Banach Lie algebras with Lie subalgebras of finite codimension: Their invariant subspaces and Lie ideals

Edward Kissin a,*, Victor S. Shulman a, Yurii V. Turovskii b,1

a Department of Computing, Communications Technology and Mathematics, London Metropolitan University, 166-220 Holloway Road, London N7 8DB, Great Britain
b Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 F. Agayev Street, Baku AZ1141, Azerbaijan

Received 26 February 2008; accepted 7 October 2008
Available online 7 November 2008
Communicated by K. Ball

Abstract

The paper studies the existence of closed invariant subspaces for a Lie algebra $\mathcal{L}$ of bounded operators on an infinite-dimensional Banach space $X$. It is assumed that $\mathcal{L}$ contains a Lie subalgebra $\mathcal{L}_0$ that has a non-trivial closed invariant subspace in $X$ of finite codimension or dimension. It is proved that $\mathcal{L}$ itself has a non-trivial closed invariant subspace in the following two cases: (1) $\mathcal{L}_0$ has finite codimension in $\mathcal{L}$ and there are Lie subalgebras $\mathcal{L}_0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_p = \mathcal{L}$ such that $\mathcal{L}_{i+1} = \mathcal{L}_i + [\mathcal{L}_i, \mathcal{L}_i+1]$ for all $i$; (2) $\mathcal{L}_0$ is a Lie ideal of $\mathcal{L}$ and $\dim(\mathcal{L}_0) = \infty$. These results are applied to the problem of the existence of non-trivial closed Lie ideals and closed characteristic Lie ideals in an infinite-dimensional Banach Lie algebra $\mathcal{L}$ that contains a non-trivial closed Lie subalgebra of finite codimension.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Invariant subspaces; Lie algebras of bounded operators

* Corresponding author.
E-mail addresses: e.kissin@londonmet.ac.uk (E. Kissin), shulman_v@yahoo.com (V.S. Shulman), yuri.turovskii@gmail.com (Y.V. Turovskii).
1 The support received from INTAS project No. 06-1000017-8609 is gratefully acknowledged by the third author.
1. Introduction

The link between the structure of Lie subalgebras and Lie ideals of Lie algebras attracted attention of researchers both in the classical situation of finite-dimensional Lie algebras and in the general case. Barnes [2] and Towers [20] studied finite-dimensional Lie algebras possessing the property that all their maximal Lie subalgebras have codimension 1. In [9] the first author considered finite-dimensional Lie algebras $L$ over $\mathbb{C}$ that have "sufficiently" many Lie subalgebras of codimension 1. Amayo in [1] studied Lie subalgebras $L_0$ of codimension 1 in finite-dimensional and infinite-dimensional Lie algebras $L$ over a field $F$. He showed that there is the largest Lie ideal $I(L_0)$ of $L$ contained in $L_0$ and that, for $\text{char}(F) = 0$, either $\dim(L/I(L_0)) \leq 3$, or $\dim(L/I(L_0)) = \infty$. He also constructed a simple infinite-dimensional Lie algebra $L$ with a Lie subalgebra $L_0$ of codimension 1, so that $I(L_0) = \{0\}$ and therefore $\dim(L/I(L_0)) = \infty$.

The situation changes if $L$ is a complex Banach Lie algebra. The first author in [10] investigated the structure of Banach Lie algebras $L$ with "sufficiently" many Lie subalgebras of codimension 1 and showed that $\dim(L/I(L_0)) \leq 3$ for each Lie subalgebra $L_0$ of codimension 1. The question was also raised as to whether $\dim(L/I(L_0)) < \infty$ for each closed Lie subalgebra $L_0$ of $L$ of finite codimension. In the last section of this paper we show that already when $\dim(L_0) = 2$ the answer to the above question is negative, but nevertheless $L$ in this case has a closed proper Lie ideal of finite codimension. For comparison it should be noted that associative algebras $A$ with associative subalgebras $A_0$ of finite codimension always have two-sided ideals of finite codimension that lie in $A_0$ (Laffey [12]).

Lie subalgebras and Lie ideals of finite codimension of associative Banach algebras and, in particular, of the algebra $C(H)$ of all compact operators on a separable Hilbert space $H$, of Schatten classes $C_p$, $1 \leq p < \infty$, and of uniformly hyperfinite $C^*$-algebras were studied in [5,13,14].

Note that Lie ideals of Banach Lie algebras $L$ are just invariant subspaces with respect to the adjoint representation:

$$\text{ad}(a)(x) = [a, x] \quad \text{for } a, x \in L.$$  

Using this link, we will study the question about the existence of Lie ideals in the much broader context of Invariant Subspace theory. Throughout the paper $X$ denotes an infinite-dimensional complex Banach space and $B(X)$ the algebra of all bounded operators on $X$. A Lie subalgebra of $B(X)$ is called reducible if it has a non-trivial closed invariant subspace; otherwise, it is irreducible. We consider a Lie subalgebra $L$ of $B(X)$ and assume that it contains a Lie subalgebra $L_0$ that has a closed invariant subspace $X_0$ in $X$. We will always assume that $X_0$ is a proper subspace of $X$ and that $L_0$ is a proper Lie subalgebra of $L$ closed in $L$:

\[ \{0\} \neq X_0 \neq X, \quad \{0\} \neq L_0 \neq L \quad \text{and} \quad \overline{L_0} \cap L = L_0, \]

where $\overline{L_0}$ is the closure of $L_0$ in $B(X)$. We investigate the problem whether $L$ itself has a non-trivial closed invariant subspace. For finite-dimensional $X$, the problem has a negative solution. Indeed, take $L = B(X)$ and $L_0 = C1$.

If $L$ and $L_0$ are associative algebras and $L_0$ has finite codimension in $L$, then this problem has a simple positive solution (see Section 8) based on the already mentioned result of [12]. If, however, both of them are Lie algebras, then the problem is much more delicate and requires a more careful investigation. Beltiţă and Şabac [3] proved that $L$ is reducible in the case when it
contains a finite-dimensional Lie ideal of nilpotent operators; the second and the third authors [19] proved this in the case when \( L \) contains a finite-dimensional Lie ideal of compact operators. Another sufficient condition of reducibility of \( L \) considered in [19, Theorem 1.1] is the assumption that \( L \) consists of compact operators and \( L_0 \) is a Lie ideal of \( L \) of some special type: for example, it is E-solvable, or consists of quasinilpotent operators (a Volterra Lie ideal).

