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Geometry of integrable billiards and pencils of quadrics

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Abstract

We study the deep interplay between geometry of quadrics in d-dimensional space and the dynamics of related integrable billiard systems. Various generalizations of Poncelet theorem are reviewed. The corresponding analytic conditions of Cayley's type are derived giving the full description of periodical billiard trajectories; among other cases, we consider billiards in arbitrary dimension d with the boundary consisting of arbitrary number k of confocal quadrics. Several important examples are presented in full details proving the effectiveness of the obtained results. We give a thorough analysis of classical ideas and results of Darboux and methodology of Lebesgue; we prove their natural generalizations, obtaining new interesting properties of pencils of quadrics. At the same time, we show essential connections between these classical ideas and the modern algebro-geometric approach in the integrable systems theory.

Résumé

Nous étudions l'interaction profonde entre la géométrie des quadriques dans l'espace de dimension d et la dynamique associée des systèmes intégrables de billards. Nous revisitons les généralisations diverses du théorème de Poncelet. Les conditions analytiques du type Cayley sont dérivées donnant une description complète des trajectoires périodiques du billard ; en particulier, nous considérons des billards de dimension d dont la frontière consiste en un nombre arbitraire k de quadriques homofocales. Quelques exemples importants sont présentés en détail, démontrant l'efficacité des résultats obtenus. Nous donnons une analyse exhaustive des idées classiques, des résultats de Darboux, de la méthodologie de Lebesgue, et nous démontrons leurs généralisations naturelles, obtenant des propriétés nouvelles et intéressantes de faisceaux de quadriques. Nous démontrons également des connections essentielles entre ces idées classiques et la méthode algébro-géométrique moderne de la théorie des systèmes integrables.

Keywords: Pencils of quadrics; Poncelet theorem; Periodic billiard trajectories

1. Introduction

In his *Traité des propriétés projectives des figures* [34], Poncelet proved one of most beautiful and most important claims of the 19th century geometry. Suppose that two ellipses are given in the plane, together with a closed polygonal

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line inscribed in one of them and circumscribed about the other one. Then, Poncelet theorem states that infinitely many such closed polygonal lines exist—every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides. Poncelet's proof was purely geometrical, synthetic. Later, using the addition theorem for elliptic functions, Jacobi gave another proof of the theorem [27]. Essentially, Poncelet theorem is equivalent to the addition theorem for elliptic functions and Poncelet's proof represents a synthetic way of deriving the group structure on an elliptic curve. Another proof, in a modern, algebro-geometrical manner, can be found in Griffiths' and Harris' paper [23]. There, they also gave an interesting generalization of the Poncelet theorem to the three-dimensional case, considering polyhedral surfaces both inscribed and circumscribed about two quadrics.

A natural question connected with Poncelet theorem is to find an analytical condition determining, for two given conics, if an n-polygon inscribed in one and circumscribed about the second conic exists. In a short paper [10], Cayley derived such a condition, using the theory of Abelian integrals. Inspired by this paper, Lebesgue translated Cayley's proof to the language of geometry. Lebesgue's proof of Cayley's condition, derived by methods of projective geometry and algebra, can be found in his book *Les coniques* [31]. Griffiths and Harris derived Cayley theorem by finding an analytical condition for points of finite order on an elliptic curve [24].

It is worth emphasizing that Poncelet, in fact, proved a statement that is much more general than the famous Poncelet theorem [7,34], then deriving the latter as a corollary. Namely, he considered $n+1$ conics of a pencil in the projective plane. If there exists an n-polygon with vertices lying on the first of these conics and each side touching one of the other n conics, then infinitely many such polygons exist. We shall refer to this statement as *Complete Poncelet theorem* (CPT) and call such polygons *Poncelet polygons*. We are going to follow here mostly the presentation of Lebesgue from [31], which is, as we learned from M. Berger, quite close to one of two Poncelet's original proofs.

A nice historical overview of the Poncelet theorem, together with modern proofs and remarks is given in [9]. Various classical theorems of Poncelet type with short modern proofs reviewed in [5], while the algebro-geometrical approach to families of Poncelet polygons via modular curves is given in [6,28].

Poncelet theorem has a nice mechanical interpretation. *Elliptical billiard* [30] is a dynamical system where a material point of the unit mass is moving with a constant velocity inside an ellipse and obeying the reflection law at the boundary, i.e., having congruent impact and reflection angles with the tangent line to the ellipse at any bouncing point. It is also assumed that the reflection is absolutely elastic. It is well known that any segment of a given elliptical billiard trajectory is tangent to the same conic, confocal with the boundary [11]. If a trajectory becomes closed after n reflections, then Poncelet theorem implies that any trajectory of the billiard system, which shares the same caustic curve, is also periodic with the period n.

Complete Poncelet theorem also has a mechanical meaning. The configuration dual to a pencil of conics in the plane is a family of confocal second order curves [4]. Let us consider the following, a little bit unusual billiard. Suppose n confocal conics are given. A particle is bouncing on each of these n conics respectively. Any segment of such a trajectory is tangent to the same conic confocal with the given n curves. If the motion becomes closed after n reflections, then, by Complete Poncelet theorem, any such a trajectory with the same caustic is also closed.

The statement dual to Complete Poncelet theorem can be generalized to the d-dimensional space [11]. Suppose vertices of the polygon $x_1x_2\dots x_n$ are respectively placed on confocal quadric hyper-surfaces $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n$ in the d-dimensional Euclidean space, with consecutive sides obeying the reflection law at the corresponding hyper-surface. Then all sides are tangent to some quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{d-1}$ confocal with $\{\mathcal{Q}_i\}$; for the hyper-surfaces $\{\mathcal{Q}_i, \mathcal{Q}^j\}$, an infinite family of polygons with the same properties exist.

But, more than one century before these quite recent results, Darboux proved the generalization of Poncelet theorem for a billiard within an ellipsoid in the three-dimensional space [13]. It seems that his work on this topic is completely forgot nowadays.

It is natural to search for a Cayley-type condition related to some of generalizations of Poncelet theorem. The authors derived such conditions for the billiard system inside an ellipsoid in the Euclidean space of arbitrary finite dimension [19,20]. In our recent note [21], algebro-geometric conditions for existence of periodical billiard trajectories within k quadrics in d-dimensional Euclidean space were announced. The aim of the present paper is to give full explanations and proofs of these results together with several important examples and improvements. The second important goal of this paper is to offer a thorough historical overview of the subject with a special attention on the detailed analysis of ideas and contributions of Darboux and Lebesgue. While Lebesgue's work on this subject has been, although rarely, mentioned by experts, on the other hand, it seems to us that relevant Darboux's ideas are practically unknown in the contemporary mathematics. We give natural higher-dimensional generalizations of the ideas and results of Darboux and Lebesgue, providing the proofs also in the low-dimensional cases if they were omitted in the original works. Beside other results, interesting new properties of pencils of quadrics are established–see Theorems 9 and 10. The latter gives a nontrivial generalization of the Basic Lemma [31].

This paper is organized as follows. In the next section, a short review of Lebesgue's results from [31] is given, followed by their application to the case of the billiard system between two confocal ellipses. Section 3 contains algebro-geometric discussions which will be applied in the rest of the paper. In Section 4, we give analytic conditions for periodicity of billiard motion inside a domain bounded by several confocal quadrics in the Euclidean space of arbitrary dimension. The complexity of the problem of billiard motion within several quadrics is well known, even in the real case, and it is induced by multivaluedness of the billiard mapping. Thus, to establish a correct setting of the problem, we introduce basic notions of reflections *from inside* and *from outside* a quadric hyper-surface, and we define *the billiard ordered game*. The corresponding closeness conditions are derived, together with examples and discussions. In Section 5, we consider the elliptical billiard as a discrete-time dynamical system, and, applying the Veselov's and Moser's algebro-geometric integration procedure, we derive the periodicity conditions. The obtained results are compared with those from Section 4. In Section 6, we give an algebro-geometric description of periodical trajectories of the billiard motion on quadric hyper-surfaces, we study the behaviour of geodesic lines after the reflection at a confocal quadric and derive a new porism of Poncelet type. In Section 7, we define the virtual reflection configuration, prove Darboux's statement on virtual billiard trajectories, generalize it to arbitrary dimension and study related geometric questions. In Section 8, we formulate and prove highly nontrivial generalization of the Basic Lemma (Lemma 1), giving a new important geometric property of dual pencils of quadrics. In that section, we also introduce and study the *generalized Cayley curve*, a natural higher-dimensional generalization of *the Cayley cubic* studied by Lebesgue. In this way, in Section 8 the most important tools of Lebesgue's study are generalized. Further development of this line will be presented in separate publication [22]. In Appendix A, we review some known classes of integrable potential perturbations of elliptical billiards, emphasizing connections with Appell hypergeometric functions and Liouville surfaces. Finally, in Appendix B, we present the related Darboux's results considering a generalization of Poncelet theorem to Liouville surfaces, giving a good basis for a study of the geometry of periodic trajectories appearing in the perturbed systems from Appendix A.

2. Planar case: $d = 2$, k—arbitrary

First of all, we consider the billiard system within k confocal ellipses in the 2-dimensional plane. In such a system, the billiard particle bounces sequentially of these confocal ellipses. We wish to get the analytical description of periodical trajectories of such a system.

Following Lebesgue, let us consider polygons inscribed in a conic Γ, whose sides are tangent to $\Gamma_1, \dots, \Gamma_k$, where $\Gamma, \Gamma_1, \dots, \Gamma_k$ all belong to a pencil of conics. In the dual plane, such polygons correspond to billiard trajectories having caustic Γ^* with bounces on $\Gamma_1^*, \dots, \Gamma_k^*$. The main object of Lebesgue's analysis is the cubic Cayley curve, which parametrizes contact points of tangents drawn from a given point to all conics of the pencil.

2.1. Full Poncelet theorem

Basic Lemma. Next lemma is the main step in the proof of full Poncelet theorem. If one Poncelet polygon is given, this lemma enables us to construct every Poncelet polygon with given initial conditions. Also, the lemma is used in deriving of a geometric condition for the existence of a Poncelet polygon.

Lemma 1. ([31]) *Let $\mathcal{F}$ be a pencil of conics in the projective plane and Γ a conic from this pencil. Then there exist quadrangles whose vertices A, B, C, D are on Γ such that three pairs of its non-adjacent sides AB, CD; AC, BD; AD, BC are tangent to three conics of $\mathcal{F}$. Moreover, the six contact points all lie on a line Δ. Any such a quadrangle is determined by two sides and the corresponding contact points.*

Let Γ, Γ_1, Γ_2, Γ_3 be conics of a pencil and ABC a Poncelet triangle corresponding to these conics, such that its vertices lie on Γ and sides AB, BC, CA touch Γ_1, Γ_2, Γ_3 respectively. This lemma gives us a possibility to construct triangle ABD inscribed in Γ whose sides AB, BD, DA touch conics Γ_1, Γ_3, Γ_2 respectively. In a similar fashion, for a given Poncelet polygon, we can, applying Lemma 1, construct another polygon which corresponds to the same conics, but its sides are tangent to them in different order.

Circumscribed and tangent polygons. Let a triangle ABC be inscribed in a conic Γ and sides BC, AC, AB touch conics Γ_1, Γ_2, Γ_3 of the pencil $\mathcal{F}$ at points M, N, P respectively. According to the lemma, there are two possible cases: either points M, N, P are collinear, then we will say that the triangle is *tangent* to Γ_1, Γ_2, Γ_3; or the line MN is intersecting AB at a point S which is a harmonic conjugate to P with respect to the pair A, B, then we say that the triangle is *circumscribed* about Γ_1, Γ_2, Γ_3.

Let $ABCD\dots KL$ be a polygon inscribed in Γ whose sides touch conics $\Gamma_1,\dots,\Gamma_n$ of the pencil $\mathcal{F}$ respectively. Denote by (AC) a conic such that $\triangle ABC$ is circumscribed about Γ_1, Γ_2, (AC), by (AD) a conic such that $\triangle ACD$ is circumscribed about (AC), Γ_3, (AD). Similarly, we find conics $(AE),\dots,(AK)$. The triangle AKL can be tangent to the conics (AK), Γ_{n-1}, Γ_n or circumscribed about them, and we will say that $ABCD\dots KL$ is *tangent* or, respectively, *circumscribed* about conics $\Gamma_1,\dots,\Gamma_n$.

Further, we will be interested only in circumscribed polygons. The Poncelet theorem does not hold for tangent triangles nor, hence, for tangent polygons with greater number of vertices.

Theorem 1 *(Complete Poncelet theorem). Let conics Γ, $\Gamma_1,\dots,\Gamma_n$ belong to a pencil $\mathcal{F}$. If a polygon inscribed in Γ and circumscribed about $\Gamma_1,\dots,\Gamma_n$ exists, then infinitely many such polygons exist.*

To determine such a polygon, it is possible to give arbitrarily:

(1) *the order which its sides touch $\Gamma_1,\dots,\Gamma_n$ in*; *let the order be*: $\Gamma_1',\dots,\Gamma_n'$;
(2) *a tangent to Γ_1' containing one side of the polygon*;
(3) *the intersecting point of this tangent with Γ which will belong to the side tangent to Γ_2'.*

The proof is given in [31].

2.2. Cayley's condition

Representation of conics of a pencil by points on a cubic curve. Let pencil $\mathcal{F}$ of conics be determined by the curves $C=0$ and $\Gamma=0$. The equation of an arbitrary conic of the pencil is $C+\lambda\Gamma=0$.

Let $P+\lambda\Pi=0$ be the equation of the corresponding polar lines from the point $A\in\Gamma$. The geometric place of contact points of tangents from A with conics of the pencil is the cubic $\mathcal{C}$: $C\Pi-\Gamma P=0$. On this cubic, any conic of $\mathcal{F}$ is represented by two contact points, which we will call *representative points* of the conic. The line determined by these two points passes through point Z: $P=0$, $\Pi=0$. There exist exactly four conics of the pencil whose representative points coincide: the conic Γ and three degenerate conics with representative points A, α, β, γ. Lines ZA, $Z\alpha$, $Z\beta$, $Z\gamma$ are tangents to $\mathcal{C}$ constructed from Z. The tangent line to cubic $\mathcal{C}$ at point Z is a polar of point A with respect to the conic of the pencil which contains Z.

Condition for existence of a Poncelet triangle. If triangles inscribed in Γ and circumscribed about Γ_1, Γ_2, Γ_3 exist, we will say that the conics Γ_1, Γ_2, Γ_3 are *joined* to Γ. In this case, CPT states there is six such triangles with the vertex $A\in\Gamma$. Let ABC be one of them. Side AB, denote it by *1*, touches Γ_1 in point m_1. Also, it touches another conic of the pencil, denote it by *(1)*, in point M. Side AC, denote it by *2′*, touches Γ_2 in m_2'. Consider the quadrangle $ABCD$ determined by AB, AC and contact points M, m_2'. Line Mm_2' meets BC at μ_3, its point of tangency to Γ_3, and meets AD (which we will denote by *3′*) at the point m_3' of tangency to Γ_3. Triangle ABD is circumscribed about Γ_1, Γ_2, Γ_3.

Similarly, triangle ACE can be obtained by construction of quadrangle $ABCE$ determined by AB, BC and contact points m_1, μ_3. Line AE touches Γ_3 at point $m_3\in m_1\mu_3$. Denote this line by *3*. Triangles with sides *3′*, *2* and *3*, *1′* are constructed analogously.

There is exactly six tangents from A to conics Γ_1, Γ_2, Γ_3. We have divided these six lines into two groups: *1, 2, 3* and *1′*, *2′*, *3′*. Two tangents enumerated by different numbers and do not belong to the same group, determine a Poncelet triangle.

Cubic $\mathcal{C}$ and the cubic consisting of lines m_1M, m_2m_2', m_3m_3' have simple common points m_1, M, m_2, m_2', m_3, m_3', A, and point Z as a double one. A pencil determined by these two cubics contains a curve that passes through a given point of line Mm_2', different from M, m_2', m_3'. This cubic has four common points with line Mm_2', so it decomposes into the line and a conic. Thus, *m_1, m_2, m_3 are intersection points, different from A and Z, of a conic which contains A and touches cubic $\mathcal{C}$ at Z.*

Converse also holds.

