemma 2.4.** Let $a$ be a nonzero real number and let $G$ be a graph such that, for every nonisolated vertex $i$, the quantity $(1/d_i) \sum_{j \in N(i)} (d_j d_i)^a$, where $N(i)$ denotes the neighbourhood of $i$, is a constant $k$. Then, for every edge $ij$, the quantity $d_i d_j$ is a constant $C = k^{1/a}$ and the components of $G$ are either $C$-regular or bipartite semiregular.

**Proof.** We only give the proof in the case where $a > 0$ since it is similar in the other case. Let us consider a connected component $G'$ of $G$. If $i$ is a vertex of minimum degree $\delta$ we find $k = (1/\delta) \sum_{j \in N(i)} (\delta d_j)^a \leq (\delta A)^a$, whereas if $i$ is of maximum degree $A$ we find $k \geq (\delta A)^a$. Thus, $k = (\delta A)^a$ and every neighbourhood of a vertex of minimum degree is $\delta$-regular and conversely. Since $G'$ is connected, proceeding from one vertex to any neighbour shows that every vertex is of degree either $\delta$ or $A$, with neighbourhood regular of the other degree.

For $a = 1$, the hypothesis means that $G$ is $dd^*$-regular. Thus, the set of connected $dd^*$-regular graphs is the union of the set of regular connected graphs and of the set of bipartite semiregular ones.

Note that, if $t_i$ denotes the number of triangles in $G$ having $i$ as one of its vertices, we have $d_i d_i^* \leq m + t_i$. Thus, as a Corollary of 2.3, we again obtain a result of Nosal [13] (cf. also [3, p. 86]).

**Corollary 2.5.** For any $K_3$-free graph, $\lambda_1 \leq \sqrt{m}$.

If $G$ is, furthermore, of girth $g \geq 5$ we have $d_i d_i^* < n - 1$ since $i$ is the only common neighbour of its neighbours. Therefore, we obtain the following corollary.

**Corollary 2.6.** For any graph with $g \geq 5$, $\lambda_1 \leq \sqrt{n - 1}$.
Equality holds if and only if $G$ is of diameter 2 and is, moreover, regular or bipartite semiregular according to the previous lemma applied with $a = \frac{1}{2}$. The only cases of equality are, thus, the graphs isomorphic to $K_{1,p}$ or to one of the Moore graphs of diameter 2: the cycle $C_5$, the Petersen graph, the Hoffmann–Singleton graph, and the 57-regular graph on 3250 vertices, if it exists.

Combining the result of Corollary 2.6 with Proposition 1.7, we obtain an upper bound on the size of a graph of girth $g \geq 5$ which can be compared to the bound $1 + n + (n/2)\sqrt{n}$ for the graphs without cycles of length 4, given in [1].

Corollary 2.7. For any graph with $g \geq 5$, $m \leq (n/2)\sqrt{n} - 1$.

For this new bound, equality holds for the 5-cages previously described and for $K_2$.

Similarly, as in Proposition 1.7, one can obtain better lower bounds on $\lambda_1$ that are currently less known.

Proposition 2.8. For any graph,

$$\lambda_1 \geq \sqrt{\frac{1}{n} \sum_{i \in V} d_i^2}$$

and

$$\lambda_1 \geq \frac{1}{m} \sum_{i,j \in E} \sqrt{d_id_j}.$$

The latter is obtained by taking the value of $Q$ on the vector of coordinates $\sqrt{d_i}$, the norm of which is $\sqrt{2in}$, and the former by considering the matrix $A^2$ which has as eigenvalues the squares of those of $A$ and, thus, has maximum eigenvalue $\lambda^2_1$ by Proposition 1.2. Applying Proposition 1.1 to its associate quadratic form on the vector $1 = (1, \ldots, 1)$ gives the inequality. Both bounds are better than the bound of Proposition 1.7 since, by Schwarz's inequality, we have $\sum_{i \in V} d_i^2 \geq (1/n) (\sum_{i \in V} d_i)^2$ and $\sum_{i,j \in E} \sqrt{d_id_j} \leq \sum_{i,j \in E} 1/\sqrt{d_id_j} \geq m^2$ and since $\sum_{j \in E} 1/\sqrt{d_i d_j} \leq \frac{1}{2} \sum_{j \in E} (1/d_i + 1/d_j) = \frac{1}{2} \sum_{i \in V} 1 = n/2$.

According to Proposition 1.3, equality holds in the former if and only if $1$ is an eigenvector of $A^2$, that is, if and only if, for every vertex $i$, the quantity $\sum_{j \in N(i)} d_j = (1/d_i) \sum_{j \in N(i)} d_id_j$ (where $N(i)$ denotes the neighbourhood of $i$) is a constant independent of $i$. Similarly, equality holds in the latter if and only if the vector $(\sqrt{d_1}, \ldots, \sqrt{d_n})$ is an eigenvector of $A$, that is, if and only if $(1/d_i) \sum_{j \in N(i)} \sqrt{d_id_j}$ is constant. Actually, these two conditions are equivalent by Lemma 2.4, which, moreover, gives a characterization of the graphs realizing equality.

The two lower bounds $\lambda_1 = \sqrt{(1/n) \sum_{i \in V} d_i^2}$ and $\lambda_2 = (1/m) \sum_{i,j \in E} \sqrt{d_id_j}$ are incomparable, as one can see by means of the two following examples.

Let $S_n(p)$ consist of a path of length $p$ on the vertices $x_1, \ldots, x_{p+1}$ and $p-1$ extra vertices $x_{p+2}, \ldots, x_{2p}$, each of them adjacent to $x_{p+1}$. We obtain a tree of order $n-2p$,
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Thus, \( b_2 < b_1 \) for \( p \) sufficiently large.

On the other hand, consider two \( K_p \)'s and a path on \( 2p + 2 \) vertices. We identify one extremity of the path with one vertex of a clique and the second extremity with one vertex of the other clique in order to obtain a barbell \( G \). This graph is of order \( n = 4p \) and size \( m = p^2 + p + 1 \) and we get

\[
\frac{b_1}{\sqrt{p}} \xrightarrow{p \to \infty} \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{b_2}{\sqrt{p}} \xrightarrow{p \to \infty} \frac{1}{2}.
\]

Thus, \( b_1 < b_2 \) for \( p \) sufficiently large.

**Corollary 2.9.** For any graph of diameter 2, \( \lambda_1 \geq \sqrt{n-1} \).

**Proof.** Since the diameter is 2, we have, for any vertex \( i, \sum_{j \in N(i)} d_j = d_i d_i^* \geq n - 1 \); thus, by summation on the set of vertices, we obtain \( \sum_{i \in V} d_i^2 = \sum_{i \in V} d_i d_i^* \geq n(n - 1) \). The corollary now results from the first inequality of Proposition 2.8. \( \square \)

It is interesting to connect this result to that of Corollary 2.6. The cases of equality are obviously the same.

**Corollary 2.10 (Conjecture 617).** For any graph of diameter 2, \( n^{1/2} \leq \text{scope of eigenvalues} \).

**Proof.** Since \( G \) is not a stable set, \( |\lambda| \geq 1 \) and, from the previous theorem, we obtain a scope of eigenvalues \( \geq \sqrt{n-1} + 1 \geq \sqrt{n} \). \( \square \)

Equality never holds for a graph of diameter 2.