In this paper we neither assume that a Lie algebra \( L \) consists of compact operators, nor that \( L_0 \) is a Lie ideal of a special type. Instead, we assume that \( L_0 \) has a closed invariant subspace \( X_0 \) in \( X \) of finite codimension (sometimes these assumptions can be weakened). Our approach is based on the study of the natural representation \( \theta \) of \( L_0 \) on the quotient space \( L/L_0 \). To describe the results we need the following definition.

**Definition 1.1.** (i) A Lie subalgebra \( L_0 \) of a Lie algebra \( L \) is called non-degenerate in \( L \), if \( L_0 \) has finite codimension in \( L \) and \( L = L_0 + [L_0, L] \).

(ii) A Lie subalgebra \( L_0 \) is related to \( L \), if there are Lie subalgebras \( L_0 = L^0 \subseteq L^1 \subseteq \cdots \subseteq L^p = L \) such that \( L^i \) is non-degenerate in \( L^{i+1} \).

The term “non-degenerate” in (i) is due to the fact that \( L_0 \) is non-degenerate if and only if the representation \( \theta \) is non-degenerate: \( \theta(L_0)(L/L_0) = L/L_0 \). Note also that if \( L_0 \) lies in a Lie ideal of \( L \), then \( L_0 \) is not related to \( L \).

We will prove that \( L \) is reducible in two “opposite extreme” cases:

**Case 1.** \( L_0 \) is related to \( L \);

**Case 2.** \( L_0 \) is a Lie ideal of \( L \) and \( \text{dim}(L_0) = \infty \).

Moreover, in Case 1 (see Section 5) a non-trivial closed invariant subspace \( Y \) of \( L \) can be chosen so that \( \text{codim}(Y) < \infty \) and \( Y \subseteq X_0 \). In particular, if \( L_0 \) is a Lie subalgebra (and not a Lie ideal) of codimension 1 and \( \text{codim}(X_0) < \infty \), then \( L \) always has a non-trivial closed invariant subspace of finite codimension contained in \( X_0 \).

In Case 2 (see Section 6) \( L \) may have no invariant subspaces of finite codimension. However, if \( X_0 \) is a maximal subspace invariant for \( L_0 \), then the subspace \( Y \) can be chosen so that it lies in \( X_0 \) (if \( L_0 \) has a non-trivial closed invariant subspace of finite codimension, it has a maximal one).

We will construct examples which show that if the conditions in Cases 1 and 2 are weakened in some way, then \( L \) may become irreducible; or even if it has invariant subspaces, they do not necessarily lie in \( X_0 \). For example, it follows from Example 9.6 that if \( \text{codim}(X_0) < \infty \) and \( \text{codim}(L_0) < \infty \), but \( [L_0, L] + L_0 \neq L \) (cf. Case 1), then non-trivial closed subspaces invariant for \( L \) (even if they exist) may not lie in \( X_0 \). We also show in Example 6.6 that if \( L_0 \) is a Lie ideal of \( L \) but either \( \text{codim}(L_0) < \infty \) or \( \text{codim}(X_0) = \infty \) (cf. Case 2), then \( L \) may be irreducible.

In Section 7 we consider the case when \( X_0 \) is a finite-dimensional subspace invariant for \( L_0 \). Using duality, we establish that \( L \) has a non-trivial closed invariant subspace that contains \( X_0 \), if either \( L_0 \) is related to \( L \), or \( L_0 \) is an infinite-dimensional Lie ideal of \( L \).

In Section 8 we combine the results of the previous sections and study the case when \( L_0 \subseteq L \subseteq L \) are Lie subalgebras of \( B(X) \), \( \text{dim}(L_0) = \infty \) and \( L_0 \) has a closed invariant subspace of finite codimension or dimension. We show that if \( L_0 \) is related to \( L \) while \( L \) is a Lie ideal of \( L \), or if \( L_0 \) is a Lie ideal of \( L \) while \( L \) is related to \( L \), then \( L \) is reducible. We also consider the
situation when \( \mathcal{L} \) and \( \mathcal{L}_0 \) are associative algebras, \( \mathcal{L}_0 \) is reducible and \( \mathcal{L} \) is finitely generated as a module over \( \mathcal{L}_0 \).

In the last section we apply the results of the previous sections to the problem of the existence of Lie ideals in infinite-dimensional Banach Lie algebras \( \mathcal{L} \) with closed Lie subalgebras \( \mathcal{L}_0 \) of finite codimension. We show that if \( \mathcal{L}_0 \) is related to \( \mathcal{L} \), then \( \mathcal{L} \) has a closed Lie ideal of finite codimension contained in \( \mathcal{L}_0 \). From this we infer that if \( \text{codim}(\mathcal{L}_0) \leq 2 \), then \( \mathcal{L} \) always has a closed Lie ideal \( \mathcal{K} \) of finite codimension and, if \( \text{codim}(\mathcal{L}_0) = 1 \) then \( \mathcal{K} \subseteq \mathcal{L}_0 \) and \( \text{codim}(\mathcal{K}) \leq 3 \). The question whether \( \mathcal{L} \) always has a closed ideal if \( \text{codim}(\mathcal{L}_0) > 2 \) is still open. Moreover, we show in Corollary 9.5 that if \( \text{codim}(\mathcal{L}_0) \geq 2 \) then even if \( \mathcal{L} \) has a non-trivial closed Lie ideal, it does not necessarily lie in \( \mathcal{L}_0 \).

At the end of Section 9 we prove that if an infinite-dimensional, non-commutative Banach Lie algebra \( \mathcal{L} \) has a proper closed Lie subalgebra related to \( \mathcal{L} \), or a non-trivial closed Lie ideal of finite codimension or dimension, then it has a closed characteristic Lie ideal—the Lie ideal invariant for all bounded derivations of \( \mathcal{L} \).

Sections 2 and 3 contain some results about finite-dimensional representations of Lie algebras that we use in later sections (in spite of the general character of these results we could not find them in the literature).