Let an arbitrary conic that contains point A and touches cubic $\mathcal{C}$ at Z be given. Denote by m_1, m_2, m_3 remaining intersection points of the curve $\mathcal{C}$ with this conic. Each of the lines m_1Z, m_2Z, m_3Z has another common point with the cubic $\mathcal{C}$; denote them by m_1', m_2', m_3' respectively. By definition of the curve $\mathcal{C}$, we have that m_1, m_1'; m_2, m_2'; m_3, m_3' are pairs of representative points of some conics Γ_1, Γ_2, Γ_3 from the pencil $\mathcal{F}$. Line Am_1, besides being tangent to Γ_1 at m_1, has to touch another conic from the pencil $\mathcal{F}$. Take that it is tangent to a conic (I) at M.

Now, in a similar fashion as before, we can conclude that points M, m_2', m_3' are colinear. Applying Lemma 1, it is easily deduced that conics Γ_1, Γ_2, Γ_3 are joined to Γ.

So, we have shown the following: *Systems of three joined conics are determined by systems of three intersecting points of cubic $\mathcal{C}$ with conics that contain point A and touch the curve $\mathcal{C}$ at Z.*

Cayley's cubic. Let $D(\lambda)$ be the discriminant of conic $C + \lambda\Gamma = 0$. We will call the curve,

$$\mathcal{C}_0\colon\ Y^2 = D(X).$$

Cayley's cubic. Representative points of conic $C + \lambda\Gamma = 0$ on Cayley's cubic are two points that correspond to the value $X = \lambda$.

The polar conic of the point Z with respect to cubic $\mathcal{C}$ passes through the contact points of the tangents ZA, $Z\alpha$, $Z\beta$, $Z\gamma$ from Z to $\mathcal{C}$. Thus, points α, β, γ are representative points of three joined conics from the pencil $\mathcal{F}$. Those three conics are obviously the decomposable ones. Corresponding values λ diminish $D(\lambda)$, and these three representative points on Cayley's cubic $\mathcal{C}_0$ lie on the line $Y = 0$.

Using the Sylvester's theory of residues, we will show the following:

Let three representative points of three conics of pencil $\mathcal{F}$ be given on the Cayley's cubic $\mathcal{C}_0$. Condition for these conics to be joined to the conic Γ is that their representative points are collinear.

Sylvester's theory of residues. When considering algebraic curves of genus 1, like the Cayley's cubic is here, Abel's theorem can always be replaced by application of this theory.

Proposition 1. *Let a given cubic and an algebraic curve of degree $m + n$ meet at $3(m + n)$ points. If there is $3m$ points among them which are placed on a curve of degree m, then the remaining $3n$ points are placed on a curve of degree n.*

If the union of two systems of points is the complete intersection of a given cubic and some algebraic curve, then we will say that these two systems are *residual* to each other. Now, the following holds:

Proposition 2. *If systems $\mathcal{A}$ and $\mathcal{A}'$ of points on a given cubic curve have a common residual system, then they share all residual systems.*

Proof. Suppose $\mathcal{B}$ is a system residual to both $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}'$ is residual to $\mathcal{A}$. Then $\mathcal{A} \cup \mathcal{A}'$ is residual to $\mathcal{B} \cup \mathcal{B}'$, i.e., the system $\mathcal{A} \cup \mathcal{A}' \cup \mathcal{B} \cup \mathcal{B}'$ is a complete intersection of the cubic with an algebraic curve. Since $\mathcal{A} \cup \mathcal{B}$ is also such an intersection, it follows, by the previous proposition, that $\mathcal{A}'$ and $\mathcal{B}'$ are residual to each other. □

Let us note that this proposition can be derived as a consequence of Abel's theorem, for a plane algebraic curve of arbitrary degree. However, if the degree is equal to three, i.e., the curve is elliptic, Proposition 2 is equivalent to Abel's theorem.

Condition for existence of a Poncelet polygon. Let conics $\Gamma, \Gamma_1, \dots, \Gamma_n$ be from a pencil. If there exists a polygon inscribed in Γ and circumscribed about $\Gamma_1, \dots, \Gamma_n$, we are going to say that conics $\Gamma_1, \dots, \Gamma_n$ are *joined* to Γ. Then, similarly as in the case of the triangle, it can be proved that tangents from the point $A \in \Gamma$ to $\Gamma_1, \dots, \Gamma_n$ can be divided into two groups such that any Poncelet n-polygon with vertex A has exactly one side in each of the groups.

This division of tangents gives a division of characteristic points of conics $\Gamma_1, \dots, \Gamma_n$ into two groups on $\mathcal{C}$ and, therefore, a division into two groups on Cayley's cubic $\mathcal{C}_0$: *1, 2, 3,* ... and *1′, 2′, 3′*, ...

Let $ABCD \dots KL$ be a Poncelet polygon, and let (AC), (AD), ... be conics determined like in the definition of a circumscribed polygon. Let $c, \gamma; d, \delta; \dots$ be a corresponding characteristic points on $\mathcal{C}_0$, such that triples *1, 2,* c; γ, *3,* d; δ, *4,* e are characteristic points of the same group with respect to corresponding conics.

Points *1, 2,* c are collinear, as γ, *3,* d are. Thus, *1, 2, 3,* d are residual with c, γ. Line $c\gamma$ contains point Z, so system c, γ is residual with Z, too. It is possible to show that Z is a triple point of curve $\mathcal{C}_0$ and it follows that it is residual with system Z, Z. This implies that points *1, 2, 3,* d, Z, Z are placed on a conic.

If we take a coordinate system such that the tangent line to $\mathcal{C}_0$ at Z is the infinite line and the axis Oy is line AZ, we will have:

four conics are joined to Γ if and only if their characteristic points of the same group are on a parabola with the asymptotic direction Oy.

Continuing deduction in the same manner, we can conclude: $3n - p$ *points of the cubic* $\mathcal{C}_0$ *are characteristic points of same group for* $3n - p$ *conics* $(1 \leqslant p \leqslant 3)$ *joined to* Γ*, if and only if these points are placed on a curve of degree* n *which has* Oy *as an asymptotic line of the order* p.

Cayley's condition. Let $\mathcal{C}_0$: $y^2 = D(x)$ be the Cayley's cubic, where $D(x)$ is the discriminant of the conic $C + x\Gamma = 0$ from pencil $\mathcal{F}$. A system of n conics joined to Γ is determined by n values x if and only if these n values are abscissae of intersecting points of $\mathcal{C}_0$ and some algebraic curve. Plugging $D(x)$ instead of y^2 in the equation of this curve, we obtain:

$$P(x)y + Q(x) = 0,$$

that is

$$P(x)\sqrt{D(x)} + Q(x) = 0.$$

From there:

$$\sqrt{D(x)}(a_0x^{p-2} + a_1x^{p-3} + \cdots + a_{p-2}) + (b_0x^p + b_1x^{p-1} + \cdots + b_p) = 0, \quad n = 2p;$$
$$\sqrt{D(x)}(a_0x^{p-1} + a_1x^{p-2} + \cdots + a_{p-1}) + (b_0x^p + b_1x^{p-1} + \cdots + b_p) = 0, \quad n = 2p + 1.$$

If $\lambda_1, \dots, \lambda_k$ denote parameters corresponding to $\Gamma_1, \dots, \Gamma_k$ respectively, then existence of a Poncelet polygon inscribed in Γ and circumscribed about $\Gamma_1, \dots, \Gamma_k$ is equivalent to:

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^p & \sqrt{D(\lambda_1)} & \lambda_1\sqrt{D(\lambda_1)} & \dots & \lambda_1^{p-2}\sqrt{D(\lambda_1)} \\ \dots \\ \dots \\ 1 & \lambda_k & \lambda_k^2 & \dots & \lambda_k^p & \sqrt{D(\lambda_k)} & \lambda_k\sqrt{D(\lambda_k)} & \dots & \lambda_k^{p-2}\sqrt{D(\lambda_k)} \end{vmatrix} = 0 \quad \text{for } k = 2p;$$

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^p & \sqrt{D(\lambda_1)} & \lambda_1\sqrt{D(\lambda_1)} & \dots & \lambda_1^{p-1}\sqrt{D(\lambda_1)} \\ \dots \\ \dots \\ 1 & \lambda_k & \lambda_k^2 & \dots & \lambda_k^p & \sqrt{D(\lambda_k)} & \lambda_k\sqrt{D(\lambda_k)} & \dots & \lambda_k^{p-1}\sqrt{D(\lambda_k)} \end{vmatrix} = 0 \quad \text{for } k = 2p + 1.$$

There exists an n-polygon inscribed in Γ and circumscribed about C if and only if it is possible to find coefficients $a_0, a_1, \dots; b_0, b_1, \dots$ such that function $P(x)\sqrt{D(x)} + Q(x)$ has $x = 0$ as a root of the multiplicity n.

For $n = 2p$, this is equivalent to the existence of a non-trivial solution of the following system:

$$\begin{aligned} a_0C_3 &+ a_1C_4 + \cdots + a_{p-2}C_{p+1} = 0, \\ a_0C_4 &+ a_1C_5 + \cdots + a_{p-2}C_{p+2} = 0, \\ &\dots \\ a_0C_{p+1} &+ a_1C_{p+2} + \cdots + a_{p-2}C_{2p-1} = 0, \end{aligned}$$

where

$$\sqrt{D(x)} = A + Bx + C_2x^2 + C_3x^3 + \cdots.$$

Finally, for $n = 2p$, we obtain the Cayley's condition:

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{p+1} \\ C_4 & C_5 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p-1} \end{vmatrix} = 0.$$

Similarly, for $n = 2p + 1$, we obtain:

$$\begin{vmatrix} C_2 & C_3 & \dots & C_{p+1} \\ C_3 & C_4 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p} \end{vmatrix} = 0.$$

These results can be directly applied to the billiard system within an ellipse: to determine whether a billiard trajectory with a given confocal caustic is periodic, we need to consider the pencil determined by the boundary and the caustic curve.

2.3. Some applications of Lebesgue's results

Now, we are going to apply the Lebesgue's results to billiard systems within several confocal conics in the plane.

Consider the dual plane. The case with two ellipses, when the billiard trajectory is placed between them and particle bounces to one and another of them alternately, is of special interest.

Corollary 1. *The condition for the existence of* $2m$*-periodic billiard trajectory which bounces exactly* m *times to the ellipse* $\Gamma_1^* = C^*$ *and* m *times to* $\Gamma_2^* = (C + \gamma\Gamma)^*$*, having* Γ^* *for the caustic, is*:

$$\det\begin{pmatrix} f_0(0) & f_1(0) & \dots & f_{2m-1}(0) \\ f_0'(0) & f_1'(0) & \dots & f_{2m-1}'(0) \\ & & \dots & \\ f_0^{(m-1)}(0) & f_1^{(m-1)}(0) & \dots & f_{2m-1}^{(m-1)}(0) \\ f_0(\gamma) & f_1(\gamma) & \dots & f_{2m-1}(\gamma) \\ f_0'(\gamma) & f_1'(\gamma) & \dots & f_{2m-1}'(\gamma) \\ & & \dots & \\ f_0^{(m-1)}(\gamma) & f_1^{(m-1)}(\gamma) & \dots & f_{2m-1}^{(m-1)}(\gamma) \end{pmatrix} = 0,$$

where $f_j = x^j$ $(0 \leqslant j \leqslant m)$, $f_{m+i} = x^{i-1}\sqrt{D(x)}$ $(1 \leqslant i \leqslant m-1)$.

We consider an example with four bounces on each of the two conics.

Example 1. The condition on a billiard trajectory placed between ellipses Γ_1^* and Γ_2^*, to be closed after 4 alternate bounces to each of them is:

$$\det X = 0,$$

where the elements of the 3×3 matrix X are:

$$\begin{aligned}
X_{11} &= -4B_0 + B_1\gamma + 4C_0 + 3C_1\gamma + 2C_2\gamma^2 + C_3\gamma^3, \\
X_{12} &= -3B_0 + B_1\gamma + 3C_0 + 2C_1\gamma + C_2\gamma^2, \\
X_{13} &= -2B_0 + B_1\gamma + 2C_0 + C_1\gamma, \\
X_{21} &= -6B_0 + B_2\gamma^2 + 6C_0 + 6C_1\gamma + 4C_2\gamma^2 + 3C_3\gamma^3, \\
X_{22} &= -6B_0 + B_1\gamma + B_2\gamma^2 + 6C_0 + 4C_1\gamma + 3C_2\gamma^2, \\
X_{23} &= -5B_0 + 2B_1\gamma + B_2\gamma^2 + 5C_0 + 3C_1\gamma, \\
X_{31} &= -4B_0 + B_3\gamma^3 + 4C_0 + 4C_1\gamma + 4C_2\gamma^2 + 3C_3\gamma^3, \\
X_{32} &= -4B_0 + B_2\gamma^2 + B_3\gamma^3 + 4C_0 + 4C_1\gamma + 3C_2\gamma^2, \\
X_{33} &= -4B_0 + B_1\gamma + B_2\gamma^2 + B_3\gamma^3 + 4C_0 + 3C_1\gamma,
\end{aligned}$$

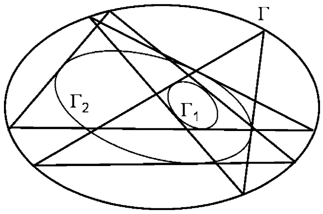

Fig. 1. A Poncelet octagon whose sides touch the conics Γ_1 and Γ_2 alternately.

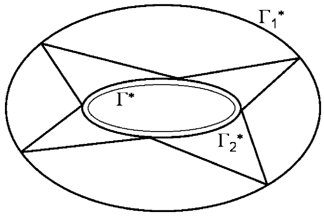

Fig. 2. A closed billiard trajectory in the domain bounded by two confocal ellipses.

with C_i, B_i being coefficients in the Taylor expansions around $x=0$ and $x=\gamma$ respectively:

$$\sqrt{D(x)} = C_0 + C_1x + C_2x^2 + \cdots, \qquad \sqrt{D(x)} = B_0 + B_1(x-\gamma) + B_2(x-\gamma)^2 + \cdots.$$

On Fig. 1, we see a Poncelet octagon inscribed in Γ and circumscribed about Γ_1 and Γ_2. In the dual plane, the billiard trajectory that corresponds to this octagon, has the dual conic Γ^* as the caustic (see Fig. 2).

3. Points of finite order on the Jacobian of a hyperelliptic curve

In order to prepare the algebro-geometric background for the rest of the article, in this section we are going to give the analytical characterization of some classes of finite order divisors on a hyperelliptic curve.

Let the curve $\mathcal{C}$ be given by:

$$y^2 = (x-x_1)\dots(x-x_{2g+1}), \quad x_i \neq x_j \text{ when } i \neq j.$$

It is a regular hyperelliptic curve of genus g, embedded in P^2. Let $\mathcal{J}(\mathcal{C})$ be its Jacobian variety, and

$$\mathcal{A}: \mathcal{C} \to \mathcal{J}(\mathcal{C})$$

the Abel–Jacobi map.