**Corollary 2.11.** For any graph, \( \lambda_1 \geq m/R_c \).

**Proof.** This results from the second lower bound \( b_2 \) of Proposition 2.8 via Schwartz's inequality \((\sum_{i \in E} \sqrt{d_i d_j})(\sum_{i \in E} 1/\sqrt{d_i d_j}) \geq m^2 \), thus giving \( b_2 \geq m/R_c \). \( \square \)

Equality in this corollary holds if and only if \( G \) is \( dd^* \)-regular.

**Corollary 2.12 (Conjecture 213).** In any \( K_3 \)-free graph, \( \sqrt{m} \leq R_c \).

**Proof.** By Corollary 2.5, \( \lambda_1 \sqrt{m} \leq m \) and, by Corollary 2.11, \( m/\lambda_1 \leq R_c \); thus, \( \sqrt{m} \leq R_c \). \( \square \)
We have equality in the second relation if and only if $G$ is $dd^*$-regular, and total equality requires that $\lambda_1 = \sqrt{m}$ and that $G$ is regular or semiregular in each component. Then in the first case $\lambda_1 = d = 2m/n$, $n$ is even, $m = n^2/4$ and $G \cong K_{n/2,n/2}$. In the second case $m = n_1 d_1 = n_2 d_2$, $\lambda_1 = \sqrt{d_1 d_2} = \sqrt{m}$ and $G \cong K_{d_1,d_2}$. In both cases $\lambda_1 = \sqrt{m} = R_c$. Note that this result can be proved directly without using $\lambda_1$, by induction on $n$.

Using again Corollary 2.5, we obtain by this result a proof of Conjecture 116, saying that $\lambda_1 \leq R_c$, for any $K_3$-free graph. Equality holds in this last result if and only if $G$ is the disjoint union of an empty graph and of complete bipartite graphs.

We will now establish some inequalities also involving the other eigenvalues.

**Proposition 2.13.** For any graph $G$, $\lambda_1 + |\lambda_i| \leq 2\sqrt{m}$ for $2 \leq i \leq n$.

**Proof.** From property (3) in Section 1.2 and Schwarz’s inequality, $\lambda_1 + |\lambda_i| \leq \sqrt{2(\lambda_1^2 + \lambda_i^2)} \leq 2\sqrt{m}$. □

**Corollary 2.14.** For any graph $G$, $|\lambda_i| \leq \sqrt{m}$ for $2 \leq i \leq n$.

The case $i = n$ (also given by Powers [15]) was the question of Conjecture 160.

**Corollary 2.15.** For any graph $G$, $|\lambda_i| \leq m/\lambda_1$.

**Proof.** This inequality follows from Proposition 2.13 and from the trivial one $\lambda_1 + |\lambda_i| \geq 2\sqrt{\lambda_1 |\lambda_i|}$. □

**Corollary 2.16.** For any graph $G$, $|\lambda_i| \leq R_c$ for $2 \leq i \leq n$.

**Proof.** Combining Corollary 2.10 and 2.14, we obtain $|\lambda_i| \leq m/\lambda_1 \leq R_c$. □

In the four preceding results, equality holds for $\lambda_n$ if and only if $G$ is the union of a complete bipartite graph and isolated vertices and for $\lambda_2$ if and only if $G$ is isomorphic to $K_2$.

For the special cases $i = 2, n$, this inequality answers Conjectures 556 and 19, respectively, and also proves Conjecture 713 stating that the absolute average of nonpositive eigenvalues is no more than the Randic. Moreover Conjecture 200, stating that minimum positive eigenvalue $\leq R_c$, is proved if $G$ is not the union of a complete multipartite graph and isolated vertices.

**Theorem 2.17.** For any graph $G$, $-\lambda_n \leq \max_{V_1 \cup V_2 = V} \sqrt{e(V_1, V_2)}$, where $e(V_1, V_2)$ is the number of edges between two sets $V_1$ and $V_2$ making up a partition of $V$. 

Proof. If $G = \overline{K}_n$, we have $\lambda_n = 0$ and the result is obviously true. Otherwise, we may suppose that $G$ is connected since $\lambda_n(G) = \lambda_n(G')$, where $G'$ is one of the connected components of $G$.

Let $X = (x_1, \ldots, x_n)$ be a unitary eigenvector of $\lambda_n$. We label the vertices of $V$ in order to have $x_1, \ldots, x_p > 0$ and $x_{p+1}, \ldots, x_n \leq 0$. Let $|X|$ denote the unitary vector $(|x_1|, \ldots, |x_n|)$.

We have

$$-\lambda_n = 2 \sum_{1 \leq i < j \leq n} a_{ij} |x_i||x_j| - 2 \sum_{1 \leq i < j \leq n} a_{ij} |x_i||x_j|$$

so,

$$-\lambda_n \leq 2 \sum_{1 \leq i < j \leq n} a_{ij} |x_i||x_j| = Q_1(|X|),$$

where $Q_1$ is the quadratic form associated with the subgraph $G_1$ obtained from $G$ by deleting every edge $ij$ with $i$ and $j$ both in $[1 \ldots p]$ or both in $[p+1 \ldots n]$. Since $G_1$ is bipartite, we have

$$-\lambda_n \leq \lambda_1(G_1) \leq \sqrt{m(G_1)}.$$ 

Then the theorem is true. \qed

Corollary 2.18. $-\lambda_n \leq \left\{ \begin{array}{ll} n/2 \\ \sqrt{(n^2 - 1)/2} \end{array} \right.$ if $n$ even, if $n$ odd.

Proof. This follows immediately from the theorem since $m(G_1) \leq \lceil n^2/4 \rceil$. \qed

Equality holds if and only if $G$ is a complete bipartite graph $K_{n/2, n/2}$ with $n$ even, or $K_{(n-1)/2, (n+1)/2}$ with $n$ odd.

As a corollary we obtain the inequality of Conjecture 199, which says that

$$-\lambda_n \leq \text{mean (Even)} \text{ when } \sum (\text{Odd}) \leq \sum (\text{Even}).$$

Corollary 2.19 (Conjecture 257). For any graph, $-\lambda_n \leq \nu + \overline{\nu}$.

Proof. It results from a slight refinement, given in the following lemma, to the well-known bound $\nu + \overline{\nu} \geq \lceil n/2 \rceil$. Details of the proof can be found in [8]. \qed

Lemma 2.20. If $G$ is $K_n$ or $\overline{K}_n$, with $n$ odd, we have $\nu + \overline{\nu} = (n-1)/2$; otherwise, $\nu + \overline{\nu} \geq \lceil n/2 \rceil$.

Equality in Corollary 2.19 holds only when $G = K_1$ for the first case of Lemma 2.20. Whereas for the second case, it must hold for Corollary 2.18. Since for $G = K_{n/2, n/2}$ with $n$ even we have $\nu + \overline{\nu} = n/2 + \lceil n/4 \rceil$ and for $G = K_{(n-1)/2, (n+1)/2}$ with $n$ odd we have

$$\nu + \overline{\nu} = (n-1)/2 + \lceil (n-1)/4 \rceil + (n+1)/4,$$

equality holds only for $G = K_1$ and $G = K_2$.