In Section 4 we introduce and study properties of \((\mathcal{L}, X_0)\)-filtrations of Banach spaces \( X \) with respect to closed subspaces \( X_0 \) of \( X \) and Lie subalgebras \( \mathcal{L} \) of \( \mathcal{B}(X) \). These filtrations provide one of the main tools for our study of invariant subspaces of \( \mathcal{L} \).

2. Eigen-representations and non-degenerate representations of Lie algebras on finite-dimensional spaces

Throughout this section \( V \) is a finite-dimensional linear space. Let \( \mathcal{L}_0 \) be a Lie algebra. A representation \( \theta \) of \( \mathcal{L}_0 \) on \( V \) is a Lie homomorphism from \( \mathcal{L}_0 \) into the algebra \( L(V) \) of all operators on \( V \). It is irreducible, if the Lie subalgebra \( \theta(\mathcal{L}_0) \) of \( L(V) \) has no non-zero invariant proper subspaces in \( V \). It is cyclic if there is \( u \in V \) such that \( V = \theta(\mathcal{L}_0)u \). A representation of a Lie algebra can be irreducible but not cyclic; it can also be cyclic but not irreducible.

**Example 2.1.** Consider the algebra \( M_3(\mathbb{C}) \) of all \( 3 \times 3 \) matrices as the algebra of all operators on a 3-dimensional space \( V \). Then \( L_1 = \{ A = (a_{ij}) \in M_3(\mathbb{C}) : a_{11} = a_{12} = 0 \text{ for all } i \} \) and \( L_2 = \{ A \in M_3(\mathbb{C}) : A^t = -A \} \), where \( A^t \) is the transposed matrix, are Lie subalgebras of \( M_3(\mathbb{C}) \).

Clearly, the identity representation of \( L_1 \) is cyclic but not irreducible.

On the other hand, the identity representation of \( L_2 \) is irreducible but not cyclic. Indeed, the enveloping algebra of \( L_2 \) coincides with \( M_3(\mathbb{C}) \). Thus the identity representation of \( L_2 \) is irreducible. Let us show that it is not cyclic. Consider \( u \in V \) as a \( 3 \times 1 \) matrix and denote by \( u^t \) its transposed matrix. Then \( u^tAu \in \mathbb{C} \) for \( A \in L_2 \). As \( (u^tAu)^t = u^tA^tAu = -u^tAu \), we have \( u^tAu = 0 \) for all \( A \in L_2 \) and \( u \in V \). Fix \( u \). As \( u^tAu = 0 \), for all \( A \in L_2 \), we have that \( \text{dim}(L_2u) \leq 2 \), so that the identity representation of \( L_2 \) is not cyclic.

For \( 0 \neq \lambda \in \mathbb{C} \), set \( E_\lambda = \{ x \in V : \theta(h)x = \lambda x \text{ for some } h \in \mathcal{L}_0 \} \). Denote by \( V_\lambda \), the linear span in \( V \) of all elements in \( E_\lambda \) and by \( V_e \) the linear span of all elements from all \( E_\lambda, 0 \neq \lambda \in \mathbb{C} \).

For a subset \( U \) of \( V \), denote by \( \theta(\mathcal{L}_0)U \) the linear span of all \( \theta(h)u \), where \( h \in \mathcal{L}_0, u \in U \).

**Lemma 2.2.** For all \( 0 \neq \lambda \in \mathbb{C} \), we have \( \theta(\mathcal{L}_0)V_\lambda = V_\lambda \) and \( \theta(\mathcal{L}_0)V_e = V_e \).
Proof. Set \( L = \theta(\mathcal{L}_0) \). For \( a, b \in L \) and \( t \in \mathbb{C} \), \( \exp(tb)a \exp(-tb) = \exp(t \text{ad}(b))(a) \in L \), where \( \text{ad}(b)(a) = [b, a] \). Let \( x \in E_{\lambda} \) and \( ax = \lambda x \) for some \( a \in L \). Then
\[
(\exp(tb)a \exp(-tb)) \exp(tb)x = \exp(tb)ax = \lambda \exp(tb)x.
\]
Hence \( \exp(tb)x \in E_{\lambda} \), so that \( bx = \lim_{t \to 0} \frac{1}{t} (\exp(tb)x - x) \in V_{\lambda} \). Thus \( V_{\lambda} \) is invariant for \( L \).

As \( E_{\lambda} \subseteq LE_{\lambda} \), we have \( LV_{\lambda} = V_{\lambda} \). Therefore \( LV_{e} = V_{e} \). \( \square \)

Definition 2.3. Let \( \theta \) be a representation of a Lie algebra \( \mathcal{L}_0 \) on \( V \).

(i) \( \theta \) is nilpotent if \( \theta(h) \) is nilpotent for each \( h \in \mathcal{L}_0 \), that is, \( \theta(h)^n = 0 \) for some \( n \) that depends on \( h \).

(ii) \( \theta \) is an eigen-representation if \( V = V_e \).

(iii) \( \theta \) is non-degenerate if \( \theta(\mathcal{L}_0)V = V \).

Clearly, \( \theta \) is nilpotent if and only if all operators \( \theta(h) \), \( h \in \mathcal{L}_0 \), have only zero eigenvalues.

Proposition 2.4. Every representation of a Lie algebra on a finite-dimensional space that decomposes into the direct sum of non-zero irreducible or cyclic representations is an eigen-representation.

Proof. Let \( \theta \) be a representation of \( \mathcal{L}_0 \) on \( V \) and \( \dim(V) = n \). Set \( L = \theta(\mathcal{L}_0) \).

(1) If \( \theta \) is irreducible then, by Lemma 2.2, either \( V_e = \{0\} \) or \( V_e = V \). In the first case all eigenvalues of all \( a \in L \) are zero. Hence all \( a \) are nilpotent, that is, \( a^n = 0 \). By Engel’s theorem, there is \( 0 \neq x \in V \) such that \( ax = 0 \) for all \( a \in L \). As \( L \) has no invariant subspaces in \( V \), \( V = Cx \) and \( L = \{0\} \), so \( \theta \) is the zero representation. Thus \( V_e = V \), so \( \theta \) is an eigen-representation.

(2) If \( \theta \) is cyclic then \( Lu = V \) for some \( u \in V \). Then \( u \in V_e \). By Lemma 2.2, \( V = Lu \subseteq LV_e \subseteq V_e \). Hence \( V_e = V \), so \( \theta \) is an eigen-representation.