Take E to be the point which corresponds to the value $x=\infty$, and choose $\mathcal{A}(E)$ to be the neutral in $\mathcal{J}(\mathcal{C})$. According to the Abel's theorem [25], $\mathcal{A}(P_1)+\cdots+\mathcal{A}(P_n)=0$ if and only if there exists a meromorphic function f with zeroes $P_1,\dots,P_n$ and a pole of order n at the point E. Let $\mathcal{L}(nE)$ be the vector space of meromorphic functions on $\mathcal{C}$ with a unique pole E of order at most n, and $f_1,\dots,f_k$ a basis of $\mathcal{L}(nE)$. The mapping,

$$F: \mathcal{C} \to P^{k-1}, \quad X \mapsto \big[f_1(X),\dots,f_k(X)\big],$$

is a projective embedding whose image is a smooth algebraic curve of degree n. Hyperplane sections of this curve are zeroes of functions from $\mathcal{L}(nE)$. Thus, the equality $n\mathcal{A}(P)=0$ is equivalent to:

$$\operatorname{rank}\begin{pmatrix} f_1(P) & f_2(P) & \dots & f_k(P) \\ f_1'(P) & f_2'(P) & \dots & f_k'(P) \\ & & \dots & \\ f_1^{(n-1)}(P) & f_2^{(n-1)}(P) & \dots & f_k^{(n-1)}(P) \end{pmatrix} < k. \tag{1}$$

Lemma 2. *For $n \leqslant 2g$, there does not exist a point P on the curve $\mathcal{C}$, such that $n\mathcal{A}(P)=0$ and $P \neq E$.*

Proof. Let P be a point on $\mathcal{C}$, $P \neq E$, and $x=x_0$ its corresponding value. Consider the case of n even. Since E is a branch point of a hyperelliptic curve, its Weierstrass gap sequence is $1,3,5,\dots,2g-1$ [25]. Now, applying the Riemann–Roch theorem, we obtain $\dim\mathcal{L}(nE)=n/2+1$. Choosing a basis $1,x,\dots,x^{n/2}$ for $\mathcal{L}(nE)$, and substituting in (1), we come to a contradiction. □

Lemma 3. *Let $P(x_0,y_0)$ be a non-branching point on the curve $\mathcal{C}$. For $n>2g$, equality $n\mathcal{A}(P)=0$ is equivalent to*:

$$\operatorname{rank}\begin{pmatrix} B_{m+1} & B_m & \dots & B_{g+2} \\ B_{m+2} & B_{m+1} & \dots & B_{g+3} \\ & & \dots & \\ B_{2m-1} & B_{2m-2} & \dots & B_{m+g} \end{pmatrix} < m-g, \quad \textit{when } n=2m, \tag{2}$$

$$\operatorname{rank}\begin{pmatrix} B_{m+1} & B_m & \dots & B_{g+1} \\ B_{m+2} & B_{m+1} & \dots & B_{g+2} \\ & & \dots & \\ B_{2m} & B_{2m-1} & \dots & B_{m+g} \end{pmatrix} < m-g+1, \quad \textit{when } n=2m+1,$$

and

$$\sqrt{(x-x_1)\cdots(x-x_{2g+1})} = B_0 + B_1(x-x_0) + B_2(x-x_0)^2 + B_3(x-x_0)^3 + \cdots.$$

Proof. This claim follows from previous results by choosing a basis for $\mathcal{L}(nE)$:

$$1,\ x,\ \dots,\ x^m,\ y,\ xy,\ \dots,\ x^{m-g-1}y \quad \text{if } n=2m,$$
$$1,\ x,\ \dots,\ x^m,\ y,\ xy,\ \dots,\ x^{m-g}y \quad \text{if } n=2m+1,$$

similarly as in [24]. □

In the next lemma, we are going to consider the case when the curve $\mathcal{C}$ is singular, i.e., when some of the values $x_1, x_2, \dots, x_{2g+1}$ coincide.

Lemma 4. *Let the curve $\mathcal{C}$ be given by*:

$$y^2 = (x-x_1)\cdots(x-x_{2g+1}), \quad x_1\cdot x_2\cdots x_{2g+1} \neq 0,$$

P_0 one of the points corresponding to the value $x=0$ and E the infinite point on $\mathcal{C}$. Then $2nP_0 \sim 2nE$ is equivalent to (2)*, where*

$$y = \sqrt{(x-x_1)\cdots(x-x_{2g+1})} = B_0 + B_1x + B_2x^2 + \cdots$$

is the Taylor expansion around the point P_0.

Proof. Suppose that, among $x_1, \dots, x_{2g+1}$, only x_{2g} and x_{2g+1} have same values. Then $(x_{2g}, 0)$ is an ordinary double point on $\mathcal{C}$. The normalization of the curve $\mathcal{C}$ is the pair $(\widetilde{\mathcal{C}}, \pi)$, where $\widetilde{\mathcal{C}}$ is the curve given by:

$$\widetilde{\mathcal{C}}\colon\ \tilde{y}^2 = (\tilde{x}-x_1)\cdots(\tilde{x}-x_{2g-1}),$$

and $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ is the projection:

$$(\tilde{x}, \tilde{y}) \stackrel{\pi}{\longmapsto} \big(x=\tilde{x},\ y=(\tilde{x}-x_{2g})\tilde{y}\big).$$

The genus of $\widetilde{\mathcal{C}}$ is $g-1$.

Denote by A and B points on $\widetilde{\mathcal{C}}$ which are mapped to the singular point $(x_{2g}, 0) \in \mathcal{C}$ by the projection π. Any other point on $\mathcal{C}$ is the image of a unique point of the curve $\widetilde{\mathcal{C}}$. Let:

$$\pi\big(\widetilde{E}\big) = E, \quad \pi\big(\widetilde{P}_0\big) = P_0.$$

The relation $2nP_0 \sim 2nE$ holds if and only if there exists a meromorphic function f on $\widetilde{\mathcal{C}}$, $f \in \mathcal{L}(2n\widetilde{E})$, having a zero of order $2n$ at $\widetilde{P}_0$ and satisfying $f(A) = f(B)$.

For $n \leqslant g-1$, according to Lemma 3, $2n\widetilde{E} \sim 2n\widetilde{P}_0$ cannot hold. For $n \geqslant g$, choose the following basis of the space $\mathcal{L}(2n\widetilde{E})$:

$$1,\ \tilde{y},\ f_1\circ\pi,\ \dots,\ f_{n-g-1}\circ\pi,$$

where $1, f_1, \dots, f_{n-g-1}$ is a basis of $\mathcal{L}(2nE)$ as in the proof of Lemma 3.

Since $\tilde{y}$ is the only function in the basis which has different values at points A and B, we obtain that the condition,

$$2n\widetilde{E} \sim 2n\widetilde{P}_0,$$

is equivalent to (2).

Cases when $\mathcal{C}$ has more singular points or singularities of higher order, can be discussed in the same way. □

Lemma 5. *Let the curve $\mathcal{C}$ be given by*:

$$y^2 = (x - x_1) \cdots (x - x_{2g+2}),$$

with all x_i distinct from 0, *and Q_+, Q_- the two points on $\mathcal{C}$ over the point $x = 0$. Then $nQ_+ \sim nQ_-$ is equivalent to*:

$$\operatorname{rank}\begin{pmatrix} B_{g+2} & B_{g+3} & \dots & B_{n+1} \\ B_{g+3} & B_{g+4} & \dots & B_{n+2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ B_{g+n} & \dots & \dots & B_{2n-1} \end{pmatrix} < n - g \quad \textit{and} \quad n > g, \tag{3}$$

where $y = \sqrt{(x - x_1) \cdots (x - x_{2g+2})} = B_0 + B_1 x + B_2 x^2 + \cdots$ is the Taylor expansion around the point Q_-.

Proof. $\mathcal{C}$ is a hyperelliptic curve of genus g. The relation $nQ_+ \equiv nQ_-$ means that there exists a meromorphic function on $\mathcal{C}$ with a pole of order n at the point Q_+, a zero of the same order at Q_- and neither other zeros nor poles. Denote by $\mathcal{L}(nQ_+)$ the vector space of meromorphic functions on $\mathcal{C}$ with a unique pole Q_+ of order at most n. Since Q_+ is not a branching point on the curve, $\dim \mathcal{L}(nQ_+) = 1$ for $n \leqslant g$, and $\dim \mathcal{L}(nQ_+) = n - g + 1$, for $n > g$. In the case $n \leqslant g$, the space $\mathcal{L}(nQ_+)$ contains only constant functions, and the divisors nQ_+ and nQ_- can not be equivalent. If $n \geqslant g + 1$, we choose the following basis for $\mathcal{L}(nQ_+)$:

$$1,\ f_1,\ \dots,\ f_{n-g},$$

where

$$f_k = \frac{y - B_0 - B_1 x - \cdots - B_{g+k-1} x^{g+k-1}}{x^{g+k}}.$$

Thus, $nQ_+ \equiv nQ_-$ if there is a function $f \in \mathcal{L}(nQ_+)$ with a zero of order n at Q_-, i.e., if there exist constants $\alpha_0, \dots, \alpha_{n-g}$, not all equal to 0, such that

$$\begin{array}{lllll} \alpha_0 + & \alpha_1 f_1(Q_-) & + \cdots + & \alpha_{n-g} f_{n-g}(Q_-) & = 0, \\ & \alpha_1 f_1'(Q_-) & + \cdots + & \alpha_{n-g} f_{n-g}'(Q_-) & = 0, \\ \dots & & & & \\ \dots & & & & \\ & \alpha_1 f_1^{(n-1)}(Q_-) & + \cdots + & \alpha_{n-g} f_{n-g}^{(n-1)}(Q_-) & = 0. \end{array}$$

Existence of a non-trivial solution to this system of linear equations is equivalent to the condition (3).

When some of the values $x_1, \dots, x_{2g+2}$ coincide, the curve $\mathcal{C}$ is singular. This case can be considered by the procedure of normalization of the curve, as in Lemma 4. The condition for the equivalence of the divisors nQ_+ and nQ_-, in the case when $\mathcal{C}$ is singular, is again (3). □

4. Periodic billiard trajectories inside *k* confocal quadrics in $\mathbb{R}^d$

Darboux was the first who considered a higher-dimensional generalization of Poncelet theorem. Namely, he investigated light-rays in the three-dimensional case ($d = 3$) and announced the corresponding complete Poncelet theorem in [13] in 1870.

Higher-dimensional generalizations of CPT ($d \geqslant 3$) were obtained quite recently in [11], and the related Cayley-type conditions were derived by the authors [21].

The main goal of this section is to present detailed proof of Cayley-type condition for generalized CPT, together with discussions and examples.

Consider an ellipsoid in $\mathbb{R}^d$:

$$\frac{x_1^2}{a_1} + \cdots + \frac{x_d^2}{a_d} = 1, \quad a_1 > \cdots > a_d > 0,$$

and the related system of Jacobian elliptic coordinates: $(\lambda_1,\dots,\lambda_d)$ ordered by the condition

$$\lambda_1>\lambda_2>\cdots>\lambda_d.$$

If we denote:

$$Q_\lambda(x)=\frac{x_1^2}{a_1-\lambda}+\cdots+\frac{x_d^2}{a_d-\lambda},$$

then any quadric from the corresponding confocal family is given by the equation of the form:

$$\mathcal{Q}_\lambda:\ Q_\lambda(x)=1. \tag{4}$$

The famous Chasles theorem states that any line in the space $\mathbb{R}^d$ is tangent to exactly $d-1$ quadrics from a given confocal family. Next lemma gives an important condition on these quadrics.

Lemma 6. *Suppose a line ℓ is tangent to quadrics $\mathcal{Q}_{\alpha_1},\dots,\mathcal{Q}_{\alpha_{d-1}}$ from the family* (4)*. Then Jacobian coordinates $(\lambda_1,\dots,\lambda_d)$ of any point on ℓ satisfy the inequalities $\mathcal{P}(\lambda_s)\geqslant 0$, $s=1,\dots,d$, where*

$$\mathcal{P}(x)=(a_1-x)\cdots(a_d-x)(\alpha_1-x)\cdots(\alpha_{d-1}-x).$$

Proof. Let x be a point of ℓ, $(\lambda_1,\dots,\lambda_d)$ its Jacobian coordinates, and y a vector parallel to ℓ. The equation $Q_\lambda(x+ty)=1$ is quadratic with respect to t. Its discriminant is:

$$\Phi_\lambda(x,y)=Q_\lambda(x,y)^2-Q_\lambda(y)\big(Q_\lambda(x)-1\big),$$

where

$$Q_\lambda(x,y)=\frac{x_1y_1}{a_1-\lambda}+\cdots+\frac{x_dy_d}{a_d-\lambda}.$$

By [33] we have:

$$\Phi_\lambda(x,y)=\frac{(\alpha_1-\lambda)\cdots(\alpha_{d-1}-\lambda)}{(a_1-\lambda)\cdots(a_d-\lambda)}.$$

For each of the coordinates $\lambda=\lambda_s$ $(1\leqslant s\leqslant d)$, the quadratic equation has a solution $t=0$; thus, the corresponding discriminants are non-negative. This is obviously equivalent to $\mathcal{P}(\lambda_s)\geqslant 0$. □

4.1. Billiard inside a domain bounded by confocal quadrics

Suppose that a bounded domain $\Omega\subset\mathbb{R}^d$ is given such that its boundary $\partial\Omega$ lies in the union of several quadrics from the family (4). Then, in elliptic coordinates, Ω is given by:

$$\beta_1'\leqslant\lambda_1\leqslant\beta_1'',\quad\dots,\quad\beta_d'\leqslant\lambda_d\leqslant\beta_d'',$$

where $a_{s+1}\leqslant\beta_s'<\beta_s''\leqslant a_s$ for $1\leqslant s\leqslant d-1$ and $-\infty<\beta_d'<\beta_d''\leqslant a_d$.

Consider a billiard system within Ω and let $\mathcal{Q}_{\alpha_1},\dots,\mathcal{Q}_{\alpha_{d-1}}$ be caustics of one of its trajectories. For any $s=1,\dots,d$, the set Λ_s of all values taken by the coordinate λ_s on the trajectory is, according to Lemma 6, included in $\Lambda_s'=\{\lambda\in[\beta_s',\beta_s'']:\ \mathcal{P}(\lambda)\geqslant 0\}$. By [29], each of the intervals (a_{s+1},a_s) $(2\leqslant s\leqslant d)$ contains at most two of the values $\alpha_1,\dots,\alpha_{d-1}$, the interval $(-\infty,a_d)$ contains at most one of them, while none is included in $(a_1,+\infty)$. Thus, for each s, the following three cases are possible:

First case: $\alpha_i,\alpha_j\in[\beta_s',\beta_s'']$, $\alpha_i<\alpha_j$. Since any line which contains a segment of the trajectory touches $\mathcal{Q}_{\alpha_i}$ and $\mathcal{Q}_{\alpha_j}$, the whole trajectory is placed between these two quadrics. The elliptic coordinate λ_s has critical values at points where the trajectory touches one them, and remains monotonous elsewhere. Hence, meeting points with $\mathcal{Q}_{\alpha_i}$ and $\mathcal{Q}_{\alpha_j}$ are placed alternately along the trajectory and $\Lambda_s=\Lambda_s'=[\alpha_i,\alpha_j]$.

Second case: Among $\alpha_1,\dots,\alpha_{d-1}$, only α_i is in $[\beta_s',\beta_s'']$. $\mathcal{P}$ is non-negative in exactly one of the intervals: $[a_{s+1},\alpha_i]$, $[\alpha_i,a_s]$, let us take in the first one. Then the trajectory has bounces only on $\mathcal{Q}_{\beta_s'}$. If $\alpha_i\neq\beta_s''$, the billiard particle never reaches the boundary $\mathcal{Q}_{\beta_s''}$. The coordinate λ_s has critical values at meeting points with $\mathcal{Q}_{\beta_s'}$ and the caustic $\mathcal{Q}_{\alpha_i}$, and remains monotonous elsewhere. Hence, $\Lambda_s=\Lambda_s'=[\beta_s',\alpha_i]$. If $\mathcal{P}$ is non-negative in $[\alpha_i,a_s]$, then we obtain $\Lambda_s=\Lambda_s'=[\alpha_i,\beta_s'']$.

Third case: The segment $[\beta_s', \beta_s'']$ does not contain any of values $\alpha_1, \dots, \alpha_{d-1}$. Then $\mathcal{P}$ is non-negative in $[\beta_s', \beta_s'']$. The coordinate λ_s has critical values only at meeting points with boundary quadrics $\mathcal{Q}_{\beta_s'}$ and $\mathcal{Q}_{\beta_s''}$, and changes monotonously between them. This implies that the billiard particle bounces of them alternately. Obviously, $\Lambda_s = \Lambda_s' = [\beta_s', \beta_s'']$.

Denote $[\gamma_s', \gamma_s''] := \Lambda_s = \Lambda_s'$. Notice that the trajectory meets quadrics of any pair $\mathcal{Q}_{\gamma_s'}, \mathcal{Q}_{\gamma_s''}$ alternately. Thus, any periodic trajectory has the same number of intersection points with each of them.

Let us make a few remarks on the case when $\gamma_s' = a_{s+1}$ or $\gamma_s'' = a_s$. This means that either a part of $\partial\Omega$ is a degenerate quadric from the confocal family or Ω is not bounded, from one side at least, by a quadric of the corresponding type. Discussion of the case when Ω is bounded by a coordinate hyperplane does not differ from the one we have just made. On the other hand, non-existence of a part of the boundary means that the coordinate λ_s will have extreme values at the points of intersection of the trajectory with the corresponding hyperplane. Since a closed trajectory intersects any hyperplane even number of times, it follows that the coordinate λ_s is taking each of its extreme values even number of times during the period.