Corollary 2.21. For any graph, $\overline{\nu} \leq n/2 - 1$. 
**Proof.** By Corollary 1.16, we have \( \lambda_2 \leq -\lambda_n(\bar{G}) - 1 \) and the inequality results from Theorem 2.17 applied to the complementary graph \( \bar{G} \). \( \square \)

This result provides an answer to Conjecture 196, which says that for every graph \( G \) such that \( \sum(\text{Odd}) \leq \sum(\text{Even}) \), \( \lambda_2 \leq \text{mean of Even} \).

Note that the results of Corollaries 2.18 and 2.21 are already known [14]. In the case of regular graphs we can improve the result of Corollary 2.19.

**Proposition 2.22.** If \( G \) is a d-regular graph, then \(-\lambda_n \leq \min(d, n - d) \leq \lfloor n/2 \rfloor\).

**Proof.** Using Propositions 1.5 and 1.2 yields \(-\lambda_n \leq \lambda_1 = d\), and \(-\lambda_n = \lambda_2 + 1 \leq n - d\). \( \square \)

The cases of equality \(-\lambda_n = \lfloor n/2 \rfloor\) are the following:

- If \( n \) is even, \(-\lambda_n = n/2 = \lambda_1 = d\) and \( G \cong K_{n/2, n/2} \).
- If \( n \) is odd, \(-\lambda_n = (n - 1)/2\) and either \( d = (n - 1)/2\) or \( d = (n + 1)/2\). But the first case is impossible and then the case of equality is described by \( V(G) = \{x_1, \ldots, x_{(n-1)/2}, y_1, \ldots, y_{(n+1)/2}\} \), with \( n \equiv 3 \pmod{4} \) and \( E(G) \) contains all the edges \( x_i y_j \) and the edges \( y_k y_{k+1} \) only for \( k \) odd.

**Corollary 2.23 (Conjecture 43).** If \( G \) is d-regular then \(-\lambda_n \leq v\).

**Proof.** By Vizing's theorem, \( v \geq nd / (d + 1) \). On the other hand, \( v \leq \lfloor n/2 \rfloor \).

- If \( d \leq n/2 - 1 \) then \( d \leq nd / (d + 1) \) and \(-\lambda_n \leq d \leq v\). Equality is impossible.
- Otherwise, \( n/2 - 1 < nd / (d + 1) \); thus, \( v = \lfloor n/2 \rfloor \). So, \(-\lambda_n \leq v\) by Corollary 2.22. \( \square \)

The cases of equality of Conjecture 43 are those described in Corollary 2.22 since they also satisfy \( v = -\lambda_n = n/2 \) if \( n \) even and \( v = -\lambda_n = (n - 1)/2 \) if \( n \equiv 3 \pmod{4} \).

**Theorem 2.24 (Conjecture 138).** For any graph \( G \neq K_2 \), \( |\lambda_2| \leq m/\omega \).

**Proof.** The result is obvious if \( G \) is a clique. So, we may suppose \( \omega \leq n - 1 \); thus, \( \lambda_2 \geq 0 \). From the inequality \( \lambda_2 \leq m/\lambda_1 \) of Corollary 2.13 the theorem is true if \( \lambda_1 \geq \omega \). Suppose now \( \lambda_1 < \omega \). Proposition 1.14 gives, in this case, \( \chi = \omega \). Let \( A \) be the adjacency matrix of \( G \), \( J_{\omega} \) that of \( K_\omega \), and \( A' \) that of the graph \( G' = G \setminus E(K_\omega) \). We may write \( A = A_1 + A' \), where

\[
A_1 = \begin{pmatrix}
J_\omega & 0 \\
0 & 0
\end{pmatrix}.
\]

From Theorem 1.15, we obtain \( \lambda_2 \leq \lambda_2(A_1) + \lambda_1(A') = \lambda_1(A') \) since \( \omega \leq n - 1 \). From Theorem 1.10, we get \( \lambda_1(A')^2 \leq 2m'(\chi' - 1)/\chi' \), where \( m' \) is the size of \( G' \), \( m' = m - \omega(\omega - 1)/2 \), and \( \chi' \) is the chromatic number of \( G' \). Therefore, \( \lambda_2^2 \leq 2m'(\chi' - 1)/\chi' \leq 2m'(\chi' - 1)/\chi = 2m(\omega - 1)/\omega(\omega - 1)^2 \leq m^2/\omega^2 \). \( \square \)

We also obtain a result similar to the previous one.
Theorem 2.25. For any graph $G$ of order at least 3, $(\lambda_2 + |\lambda_n|)/2 \leq m/\omega$.

Proof. In this proof and that of the following lemma, we may obviously suppose that $G$ has no isolated vertices.

Lemma. 2.26. For any graph, $\lambda_1 \leq 2m/\omega$.

Proof of the Lemma. If $G$ is a clique, $2m/\omega = \omega - 1 = \lambda_1$. Otherwise, by the interlacing theorem, we have, for $n-\omega + 2 \leq i \leq n$, $\lambda_i \leq 1$. Moreover, either $G$ contains a disjoint union of two cliques, in which case $\lambda_{m-\omega+1} = -1$, either $G$ contains a path $P_2$ of length 2 as induced subgraph and $\lambda_n \leq -\sqrt{2} = \lambda_2(P_2)$ by the interlacing Corollary 1.12. In both cases we have $\lambda_1^2 \leq 2m - \omega$ and the result is immediate. □

Equality holds if and only if $G$ is a clique or a union of a clique and a stable set.

Let us now prove the theorem. First of all we prove it for the case $\omega = 2$. We have, by Corollaries 2.12 and 2.14, $(\lambda_2 + |\lambda_n|)/2 \leq (n-1)/2$, which is at most $m/2$ if $G$ is connected. If $G$ is not connected, but if $\lambda_2$ and $\lambda_n$ belong to the spectrum of the same component $G_i$ of size $m_i$, the result is a fortiori true. Otherwise, let $\lambda_2$ belong to the spectrum of a component $G_1$ of size $m_1$, and $\lambda_n$ belong to that of $G_2$ of size $m_2$. We deduce from the previous lemma that $\lambda_2 \leq \lambda_1(G_1) \leq m_1/2$, $|\lambda_n| \leq \lambda_1(G_2) \leq m_2/2$; therefore, $(\lambda_2 + |\lambda_n|)/2 \leq (m_1 + m_2)/2 \leq m/2$ and the proof is complete in the case $\omega = 2$.

Suppose now that we have $\omega \geq 3$. From Corollary 1.12 applied to the induced subgraph $K_\omega$ of $G$, we get $\lambda_1 \geq \omega - 1$ and $\lambda_i \leq 1$ for $n-\omega + 2 \leq i \leq n-1$; therefore, $\lambda_2^2 + \lambda_n^2 \leq 2m-(\omega - 1)^2-(\omega - 2) = 2m - \omega^2 + \omega + 1$ and, finally, $((\lambda_2 + |\lambda_n|)/2)^2 \leq \frac{1}{2}(\lambda_2^2 + \lambda_n^2) \leq m - (\omega^2 - \omega - 1)/2$, which is $\leq m^2/\omega^2$ for $\omega \geq 3$. □

Corollary 2.29 of the next theorem gives a partial answer to Conjecture 27, which says that, for any graph, the deviation of the Degree is at most the Randic.

Theorem 2.27. Let $Q$ be a real quadratic form with polar form $B$, $\lambda_1$ the largest and $\lambda_n$ the smallest eigenvalues of $Q$. Let $\| \cdot \|$ denote the euclidean norm in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ the associated scalar product. If $\langle X, Y \rangle = 0$ then $\|B(X, Y)\| \leq (\lambda_1 - \lambda_n)/2 \| X \| \| Y \|$.