The general case follows immediately from (1) and (2). \( \square \)

Proposition 2.5. A representation \( \theta \) of \( \mathcal{L}_0 \) on \( V \) is nilpotent if and only if \( V \) has no non-zero subspaces \( M \) invariant for \( \theta(\mathcal{L}_0) \) such that the restrictions of \( \theta \) to \( M \) are non-degenerate.

Proof. By Lemma 2.2, the restriction of \( \theta \) to each \( V_{\lambda} \) is non-degenerate. If \( V \) has no non-zero subspaces invariant for \( \theta(\mathcal{L}_0) \) such that the restrictions of \( \theta \) to them are non-degenerate, all \( V_{\lambda} = \{0\} \). Thus all \( \theta(h) \), \( h \in \mathcal{L}_0 \), have only zero eigenvalues, so that \( \theta \) is nilpotent.

Conversely, if \( \theta \) is nilpotent, its restriction \( \theta_M \) to any invariant subspace \( M \) of \( V \) is nilpotent. It follows from the Engel theorem (see [8, Theorem II.2.1’]) that there are subspaces \( \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_p = M \) invariant for \( \theta \) such that each quotient representation of \( \mathcal{L}_0 \) on \( V_i/V_{i-1} \) is a one dimensional zero representation. Hence \( \theta(\mathcal{L}_0)M \neq M \). \( \square \)

Let subspaces \( K \) and \( M \) of \( V \) be invariant for a representation \( \theta \) of \( \mathcal{L}_0 \) and \( K \subseteq M \). We will call the corresponding representation of \( \mathcal{L}_0 \) on the quotient space \( M/K \) a quotient representation.

Theorem 2.6. Let \( \theta \) be a representation of \( \mathcal{L}_0 \) on \( V \). The following conditions are equivalent:

(i) \( V \) has no subspaces \( M \neq V \) invariant for \( \theta(\mathcal{L}_0) \) such that the quotient representations of \( \mathcal{L}_0 \) on \( V/M \) are nilpotent;
(ii) $\theta$ is non-degenerate;
(iii) there are subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_p = V$ invariant for $\theta$ such that each quotient representation $\theta_i$ of $\mathcal{L}_0$ on $V_i/V_{i-1}$ is an eigen-representation.

**Proof.** (i) $\Rightarrow$ (iii). As $\theta$ is not nilpotent, there is $h \in \mathcal{L}_0$ with a non-zero eigenvalue. Hence $V_e \neq \{0\}$. Set $V_1 = V_e$. By Lemma 2.2, $V_1$ is invariant for $\theta$ and $\theta_1 = \theta|V_1$ is an eigen-representation. The quotient representation $\psi$ of $\mathcal{L}_0$ on $V/V_1$ is not nilpotent, so $(V/V_1)_e \neq \{0\}$. By Lemma 2.2, $(V/V_1)_e$ is invariant for $\psi$ and $\psi|(V/V_1)_e$ is an eigen-representation. Let $V_2$ be the subspace of $V$ such that $V_1 \subset V_2$ and $V_2/V_1$ is isomorphic to $(V/V_1)_e$. Then $V_2$ is invariant for $\theta$ and the quotient representation $\theta_2$ on $V_2/V_1$ is equivalent to $\psi|_{(V/V_1)_e}$ and, therefore, is an eigen-representation. Continuing this process, we conclude the proof.

(iii) $\Rightarrow$ (ii). As $\theta_i$ are eigen-representations, $\theta(\mathcal{L}_0)V_1 = \theta_1(\mathcal{L}_0)V_1 = V_1$ and $\theta_i(\mathcal{L}_0)(V_i/V_{i-1}) = V_i/V_{i-1}$, for $i > 1$. Therefore $\theta(\mathcal{L}_0)V_i = V_i$, for all $i$. So $\theta(\mathcal{L}_0)V = \theta(\mathcal{L}_0)V_0 = V_p = V$.

(ii) $\Rightarrow$ (i). Suppose that there is a subspace $M$ in $V$ invariant for $\theta$ such that the quotient representation $\tilde{\theta}$ of $\mathcal{L}_0$ on $V/M$ is nilpotent. By Proposition 2.5, $\tilde{\theta}$ is not non-degenerate. Hence $\tilde{\theta}(\mathcal{L}_0)(V/M) \neq V/M$. Therefore $\tilde{\theta}(\mathcal{L}_0)V \neq V$.  

3. Nilpotent part of spaces of commuting operators

Let $X$ be a Banach space and let $M$ be a linear subspace of $B(X)$ that consists of mutually commuting operators. Denote by $N_M$ the set of all nilpotent operators in $M$:

$$N_M = \{ A \in M : A^n = 0 \text{ for some } n \text{ that depends on } A \}. \tag{3.1}$$

As operators in $M$ commute, $N_M$ is a linear subspace of $M$. Let $\overline{M}$ be the closure of $M$. For $n \in \mathbb{N}$, let $M^n$ be the linear subspace of $B(X)$ generated by all products of $n$ elements from $M$.

**Proposition 3.1.** Let $B \in B(X)$ be such that $[B, A] \in \overline{M}$ for all $A \in M$. Then

(i) $[B, Y] \in N_M$ for all $Y \in N_M$.
(ii) If $[B, A] - \mu A \in \overline{N_M}$, for some $A \in M$ and $\mu \neq 0$, then $A^n \in \overline{N_M}$ for some $n$.

If $\dim(M) < \infty$ then $A \in N_M$.

**Proof.** For $S \in B(X)$, denote by $\delta_S$ the operator on $B(X)$ that acts by the formula $\delta_S(C) = [S, C]$ for $C \in B(X)$.

Let $Y \in N_M$. As $\delta_Y(B) = [Y, B] \in \overline{M}$, we have $\delta_Y^n(B) = [Y, [Y, B]] = 0$. The fact that $[Y, B]$ is nilpotent, if $[Y, [Y, B]] = 0$ and $Y$ is nilpotent is well known (see [7]); we will give the proof of it for the sake of completeness. Since $\delta_Y^k = 0$, for some $k$, and all terms in $\delta_Y^{2k-1}(C)$, for each $C \in B(X)$, contain $Y^m$ with $m \geq k$, we have $\delta_Y^m(C) = 0$. Then (see [6, p. 335]), $n!(\delta_Y(B))^n = \delta_Y^n(B^n)$ for all $n$. Hence $(\delta_Y(B))^m = 0$. Thus $[Y, B] \in N_M$. Part (i) is proved.