Theorem 2. *A trajectory of the billiard system within Ω with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ is periodic with exactly n_s points at $\mathcal{Q}_{\gamma_s'}$ and n_s points at $\mathcal{Q}_{\gamma_s''}$ $(1 \leqslant s \leqslant d)$ if and only if*

$$\sum_{s=1}^{d} n_s \big(\mathcal{A}(P_{\gamma_s'}) - \mathcal{A}(P_{\gamma_s''})\big) = 0$$

on the Jacobian of the curve:

$$\Gamma\colon\ y^2 = \mathcal{P}(x) := (a_1 - x)\cdots(a_d - x)(\alpha_1 - x)\cdots(\alpha_{d-1} - x).$$

Here, $\mathcal{A}$ denotes the Abel–Jacobi map, where $P_{\gamma_s'}$, $P_{\gamma_s''}$ are points on Γ with coordinates $P_{\gamma_s'} = (\gamma_s', (-1)^s\sqrt{\mathcal{P}(\gamma_s')})$, $P_{\gamma_s''} = (\gamma_s'', (-1)^s\sqrt{\mathcal{P}(\gamma_s'')})$.

Proof. Following Jacobi [27] and Darboux [15], let us consider the equations:

$$\sum_{s=1}^{d} \frac{\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0, \quad \sum_{s=1}^{d} \frac{\lambda_s\,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0, \quad \dots, \quad \sum_{s=1}^{d} \frac{\lambda_s^{d-2}\,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0, \tag{5}$$

where, for any fixed s, the square root $\sqrt{\mathcal{P}(\lambda_s)}$ is taken with the same sign in all of the expressions. Then (5) represents a system of differential equations of a line tangent to $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$. Besides that,

$$\sum_{s=1}^{d} \frac{\lambda_s^{d-1}\,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 2\,\mathrm{d}\ell, \tag{6}$$

where $\mathrm{d}\ell$ is the element of the line length.

Attributing all possible combinations of signs to $\sqrt{\mathcal{P}(\lambda_1)}, \dots, \sqrt{\mathcal{P}(\lambda_d)}$, we can obtain 2^{d-1} non-equivalent systems (5), which correspond to 2^{d-1} different tangent lines to $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ from a generic point of the space. Moreover, the systems corresponding to a line and its reflection to a given hyper-surface $\lambda_s = \text{const}$ differ from each other only in signs of the roots $\sqrt{\mathcal{P}(\lambda_s)}$.

Solving (5) and (6) as a system of linear equations with respect to $\frac{\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}}$, we obtain:

$$\frac{\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = \frac{2\,\mathrm{d}\ell}{\prod_{i\neq s}(\lambda_s - \lambda_i)}.$$

Thus, along a billiard trajectory, the differentials $(-1)^{s-1}\frac{\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}}$ stay always positive, if we assume that the signs of the square roots are chosen appropriately on each segment.

From these remarks and the discussion preceding this theorem, it follows that the value of the integral $\int \frac{\lambda_s^i \, d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}}$ between two consecutive common points of the trajectory and the quadric $\mathcal{Q}_{\gamma_s'}$ (or $\mathcal{Q}_{\gamma_s''}$) is equal to:

$$2(-1)^{s-1} \int_{\gamma_s'}^{\gamma_s''} \frac{\lambda_s^i \, d\lambda_s}{+\sqrt{\mathcal{P}(\lambda_s)}}.$$

Now, if $\mathbf{p}$ is a finite polygon representing a billiard trajectory and having exactly n_s points at $\mathcal{Q}_{\gamma_s'}$ and n_s at $\mathcal{Q}_{\gamma_s''}$ $(1 \leqslant s \leqslant d)$, then

$$\sum \int^{\mathbf{p}} \frac{\lambda_s^i \, d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 2 \sum (-1)^{s-1} n_s \int_{\gamma_s'}^{\gamma_s''} \frac{\lambda_s^i \, d\lambda_s}{+\sqrt{\mathcal{P}(\lambda_s)}} \quad (1 \leqslant i \leqslant d).$$

Finally, the polygonal line is closed if and only if

$$\sum (-1)^s n_s \int_{\gamma_s'}^{\gamma_s''} \frac{\lambda_s^i \, d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0 \quad (1 \leqslant i \leqslant d-1),$$

which was needed. □

Example 2. Consider two domains Ω' and Ω'' in $\mathbb{R}^3$. Let Ω' be bounded by the ellipsoid $\mathcal{Q}_0$ and the two-folded hyperboloid $\mathcal{Q}_\beta$, $a_2 < \beta < a_1$, in such a way that Ω' is placed between the branches of $\mathcal{Q}_\beta$. On the other hand, suppose Ω'' is bounded by $\mathcal{Q}_0$, the righthand branch of $\mathcal{Q}_\beta$ (this one which is placed in the half-space $x_1 > 0$) and the plane $x_1 = 0$. Elliptic coordinates of points inside both Ω' and Ω'' satisfy:

$$0 \leqslant \lambda_3 \leqslant a_3, \qquad \beta \leqslant \lambda_1 \leqslant a_1.$$

Consider billiard trajectories within these two domains, with caustics $\mathcal{Q}_{\mu_1}$ and $\mathcal{Q}_{\mu_2}$, $a_3 < \mu_1 < a_2$, $a_2 < \mu_2 < a_1$. Since $\mu_2 \leqslant \beta$, the segments Λ_s $(s \in \{1, 2, 3\})$ of all possible values of elliptic coordinates along a trajectory are, for both domains:

$$\Lambda_1 = [\beta, a_1], \qquad \Lambda_2 = [\mu_1, a_2], \qquad \Lambda_3 = [0, a_3].$$

In Ω'', existence of a periodic trajectory with caustics $\mathcal{Q}_{\mu_1}$ and $\mathcal{Q}_{\mu_2}$, which becomes closed after n bounces at $\mathcal{Q}_0$ and $2m$ bounces at $\mathcal{Q}_\beta$ is equivalent to the equality:

$$n\big(\mathcal{A}(P_0) - \mathcal{A}(P_{a_3})\big) + 2m\big(\mathcal{A}(P_\beta) - \mathcal{A}(P_{\mu_1})\big) = 0,$$

on the Jacobian of the corresponding hyperelliptic curve. In Ω', existence of a trajectory with same properties is equivalent to:

$$n\big(\mathcal{A}(P_0) - \mathcal{A}(P_{a_3})\big) + 2m\big(\mathcal{A}(P_\beta) - \mathcal{A}(P_{\mu_1})\big) = 0 \quad \text{and} \quad n \text{ is even}.$$

The fact that the second equality implies the first one is due to the following geometrical fact: any billiard trajectory in Ω' can be transformed to a trajectory in Ω'' applying the symmetry with respect to the x_2x_3-plane to all its points placed under this plane. Notice that this correspondence of trajectories is 2 to 1—in such a way, a generic billiard trajectory in Ω'' corresponds to exactly two trajectories in Ω'. An example of such corresponding billiard trajectories is shown on Figs. 3 and 4.

4.2. Billiard ordered game

Our next step is to introduce a notion of bounces "from outside" and "from inside". More precisely, let us consider an ellipsoid $\mathcal{Q}_\lambda$ from the confocal family (4) such that $\lambda \in (a_{s+1}, a_s)$ for some $s \in \{1, \dots, d\}$, where $a_{d+1} = -\infty$.

Observe that along a billiard ray which reflects at $\mathcal{Q}_\lambda$, the elliptic coordinate λ_i has a local extremum at the point of reflection.

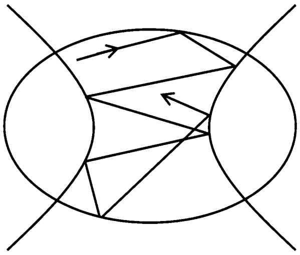

Fig. 3. A trajectory inside Ω'.

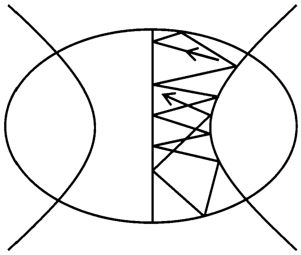

Fig. 4. A trajectory inside Ω''.

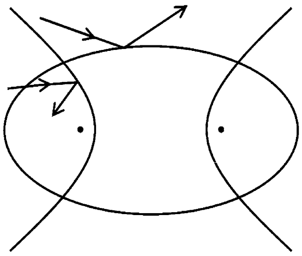

Fig. 5. Reflection from outside.

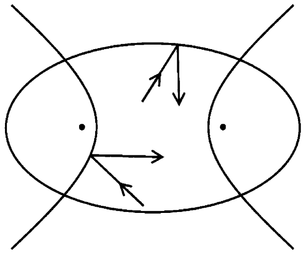

Fig. 6. Reflection from inside.

Definition 1. A ray reflects *from outside* at the quadric $\mathcal{Q}_\lambda$ if the reflection point is a local maximum of the Jacobian coordinate λ_s, and it reflects *from inside* if the reflection point is a local minimum of the coordinate λ_s.

On Figs. 5 and 6 reflections from inside and outside an ellipse and a hyperbola are sketched.

Let us remark that in the case when $\mathcal{Q}_\lambda$ is an ellipsoid, the notions introduced in Definition 1 coincide with the usual ones.

Assume now a k-tuple of confocal quadrics $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ from the confocal pencil (4) is given, and $(i_1, \dots, i_k) \in \{-1, 1\}^k$.

Definition 2. *The billiard ordered game* joined to quadrics $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$, with the *signature* $(i_1, \dots, i_k)$ is the billiard system with trajectories having bounces at $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ respectively, such that

$$\text{the reflection at } \mathcal{Q}_{\beta_s} \text{ is from inside if } i_s = +1;$$
$$\text{the reflection at } \mathcal{Q}_{\beta_s} \text{ is from outside if } i_s = -1.$$

Note that any trajectory of a billiard ordered game has $d-1$ caustics from the same family (4).

Suppose $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ are ellipsoids and consider a billiard ordered game with the signature $(i_1, \dots, i_k)$. In order that trajectories of such a game stay bounded, the following condition has to be satisfied:

$$i_s = -1 \Rightarrow i_{s+1} = i_{s-1} = 1 \quad \text{and} \quad \beta_{s+1} < \beta_s, \ \beta_{s-1} < \beta_s.$$

(Here, we identify indices 0 and $k+1$ with k and 1 respectively.)

Example 3. On Fig. 7, a trajectory corresponding to the 7-tuple,

$$(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_1, \mathcal{Q}_3, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_1),$$

with the signature $(1, -1, 1, -1, 1, 1, 1)$, is shown.

Theorem 3. *Given a billiard ordered game within k ellipsoids $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ with the signature $(i_1, \dots, i_k)$. Its trajectory with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ is k-periodic if and only if*

$$\sum_{s=1}^{k} i_s \big(\mathcal{A}(P_{\beta_s}) - \mathcal{A}(P_\alpha)\big)$$

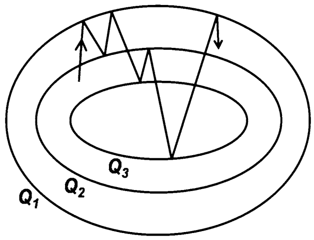

Fig. 7. Billiard ordered game.

is equal to a sum of several expressions of the form: $(\mathcal{A}(P_{\alpha_p}) - \mathcal{A}(P_{\alpha_{p'}}))$ *on the Jacobian of the curve* Γ: $y^2 = \mathcal{P}(x)$, *where* $P_{\beta_s} = (\beta_s, +\sqrt{\mathcal{P}(\beta_s)})$, $\alpha = \min\{a_d, \alpha_1, \dots, \alpha_{d-1}\}$ *and* $\mathcal{Q}_{\alpha_p}$, $\mathcal{Q}_{\alpha_{p'}}$ *are pairs of caustics of the same type.*

When $\mathcal{Q}_{\beta_1} = \cdots = \mathcal{Q}_{\beta_k}$ and $i_1 = \cdots = i_k = 1$ we obtain the Cayley-type condition for the billiard motion inside an ellipsoid in $\mathbb{R}^d$.

We are going to treat in more detail the case of the billiard motion between two ellipsoids.

Proposition 3. *The condition that there exists a closed billiard trajectory between two ellipsoids* $\mathcal{Q}_{\beta_1}$ *and* $\mathcal{Q}_{\beta_2}$, *which bounces exactly m times to each of them, with caustics* $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$, *is*:

$$\operatorname{rank}\begin{pmatrix} f_1'(P_{\beta_2}) & f_2'(P_{\beta_2}) & \dots & f_{m-d+1}'(P_{\beta_2}) \\ f_1''(P_{\beta_2}) & f_2''(P_{\beta_2}) & \dots & f_{m-d+1}''(P_{\beta_2}) \\ & & \dots & \\ & & \dots & \\ f_1^{(m-1)}(P_{\beta_2}) & f_2^{(m-1)}(P_{\beta_2}) & \dots & f_{m-d+1}^{(m-1)}(P_{\beta_2}) \end{pmatrix} < m - d + 1.$$

Here

$$f_j = \frac{y - B_0 - B_1(x-\beta_1) - \cdots - B_{d+j-2}(x-\beta_1)^{d+j-2}}{x^{d+j-1}}, \quad 1 \leqslant j \leqslant m-d+1,$$

and $y = B_0 + B_1(x-\beta_1) + \cdots$ *is the Taylor expansion around the point symmetric to* P_{β_1} *with respect to the hyperelliptic involution of the curve* Γ. (All notations are as in Theorem 3.)

Example 4. Consider a billiard motion in the three-dimensional space, with ellipsoids $\mathcal{Q}_0$ and $\mathcal{Q}_\gamma$ as boundaries $(0 < \gamma < a_3)$ and caustics $\mathcal{Q}_{\alpha_1}$ and $\mathcal{Q}_{\alpha_2}$. Such a motion closes after 4 bounces from inside to $\mathcal{Q}_0$ and 4 bounces from outside to $\mathcal{Q}_\gamma$ if and only if:

$$\operatorname{rank} X < 2.$$

The matrix X is given by:

$$\begin{aligned}
X_{11} &= -3C_0 + C_1\gamma + 3B_0 + 2B_1\gamma + B_2\gamma^2, \\
X_{12} &= -4C_0 + C_1\gamma + 4B_0 + 3B_1\gamma + 2B_2\gamma^2 + B_3\gamma^3, \\
X_{21} &= 6C_0 - 3C_1\gamma + C_2\gamma^2 - 6B_0 - 3B_1\gamma - B_2\gamma^2, \\
X_{22} &= 10C_0 - 4C_1\gamma - 10B_0 - 6B_1\gamma - 3B_2\gamma^2 - B_3\gamma^3, \\
X_{31} &= -10C_0 + 6C_1\gamma - 3C_2\gamma^2 + C_3\gamma^3 + 10B_0 + 4B_1\gamma + B_2\gamma^2, \\
X_{32} &= -20C_0 - 10C_1\gamma - 4C_2\gamma^2 + C_3\gamma^3 + 20B_0 + 10B_1\gamma + 4B_2\gamma^2 + B_3\gamma^3,
\end{aligned}$$

and the expressions

$$\begin{aligned}
-\sqrt{(a_1-x)(a_2-x)(a_3-x)(\alpha_1-x)(\alpha_2-x)} &= B_0 + B_1x + B_2x^2 + \cdots, \\
+\sqrt{(a_1-x)(a_2-x)(a_3-x)(\alpha_1-x)(\alpha_2-x)} &= C_0 + C_1(x-\gamma) + \cdots
\end{aligned}$$

are Taylor expansions around points $x = 0$ and $x = \gamma$ respectively.

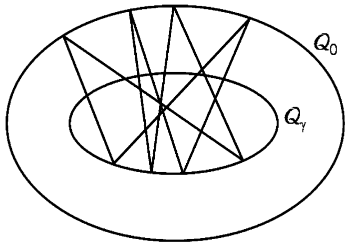

Fig. 8. A closed trajectory of the billiard ordered game with 8 alternate bounces from inside of two ellipses.

Example 5. Using the same notations as in the previous example, let us consider trajectories with 4 bounces from inside to each of $\mathcal{Q}_0$ and $\mathcal{Q}_\gamma$, as shown on Fig. 8.