Proof. The quadratic forms $Q_1(X) = \lambda_1 \| X \|^2 - Q(X)$ and $Q_n(X) = Q(X) - \lambda_n \| X \|^2$ are positive and, thus, verify Schwarz’s inequality $\|B(X, Y) - \lambda_i \langle X, Y \rangle\| \leq \sqrt{Q_i(X)Q_i(Y)}$ for $i = 1, n$. Applying these two inequalities to a pair $X, Y$ orthogonal in $\mathbb{R}^n$ and multiplying term by term, we get $\|B(X, Y)\|^2 \leq \sqrt{Q_1(X)Q_n(X)) \sqrt{Q_1(Y)Q_n(Y)}}$.

On the other hand, $\sqrt{Q_1(X)Q_n(X)} \leq \frac{1}{2}(Q_1(X) + Q_n(X)) = \frac{1}{2}(\lambda_1 - \lambda_n) \| X \|^2$. Thus, we obtain the desired inequality. □

Corollary 2.28. For any graph $G$, deviation(Degree) $\leq \frac{1}{2}(\lambda_1 - \lambda_n)$. 

Proof. Let us consider the quadratic form associated to the adjacency matrix of $G$. Let $1$ be the vector of $\mathbb{R}^n$ with all coordinates equal to 1 and $D$ the vector Degree. Note that we have $A1 = D$. Let $\bar{d}$ denote the mean degree (equal to $2m/n$) and $Y$ denote the vector $D - \bar{d}1$. So, $\langle 1, Y \rangle = n\bar{d} - \bar{d}111 = 0$ and $B(1, Y) = \sum_{i} d_i^2 - n\bar{d}^2 = nVar(D)$. The theorem then gives $nVar(D) \leq \frac{1}{n} \| \lambda_1 - \lambda_n \|_1 \| Y \| = \frac{1}{n} (\lambda_1 - \lambda_n) \sqrt{nVar(D)}$; hence, we obtain the desired result. \qed

Corollary 2.29. For every $K_3$-free graph $G$, deviation(Degree) $\leq Rc$.

Proof. This is a consequence of Corollary 2.12, Proposition 2.13 and Corollary 2.28. \qed

Theorem 2.30. For any graph, $n - \bar{x} \leq \lambda_1 + \cdots + \lambda_{\bar{x}}$.

Proof. By the definition of the mischromatic number, there exists a partition of $V$ into $\bar{x}$ sets $V_i$ such that the induced subgraphs $G(V_i)$ are cliques. In the adjacency matrix of $G$, for a suitable order of the vertices, the adjacency matrices of these subgraphs appear as principal submatrices $A_i$. Applying Theorem 1.17, we obtain

$$\lambda_1 + \cdots + \lambda_{\bar{x}} \geq \lambda_1(A_1) + \cdots + \lambda_1(A_{\bar{x}}) = (|V_1| - 1) + \cdots + (|V_{\bar{x}}| - 1) = n - \bar{x}.$$

\qed

Corollary 2.31 (Conjecture 706). For any graph, $v \leq \text{sum of positive eigenvalues}$.

Proof. Since $\bar{x} \leq n - v$ for any graph, Theorem 2.30 gives a better lower bound for the sum of positive eigenvalues, namely $n - \bar{x}$. \qed

Note that, in any graph, $n - \bar{x} \leq n - \alpha$. Since, by Proposition 1.13, the number $p_+$ of positive eigenvalues is at most equal to $n - \alpha$, Corollary 2.31 is a partial answer, in the case $\bar{x} = \alpha$, to Conjecture 20, which asks if the sum of the positive eigenvalues is at least equal to their number. Another partial result was proved by Fajtlowicz [5], who established that the sum of the absolute values of the nonzero eigenvalues is at least equal to their number.

Other conjectures concern the number $p_+$ of positive eigenvalues, among which we prove the following theorem.

Theorem 2.32 (Conjecture 258). For any graph $G$, $p_+ \leq v + \bar{v}$.

Proof. If $p_+ = 0$, $G$ is a stable set and the result is obvious. If $p_+ = 1$, $G$ is not a stable set; so, $v \geq 1$ and the result is also true. Suppose now $p_+ \geq 2$. By Proposition 1.13, $p_+ \leq n - \alpha$ and, since the complement of the set of vertices saturated by a maximum matching is a stable set, $n - \alpha \leq 2v$. By Corollary 1.16, we have, for $i \geq 2$, the inequalities $\lambda_i \leq -\lambda_{n+2-i} - 1$ relating the eigenvalues of $G$ and its complement $\bar{G}$. Since $p_+ \geq 2$, we obtain $0 < \lambda_{p_+} \leq -\lambda_{n+2-p_+} - 1$, implying $\lambda_{n+2-p_+} < -1 < 0$. Thus, the number of
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negative eigenvalues of $\bar{G}$ is at least $p_+ - 1$ and we have $p_+ - 1 \leq n - \bar{V} \leq 2\bar{V}$. So, we get $2p_+ - 1 \leq 2(v + \bar{V})$ but, since the two sides are integers of different parities, we obtain $2p_+ \leq 2(v + \bar{V})$ and the proof is complete. $\square$

The end of this section is devoted to some properties of the Laplacian spectrum. Let us first remark that Conjecture 297, saying that in a tree, $A_2 \leq n/\alpha$, is obvious since $A_2 \leq 1$ (see [12]).

Theorem 2.33. For any nonempty graph, $(1/2m)\sum_{i\in V} d_i^2 \leq A_n - 1$.

Proof. Let $ij$ be an arbitrary edge of $G$ and $X_{ij}$ the vector with the $i$th coordinate equal to 1, $j$th equal to $-1$ and other coordinates null. Applying Proposition 1.1 to the value of the laplacian quadratic form on $X_{ij}$, we obtain $d_i + d_j \leq 2(A_n - 1)$. Thus, for every edge, we have the inequality $d_i + d_j \leq 2(A_n - 1)$. Summing over the set of edges, we get $\sum_{i\in V} d_i^2 \leq 2m(A_n - 1)$. $\square$

Corollary 2.34. For any nonempty graph, $\bar{d} \leq A_n - 1$.

Proof. This results from Theorem 2.33 and Schwarz's inequality $(\bar{d})^2 \leq (1/n)\sum_{i\in V} d_i^2$. $\square$

Corollary 2.35 (Conjecture 378). For any nonempty graph, $n/\alpha \leq A_n$.

Proof. This results from the previous one and from Wei's theorem $\alpha \geq n/(\bar{d} + 1)$. $\square$

(The conjecture was proposed for a particular class of graphs, but is always true.)

The last two conjectures involve the deviation of eigenvalues of the Laplacian of a graph. Actually, the value of this parameter can be obtained by simply knowing the degree sequence. In fact, the sum of eigenvalues of the Laplacian, equal to its trace, is $2m$ and the sum of the squares is similarly, $\sum_{i\in V} d_i^2 + 2m$. Thus, we have

$$\text{deviation of eigenvalues of the Laplacian} = \sigma(A) = \left(\frac{1}{n}\sum_{i\in V} d_i^2 + \bar{d} - \bar{\bar{d}}^2\right)^{1/2}.$$ 

Theorem 2.36. For any graph, $\sigma(A) \leq n/2$.