Assume now that $\delta_B(A) = \mu A + Y$, where $Y \in \overline{N_M}$ and $\mu \neq 0$. As $\delta_B$ is a derivation,

$$\delta_B(CD) = C\delta_B(D) + \delta_B(C)D \quad \text{for } C, D \in B(X). \tag{3.2}$$
Hence, since all operators in \( M \) commute and \( \delta_B(A) \in M \), \( \delta_B(A^2) = 2A\delta_B(A) \). Continuing this, we obtain that \( \delta_B(A^n) = nA^{n-1}\delta_B(A) \). Hence

\[
\delta_B(A^n) = n\mu A^n + nA^{n-1}Y. \tag{3.3}
\]

As \( \|\delta_B\| \leq 2\|B\| \) and \( \mu \neq 0 \), the operator \( T_n = \delta_B - n\mu \mathbf{1} \) on \( B(X) \) is invertible for \( n > \frac{2\|B\|}{\mu} \).

Fix such an \( n \). All operators of \( M^n \) mutually commute. As \( \delta_B \) is bounded and maps \( M \) into itself, it follows from (3.2) that \( \delta_B \) also maps \( M^n \) into itself.

Applying (i) to \( M^n \), we have that \( \delta_B \) maps \( N_{M^n} \) into itself. As \( T_n^{-1} = -n\mu \sum_{k=0}^{\infty} \delta_{B}^k/(n\mu)^k \), it maps \( \overline{N_{M^n}} \) into itself. By (3.3), \( T_n(A^n) = nA^{n-1}Y \). Since \( Y \in \overline{N_M} \), there are \( Y_\mu \in \overline{N_M} \) that converge to \( Y \). As \( A \) and \( Y_\mu \) commute, all \( nA^{n-1}Y_\mu \in \overline{N_{M^n}} \).

Therefore \( A^n \in \overline{N_{M^n}} \).

If \( \dim(M) < \infty \), then \( \overline{N_{M^n}} = N_{M^n} \). Hence \( A \) is nilpotent, so that \( A \in N_M \).

Let \( V \) be a finite-dimensional space and assume that there is a linear map \( u \mapsto E^u \) from \( V \) into \( B(X) \) such that

\[
E^v E^u = E^u E^v \quad \text{for all } u, v \in V.
\]

Set

\[
V^{\mathrm{nil}} = \{ u \in V : E^u \text{ is nilpotent} \}. \tag{3.4}
\]

Then \( V^{\mathrm{nil}} \) is a linear subspace of \( V \).

Let \( \theta \) be a representation of a Lie algebra \( \mathcal{L}_0 \) on \( V \) and let \( \rho \) be a representation of \( \mathcal{L}_0 \) on \( X \) that satisfy

\[
[\rho(h), E^u] = E^{\theta(h)u} \quad \text{for } h \in \mathcal{L}_0 \text{ and } u \in V. \tag{3.5}
\]

**Proposition 3.2.**

(i) The subspace \( V^{\mathrm{nil}} \) is invariant for all \( \theta(h), h \in \mathcal{L}_0 \).

(ii) Let \( u \in V \) and let there exist \( h \in \mathcal{L}_0 \) such that \( \theta(h)u - \mu u \in V^{\mathrm{nil}} \), for some \( \mu \neq 0 \). Then \( u \in V^{\mathrm{nil}} \).

**Proof.** \( M = \{ E^u : u \in V \} \) is a finite-dimensional linear subspace of \( B(X) \), all its elements commute and \( N_M = \{ E^u : u \in V^{\mathrm{nil}} \} \). Hence part (i) follows from (3.5) and Proposition 3.1(i). If \( \theta(h)u - \mu u \in V^{\mathrm{nil}} \), for some \( \mu \neq 0 \), then \( [\rho(h), E^u] = E^{\theta(h)u} = \mu E^u + E^v \), where \( v \in V^{\mathrm{nil}} \). Hence part (ii) follows from Proposition 3.1(ii). \( \square \)

**Corollary 3.3.** If \( \theta \) is non-degenerate then \( V = V^{\mathrm{nil}} \).

**Proof.** If \( V \neq V^{\mathrm{nil}} \), it follows from Proposition 3.2 that the quotient representation \( \overline{\theta} \) of \( \mathcal{L}_0 \) on \( V/V^{\mathrm{nil}} \) is such that each \( \overline{\theta}(h), h \in \mathcal{L}_0 \), has only zero eigenvalues. Hence \( \overline{\theta} \) is nilpotent. By Theorem 2.6, \( \theta \) is not non-degenerate. This contradiction shows that \( V = V^{\mathrm{nil}} \). \( \square \)
4. Filtrations of spaces and Lie algebras

Let \( \mathcal{L} \) be a Lie algebra of bounded operators on a Banach space \( X \), that is, a linear subspace of \( \mathcal{B}(X) \) closed under the Lie multiplication \( [a, b] = ab - ba \). For linear subspaces \( \mathcal{K} \) and \( \mathcal{M} \) of \( \mathcal{L} \), set

\[
[\mathcal{K}, \mathcal{M}] = \text{span}\{[b, a]: b \in \mathcal{K}, a \in \mathcal{M}\} \quad \text{and} \quad \mathcal{K}\mathcal{M} = \text{span}\{ba: b \in \mathcal{K}, a \in \mathcal{M}\}.
\]

If \( [\mathcal{K}, \mathcal{L}] \subseteq \mathcal{K} \) then \( \mathcal{K} \) is a Lie ideal of \( \mathcal{L} \); in this case we will write \( \mathcal{K} \triangleleft \mathcal{L} \).

Let \( Y \) be a subspace of \( X \) and \( x \in X \). Set

\[
\mathcal{L}x = \{ax: a \in \mathcal{L}\} \quad \text{and} \quad \mathcal{L}Y = \text{span}\{ax: a \in \mathcal{L}, x \in Y\}.
\]

The subspace \( Y \) is called invariant for \( \mathcal{L} \) if \( \mathcal{L}Y \subseteq Y \).