The explicit condition for periodicity of such a trajectory is:

$$\operatorname{rank} X < 2,$$

with

$$\begin{aligned}
X_{11} &= -4C_0 + C_1\gamma + 3B_1\gamma + 2B_2\gamma^2 + B_3\gamma^3,\\
X_{12} &= -3C_0 + C_1\gamma + 3B_0 + 2B_1\gamma + B_2\gamma^2,\\
X_{21} &= -6C_0 + C_2\gamma^2 + 6B_0 + 6B_1\gamma + 5B_2\gamma^2 + 3B_3\gamma^3,\\
X_{22} &= -6C_0 + C_1\gamma + C_2\gamma^2 + 6B_0 + 5B_1\gamma + 3B_2\gamma^2,\\
X_{31} &= -4C_0 + C_3\gamma^3 + 4B_0 + 4B_1\gamma + 4B_2\gamma^2 + 3B_3\gamma^3,\\
X_{32} &= -4C_0 + C_2\gamma^2 + C_3\gamma^3 + 4B_0 + 4B_1\gamma + 3B_2\gamma^2.
\end{aligned}$$

5. Elliptical billiard as a discrete time system: *d*—arbitrary, *k* = 1

Another approach to the description of periodic billiard trajectories is based on the technique of discrete Lax representation.

In this section, first, we are going to list the main steps of algebro-geometric integration of the elliptic billiard, following [32]. Then, the connection between periodic billiard trajectories and points of finite order on the corresponding hyperelliptic curve will be established, and, using results from Section 3, the Cayley-type conditions will be derived, as they were obtained by the authors in [19,20]. In addition, here we provide a more detailed discussion concerning trajectories with the period not greater than d and the cases of the singular isospectral curve.

Following [32], the billiard system will be considered as a system with the discrete time. Using its integration procedure, the connection between periodic billiard trajectories and points of finite order on the corresponding hyperelliptic curve will be established.

5.1. XYZ model and isospectral curves

Elliptical billiard as a mechanical system with the discrete time. Let the ellipsoid in $\mathbb{R}^d$ be given by:

$$(Ax, x) = 1.$$

We can assume that A is a diagonal matrix, with different eigenvalues. The billiard motion within the ellipsoid is determined by the following equations:

$$x_{k+1} - x_k = \mu_k y_{k+1}, \qquad y_{k+1} - y_k = \nu_k A x_k,$$

where

$$\mu_k = -\frac{2(Ay_{k+1}, x_k)}{(Ay_{k+1}, y_{k+1})}, \qquad \nu_k = -\frac{2(Ax_k, y_k)}{(Ax_k, Ax_k)}.$$

Here, x_k is a sequence of points of billiard bounces, while $y_k = \frac{x_k - x_{k-1}}{|x_k - x_{k-1}|}$ are the momenta.

Connection between billiard and XYZ model. To the billiard system with the discrete time, Heisenberg XYZ model can be joined, in the way described by Veselov and Moser in [32] and which is going to be presented here.

Consider the mapping $\varphi:(x,y)\mapsto(x',y')$ given by:

$$x'_k = Jy_{k+1} = J(y_k + \nu_k A x_k), \quad y'_k = -J^{-1}x_k, \quad J = A^{-1/2}.$$

Notice that the dynamics of φ contains the billiard dynamics,

$$x''_k = Jy'_{k+1} = -x_{k+1}, \qquad y''_k = -J^{-1}x'_k = -y_{k+1},$$

and define the sequence $(\bar{x}_k, \bar{y}_k)$,

$$(\bar{x}_0, \bar{y}_0) := (x_0, y_0), \qquad (\bar{x}_{k+1}, \bar{y}_{k+1}) := \varphi(\bar{x}_k, \bar{y}_k),$$

which obeys the following relations:

$$\bar{x}_{k+1} = J\bar{y}_k + \nu_k J^{-1}\bar{x}_k, \qquad \bar{y}_{k+1} = -J^{-1}\bar{x}_k,$$

where the parameter ν_k is such that $|\bar{y}_k| = 1$, $(A\bar{x}_k, \bar{x}_x) = 1$. This can be rewritten in the following way:

$$\bar{x}_{k+1} + \bar{x}_{k-1} = \nu_k J^{-1}\bar{x}_k.$$

Now, for the sequence $q_k := J^{-1}\bar{x}_k$, we have:

$$q_{k+1} + q_{k-1} = \nu_k J^{-1} q_k, \quad |q_k| = 1.$$

These equations represent the equations of the discrete Heisenberg XYZ system.

Theorem 4. ([32]) *Let $(\bar{x}_k, \bar{y}_k)$ be the sequence connected with elliptical billiard in the described way. Then $q_k = J^{-1}\bar{x}_k$ is a solution of the discrete Heisenberg system.*

Conversely, if q_k is a solution to the Heisenberg system, then the sequence $x_k = (-1)^k Jq_{2k}$ is a trajectory of the discrete billiard within an ellipsoid.

Integration of the discrete Heisenberg XYZ system. Usual scheme of algebro-geometric integration contains the following [32]. First, the sequence $L_k(\lambda)$ of matrix polynomials has to be determined, together with a factorization,

$$L(\lambda) = B(\lambda)C(\lambda) \mapsto C(\lambda)B(\lambda) = B'(\lambda)C'(\lambda) = L'(\lambda),$$

such that the dynamics $L \mapsto L'$ corresponds to the dynamics of the system q_k. For each problem, finding this sequence of matrices requires a separate search and a mathematician with the excellent intuition. All matrices L_k are mutually similar, and they determine the same *isospectral curve*:

$$\Gamma:\ \det\bigl(L(\lambda) - \mu I\bigr) = 0.$$

The factorization $L_k = B_k C_k$ gives splitting of spectrum of L_k. Denote by ψ_k the corresponding eigenvectors. Consider these vectors as meromorphic functions on Γ and denote their pole divisors by D_k.

The sequence of divisors is linear on the Jacobian of the isospectral curve, and this enables us to find, conversely, eigenfunctions ψ_k, then matrices L_k, and, finally, the sequence (q_k).

Now, integration of the discrete XYZ system by this method will be shortly presented. Details of the procedure can be found in [32].

The equations of discrete XYZ model are equivalent to the isospectral deformation:

$$L_{k+1}(\lambda) = A_k(\lambda)L_k(\lambda)A_k^{-1}(\lambda),$$

where

$$L_k(\lambda) = J^2 + \lambda q_{k-1} \wedge Jq_k - \lambda^2 q_{k-1} \otimes q_{k-1},$$
$$A_k(\lambda) = J - \lambda q_k \otimes q_{k-1}.$$

The equation of the isospectral curve $\Gamma:\ \det(L(\lambda) - \mu I) = 0$ can be written in the following form:

$$\nu^2 = \prod_{i=1}^{d-1}(\mu - \mu_i)\prod_{j=1}^{d}\bigl(\mu - J_j^2\bigr), \tag{7}$$

where $\nu = \lambda \prod_{i=1}^{d-1}(\mu - \mu_i)$ and $\mu_1, \dots, \mu_{d-1}$ are zeroes of the function:

$$\phi_\mu(x, Jy) = \sum_{i=1}^{d} \frac{F_i(x, y)}{\mu - J_i^2},$$

$$F_i = x_i^2 + \sum_{j \neq i} \frac{(x \wedge Jy)_{ij}^2}{J_i^2 - J_j^2}, \quad x = q_{k-1},\ y = q_k.$$

It can be proved that $\mu_1, \dots, \mu_{d-1}$ are parameters of the caustics corresponding to the billiard trajectory [33]. Another way for obtaining the same conclusion is to calculate them directly by taking the first segment of the billiard trajectory to be parallel to a coordinate axe.

If eigenvectors ψ_k of matrices $L_k(\lambda)$ are known, it is possible to determine uniquely members of the sequence q_k. Let D_k be the divisor of poles of function ψ_k on curve Γ. Then [32]:

$$D_{k+1} = D_k + P_\infty - P_0,$$

where P_∞ is the point corresponding to the value $\mu = \infty$ and P_0 to $\mu = 0$, $\lambda = (q_k, J^{-1}q_{k+1})^{-1}$.

5.2. Characterization of periodical billiard trajectories

In the next lemmae, we establish a connection between periodic billiard sequences q_k and periodic divisors D_k.

Lemma 7. ([19]) *Sequence of divisors D_k is n-periodic if and only if the sequence q_k is also periodic with the period n or $q_{k+n} = -q_k$ for all k.*

Proof. If, for all k, $q_{k+n} = q_k$, or $q_{k+n} = -q_k$ for all k, then, obviously, $L_{k+n} = L_k$. Thus, the sequence of eigenvectors ψ_k is periodic with the period n. It follows that the sequence of divisors is also periodic.

Suppose now that $D_{k+n} = D_k$ for all k. This implies that $\psi_{k+n} = c_k\psi_k$. We have:

$$\psi_{k+1} = A_k(\lambda)\psi_k.$$

Let μ_1 and μ_2 be values of parameter μ which correspond to the value $\lambda = 1$ on the curve Γ, and

$$\Psi_k = \big(\psi_k(1, \mu_1), \psi_k(1, \mu_2)\big).$$

From $\psi_{k+1} = A_k(\lambda)\psi_k$, we obtain $A_k(1) = \Psi_{k+1}\Psi_k^{-1}$. It follows that

$$A_k(1) = \frac{c_{k+1}}{c_k} A_{k+n}(1).$$

From the condition $\det A_k = \det A_{k+1}$ for all k, we have $c_k = c_{k+1}$. Thus, the sequence

$$A_k(1) = J - q_k \otimes q_{k-1}$$

is n-periodic. From there,

$$q_{k+n} = \alpha_k q_k, \qquad q_{k+n-1} = \frac{1}{\alpha_k} q_{k-1}.$$

Since $|q_k| = 1$, we have $\alpha_k = 1$ or $\alpha_k = -1$, where all α_k are equal to each other, which proves the assertion. □

Lemma 8. ([19]) *The billiard is, up to the central symmetry, periodic with the period n if and only if the divisor sequence D_k joined to the corresponding Heisenberg XYZ system is also periodic, with the period $2n$.*

Proof. Let $x_{k+n} = \alpha x_k$ for all k, $\alpha \in \{-1, 1\}$. Join to a billiard trajectory (x_k, y_k) the corresponding flow $(\bar{x}_k, \bar{y}_k)$. Since

$$(\bar{x}_{2k}, \bar{y}_{2k}) = (-1)^k(x_k, y_k), \qquad (\bar{x}_{2k+1}, \bar{y}_{2k+1}) = \phi(\bar{x}_{2k}, \bar{y}_{2k}),$$

where the mapping ϕ is linear, we obtain:

$$\bar{x}_{k+2n} = \alpha(-1)^n \bar{x}_k.$$

From there, immediately follows that $q_{k+2n} = \alpha(-1)^n q_k$. According to Lemma 3, the divisor sequence D_k is $2n$-periodic. □

Applying the previous lemma, we obtain the main statement of this section:

Theorem 5. ([20]) *A condition on a billiard trajectory inside ellipsoid $\mathcal{Q}_0$ in $\mathbb{R}^d$, with non-degenerate caustics $\mathcal{Q}_{\mu_1}, \ldots, \mathcal{Q}_{\mu_{d-1}}$, to be periodic, up to the central symmetry, with the period $n \geqslant d$ is*:

$$\operatorname{rank}\begin{pmatrix} B_{n+1} & B_n & \ldots & B_{d+1} \\ B_{n+2} & B_{n+1} & \ldots & B_{d+2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{2n-1} & B_{2n-2} & \ldots & B_{n+d-1} \end{pmatrix} < n - d + 1,$$

where

$$\sqrt{(x-\mu_1)\cdots(x-\mu_{d-1})(x-a_1)\cdots(x-a_d)} = B_0 + B_1 x + B_2 x^2 + \cdots.$$

Proof. The trajectory is periodic with period n if, by Lemma 8, the corresponding divisor sequence on the curve Γ has the period $2n$, i.e., $2n(P_\infty - P_0) = 0$ on $\mathcal{J}(\Gamma)$. Curve Γ is hyperelliptic with genus $g = d - 1$. Taking $\mathcal{A}(P_\infty)$ to be the neutral on $\mathcal{J}(\Gamma)$ we get the desired result by applying Lemma 3. □

Cases of singular isospectral curve. When all $a_1, \ldots, a_d, \mu_1, \ldots, \mu_{d-1}$ are mutually different, then the isospectral curve has no singularities in the affine part. However, singularities appear in the following three cases and their combinations:

(i) $a_i = \mu_j$ for some i, j. The isospectral curve (7) decomposes into a rational and a hyperelliptic curve. Geometrically, this means that the caustic corresponding to μ_i degenerates into the hyperplane $x_i = 0$. The billiard trajectory can be asymptotically tending to that hyperplane (and therefore cannot be periodic), or completely placed in this hyperplane. Therefore, closed trajectories appear when they are placed in a coordinate hyperplane. Such a motion can be discussed like in the case of dimension $d - 1$.
(ii) $a_i = a_j$ for some $i \neq j$. The boundary $\mathcal{Q}_0$ is symmetric.
(iii) $\mu_i = \mu_j$ for some $i \neq j$. The billiard trajectory is placed on the corresponding confocal quadric hyper-surface.

In the cases (ii) and (iii) the isospectral curve Γ is a hyperelliptic curve with singularities. In spite of their different geometrical nature, they both need the same analysis of the condition $2nP_0 \sim 2nE$ for the singular curve (7).

Immediate consequence of Lemma 4 is that Theorem 5 can be applied not only for the case of the regular isospectral curve, but in the cases (ii) and (iii), too. Therefore, the following interesting property holds.

Theorem 6. *If the billiard trajectory within an ellipsoid in d-dimensional Euclidean space is periodic, up to the central symmetry, with the period $n < d$, then it is placed in one of the n-dimensional planes of symmetry of the ellipsoid.*

Proof. This follows immediately from Theorem 5 and the fact that the section of a confocal family of quadrics with a coordinate hyperplane is again a confocal family. □

Note that all trajectories periodic with period n up to the central symmetry, are closed after $2n$ bounces. A statement sharper than Theorem 6 for the trajectories closed after n bounces, where $n \leqslant d$, can be obtained in the elementary fashion:

Proposition 4. *If the billiard trajectory within an ellipsoid in d-dimensional Euclidean space is periodic with period $n \leqslant d$, then it is placed in one of the $(n-1)$-dimensional planes of symmetry of the ellipsoid.*

Proof. First, consider the case $n = d$. Let $x_1 \cdots x_d$ be a periodic trajectory, and $(N, x) = \alpha$ the equation of the hyperplane spanned by its vertices. Here, N is a vector normal to the hyperplane and α is a constant. Since all lines normal to the surface of the ellipsoid at the points of reflection belong to this hyperplane, it follows that $(AN, x_i) = (N, Ax_i) = 0$. Thus, $(AN, x) = 0$ is also an equation of the hyperplane, so $\alpha = 0$ and the vectors N, AN are collinear. From here, the claim follows immediately.

The case $n < d$ can be proved similarly, or applying Theorem 6 and the previous case of this proposition. □

This property can be seen easily for $d = 3$.

Example 6. Consider the billiard motion in an ellipsoid in the 3-dimensional space, with $\mu_1 = \mu_2$, when the segments of the trajectory are placed on generatrices of the corresponding one-folded hyperboloid, confocal to the ellipsoid. If there existed a periodic trajectory with period $n = d = 3$, the three bounces would have been coplanar, and the intersection of that plane and the quadric would have consisted of three lines, which is impossible. It is obvious that any periodic trajectory with period $n = 2$ is placed along one of the axes of the ellipsoid. So, there is no periodic trajectories contained in a confocal quadric surface, with period less or equal to 3.

6. Periodic trajectories of billiards on quadrics in $\mathbb{R}^d$

In [12], the billiard systems on a quadric $\mathcal{Q}_0$ in $\mathbb{R}^d$,

$$\frac{x_1^2}{a_1} + \cdots + \frac{x_d^2}{a_d} = 1, \quad a_1 > \cdots > a_d,$$

are defined as limits of corresponding billiards within $\mathcal{Q}_0$, when one of the caustics tends to $\mathcal{Q}_0$. The boundary of such a billiard consists of the intersection of $\mathcal{Q}_0$ with certain confocal quadrics $\mathcal{Q}_{\beta_1}, \ldots, \mathcal{Q}_{\beta_k}$ of the family (4). Between the bounces, the billiard trajectories follow geodesics of the surface of $\mathcal{Q}_0$, while obeying the reflection law at the points of the boundary. A tangent line issued from any point of a given trajectory touches, besides $\mathcal{Q}_0$, also $d - 2$ confocal quadrics $\mathcal{Q}_{\alpha_1}, \ldots, \mathcal{Q}_{\alpha_{d-2}}$ from the family (4). These $d - 2$ quadrics are fixed for each trajectory, and we shall refer to them as caustics.