Proof. It is obvious that $\sum_{i\in V} d_i^2 \leq (n - 1)\sum_{i\in V} d_i = 2m(n - 1)$; therefore,

$$\sigma(A) \leq (n - 1)\bar{d} + \bar{\bar{d}} - \bar{\bar{d}}^2)^{1/2} = (\bar{d}(n - \bar{d}))^{1/2} \leq n/2.$$ $\square$

Corollary 2.37 (Conjecture 190). For every graph such that $\sum_{\text{Odd}} \leq \sum_{\text{Even}}$, $\sigma(A) \leq \text{mean}(\text{Even})$. 


Proof. Obvious since the hypothesis is equivalent to \( \text{mean (Even)} \geq n/2 \). \( \square \)

Theorem 2.38 (Conjecture 253). For any graph, \( \sigma(A) \leq \text{maximum (Dual Degree)} \).

Proof. If the graph \( G \) has no isolated vertices,

\[
\bar{d} \geq 1 \quad \text{and} \quad \sigma(A) \leq \left( \frac{1}{n} \sum_{i \in V} d_i^2 \right)^{1/2},
\]

which is the first lower bound on \( \bar{\lambda}_1 \) given in Corollary 2.7.

By Corollary 2.2, we have also \( \bar{\lambda}_1 \leq \text{max (Dual Degree)} \) and in this case we are done.

If \( G \) has \( k \) isolated vertices we take 0 as the value of their dual degree. The previous proof now remains valid since we have \( \left( \frac{1}{n} \sum_{i \in V} d_i^2 + d - \bar{d} \right)^{1/2} \leq 1/(n-k) \sum_{i \in V} d_i^2 \). \( \square \)

3. Some particular spectra

Many counterexamples which are given in the following section involve simple constructions on graphs having known spectra. Such constructions make possible the calculation of the spectrum of the resulting graph. For instance, let us recall the known result on the cartesian sum. Let \( G' \) and \( G'' \) be two graphs. Their cartesian sum \( G' + G'' \) is the graph with vertex set \( V' \times V'' \) in which \( ((i', i'') (j', j'')) \) is an edge if and only if either \( i' = j' \) and \( i'' j'' \in E'' \) or \( j' = j'' \) and \( i' j' \in E' \).

Proposition 3.1. The spectrum of \( G' + G'' \) is the set \( \{ \lambda' + \lambda'' | \lambda' \in \text{Spec}(G'), \lambda'' \in \text{Spec}(G'') \} \).

Proof. Let \( X'=(x'_1, \ldots, x'_p) \) \( (X''=(x''_1, \ldots, x''_q)) \) be an eigenvector associated with \( \lambda' \) \( (\lambda'') \). Then it is easy to verify that the vector of coordinates \( x'_i x''_j \) is an eigenvector for \( G' + G'' \) associated with \( \lambda' + \lambda'' \). \( \square \)

We define the complete product \( G' \ast G'' \) of two graphs to be the graph obtained from their disjoint union by adding all the edges between them. In the case when the two graphs are regular, one obtains a simple result [9] (see also [3, p. 57]). For the sake of convenience, we denote by \( \text{Spec}'(G) \) the set \( \{ \lambda_2(G), \ldots, \lambda_n(G) \} \).

Proposition 3.2. Let \( G' \) \( (G'') \) be a \( d' \)-regular \( (d'' \)-regular) graph of order \( n' \) \( (n'') \). The spectrum of their complete product is the union of \( \text{Spec}'(G') \) and \( \text{Spec}'(G'') \) together with the two roots of the polynomial \( P(x)=x^2 -(d' +d'')x +(d'd'' -n'n'') \).

Proof. Let \( A'=(a'_{ij}) \) and \( A''=(a''_{ij}) \) be the adjacency matrices of the two graphs. Let \( x'_i, 1 \leq i \leq n' \), and \( x''_i, 1 \leq i \leq n'' \), denote the coordinates of a vector on the vertices of these
Some eigenvalue properties in graphs: We obtain the following equations for an eigenvector:

\[
\sum_{1 \leq j \leq n'} a_{ij} x_j + \sum_{1 \leq j \leq n''} c x_j = \lambda x_i, \quad 1 \leq i \leq n',
\]

\[
\sum_{1 \leq j \leq n'} x_j + \sum_{1 \leq j \leq n''} a_{ij} x_j = \lambda x_i, \quad 1 \leq i \leq n''.
\]

Letting \(x_j = 0\) in this system reduces it to the eigensystem of \(G'\) together with the equation \(\sum_{1 \leq j \leq n'} x_j = 0\), thus giving, by Proposition 1.5, the eigenvalues of \(\text{Spec}'(G')\). Similarly, by letting \(x_j = 0\), one obtains the eigenvalues of \(\text{Spec}'(G'')\). The remaining eigenvectors are in the orthogonal subspace \(x_j = x',\ x_j = x''\), in which the system reduces to

\[
\begin{align*}
\{ & d'x' + n''x'' = \lambda x', \\
& n'x' + d''x'' = \lambda x'',
\}
\]

thus, giving as eigenvalues the roots of \(P(x)\), which is the characteristic polynomial of the matrix

\[
\begin{pmatrix}
 d' & n'' \\
 n' & d''
\end{pmatrix}.
\]

Note that, since \(P(d' + d'') = d'd'' - n'n''\) is negative, the positive root of \(P\) is greater than \(d' + d''\); thus, by Proposition 1.5, this root is \(\lambda_1(G)\). The second root is negative but not necessarily equal to the smallest eigenvalue of \(G\).

Similar methods can be used to prove the following two results. Details may be found in [8].

**Proposition 3.3.** Let \(G'\) and \(G''\) be as in the previous proposition and \(G\) be the graph obtained from the disjoint union of these two graphs by adding an extra dominating vertex. The spectrum of \(G\) is the union of the two sets \(\text{Spec}'(G')\) and \(\text{Spec}'(G'')\) together with the roots of the polynomial

\[
P(x) = x^3 - (d' + d'')x^2 - (n' + n'' - d'd'')x + (n'd'' + n''d').
\]

**Proposition 3.4.** Let \(G'\) be a \(d'\)-regular graph of order \(n'\), \(G''\) be the bipartite complete graph \(K_p,q\), and \(G\) be the graph obtained from their disjoint union by adding a dominating vertex. The spectrum of \(G\) consists of \(\text{Spec}'(G')\), 0 with multiplicity \(p+q-1\) together with the roots of the polynomial

\[
P(x) = x^4 - d'x^3 - (n' + p + q + pq)x^2 + (d'(p + q + pq) - 2pq)x + (n' + 2d')pq.
\]

4. Disproved conjectures

We shall describe some infinite families of graphs, each of which disproves one or more conjectures. Details may be found in [8].
The family \( C_{n,k} \). We denote by \( C_{n,k} \) with \( 2 \leq k \leq n-2 \), the complete product of a clique \( K_k \) and an independent set \( K_{n-k} \).

We have even \( (x) \) equal to 1 for \( k \) vertices and to \( n-k \) for the others, giving \( \sum(\text{Even}) = k + (n-k)^2 \), whereas \( \sum(\text{Odd}) = n^2 - \sum(\text{Even}) \). Thus, \( \sum(\text{Even}) \leq \sum(\text{Odd}) \) if and only if \( f(k) \leq 0 \), with \( f(k) = 2k^2 - 2(2n-1)k + n^2 \), that is, if and only if \( k \) is between the two positive roots \( k_0 \) and \( k_1 \) of \( f(k) \). Note that, for \( n \geq 10 \), \( 3 < k_0 \leq n/2 - 1 < n < k_1 \).