Throughout the paper \( X_0 \) will be a proper subspace of \( X \). Set \( X_{-1} = X \) and

\[
X_n = \{x \in X_{n-1}: \mathcal{L}x \subseteq X_{n-1}\} \quad \text{for} \ n \geq 1. \tag{4.1}
\]

Then \( X_n \subseteq X_{n-1} \) and, for \( n \geq 1 \), \( X_n \) is the largest linear subspace of \( X_{n-1} \) such that

\[
\mathcal{L}X_n \subseteq X_{n-1}. \tag{4.2}
\]

We denote by \( \mathbb{N} \) the set of all non-negative integers.

**Definition 4.1.** The decreasing sequence \( \{X_n\}_{n=-1}^\infty \) of subspaces of \( X \) constructed in (4.1) is called \((\mathcal{L}, X_0)\)-filtration of \( X \). A filtration is non-trivial if \( X_n \neq \{0\} \) for all \( n \in \mathbb{N} \).

For a non-trivial filtration \( \{X_n\} \) of \( X \), the following three cases are possible:

1. \( X_n = X_{n+1} \), for some \( n \);
2. \( \bigcap X_n \neq \{0\} \) and all \( X_n \) are distinct;
3. \( \bigcap X_n = \{0\} \).

In Cases 1 and 2 the subspace \( \bigcap X_n \) is non-zero and invariant for \( \mathcal{L} \).

Throughout the paper \( \mathcal{L}_0 \) will be a proper Lie subalgebra of \( \mathcal{L} \) that leaves \( X_0 \) invariant and we assume that \( \mathcal{L}_0 \) is closed in \( \mathcal{L}: \mathcal{L}_0 \cap \mathcal{L} = \mathcal{L}_0 \). Set

\[
\mathcal{L}_1 = \{b \in \mathcal{L}_0: [b, \mathcal{L}] \subseteq \mathcal{L}_0\} \quad \text{and} \quad Z(\mathcal{L}, \mathcal{L}_0, X_0) = \{b \in \mathcal{L}_1: bX \subseteq X_0\}. \tag{4.3}
\]

We will often write \( Z(\mathcal{L}) \) instead of \( Z(\mathcal{L}, \mathcal{L}_0, X_0) \).

**Lemma 4.2.** Let \( \{X_n\} \) be \((\mathcal{L}, X_0)\)-filtration of \( X \). Then

(i) \( \mathcal{L}_0 X_n \subseteq X_n \) and \( Z(\mathcal{L})X_n \subseteq X_{n+1} \) for \( n \in \mathbb{N} \cup \{-1\} \).

(ii) \( Z(\mathcal{L}) \triangleleft \mathcal{L}_0 \).

(iii) Let \( q = \text{codim}(\mathcal{L}_0) < \infty \) and \( p = \text{codim}(X_0) < \infty \). Then \( \dim(\mathcal{L}_0/Z(\mathcal{L})) < \infty \) and \( \dim(X_{n-1}/X_n) \leq pq^n \).

(iv) If \( X_0 \) is a closed subspace of \( X \), then all \( X_n \) are closed subspaces of \( X \).
Proof. We have $L_0 X_0 \subseteq X_0$. It follows from (4.2) that, for $n \geq 1$,
\[
L L_0 X_n \subseteq L_0 LX_n + [L, L_0] X_n \subseteq L_0 X_{n-1} + LX_n \subseteq L_0 X_{n-1} + X_{n-1}.
\]
Assume, by induction, that $L_0 X_i \subseteq X_i$ for all $i \leq n - 1$. Then it follows from (4.4) that $L L_0 X_n \subseteq X_{n-1}$. Hence, by (4.2), $L_0 X_n \subseteq X_n$.

Set $Z = Z(L)$. As $Z \subseteq L_1$ and $[L_1, L] \subseteq L_0$, we have $[Z, L] \subseteq L_0$. Hence
\[
L Z X_n \subseteq Z LX_n + [L, Z] X_n \subseteq Z X_{n-1} + L_0 X_n \subseteq Z X_{n-1} + X_n.
\]
Then $L Z X_0 \subseteq Z X + X_0 \subseteq X_0$. Using this and (4.5), we obtain by induction that $L Z X_n \subseteq X_n$.

Therefore, by (4.2), $Z X_n \subseteq X_{n+1}$. Part (i) is proved.

As $Z \subseteq L_1$ and $L_1 \triangleleft L_0$, $[L_0, Z] \subseteq [L_0, L_1] \subseteq L_1$. Also
\[
[L_0, Z] X \subseteq L_0 Z X + L Z X_0 \subseteq L_0 X_0 + ZX \subseteq X_0 + X_0 = X_0.
\]
Thus $[L_0, Z] \subseteq Z$, so that $Z \triangleleft L_0$. Part (ii) is proved.

There are $e_i_{i=1}^q$ in $L \setminus L_0$ such that $L = L_0 + \mathbb{C} e_1 + \cdots + \mathbb{C} e_q$. Set $p_n = \dim(X_{n-1}/X_n)$. Fix $i$ and assume that $p_n < \infty$. By (4.2), $e_i(X_n) \subseteq X_{n-1}$. Hence the subspace $L_i = \{x \in X_n : e_i x \in X_n\}$ has codimension in $X_n$ less or equal to $p_n$. As $X_{n+1} \subseteq X_n$ and $L_0 X_n \subseteq X_n$,
\[
X_{n+1} = \{x \in X_n : L x \subseteq X_n\} = \{x \in X_n : e_i x \in X_n \text{ for } i = 1, \ldots, q\} = \bigcap_{i=1}^q L_i
\]
has codimension in $X_n$ less or equal to $qp_n$. Hence $p_{n+1} \leq qp_n$, so $p_n \leq p q^n$.

Let $Y$ be a subspace of $X$ such that $X = X_0 + Y$. Then $\dim(Y) = p$. As $L_1 X_0 \subseteq L_0 X_0 \subseteq X_0$, we have that $Z = \{b \in L_1 : b Y \subseteq X_0\}$. Therefore there is an injective linear map from $L_1/Z$ into the algebra of all operators on $Y$. Hence $\dim(L_1/Z) \leq p^2$.