The question of description of periodic trajectories of these systems was formulated as an open problem by Abenda and Fedorov [1].

Like in Section 4, we will consider two cases: the billiard inside a domain $\Omega \subset \mathcal{Q}_0$ bounded by several confocal quadrics, and the billiard ordered game within several confocal quadrics of the same type.

For the first case, the domain Ω is given by:

$$\beta_1' \leqslant \lambda_1 \leqslant \beta_1'', \quad \ldots, \quad \beta_{d-1}' \leqslant \lambda_{d-1} \leqslant \beta_{d-1}'' \quad (\beta_d = 0),$$

where $\beta_s', \beta_s'' \in [a_{s+1}, a_s]$ for $1 \leqslant s \leqslant d - 1$. Like in Section 4, Λ_s $(1 \leqslant s \leqslant d - 1)$, is defined as the set of all values taken by the coordinate λ_s on the given trajectory, and it can be shown in the same way that

$$\Lambda_s = \{\lambda \in [\beta_s', \beta_s'']\colon \mathcal{P}_1(\lambda) \geqslant 0\},$$

with

$$\mathcal{P}_1(x) = -x(a_1 - x) \cdots (a_d - x)(\alpha_1 - x) \cdots (\alpha_{d-2} - x),$$

where $\alpha_1, \ldots, \alpha_{d-2}$ are parameters of caustics of the trajectory.

Denote $[\gamma_s', \gamma_s''] := \Lambda_s$.

Before formulating the theorem, let us define the following projection of the Abel–Jacobi map on the curve,

$$\Gamma_1\colon\ y^2 = \mathcal{P}_1(x),$$

by:

$$\bar{\mathcal{A}}(P) = \begin{pmatrix} 0 \\ \int_0^P \frac{x\,\mathrm{d}x}{y} \\ \int_0^P \frac{x^2\,\mathrm{d}x}{y} \\ \cdots \\ \int_0^P \frac{x^{d-2}\,\mathrm{d}x}{y} \end{pmatrix}.$$

Theorem 7. *A trajectory of the billiard system constrained on the ellipsoid $\mathcal{Q}_0$ within Ω, with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-2}}$, is periodic with exactly n_s bounces at each of quadrics $\mathcal{Q}_{\gamma_s'}$, $\mathcal{Q}_{\gamma_s''}$ $(1 \leqslant s \leqslant d-1)$, if and only if*

$$\sum_{s=1}^{d-1} n_s\big(\bar{\mathcal{A}}(P_{\gamma_s'}) - \bar{\mathcal{A}}(P_{\gamma_s''})\big) = 0.$$

Here $P_{\gamma_s'}$, $P_{\gamma_s''}$ are the points on Γ_1 with coordinates $P_{\gamma_s'} = (\gamma_s', (-1)^s\sqrt{\mathcal{P}_1(\gamma_s')})$, $P_{\gamma_s''} = (\gamma_s'', (-1)^s\sqrt{\mathcal{P}_1(\gamma_s'')})$.

Proof. By [27], the system of differential equations of a geodesic line on $\mathcal{Q}_0$ with the caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-2}}$ is:

$$\sum_{s=1}^{d-1} \frac{\lambda_s \,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}} = 0, \quad \sum_{s=1}^{d-1} \frac{\lambda_s^2 \,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}} = 0, \quad \dots, \quad \sum_{s=1}^{d-1} \frac{\lambda_s^{d-2} \,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}} = 0,$$

with the square root $\sqrt{\mathcal{P}(\lambda_s)}$ taken with the same sign in all of the expressions, for any fixed s. Also,

$$\sum_{s=1}^{d-1} \frac{\lambda_s^{d-1} \,\mathrm{d}\lambda_s}{\sqrt{\mathcal{P}_1(\lambda_s)}} = 2\,\mathrm{d}\ell,$$

where $\mathrm{d}\ell$ is the length element.

Now, the rest of the proof is completely parallel to the proof of Theorem 2. □

In the same way as in Section 4, a billiard ordered game constrained on the ellipsoid $\mathcal{Q}_0$ within given quadrics $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ of the same type can be defined. The only difference is that now the signature $(i_1, \dots, i_k)$ can be given arbitrarily, since trajectories are bounded, lying on the compact hypersurface $\mathcal{Q}_0$. Denote by $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-2}}$ the caustics of a given trajectory of the game. Since quadrics $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ are all of the same type, there exist μ', μ'' in the set $S = \{a_1, \dots, a_d, \alpha_1, \dots, \alpha_{d-2}\}$ such that $\beta_1, \dots, \beta_k \in [\mu', \mu'']$ and $(\mu', \mu'') \cap S$ is empty.

Associate to the game the following divisors on the curve Γ_1:

$$\mathcal{D}_s = \begin{cases} P_{\mu''} & \text{if } i_s = i_{s+1} = 1, \\ 0 & \text{if } i_s = -i_{s+1} = 1, \beta_s < \beta_{s+1}, \\ & \text{or } i_s = -i_{s+1} = -1, \beta_s > \beta_{s+1}, \\ P_{\mu''} - P_{\mu'} & \text{if } i_s = -i_{s+1} = 1, \beta_s > \beta_{s+1}, \\ P_{\mu'} - P_{\mu''} & \text{if } i_s = -i_{s+1} = -1, \beta_s < \beta_{s+1}, \\ P_{\mu'} & \text{if } i_s = i_{s+1} = -1, \end{cases}$$

where $P_{\mu'}$ and $P_{\mu''}$ are the branching points with coordinates $(\mu', 0)$ and $(\mu'', 0)$ respectively.

Theorem 8. *Given a billiard ordered game constrained on $\mathcal{Q}_0$ within quadrics $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ with signature $(i_1, \dots, i_k)$. Its trajectory with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-2}}$ is k-periodic if and only if*

$$\sum_{s=1}^{k} i_s\big(\bar{\mathcal{A}}(P_{\beta_s}) - \bar{\mathcal{A}}(\mathcal{D}_s)\big)$$

is equal to a sum of several expressions of the form $\bar{\mathcal{A}}(P_{\alpha_p}) - \bar{\mathcal{A}}(P_{\alpha_{p'}})$ on the Jacobian of the curve Γ_1: $y^2 = \mathcal{P}_1(x)$, where $P_{\beta_s} = (\beta_s, +\sqrt{\mathcal{P}_1(\beta_s)})$ and $\mathcal{Q}_{\alpha_p}$, $\mathcal{Q}_{\alpha_{p'}}$ are pairs of caustics of the same type.

Example 7. Consider the case $d = 3$ and a billiard system constrained on the ellipsoid $\mathcal{Q}_0$ with the boundary $\mathcal{Q}_\gamma$ and caustic $\mathcal{Q}_\alpha$, $a_3 < \gamma < \alpha < a_2$. A sufficient condition for a corresponding billiard trajectory to be n-periodic is:

$$n\big(\mathcal{A}(P_\gamma) - \mathcal{A}(P_\alpha)\big) = 0,$$

or, equivalently, in Cayley-type form:

$$\operatorname{rank}\begin{pmatrix} C_{p+1} & C_{p+2} & \dots & C_{2p-2} \\ C_{p+2} & C_{p+3} & \dots & C_{2p-1} \\ \dots \\ C_{2p} & C_{2p+1} & \dots & C_{3p-3} \end{pmatrix} < p-2, \quad n = 2p,$$

and

$$\operatorname{rank}\begin{pmatrix} C_{p+1} & C_{p+2} & \dots & C_{2p-1} \\ C_{p+2} & C_{p+3} & \dots & C_{2p} \\ \dots & & & \\ C_{2p} & C_{2p+1} & \dots & C_{3p-2} \end{pmatrix} < p-1, \quad n = 2p+1,$$

where

$$y = C_0 + C_1\left(\tilde{x} - \frac{1}{\alpha} - \gamma\right) + C_2\left(\tilde{x} - \frac{1}{\alpha} - \gamma\right)^2 + \cdots$$

is the Taylor expansion around the point P_γ, with $\tilde{x} = 1/(\alpha - x)$. (See also [21].)

More precisely, this condition will be satisfied if and only if the trajectory is n-periodic and its length L with respect to the parameter s:

$$\mathrm{d}s = \lambda_1\lambda_2\cdots\lambda_d\,\mathrm{d}\ell$$

is such that the vector

$$\begin{pmatrix} L/2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + n\bar{\mathcal{A}}(P_\gamma) - n\bar{\mathcal{A}}(P_\alpha)$$

belongs to the period-lattice of the Jacobian of the corresponding hyper-elliptic curve.

Example 8. In a recent preprint [1], another interesting class of periodical billiard trajectories on 2-dimensional ellipsoid, associated with closed geodesics, is described. Such geodesics are closely connected with 3-elliptic KdV solutions, which historically goes back to Hermite and Halphen [26].

The corresponding genus 2 isospectral curve is:

$$\mathrm{G}\colon\ y^2 = -\frac{1}{4}\left(4x^3 - 9g_2x - 27g_3\right)\left(x^2 - 3g_2\right). \tag{8}$$

This curve is 3 : 1 tangential covering of two elliptic curves:

$$\mathrm{E}_1\colon\ y_1^2 = 4x_1^3 - g_2x_1 - g_3,$$
$$\mathrm{E}_2\colon\ y_2^2 = x_2^3 - \frac{27}{16}\left(g_2^3 + 9g_3^2\right)x_2 - \frac{243}{32}\left(3g_3^2 - g_3g_2^3\right).$$

The corresponding conditions for closed billiard trajectories are obtained in [1], by use of Lemma 3 from [20], which corresponds to Lemma 3 of the present paper. For the exact formulae and details of calculations, see [1].

Note that in the same algebro-geometric situation as in the previous example, with the application of Theorems 7 and 8, it is possible to describe a wider class of closed billiard trajectories, without the assumption that their segments belong to closed geodesics. Namely, an important property of the covering $\mathrm{G} \to \mathrm{E}_1$ from Example 8 is that the holomorphic differential $\frac{x\,\mathrm{d}x}{y}$ on G reduces to $-\frac{2}{3}\frac{\mathrm{d}x_1}{y_1}$ on E_1. Thus, whenever the isospectral curve can be represented in the form (8), it is possible to obtain conditions of Cayley's type by the application of Theorems 7, 8 and Lemma 3.

This is going to be illustrated in the next example.

Example 9. Consider the billiard motion on quadric $\mathcal{Q}_0$ in $\mathbb{R}^3$, with $a_1 = a\sqrt{3}$, $a_2 = \frac{3}{2}a$, $a_3 = -a\sqrt{3}$, with the caustic $\mathcal{Q}_\alpha$, $\alpha = -\frac{3}{2}a$, $a > 0$. The domain Ω is bounded by confocal ellipsoid $\mathcal{Q}_\beta$, $\beta < -a\sqrt{3}$. By Theorem 7, the condition for such a billiard to be n-periodic is:

$$n\left(\bar{\mathcal{A}}(P_\beta) - \bar{\mathcal{A}}(P_{-a\sqrt{3}})\right) = 0 \tag{9}$$

on the Jacobian of the curve

$$y^2 = -x\left(x^2 - \frac{9}{4}a^2\right)\left(x^2 - 3a^2\right).$$

This curve is of the form 8, with $g_2 = a^2$, $g_3 = 0$.

The covering $G \to E_1$ is given by:

$$x_1 = -\frac{1}{9}\frac{x^3}{x^2 - 3a^2}, \qquad y_1 = \frac{2}{27}\frac{x^3 - 9a^2 x}{(x^2 - 3a^2)^2}.$$

The condition (9) reduces to the following relation:

$$n\big(\mathcal{A}(Q_{\bar\beta}) - \mathcal{A}(Q_\infty)\big) = 0$$

on the Jacobian of the curve,

$$E_1\colon\ y_1^2 = 4x_1^3 - a^2 x_1,$$

with the point $Q_{\bar\beta} \in E_1$ having the coordinates $(z_1 = \bar\beta,\ w_1 = 4\bar\beta^3 - a^2\bar\beta)$, $\bar\beta = -\frac{1}{9}\frac{\beta^3}{\beta^2 - 3a^2}$ and Q_∞ being the infinite point on E_1.

By Lemma 3, the desired condition becomes:

$$\begin{vmatrix} B_3 & B_4 & \dots & B_{m+1} \\ B_4 & B_5 & \dots & B_{m+2} \\ & & \dots & \\ B_{m+1} & B_{m+2} & \dots & B_{2m-1} \end{vmatrix} = 0 \quad \text{for } n = 2m,$$

$$\begin{vmatrix} B_2 & B_3 & \dots & B_{m+1} \\ B_3 & B_4 & \dots & B_{m+2} \\ & & \dots & \\ B_{m+1} & B_{m+2} & \dots & B_{2m} \end{vmatrix} = 0 \quad \text{for } n = 2m+1,$$

where $\sqrt{4x_1^3 - a^2 x_1} = B_0 + B_1(x_1 - \bar\beta) + B_2(x_1 - \bar\beta)^2 + \cdots$ is the Taylor expansion around the point $Q_{\bar\beta}$.

We will finish this section by analysis of the behaviour of geodesic lines on quadrics after the reflection of a confocal quadric.

Denote by $\mathbf{g}$ a geodesic line on the quadric $\mathcal{Q}_0$ in $\mathbb{R}^d$, and by Ω a domain on $\mathcal{Q}_0$ bounded by a single confocal quadric $\mathcal{Q}_\beta$. We will suppose that the set $\mathbf{g} \cap \mathcal{Q}_\beta \neq \emptyset$ and that the geodesic $\mathbf{g}$ intersects $\mathcal{Q}_\beta$ transversally. Under these assumptions, the number of their intersection points will be finite if and only if the line $\mathbf{g}$ is closed. These points are naturally divided into two sets—one set, denote it by S_1, contains those points where $\mathbf{g}$ enters into Ω, while S_2 contains the points where the geodesic leaves the domain.

Consider reflections on $\mathcal{Q}_\beta$ according to the billiard rule in each point of S_2. Applying these reflections to $\mathbf{g}$, we obtain a family of geodesic segments on $\mathcal{Q}_0$. It appears that all these segments belong to one single geodesic line, see Fig. 9.

Proposition 5. *Let $\mathbf{g}'$ be a geodesic line which contains a point $P \in S_2$, such that $\mathbf{g}$ and $\mathbf{g}'$ satisfy the reflection law in P at $\mathcal{Q}_\beta$. Then $\mathbf{g}'$ contains all points of S_2 and the two lines $\mathbf{g}$, $\mathbf{g}'$ satisfy the reflection law at $\mathcal{Q}_\beta$ in each of these points.*

If $\mathbf{g}$ is closed, then $\mathbf{g}'$ is also closed and they have the same number of intersection points with $\mathcal{Q}_\beta$.

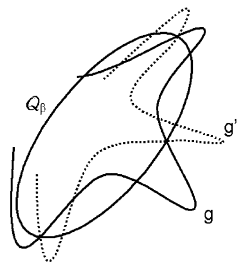

Fig. 9. Reflection of a geodesic line.

Proof. The reflection from inside at $\mathcal{Q}_\beta$ corresponds to the shift by the vector $\bar{\mathcal{A}}(P_\beta) - \bar{\mathcal{A}}(\tau P_\beta)$ on the Jacobian of the curve Γ_1, where $\tau : \Gamma_1 \to \Gamma_1$ is the hyper-elliptic involution: $(x, y) \mapsto (x, -y)$. Thus, the shift does not depend on the choice of the intersection point and it maps one geodesic into another one. This proof is essentially the same as the proof of Lemma 5.1 in [1], which covers the special case of a closed geodesic line on the two-dimensional ellipsoid. □

Observe that $\mathbf{g}'$ not necessarily contains the points of S_1, see Fig. 9.