From Proposition 3.2 the spectrum consists of \(-1\) with multiplicity \( k-1 \), \( 0 \) with multiplicity \( n-k-1 \), and the roots of \( P(A) = A^2 - (k-1)A - k(n-k) \). These two roots are \( \lambda_1 \) and \( \lambda_2 \) since they do not belong to the interval \([-1,0]\).

**Conjecture 207.** In every graph such that \( \sum(\text{Even}) \leq \sum(\text{Odd}) \), \( -\lambda_n \leq \nu \).

As a counterexample, we consider the previously described graph \( C_{n,k} \), with \( n \geq 22 \) and \( k \in [k_0,(n+1)/3] \).

The family \( J_{p,q} \). Let \( J_{p,q} \) be the graph obtained from \( K_p \cup K_1, \) by adding a dominating vertex \( x \). Let \( u \) denote the centre of the \( K_1, \) Note that \( n = p + q + 2, \) \( 2m = p^2 + p + 4q + 2, \) \( d(x) = p + q + 1, \) \( d(u) = q + 1, \) \( \forall y \in K_p, \) \( d(y) = p, \) \( \forall z \in K_1, \{u\}, \) \( d(z) = 2, \) \( \omega = p + 1. \)

According to Proposition 3.4, we find an eigenvalue \(-1\) of multiplicity \( p-1 \), an eigenvalue \( 0 \) of multiplicity \( q-1 \) and the roots of

\[
P(\lambda, p, q) = \lambda^4 - (p-1)\lambda^3 - (p + 2q + 1)\lambda^2 + (2pq + p - 4q - 1)\lambda + 3pq - 2q.
\]

**Conjecture 137.** For any graph, \( \lambda_2 \leq Hc. \)

**Conjecture 627.** For any graph of diameter 2, \( \lambda_2 \leq \sqrt{n}. \)

**Conjecture 628.** For any graph of diameter 2, \( \lambda_2 \leq Hc. \)

Consider the graph \( J_{p,q} \) with \( q \) a square and \( p = 2\sqrt{q} \). We have \( Hc/p \rightarrow 1/\sqrt{2} \), whereas \( \lambda_2/p \rightarrow 1/\sqrt{2} \); thus, this graph yields a counterexample for \( p \) sufficiently large.

**Conjecture 201.** For every graph such that \( \sum(\text{Odd}) \leq \sum(\text{Even}) \), the minimum of derivative of positive eigenvalues \( \leq m/\omega \).

The graph \( J_{2q,q} \) of the previous family belongs to the required class. For this graph, the quotient of the left-hand side by \( p \) tends to 1 as \( p \) goes to infinity, whereas the quotient of the right-hand side by \( p \) tends to \( 1/2 \). Thus, we have a counterexample for \( p \) sufficiently large.

The family \( B_{p,q} \). Let \( p \geq 2, q \geq 1 \) be two integers and let \( B_{p,q} \) be the graph consisting of a clique \( K_{p-1}, \) an independent set \( K_q \) and a dominating vertex \( x \). Thus, we obtain
a graph of diameter 2 of order $n = p + q$ and size $m = p(p - 1)/2 + q$. Easy calculations
give $\mu = 1 + q(2p + q - 3)/(p + q)(p + q - 1)$ and the value of the Randic $R_c = (p - 2)/2$
$+ \sqrt{(p - 1)/(p + q - 1)} + q/\sqrt{p + q - 1}(p - 2)/2 + q/\sqrt{p + q - 1}$.
According to Proposition 3.2, we find an eigenvalue $-1$ of multiplicity $p - 2$, an
eigenvalue 0 of multiplicity $q - 1$ and the roots of

$$P(\lambda, p, q) = -\lambda^3 + (p - 2)\lambda^2 + (p + q - 1)\lambda - q(p - 2).$$

One can verify that $\lambda_1 < -\sqrt{q}$, $0 < \lambda_2 < \min(p - 1, \sqrt{q})$, $\lambda_1 > \max(p - 1, \sqrt{q})$.

**Conjecture 145.** For any graph, $\min(\text{Derivative of positive eigenvalues}) \leq n/\mu$.

**Conjecture 557.** For any graph, separator $\leq n/\mu$.

**Conjecture 626.** For any graph of diameter 2, separator $\leq m/R_c$.

Let us consider the graph $B_{p,p}$. Since we have only two positive eigenvalues, the
separator is equal to the minimum of derivative of positive eigenvalues and is greater
than $p - 1 - \sqrt{p/2}$. As $p \to \infty$, we have $\mu \to \frac{1}{2}$ and these conjectures are false for
$p$ sufficiently large.

*The family $H_p$.* Let $H_p$ be the cartesian sum of a $K_p$ and a $K_3$. This graph is
obviously of diameter 2 and has $m = 3p(p + 1)/2$, $\alpha = 3$, $\omega = p$. From Proposition 3.1 its
spectrum is the sum of the spectra of $K_p$ and $K_3$, that is, $\lambda_1 = p + 1$, $\lambda_2 = \lambda_3 = p - 2$, $\lambda_4 = \cdots = \lambda_{p + 1} = 1$, $\lambda_{p + 2} = \cdots = \lambda_n = -2$.

This graph yields a counterexample to the next conjecture.

**Conjecture 186.** For every graph such that $\Sigma(\text{Odd}) \leq \Sigma(\text{Even})$, $m/\alpha \leq \sum_{i=1}^n |\lambda_i|$.

The complementary graph $\tilde{H}_p$ with $p$ even, $p \geq 6$, the spectrum of which is given by
Proposition 1.5, yields a counterexample to the following conjectures (recall that
$p_+, p_-$ denote the numbers of positive and negative eigenvalues and $p_0$ is the
multiplicity of 0 as eigenvalue of the graph).

**Conjecture 663.** For every graph such that $\Sigma(\text{Even}) \leq \Sigma(\text{Odd})$, $p_+ \leq p_-.$

**Conjecture 664.** For every graph such that $\Sigma(\text{Even}) \leq \Sigma(\text{Odd})$, $p_- \leq \chi + \bar{\chi}$.

**Conjecture 420.** For every graph such that $p_- \leq p_+$, $\min(\text{Odd}) \leq \bar{\chi}$.

**Conjecture 421.** For every graph such that $p_- \leq p_+$, $\text{mean}(\text{Odd}) \leq \chi + \bar{\chi}$.

**Conjecture 330.** For any graph with a perfect matching, $\min(D^*) \leq p_+ + p_0$.  
Conjecture 410. For every graph such that \( p_- \leq p_+ \), \( \delta \leq v \).

Conjecture 411. For every graph such that \( p_- \leq p_+ \), \( \delta \leq Hc \).

The family \( Km_{p,q} \). Let \( Km_{p,q} \) be the complete product of a clique \( K_q \) and the disjoint union of \( p \) copies of \( K_2 \). This graph is of order \( n = 2p + q \), size \( m = q(q-1)/2 + p(2q+1) \) and diameter 2.

We denote by \( x_i \) the vertices of the \( K_q \) and by \( y_j, y'_j \) the vertices of the \( j \)th \( K_2 \).