Let us show now that $\dim(L_0/L_1) < \infty$. Fix $i$. Then $[L_0, e_i] \subseteq L$ and the subspace $\{b \in L_0 : \{b, e_i\} \subseteq L_0\}$ has a finite codimension in $L_0$. Hence $L_1$ has finite codimension in $L_0$, as
\[
L_1 = \bigcap_{i=1}^q \{b \in L_0 : \{b, e_i\} \subseteq L_0\}.
\]
Thus $\dim(L_0/Z) = \dim(L_0/L_1) + \dim(L_1/Z) < \infty$. Part (iii) is proved.

Finally, let $X_0$ be closed. Let $x_k \in X_n$ and $x_k \to x$. Then $x \in X_0$ and, for each $a \in L$, $\|a x - a x_k\| \leq \|a\| \|x - x_k\| \to 0$. Assume, by induction, that $X_{n-1}$ is closed. As $a x_k \in X_{n-1}$, we have that $a x \in X_{n-1}$. Hence, by (4.2), $x \in X_n$. Thus $X_n$ is closed. \qed

For $n \in \mathbb{N}$ (cf. (4.3)), set
\[
L_n = \{b \in L_0 : \{b, L\} \subseteq L_{n-1}\}, \text{ so } L_n \subseteq L_{n-1} \text{ and } [L, L_n] \subseteq L_{n-1}.
\]
The sequence $\{L_n\}$ of Lie subalgebras of $L$ is called $L_0$-filtration of $L$ (see [11]). We will consider now the action of Lie subalgebras $L_0$ on subspaces $X_n$.
**Lemma 4.3.** Let $X_0$ be invariant for $L_0$, let \{X_n\} be (L,X_0)-filtration of X and \{L_n\} be $L_0$-filtration of L. If $L_m X \subseteq X_0$, for some $m \geq 1$, then
\[
L_p X_n \subseteq X_{p-m+n+1} \quad \text{when } p + n \geq m - 1. \tag{4.7}
\]

**Proof.** We have from (4.6) that $L_m L_n \subseteq L_m L + \{L, L_m\} \subseteq L_m L + L_0$, for all $n$. Therefore $L_m X_0 \subseteq L_m X + L_0 X_0 \subseteq X_0$, so (4.2) implies $L_m x_0 \subseteq X_1$. Assume that $L_m X_k \subseteq X_{k+1}$, for some $k \geq 0$. Then, by Lemma 4.2(i), $L_m X_{k+1} \subseteq L_m L X_{k+1} + L_0 X_{k+1} \subseteq L_m X_k + L_0 X_{k+1} \subseteq X_{k+1}$. Hence $L_m X_{k+1} \subseteq X_{k+1}$. By induction, $L_m X_n \subseteq X_{n+1}$ for all $n$, so (4.7) holds for $p = m$.

Assume now that (4.7) holds for some $p = k \geq m$ and all $n \in \mathbb{N}$. By (4.6),
\[
L L_{k+1} \subseteq L_{k+1} L + \{L, L_{k+1}\} \subseteq L_{k+1} L + L_k. \tag{4.8}
\]

Hence $L L_{k+1} X_{-1} \subseteq L_{k+1} X_{-1} \subseteq X_{k-m+1}$, so, by (4.2), $L_{k+1} X \subseteq X_{k-m+1}$. Using (4.8), we have $L L_{k+1} X_0 \subseteq L_{k+1} X + L_0 X_0 \subseteq X_{k-m+1}$. Hence, by (4.2), $L_{k+1} X_0 \subseteq X_{(k+1)-m+1}$. Assume that $L_{k+1} X_{l} \subseteq X_{(k+1)-m+l+1}$, for some $l \geq 0$. Then, by (4.8), $L L_{k+1} X_{l+1} \subseteq L_{k+1} X_{l} + L_0 X_{l+1} \subseteq X_{(k+1)-m+l+1}$, whence $L_{k+1} X_{l+1} \subseteq X_{(k+1)-m+l+1}$. By induction, $L_{k+1} X_n \subseteq X_{(k+1)-m+n+1}$, for all $n \geq -1$, so (4.7) holds for $p = k + 1$. Thus, by induction, (4.7) holds for all $p \geq m$ and $n \geq -1$.

Let $0 \leq p < m$. Then $L_p X_n \subseteq L_0 X_n \subseteq X_n \subseteq X_{p-m+n+1}$, as $p - m + n + 1 \leq n$. If $p = -1$ then $L_{-1} X_n \subseteq L_0 X_n \subseteq X_{n-1} \subseteq X_{p-m+n+1}$, as $p - m + n + 1 = n - m \leq n - 1$. \(\square\)

Given a Lie algebra $L$ and its subalgebra $L_0$, we can consider $L$ as a space $X$ and $L_0$ as its subspace $X_0$. Let $ad(L) = \{ad(a): a \in L\}$ be the Lie algebra of operators on $L$ where each $ad(a)$ acts by $ad(a)x = [a, x]$ for $x \in L$. Lemmas 4.2 and 4.3 yield (cf. [11]):

**Corollary 4.4.** Let $L_0$ be a Lie subalgebra of $L$ and \{L_n\} be $L_0$-filtration of $L$. Then

(i) $\{L_n\}$ coincides with (ad(L), $L_0$)-filtration of $L$ considered as a space and $Z(ad(L)$, \(ad(L_0), L_0) = ad(L_1)\);

(ii) \(ad(L_n)\) is ad($L_0$)-filtration of ad($L$);

(iii) $L_n$ are Lie ideals of $L_0$ and $L_p, L_n \subseteq L_{p+n}$ for $-1 \leq p + n$;

(iv) If $q = \text{codim}(L_n) < \infty$ then $\text{dim}(L_{n-1}/L_n) \leq q^{n+1}$.

Let $L$ be a subspace of $B(X)$ and let $X_0$ be invariant for $L$. Denote by $L^k$, $k \geq 1$, the linear subspace of $B(X)$ generated by all products $a_1 \cdots a_k$, $a_i \in L$. We say that $L$ is operator-nilpotent (to distinguish it from Lie nilpotency) on $X_0$ if $L^k X_0 = \{0\}$ for some $k \geq 1$.

We will consider now some cases when ($L, X_0$)-filtration of $X$ is non-trivial.