Denote by T_β the above mapping $\mathbf{g} \mapsto \mathbf{g}'$ in the set of geodesics on $\mathcal{Q}_0$ (on geodesic lines that are tangent to $\mathcal{Q}_\beta$ or do not intersect it at all, we define T_β to be the identity). As a consequence of Proposition 5, we obtain a new porism of Poncelet type.

Corollary 2. *Suppose that* $T_\beta^n(\mathbf{g}) = \mathbf{g}$. *Then* T_β^n *is the identity on the class of all geodesics sharing the same caustics with* $\mathbf{g}$.

Moreover, if the geodesic line $\mathbf{g}$ *is closed, then any billiard trajectory in* Ω *with the initial segment lying on* $\mathbf{g}$, *will be periodic.*

7. Virtual billiard trajectories

Apart from reflections from inside and outside of a quadric, as we defined in Section 4, which correspond to the real motion of the billiard particle, it is of interest to consider *virtual reflections*. These reflections were considered by Darboux in [15]. The aim of this section is to prove and generalize a property of virtual reflections formulated by Darboux in the three-dimensional case in the footnote [15] on pp. 320–321:

> "… Il importe de remarquer: le théorème donné dans le texte suppose essentiellement que les côtés du polygone soient formés par les parties *réelles* et non *virtuelles* du rayon réfléchi. Il existe des polygones fermés d'une tout autre nature. Étant donnés, par exemple, deux ellipsoïdes homofocaux (E_0), (E_1), si, par une droite quelconque, on leur mène des plans tangents, on aura quatre points de contact a_0, b_0 sur (E_0), a_1, b_1 sur (E_1). Le quadrilatère $a_0a_1b_0b_1$ sera tel que les bissectrices des angles a_1, b_1 soient les normales de (E_1), et les bissectrices des angles a_0, b_0 les normales de (E_0), mais il ne constituera pas une route *réelle* pour un rayon lumineux; deux de ses côtés seront formés par les parties virtuelles des rayons réfléchis. De tels polygônes mériteraient aussi d'être étudiés, leur théorie offre les rapports les plus étroits avec celle de l'addition des fonctions hyperelliptiques et de certaines surfaces algébriques …"

More formally, we can define the *virtual reflection* at the quadric $\mathcal{Q}$ as a map of a ray ℓ with the endpoint P_0 ($P_0 \in \mathcal{Q}$) to the ray complementary to the one obtained from ℓ by the real reflection from $\mathcal{Q}$ at the point P_0.

Let us remark that, in the case of real reflections, exactly one elliptic coordinate, the one corresponding to the quadric $\mathcal{Q}$, has a local extreme value at the point of reflection. On the other hand, on a virtual reflected ray, this coordinate is the only one not having a local extreme value. In the 2-dimensional case, the virtual reflection can easily be described as the real reflection from the other confocal conic passing through the point P_0. In higher-dimensional cases, the virtual reflection can be regarded as the real reflection of the line normal to $\mathcal{Q}$ at P_0.

From now on, we consider the n-dimensional case.

Let points X_1, X_2; Y_1, Y_2 belong to quadrics $\mathcal{Q}_1$, $\mathcal{Q}_2$ of a given confocal system.

Definition 3. We will say that the quadruple of points X_1, X_2, Y_1, Y_2 constitutes a *virtual reflection configuration* if pairs of lines X_1Y_1, X_1Y_2; X_2Y_1, X_2Y_2; X_1Y_1, X_2Y_1; X_1Y_2, X_2Y_2 satisfy the reflection law at points X_1, X_2 of $\mathcal{Q}_1$ and Y_1, Y_2 of $\mathcal{Q}_2$ respectively.

Now, the Darboux's statement can be generalized and proved as follows:

Theorem 9. *Let* $\mathcal{Q}_{\lambda_1}$, $\mathcal{Q}_{\lambda_2}$ *be confocal quadrics,* X_1, X_2 *points on* $\mathcal{Q}_{\lambda_1}$ *and* Y_1, Y_2 *on* $\mathcal{Q}_{\lambda_2}$. *If the tangent hyperplanes at these points to the quadrics belong to a pencil, then* X_1, X_2, Y_1, Y_2 *constitute a virtual reflection configuration.*

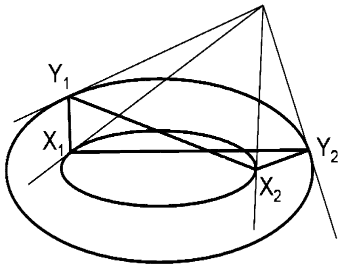

Fig. 10. Virtual reflection configuration.

Furthermore, if $\mathcal{Q}_{\lambda_1}$ and $\mathcal{Q}_{\lambda_2}$ are ellipsoids and $\lambda_1 > \lambda_2$, then the sides of the quadrilateral $X_1Y_1X_2Y_2$ obey the real reflection from $\mathcal{Q}_{\lambda_2}$ and the virtual reflection from $\mathcal{Q}_{\lambda_1}$.

Proof. Denote by ξ_1, ξ_2 and η_1, η_2 the tangent hyperplanes to $\mathcal{Q}_{\lambda_1}$ at X_1, X_2 and to $\mathcal{Q}_{\lambda_2}$ at Y_1, Y_2, respectively. All these hyperplanes belong to a pencil, thus their poles with respect to any quadric will be collinear—particularly, the pole P of ξ_1 lies on line Y_1Y_2. If $Q = Y_1Y_2 \cap \xi_1$, then pairs P, Q and Y_1, Y_2 are harmonically conjugate. It follows that the lines X_1Y_1, X_1Y_2 obey the reflection law from ξ_1. We prove similarly that all other adjacent sides of the quadrilateral $X_1Y_1X_2Y_2$ obey the law of reflection on the corresponding quadrics.

Now, suppose that $\mathcal{Q}_{\lambda_1}$, $\mathcal{Q}_{\lambda_2}$ are ellipsoids, and $\lambda_1 > \lambda_2$. This means that $\mathcal{Q}_{\lambda_1}$ is placed inside $\mathcal{Q}_{\lambda_2}$, thus the whole quadrilateral $X_1Y_1X_2Y_2$ is inside $\mathcal{Q}_{\lambda_2}$. This means that the reflections in points Y_1, Y_2 are real reflections from inside on $\mathcal{Q}_{\lambda_2}$. Besides, the ellipsoid $\mathcal{Q}_{\lambda_1}$ is completely placed inside the dihedron with the sides η_1, η_2. This ellipsoid is also inside the dihedron $\angle(\xi_1, \xi_2)$. Since planes η_1 and η_2 are outside $\angle(\xi_1, \xi_2)$, it follows that points Y_1, Y_2 are also outside this dihedron. Thus, points Y_1, Y_2 are placed at the different sides of each of the planes ξ_1, ξ_2, and reflections of $\mathcal{Q}_{\lambda_1}$ are virtual. □

We are going to conclude this section with the statement converse to the previous theorem.

Proposition 6. *Let pairs of points X_1, X_2 and Y_1, Y_2 belong to confocal ellipsoids $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and let α_1, α_2, β_1, β_2 be the corresponding tangent planes. If a quadruple X_1, X_2, Y_1, Y_2 is a virtual reflection configuration, then planes α_1, α_2, β_1, β_2 belong to a pencil.*

Proof. Consider the pencil determined by α_1 and β_1. Let α_2', β_2' be planes of this pencil, tangent respectively to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at points X_2', Y_2', and distinct from α_1 and β_1. By Theorem 9, the quadruple X_1, X_2', Y_1, Y_2' is a virtual reflection configuration. Moreover, if denote by λ_1, λ_2 parameters of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and assume $\lambda_1 > \lambda_2$, then the sides of the quadrangle obey the reflection law at points Y_1, Y_2' and the virtual reflection at X_1, X_2'. Since the ray obtained from X_1Y_1 by the virtual reflection of $\mathcal{Q}_1$ at X_1, has only one intersection with $\mathcal{Q}_2$, we have $Y_2 = Y_2'$. Points X_2 and X_2' coincide, being the intersection of rays obtained from Y_1X_1 and Y_2X_1, by the reflection at the quadric $\mathcal{Q}_2$. Now, the four tangent planes are all in one pencil. □

8. On generalization of Lebesgue's proof of Cayley's condition

In this section, the analysis of possibility of generalization of the inspiring Lebesgue's procedure to higher-dimensional cases will be given.

A higher-dimensional analogue of the crucial lemma from [31], which is Lemma 1 of this article, is the following:

Lemma 9. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be quadrics of a confocal system and let lines ℓ_1, ℓ_2 satisfy the reflection law at point X_1 of $\mathcal{Q}_1$ and ℓ_2, ℓ_3 at Y_2 of $\mathcal{Q}_2$. Then line ℓ_1 meets $\mathcal{Q}_2$ at point Y_1 and ℓ_3 meets $\mathcal{Q}_1$ at point X_2 such that pairs of lines ℓ_1, Y_1X_2 and $Y_1X_2\ell_3$ satisfy the reflection law at points Y_1, X_2 of quadrics $\mathcal{Q}_2$, $\mathcal{Q}_1$ respectively. Moreover, tangent planes at X_1, X_2, Y_1, Y_2 of these two quadrics are in the same pencil.*

This statement can be proved by the direct application of Theorem 9 on virtual reflections. Nevertheless, there is no complete analogy between Lemma 9 and the corresponding assertion in plane. Lines ℓ_1, ℓ_3 and ℓ_2, Y_1Y_2 are generically skew. Hence we do not have the third pair of planes tangent to the quadric, containing intersection points of these two pairs of lines.

Nevertheless, a complete generalization of the Basic Lemma, can be formulated as follows:

Theorem 10. *Let $\mathcal{F}$ be a dual pencil of quadrics in the three-dimensional space.*

For a given quadric $\Gamma_0 \in \mathcal{F}$, there exist quadruples α, β, γ, δ of planes tangent to Γ_0, and quadrics $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{F}$ touching the pairs of intersecting lines $\alpha\beta$ and $\gamma\delta$, $\alpha\gamma$ and $\beta\delta$, $\alpha\delta$ and $\beta\gamma$ respectively, with the tangent planes to Γ_1, Γ_2, Γ_3 at points of tangency with the lines, all being in one pencil Δ. Moreover, the six intersecting lines are in one bundle.

Every such a configuration of planes α, β, γ, δ and quadrics Γ_0, Γ_1, Γ_2, Γ_3 is determined by two of the intersecting lines, and the tangent planes to Γ_1, Γ_2, or Γ_3 corresponding to these two lines.

Proof. Suppose first that two adjacent lines $\alpha\beta$ and $\alpha\gamma$ are given. There exist two quadrics in $\mathcal{F}$ touching $\alpha\beta$—let Γ_1 be the one tangent to the given plane $\mu \supset \alpha\beta$. Γ_2 denotes the quadric that touches $\alpha\gamma$ and the given plane $\pi \supset \alpha\gamma$. The pencil Δ is determined by its planes μ and π. All three lines $\alpha\beta$, $\alpha\gamma$, Δ are in one bundle $\mathcal{B}$, see Fig. 11.

Note that lines $\alpha\beta$ and $\alpha\gamma$ determine plane α and that α touches a unique quadric Γ_0 from $\mathcal{F}$. Thus, β and γ are determined as tangent planes to Γ_0, containing lines $\alpha\beta$ and $\alpha\gamma$ respectively and being different from α.

Let ν be the plane from Δ, other than μ, that is touching Γ_1. We are going to prove that the point of tangency ν with Γ_1 belongs to γ.

Denote by Φ the dual pencil determined by quadric Γ_1 and the degenerate quadric which consists of lines $\alpha\beta$ and $\gamma\nu$. Since $\gamma\nu \in \mathcal{B}$, these two lines are coplanar.

Dual pencils $\mathcal{F}$ and Φ determine the same involution of the pencil $\alpha\gamma$, because they both determine the pair α, γ and the quadric Γ_1 is the common for both pencils. Since quadric $\Gamma_2 \in \mathcal{F}$ determines the pair π, π of coinciding planes, a quadric $\mathcal{Q}$ determining the same pair has to exist in Φ. This quadric, as well as all other quadrics in Φ, touches ν and μ. Since ν, μ, π all belong to the pencil Δ, $\mathcal{Q}$ is degenerate and contains Δ. Since any quadric of Φ touches μ, π at points of lines $\alpha\beta$, $\alpha\gamma$ respectively, the other component of $\mathcal{Q}$ also has to be Δ, thus $\mathcal{Q}$ is the double Δ.

It follows that all quadrics of Φ, and particularly Γ_1, are tangent to ν at a point of $\gamma\nu$.

Similarly, if $\kappa \neq \pi$ is the other plane in Δ tangent to Γ_2, the touching point belongs to β and to δ, the plane, other than γ, tangent to Γ_0 and containing the line $\gamma\nu$.

Now, let us note that $\mathcal{F}$ and Φ determine the same involution on pencil $\alpha\delta$, because they both determine pair α, δ and Γ_1 belongs to both pencils. Thus, the common plane ρ of pencils Δ and $\alpha\delta$ is tangent to a quadric of $\mathcal{F}$ at a point of $\alpha\delta$. Denote this quadric by Γ_3. Similarly as before, we can prove that Γ_3 is touching the line $\beta\gamma$ and the corresponding tangent plane σ is common to pencils Δ and $\beta\gamma$.

Now, suppose that two non-adjacent lines $\alpha\beta$, $\gamma\delta$, both tangent to a quadric $\Gamma_1 \in \mathcal{F}$, with the corresponding tangent planes μ, ν, are given. Similarly as above, we can prove that the plane ρ is tangent to a quadric from $\mathcal{F}$ at a point of $\alpha\delta$. So, we can consider the configuration as determined by adjacent lines $\alpha\beta$, $\alpha\delta$ and planes μ, ρ. In this way, we reduced it to the previous case. □

For the conclusion, we will generalize the notion of Cayley's cubic.

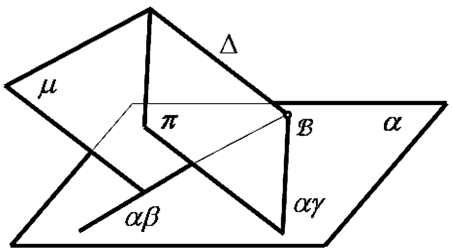

Fig. 11. Theorem 10.

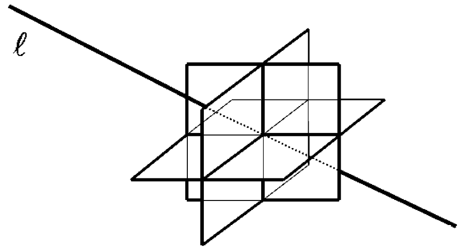

Fig. 12. Three points of the generalized Cayley curve in dimension 3.

Definition 4. The *generalized Cayley curve* is the variety of hyperplanes tangent to quadrics of a given confocal family in $\mathbb{C}P^d$ at the points of a given line ℓ.

This curve is naturally embedded in the dual space $\mathbb{C}P^{d*}$.

On Fig. 12, we see the planes which correspond to one point of the line ℓ in the 3-dimensional space.

Proposition 7. *The generalized Cayley curve in $\mathbb{C}^d$, for $d \geqslant 3$ is a hyperelliptic curve of genus $g = d - 1$. Its natural realization in $\mathbb{C}^{d*}$ is of degree $2d - 1$.*

Proof. Let us consider the projection from the generalized Cayley's curve to the line of the parameters of the confocal family. Since ℓ intersects each quadric twice, this is a two-folded covering, with branching points corresponding to quadrics touching ℓ and the degenerate ones. Since there is $(d - 1) + (d + 1)$ such quadrics, we obtain the genus directly from Riemann–Hurwitz formula.

Its degree is equal to the number of intersection points with a hyperplane in $\mathbb{C}^{d*}$. Such a hyperplane is a bundle of hyperplanes containing one point in $\mathbb{C}^d$. Take $P \in \ell$ to be this point. Since there are d quadrics from the confocal family containing P and $d - 1$ tangent to ℓ, the assertion follows. □

Let us note that this curve is isomorphic to the Veselov–Moser isospectral curve (7). Also, in the 3-dimensional case, it is isomorphic to the Jacobi hyperelliptic curve, which was used by Darboux considering the generalization of Poncelet theorem.

Further development of these ideas will be presented in the separate publication [22].