Proposition 3.2 gives an eigenvalue \(-1\) of multiplicity \( q + p - 1 \), an eigenvalue \( 1 \) of multiplicity \( (p-1) \), and the roots of \( P(\lambda, p, q) = \lambda^2 - q(2p-1)q + 1 \).

This family gives counterexamples to the following conjectures if we choose \( p = 3q \) with \( q \) even for the first one, \( p = q/2 \) for the second one and \( p = 3q/2 \) for the last one, and make \( q \) sufficiently large.

Conjecture 333. For any graph with a perfect matching, \( -\lambda_n \leq Hc \).

Conjecture 334. For any graph with a perfect matching, \( \alpha \leq n \)-scope of positive eigenvalues.

Conjecture 690. For every graph such that \( \Sigma(\text{Odd}) \leq \Sigma(\text{Even}) \), the scope of positive eigenvalues is \( \leq n - v \).

In order to evaluate the spectrum of the Laplacian of \( Km_{p,q} \), we write the equations of an eigenvector:

\[
(2p + q)x_i \sum_{j=1}^{q} x_j \sum_{j=1}^{p} (y_j + y'_j) = \lambda x_i, \quad 1 \leq i \leq q,
\]

\[
(q + 2)y_i - \sum_{j=1}^{q} x_j - (y_i + y'_i) = \lambda y_i, \quad 1 \leq i \leq p,
\]

\[
(q + 2)y'_i - \sum_{j=1}^{q} x_j - (y_i + y'_i) = \lambda y'_i, \quad 1 \leq i \leq p.
\]

A similar calculation as in Section 3 gives the spectrum sorted in nondecreasing order:

\( A_1 = 0, A_2 = \cdots = A_p = q, A_{p+1} = \cdots = A_{2p} = q + 2, A_{2p+1} = \cdots = A_n = n \).

The same family gives counterexamples to the following three conjectures if we choose \( p = 3q/2 \) for the first and second ones, \( p = 2q \) for the last one, and make \( q \) sufficiently large.

Recall that the deviation \( \sigma(A) \) of eigenvalues of the Laplacian is equal to \( ((1/n) \sum_{i \in V} d_i^2 + d^2)^{1/2} \).

Conjecture 684. For every graph such that \( \Sigma(\text{Odd}) \leq \Sigma(\text{Even}) \), mode of eigenvalues of Laplacian \( \leq \) frequency of mode of Degree.
**Conjecture 685.** For every graph such that $\Sigma(\text{Odd}) \leq \Sigma(\text{Even})$, mode of eigenvalues of Laplacian $\leq$ frequency of mode of Even.

**Conjecture 274.** For any graph, deviation of eigenvalues of Laplacian is $\leq \frac{1}{2}(\chi + \bar{\chi})$.

The following conjecture also involving the Laplacian is disproved by $K_{2,p}$ for $p$ sufficiently large.

**Conjecture 244.** For any $K_4$-free graph, $\sigma(A) \leq \sqrt{n}$.

*The family $M(p, q)$.* Let $M(p, q)$ be the complete product of two subgraphs consisting of two matchings of cardinalities $p$ and $q$. We may suppose $p \geq q$. This graph is obviously of diameter 2. Its chromatic number is equal to 4, whereas the chromatic number of the complement is equal to $p$. The degree of a vertex is $2q + 1$ or $2p + 1$; so, it is easy to compute Odd and Even. The graph is then such that $\Sigma(\text{Odd}) \leq \Sigma(\text{Even})$ if and only if

$$p^2 + q^2 - 2pq - p - 2q \geq 0.$$  

From Proposition 3.2 its spectrum consists of 1 with multiplicity $(p + q - 2)$, $-1$ with multiplicity $p + q$, together with the roots of the polynomial $P(\lambda, p, q) = \lambda^2 - (d_1 + d_2)\lambda + d_1d_2 - n_1 n_2$.

This family yields counterexamples to the following four conjectures if we choose $p = q \geq 6$ for the first and second ones, and $p = 3q/2 \geq 9$ for the others.

**Conjecture 265.** For any graph, $2 - \lambda_n \leq \chi + \bar{\chi}$.

**Conjecture 631.** For every graph of diameter 2, $p_+ \leq \chi + \bar{\chi}$.

**Conjecture 688.** For every graph such that $\Sigma(\text{Odd}) \leq \Sigma(\text{Even})$, $\lambda_n \leq \chi + \bar{\chi}$.

**Conjecture 689.** For every graph such that $\Sigma(\text{Odd}) \leq \Sigma(\text{Even})$, $\lambda_1 - \lambda_2 \leq \chi + \bar{\chi}$.

*The family $B_t, p$.* Let $B_t, p$ consist of $p$ independent edges $y_i y'_i$, $1 \leq i \leq p$, a vertex $z$ and a dominating vertex $x$. We obtain a graph of order $n = 2p + 2$ having a perfect matching and $p + 1$ as independence number. By Proposition 3.3, we get an eigenvalue $-1$ of multiplicity $p$, an eigenvalue 1 with multiplicity $p - 1$, and the zeros of the polynomial $P(\lambda, p) = -\lambda^3 + \lambda^2 + (2p + 1)\lambda - 1$ two of which are positive. For $p \geq 3$, this graph is a counterexample to the following conjecture.

**Conjecture 329.** For any graph with a perfect matching, scope(Dual Degree) $\leq p_-$.

*The family $B_{c, p, q}$.* Let $B_{c, p, q}$ consist of two cliques $K_p$ and $K_q$ with an extra dominating vertex $x$. We obtain a graph of order $n = p + q + 1$, size $m = (p(p + 1) + q(q + 1))/2$, diameter 2 and independence number 2.
From Proposition 3.3 we get $-1$ as eigenvalue of multiplicity $p+q-2$ and the zeros of $P(\lambda, p, q) = -\lambda^3 + (p+q-2)\lambda^2 - (pq - 2p - 2q + 1)\lambda - 2pq + p + q$. Note that we have $\lambda_1 \in ]p, p+1[$, $\lambda_2 \in ]q-1, q[$ for $p \geq q$ and $\lambda_n \in ]-2, -1[$. In the case where $p = q$ we have $\lambda_1 = (p-1 + \sqrt{(p-1)^2 + 8})/2$, $\lambda_2 = p-1$, $\lambda_3 = \cdots = \lambda_2p = -1$, $\lambda_{2p+1} = (p-1 - \sqrt{(p-1)^2 + 8})/2$.

The graph $Bc_{p,p}$ disproves the next two conjectures for $p$ sufficiently large.

**Conjecture 403.** For any graph with independence number 2, $n/\mu \leq \lambda_1 - \lambda_n$.

**Conjecture 630.** For any graph of diameter 2, $m/\lambda_1 \leq \sum_{i=1}^n |\lambda_i|$.

Note that this family also disproves Conjectures 186 (cf. $H_p$) and 627 (cf. $J_{p,q}$).

*The family Cir(t).* Let $Cir(t)$ be the circulant graph of order $n = 6t - 1$, with $t \geq 1$, in which vertex $i$ is adjacent to $\{i + 1, \ldots, i + 3k + 1, \ldots, i + 6t - 2\}$, where the labels are taken modulo $n$. This graph is $2t$-regular of diameter 2 and has $\chi = 2t$, $\chi = 3$ since we can easily partition $V$ into three stable sets $S_i = \{x_{3k+i} \mid 0 \leq k \leq K_i\}$, where $i = 1, 2, 3$ and $K_1 = K_2 = 2t - 1$, $K_3 = 2t - 2$. We also have $\chi = n - \nu = 3t$ since this graph is $K_3$-free.

The spectrum of a circulant graph is known. In this case we have $\lambda_1 - 2t$ of multiplicity 1 and the other eigenvalues of multiplicity 2 are given by the values $2\sum_{p=1}^t \cos 2k(3p - 2)\pi/n$ for $k = 1$ to $n - 1$. The spectrum can be written as $\lambda_1 = 2t$, $\forall k \in [1, 2t - 1]$, $\lambda_{2k} = \lambda_{2k+1} = (3 - 4\sin^2 \pi(2t-k)/(6t-1))^{-1}$ as regards the positive eigenvalues and as $\forall k \in [2t, 3t-1]$, $\lambda_{2k} = \lambda_{2k+1} = (3 - 4\sin^2 \pi(5t-k-1)/(6t-1))^{-1}$ for the negative ones.

This graph disproves for every $t$ the next three conjectures.

**Conjecture 358.** For any geotropic plant, $m/\chi \leq \chi + \bar{\chi}$.

**Conjecture 408.** For every graph such that $p_- \leq p_+$, $n - \chi \leq \chi + \bar{\chi}$.

**Conjecture 419.** For every graph such that $p_- \leq p_+$, $\max(\text{Even}) \leq \chi + \bar{\chi}$.

**Conjecture 343.** For any plant, $\lambda_1 - \lambda_n \leq \chi + \bar{\chi}$.

For the graph $Cir(t)$, we obtain by Taylor expansion $\lambda_1 - \lambda_n \approx 3.65t$, which is greater than $\chi + \bar{\chi} = 3t + 3$, and the conjecture is disproved for $t$ sufficiently large.

**Conjecture 430.** For any regular graph, the variance of positive eigenvalues is $\leq n/\chi$.

The average of the positive eigenvalues is a Riemann sum of

$$
(3/\pi) \int_0^{\pi/3} \frac{dx}{3 - 4\sin^2 x}
$$
and, thus, is of order \( \ln t \), whereas the average of the squares of the positive eigenvalues is at least \( \lambda_1^2/(4t-1) > t \). Thus, the variance is at least \( t \) as \( t \) goes to infinity and the conjecture is disproved for \( t \) sufficiently large.

The family \( Sn(p) \). Let \( Sn(p) \) consist of a path of length \( p \) on the vertices \( x_1, \ldots, x_{p+1} \) and \( p-1 \) extra vertices \( x_{p+2}, \ldots, x_{2p} \), each of them adjacent to \( x_{p+1} \). We obtain a tree of order \( n = 2p \), with a \( K_{p,1} \) induced subgraph; thus, by the interlacing Corollary 1.12, we have \( \lambda_1 = -\lambda_n \geq \lambda_1(K_{p,1}) = \sqrt{p} \).

The average distance of the graph can be compared to that of the induced path by the formula:

\[
\mu \geq \frac{(p+1)p}{2p(2p-1)} \times \frac{(p+2)}{3} > \frac{p}{12},
\]

thus, \( n/\mu < 24 \) and the following five conjectures are disproved at least for \( p > 576 \) (note that, in a tree, \( \mu \leq R \) [7]).

**Conjecture 354.** For any heliotropic plant, \( -\lambda_n \mu \leq n \).

**Conjecture 373.** For every graph such that \( \mu \leq R \), \( -\lambda_n \mu \leq n \).

**Conjecture 590.** For any tree, \( \lambda_1 \mu \leq n \).

**Conjecture 194.** For every graph such that \( \gamma(Odd) \leq \gamma(Even) \), \( \lambda_1 \mu \leq m \).

**Conjecture 414.** For every graph such that \( p_- \leq p_+ \), \( \lambda_1 \mu \leq m \).

The family \( Qz(p) \). Let \( Qz(p) \) consist of a path \( P_{2p} \) on 2p vertices labelled from 1 to 2p and \( p \) \( K_2 \) dominated by the vertex 2p. We have \( m = 5p - 1 \), \( A = 2p + 1 \) and we know from Corollary 1.9 that \( \lambda_1 \geq \sqrt{A} = \sqrt{2p+1} \). In the same way as for the family \( Sn(p) \), we obtain \( \mu \geq p/6 \). Thus, for sufficiently large \( p \), the following conjecture is disproved.

**Conjecture 332.** For any graph with a perfect matching, \( \lambda_1 \mu \leq m \).

The family \( Ks(n) \). Let \( Ks(n) \) be obtained from a \( K_n \) with \( n \) even \( \geq 8 \) by deleting two adjacent edges. From Proposition 1.7, \( \lambda_1 \geq \tilde{d} = (n(n-1) - 4)/n \geq n-2 \) and \( \lambda_2 \) is positive since it is not a complete multipartite graph. On the other hand, the smallest eigenvalue of its complement is equal to \( -\sqrt{2} \); therefore, by Corollary 1.16, \( \lambda_2 \leq \sqrt{2} - 1 \) and then the scope of positive eigenvalues is \( \geq \lambda_1 - \lambda_2 > n-3 \), whereas \( v + \bar{v} = n/2 \). Thus the following conjecture is disproved.

**Conjecture 336.** For any graph with a perfect matching, the scope of positive eigenvalues is \( \leq v + \bar{v} \).

The family \( F_p \). Let \( F_p \) be obtained from \( K_2 \cup K_p \) by adding a dominating vertex. The spectrum of \( F_1 \) is approximately \( 2.17011, 0.311, -1, -1.4812 \). Since \( F_p \) has \( F_1 \) as an induced subgraph, we obtain, from Corollary 1.12, \( \lambda_1(F_p) \geq \lambda_1(F_1) \), \( \lambda_2(F_p) \geq \lambda_2(F_1) \)
and \( \lambda_1(F_p) \neq \lambda_2(F_p) \) by Proposition 1.3 since \( F_p \) is connected. So, the range of its positive spectrum is at least 2 and \( 2 > m/x = (p + 3)/(p + 1) \) if \( p \geq 2 \); thus, the following conjecture is disproved.

**Conjecture 394.** For any graph of diameter 2, the range of positive spectrum is \( \leq m/x \).

The appendix of [2] gives the spectrum of all the graphs on seven vertices. This list yields one or more counterexamples to some conjectures. Here we give only the label of the graph we refer to, after each conjecture.

**Conjecture 141.** For any graph \( G \), the range of positive eigenvalues is \( \leq v \) (224).

**Conjecture 359.** For any geotropic plant \( G \), \( m/x \leq \text{mean(Even)} \) (524).

**Conjecture 363.** For any geotropic plant, \( \min(\text{Dual Degree}) \leq p_- \) (524).

**Conjecture 415.** For every graph such that \( p_+ \geq p_- \), \( \lambda_1 \leq \text{max(Even)} \) (788).

**Conjecture 417.** For every graph such that \( p_- \leq p_+ \), \( -\lambda_n \leq \text{mean(Even)} \) (780).

**Conjecture 655.** For every graph such that \( \Sigma(\text{Even}) \leq \Sigma(\text{Odd}) \), \( n/2 \leq p_- + p_0 \) (626).

**References**

[6] S. Fajtlowicz, Written on the wall, a list of conjectures of Graffiti, University of Houston, USA.