Acknowledgements

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Appendix A. Integrable potential perturbations of the elliptical billiard

The equation,

$$\lambda V_{xy} + 3(yV_x - xV_y) + \left(y^2 - x^2\right)V_{xy} + xy(V_{xx} - V_{yy}) = 0, \tag{10}$$

is a special case of the Bertrand–Darboux equation [8,14,36], which represents the necessary and sufficient condition for a natural mechanical system with two degrees of freedom,

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + V(x, y),$$

to be separable in elliptical coordinates or some of their degenerations.

Solutions of Eq. (10) in the form of Laurent polynomials in x, y were described in [16]. Such solutions are in [18] naturally related to the well-known hypergeometric functions of Appell. This relation automatically gives a wider class of solutions of Eq. (10)—new potentials are obtained for non-integer parameters, giving a huge family of integrable billiards within an ellipse with potentials. Similar formulae for potential perturbations for the Jacobi problem for geodesics on an ellipsoid from [16,17] are given. They show the existence of a connection between separability of classical systems on one hand, and the theory of hypergeometric functions on the other one, which is still not completely understood. Basic references for the Appell functions are [2,3,35].

The function F_4 is one of the four hypergeometric functions in two variables, which are introduced by Appell [2,3],

$$F_4(a,b,c,d;x,y)=\sum\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

where $(a)_n$ is the standard Pochhammer symbol:

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1),\quad (a)_0=1.$$

(For example $m!=(1)_m$.)

The series is convergent for $\sqrt{x}+\sqrt{y}\leqslant 1$. The functions F_4 can be analytically continued to the solutions of the equations:

$$x(1-x)\frac{\partial^2F}{\partial x^2}-y^2\frac{\partial^2F}{\partial y^2}-2xy\frac{\partial^2F}{\partial x\partial y}+\bigl[c-(a+b+1)x\bigr]\frac{\partial F}{\partial x}-(a+b+1)y\frac{\partial F}{\partial y}-abF=0,$$
$$y(1-y)\frac{\partial^2F}{\partial y^2}-x^2\frac{\partial^2F}{\partial x^2}-2xy\frac{\partial^2F}{\partial x\partial y}+\bigl[c'-(a+b+1)y\bigr]\frac{\partial F}{\partial y}-(a+b+1)x\frac{\partial F}{\partial x}-abF=0.$$

A.1. Potential perturbations of a billiard inside an ellipse

A billiard system which describes a particle moving freely within the ellipse,

$$\frac{x^2}{A}+\frac{y^2}{B}=1,$$

is completely integrable and it has an additional integral:

$$K_1=\frac{\dot x^2}{A}+\frac{\dot y^2}{B}-\frac{(\dot xy-\dot yx)^2}{AB}.$$

We are interested now in potential perturbations $V=V(x,y)$ such that the perturbed system has an integral $\widetilde{K}_1$ of the form:

$$\widetilde{K}_1=K_1+k_1(x,y),$$

where $k_1=k_1(x,y)$ depends only on coordinates. This specific condition leads to equation (10) on V with $\lambda=A-B$.

In [16,17] the Laurent polynomial solutions of Equation (10) were given. Denoting

$$V_\gamma=\tilde y^{-\gamma}\bigl((1-\gamma)\,\tilde x\,F_4(1,2-\gamma,2,1-\gamma,\tilde x,\tilde y)+1\bigr), \tag{11}$$

where $\tilde x=x^2/\lambda$, $\tilde y=-y^2/\lambda$, the more general result was obtained in [18]:

Theorem 11. *Every function V_γ given with* (11) *and $\gamma\in\mathbb{C}$ is a solution of Eq.* (10).

This theorem gives new potentials for non-integer γ. For integer γ, one obtains the Laurent solutions.

Mechanical interpretation. With $\gamma\in\mathbb{R}^-$ and the coefficient multiplying V_γ positive, we have a potential barrier along x-axis. We can consider billiard motion in the upper half-plane. Then we can assume that a cut is done along negative part of y-axis, in order to get a unique-valued real function as a potential.

Solutions of Eq. (10) are also connected with interesting geometric subjects.

A.2. The Jacobi problem for geodesics on an ellipsoid

The Jacobi problem for the geodesics on an ellipsoid,

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=1,$$

has an additional integral:

$$K_1=\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)\left(\frac{\dot{x}^2}{A}+\frac{\dot{y}^2}{B}+\frac{\dot{z}^2}{C}\right).$$

Potential perturbations $V=V(x,y,z)$, such that perturbed systems have integrals of the form,

$$\widetilde{K}_1=K_1+k(x,y,z),$$

satisfy the following system:

$$\begin{aligned}
&\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)V_{xy}\frac{A-B}{AB}-3\frac{y}{B^2}\frac{V_x}{A}+3\frac{x}{A^2}\frac{V_y}{B}\\
&\quad+\left(\frac{x^2}{A^3}-\frac{y^2}{B^3}\right)V_{xy}+\frac{xy}{AB}\left(\frac{V_{yy}}{A}-\frac{V_{xx}}{B}\right)+\frac{zx}{CA^2}V_{zy}-\frac{zy}{CB^2}V_{zx}=0,\\
&\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)V_{yz}\frac{B-C}{BC}-3\frac{z}{C^2}\frac{V_y}{B}+3\frac{y}{B^2}\frac{V_z}{C}\\
&\quad+\left(\frac{y^2}{B^3}-\frac{z^2}{C^3}\right)V_{yz}+\frac{yz}{BC}\left(\frac{V_{zz}}{B}-\frac{V_{yy}}{C}\right)+\frac{xy}{AB^2}V_{xz}-\frac{xz}{AC^2}V_{xy}=0,\\
&\left(\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}\right)V_{zx}\frac{C-A}{AC}-3\frac{x}{A^2}\frac{V_z}{C}+3\frac{z}{C^2}\frac{V_x}{A}\\
&\quad+\left(\frac{z^2}{C^3}-\frac{x^2}{A^3}\right)V_{zx}+\frac{xz}{AC}\left(\frac{V_{xx}}{C}-\frac{V_{zz}}{A}\right)+\frac{zy}{BC^2}V_{xy}-\frac{yx}{BA^2}V_{yz}=0.
\end{aligned}\tag{12}$$

This system is an analogue of the Bertrand–Darboux equation (10) (see [18]).

Let

$$\frac{x^2C(A-C)}{z^2(B-A)A}=\hat{x},\qquad \frac{y^2C(C-B)}{z^2(B-A)B}=\hat{y}.$$

The following statement was also proved in [18]:

Theorem 12. *For every $\gamma\in\mathbb{C}$, the function,*

$$V_\gamma=(-\gamma+1)\left(\frac{z^2}{x^2}\right)^{\gamma}F_4(1,-\gamma+2,2,-\gamma+1,\hat{x},\hat{y}),$$

is a solution of the system (12).

Thus, by solving Bertrand–Darboux equation and its generalizations, as it is done in Theorems 11 and 12, one get large families of separable mechanical systems with two degrees of freedom. It is well known that separable systems of two degrees of freedom are necessarily of the Liouville type, see [36].

Now the natural question of Poncelet type theorem describing periodic solutions of such perturbed billiard systems arises. It appears that again Darboux studied such a question, since in [15], he analyzed generalizations of Poncelet theorem in the case of the Liouville surfaces.

Appendix B. Poncelet theorem on Liouville surfaces

In this section, we are going to give the presentation and comments to the Darboux results on the generalization of Poncelet theorem to Liouville surfaces.

B.1. Liouville surfaces and families of geodesic conics

In this subsection, following [15], we are going to define geodesic conics on an arbitrary surface, derive some important properties of theirs and finally to obtain an important characterization of Liouville surfaces via families of geodesic conics.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two fixed curves on a given surface $\mathcal{S}$. *Geodesic ellipses and hyperbolae* on $\mathcal{S}$ are curves given by the equations:

$$\theta + \sigma = \text{const}, \qquad \theta - \sigma = \text{const},$$

where θ, σ are geodesic distances from $\mathcal{C}_1$, $\mathcal{C}_2$ respectively.

A coordinate system composed of geodesic ellipses and hyperbolae joined to two fixed curves is orthogonal. In the following proposition, we are going to describe all orthogonal coordinate systems with coordinate curves that can be regarded as a family of geodesic ellipses and hyperbolae.

Proposition 8. *Let,*

$$\mathrm{d}s^2 = A^2\,\mathrm{d}u^2 + C^2\,\mathrm{d}v^2, \tag{13}$$

be the surface element corresponding to an orthogonal system of coordinate curves. Then the coordinate curves represent a family of geodesic ellipses and hyperbolae if and only if the coefficients A, C satisfy a relation of the form:

$$\frac{U}{A^2} + \frac{V}{C^2} = 1,$$

with U and V being functions of u, v respectively.

Proof. By assumption, equations of coordinate curves are:

$$\theta + \sigma = \text{const}, \qquad \theta - \sigma = \text{const},$$

with θ, σ representing geodesic distances from a point of the surface to two fixed curves.

Thus:

$$u = F(\theta + \sigma), \qquad v = F_1(\theta - \sigma).$$

Solving these equations with respect to θ and σ, we obtain:

$$\theta = \phi(u) + \psi(v), \qquad \sigma = \phi(u) - \psi(v).$$

As geodesic distances, θ and σ need to satisfy the characteristic partial differential equation:

$$\frac{A^2(\frac{\partial \xi}{\partial v})^2 + C^2(\frac{\partial \xi}{\partial u})^2}{A^2C^2} = 1. \tag{14}$$

From there, we deduce the desired relation with $U = (\phi'(u))^2$, $V = (\psi'(v))^2$.

The converse is proved in a similar manner. □

As a straightforward consequence, the following interesting property is obtained:

Corollary 3. *If an orthogonal system of curves can be regarded as a system of geodesic ellipses and hyperbolae in two different ways, then it can be regarded as such a system in infinitely many ways.*

Now, let us concentrate to Liouville surfaces, i.e., to surfaces with the surface element of the form:

$$\mathrm{d}s^2 = (U - V)\bigl(U_1^2\,\mathrm{d}u^2 + V_1^2\,\mathrm{d}v^2\bigr), \tag{15}$$

where U, U_1 and V, V_1 depend only on u and v respectively.

Now, we are ready to present the characterization of Liouville surfaces via geodesic conics.

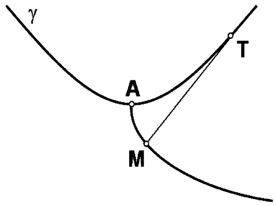

Fig. 13. Involute.

Theorem 13. *An orthogonal system on a surface can be regarded in two different manners as a system of geodesic conics if and only if it is of the Liouville form.*

Proof. Consider a surface with the element (13). If coordinate lines $u = \text{const}$, $v = \text{const}$ can be regarded as geodesic conics in two different manners then, by Proposition 6, A, C satisfy two different equations:

$$\frac{U}{A^2} + \frac{V}{C^2} = 1, \qquad \frac{U_1}{A^2} + \frac{V_1}{C^2} = 1.$$

Solving them with respect to A, C, we obtain that the surface element is of the form:

$$\mathrm{d}s^2 = \left(\frac{U}{U - U_1} + \frac{V}{V_1 - V}\right)\big((U - U_1)\,\mathrm{d}u^2 + (V_1 - V)\,\mathrm{d}v^2\big).$$

Conversely, consider the Liouville surface with the element (15) and the following solutions of Eq. (14):

$$\theta = \int U_1\sqrt{U - a}\,\mathrm{d}u + \int V_1\sqrt{a - V}\,\mathrm{d}v, \qquad \sigma = \int U_1\sqrt{U - a}\,\mathrm{d}u - \int V_1\sqrt{a - V}\,\mathrm{d}v.$$

The equations $\theta + \sigma = \text{const}$, $\theta - \sigma = \text{const}$ will define the coordinate curves. □

B.2. Generalization of graves and Poncelet theorems to Liouville surfaces

We learned from Darboux [15] that Liouville surfaces are exactly these having an orthogonal system of curves that can be regarded in two or, equivalently, infinitely many different ways, as geodesic conics. Now, we are going to present how to make a choice, among these infinitely many presentations, of the most convenient one, which will enable us to show the generalizations of theorems of Graves and Poncelet. All these ideas of enlightening beauty and profoundness belong to Darboux [15].

Consider a curve γ on a surface $\mathcal{S}$. The *involute* of γ with respect to a point $A \in \gamma$ is the set of endpoints M of all geodesic segments TM, such that

- $T \in \gamma$;
- TM is tangent to γ at T;
- the length of TM is equal to the length of the segment $TA \subset \gamma$; and
- these two segments are placed at the same side of the point T.

Involutes have the following important property, which follows immediately from the definition:

Lemma 10. *The geodesic segments TM are orthogonal to the involute, and the involute itself is orthogonal to γ at A.*

Now, we are going to find explicitly the equations of involutes of coordinate curves on a Liouville surface $\mathcal{S}$ with the surface element (15).

Lemma 11. *The curves on $\mathcal{S}$ given by the equations*:

$$\begin{aligned} \theta &= \int U_1\sqrt{U - a}\,\mathrm{d}u + \int V_1\sqrt{a - V}\,\mathrm{d}v = \text{const}, \\ \sigma &= \int U_1\sqrt{U - a}\,\mathrm{d}u - \int V_1\sqrt{a - V}\,\mathrm{d}v = \text{const}, \end{aligned} \tag{16}$$

are involutes of the coordinate curve whose parameter satisfies the equation:

$$(U-a)(V-a)=0.$$

Proof. Fix the parameter a. The equations of geodesics normal to the curves $\theta=\text{const}$, $\sigma=\text{const}$ are obtained by differentiating (16) with respect to a:

$$\frac{U_1\,\mathrm{d}u}{\sqrt{U-a}}\pm\frac{V_1\,\mathrm{d}v}{\sqrt{a-V}}=0. \tag{17}$$

Let u_0 be a solution of the equation $U-a=0$. Then, the geodesic line (17) will satisfy $\mathrm{d}u=0$ at the point of intersection with the curve $u=u_0$, i.e., it will be tangent to this coordinate curve. The statement now follows from Lemma 10. □

Proposition 9. *Coordinate curves on a Liouville surface are geodesic conics with respect to any two involutes of one of them.*

Proof. Follows from Lemma 11 and the proof of Theorem 13. □

Now, we are ready to prove the generalization of Graves' theorem.

Theorem 14. *Let $\mathcal{E}_0$: $u=u_0$ and $\mathcal{E}_1$: $u=u_1$ be coordinate curves on the Liouville surface $\mathcal{S}$. For a point $M\in\mathcal{E}_1$, denote by MP and MP' geodesic segments that touch $\mathcal{E}_0$ at Q, Q'. Then the expression,*

$$\ell(MP)+\ell(MP')-\ell(PP'),$$

is constant for all M, where $\ell(MP)$, $\ell(MP')$, and $\ell(PP')$ denote lengths of geodesic segments MP, MP', and of the segment $PP'\subset\mathcal{E}_0$ respectively.

Proof. Let $\mathcal{D}$, $\mathcal{D}'$ be involutes of the curve $\mathcal{E}_0$ with respect to points R, $R'\in\mathcal{E}_0$, and Q, Q' intersections of geodesics MP, MP' with these involutes.

Both $\mathcal{E}_0$ and $\mathcal{E}_1$ are geodesic ellipses with base curves $\mathcal{D}$, $\mathcal{D}'$, thus the sum $\ell(MQ)+\ell(MQ')$ remains constant when M moves on $\mathcal{E}_1$.

Since $\ell(PR)=\ell(MP)+\ell(MQ)$, $\ell(P'R')=\ell(MP')+\ell(MQ')$, we have:

$$\ell(MQ)+\ell(MQ')=\ell(PR)+\ell(P'R')-\ell(MP)-\ell(MP')=\ell(RR')-\big(\ell(MP)+\ell(MP')-\ell(PP')\big),$$

and the theorem is proved. □

From here, the complete analogue of the Poncelet theorem can be derived:

Theorem 15. *Let us consider a polygon on the Liouville surface $\mathcal{S}$, with all sides being geodesics tangent to a given coordinate curve, and each vertex but one moving on a coordinate curve. Then the last vertex also remains on a fixed coordinate curve.*

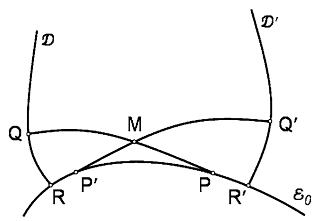

Fig. 14. Theorem 14.

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