**Corollary 4.5.** Let $X_0$ be invariant for $L_0$, let \{X_n\} be ($L, X_0$)-filtration of $X$ and \{L_n\} be $L_0$-filtration of $L$. The filtration $\{X_n\}$ is non-trivial if one of the following conditions holds:

(i) \(\text{codim}(L_0) < \infty\) and \(\text{codim}(X_0) < \infty\);

(ii) $L_m \subseteq Z(L, L_0, X_0)$, for some $m \geq 1$, and $L_m$ is not operator-nilpotent on $X_0$;

(iii) $L_m \subseteq Z(L, L_0, X_0)$, for some $m \geq 1$, and $L_p|_{X_0} \neq \{0\}$ for all $p \geq m$.

**Proof.** Part (i) follows from Lemma 4.2(iv).
Let $\mathcal{L}_m X \subseteq X_0$, for some $m$. If $\mathcal{L}_m^n X_0 \neq \{0\}$ for all $n$, repeatedly applying (4.7), we have

$$\{0\} \neq \mathcal{L}_m^n X_0 = \mathcal{L}_m^{n-1} (\mathcal{L}_m X_0) \subseteq \mathcal{L}_m^{n-1} X_1 = \mathcal{L}_m^{n-2} (\mathcal{L}_m X_1) \subseteq \mathcal{L}_m^{n-2} X_2 = \cdots \subseteq \mathcal{L}_m X_{n-1} \subseteq X_n.$$ 

If $\mathcal{L}_p|X_0 \neq \{0\}$ for all $p \geq m$, it follows from (4.7) that $\{0\} \neq \mathcal{L}_p X_0 \subseteq X_{p-m+1}$. Hence in both cases $X_n \neq \{0\}$ for all $n$. □

Let, for example, $X = \mathcal{L}$ and $X_0 = \mathcal{L}_0$. By Corollary 4.4(i), $Z(\text{ad}(\mathcal{L})) = \text{ad}(\mathcal{L}_1)$. If ad$(\mathcal{L}_1)$ is not operator-nilpotent on $\mathcal{L}_0$, it follows from Corollary 4.5(ii) that the filtration $\{\mathcal{L}_n\}$ is non-trivial.

5. Invariant subspaces of operator Lie algebras: the case when $\mathcal{L}_0$ is related to $\mathcal{L}$

Recall that a Lie subalgebra of $\mathcal{B}(X)$ is reducible if it has a non-trivial closed invariant subspace. In this section $\mathcal{L}$ is a Lie subalgebra (not necessarily infinite-dimensional) of $\mathcal{B}(X)$ and $\mathcal{L}_0$ is a non-trivial reducible Lie subalgebra of $\mathcal{L}$ of finite codimension. As $\mathcal{L}_0$ and $\overline{\mathcal{L}}_0$ have the same closed invariant subspaces, we assume without loss of generality that $\overline{\mathcal{L}}_0 \cap \mathcal{L} = \mathcal{L}_0$ (otherwise, we replace $\mathcal{L}_0$ by $\overline{\mathcal{L}}_0 \cap \mathcal{L}$).

Let $\phi$ be the canonical linear map from $\mathcal{L}$ onto $\mathcal{L}/\mathcal{L}_0$. For each $a \in \mathcal{L}_0$, denote by $\theta(a)$ the operator on $\mathcal{L}/\mathcal{L}_0$ defined by

$$\theta(a) \phi(e) = \phi([a,e]), \quad \text{for } e \in \mathcal{L}. \quad (5.1)$$

Then $\theta: a \mapsto \theta(a)$ is a representation of $\mathcal{L}_0$ on $\mathcal{L}/\mathcal{L}_0$ and Ker$(\theta) = \mathcal{L}_1$. It is easy to see that

$\theta$ is non-degenerate (Definition 2.3) $\iff$ $\mathcal{L}_0$ is non-degenerate in $\mathcal{L}$ (Definition 1.1). \quad (5.2)

Let $X_0$ be a closed subspace of $X$ invariant for $\mathcal{L}_0$. Set $X_{-1} = X$ and let $\{X_n\}_{n=-1}^{\infty}$ be $\mathcal{L}(X_0)$-filtration of $X$. For each $n \in \mathbb{N} \cup \{-1\}$, $\hat{X}_n = X_n/X_{n+1}$ is a Banach space. Denote by $\tau_n$ the canonical map from $X_n$ onto $\hat{X}_n$. By Lemma 4.2, there is a representation $\rho_n$ of $\mathcal{L}_0$ on $\hat{X}_n$ defined by

$$\rho_n(h) \tau_n(x) = \tau_n(hx), \quad \text{for } h \in \mathcal{L}_0, \ x \in X_n. \quad (5.3)$$

Each $e \in \mathcal{L}$ defines operators $E_n^e$, for all $n \in \mathbb{N}$, from $\hat{X}_n$ into $\hat{X}_{n-1}$ by the formula

$$E_n^e \tau_n(x) = \tau_{n-1}(ex), \quad \text{where } x \in X_n. \quad (5.4)$$

The maps $e \in \mathcal{L} \mapsto E_n^e$ are linear. By (4.1), for each $0 \neq x \in X_n \setminus X_{n+1}$, $n \in \mathbb{N}$, there is $e \in \mathcal{L}$ such that $\tau_{n-1}(ex) \neq 0$. From this, from (5.4) and Lemma 4.2 it follows that

$$E_n^e = 0 \quad \text{if } e \in \mathcal{L}_0, \quad \text{and} \quad \bigcap \{\text{Ker } E_n^e : e \in \mathcal{L}\} = \{0\} \quad \text{for all } n \in \mathbb{N}. \quad (5.5)$$

For each $u = \phi(e) \in \mathcal{L}/\mathcal{L}_0$ and $n \in \mathbb{N}$, set

$$E_n^u = E_n^e. \quad (5.6)$$

As $E_n^e = 0$ if $e \in \mathcal{L}_0$, the linear maps $u \mapsto E_n^u$ are well defined on $\mathcal{L}/\mathcal{L}_0$ for all $n$. 

Let \( h \in \mathcal{L}_0 \) and \( e \in \mathcal{L} \). Then, by (5.3) and (5.4), for \( x \in X_n \),

\[
\rho_{n-1}(h) E_n^e \tau_n(x) = \rho_{n-1}(h) \tau_{n-1}(e x) = \tau_{n-1}(h(e x)) + \tau_{n-1}(e h x)
\]

\[
= E_n^{[h,e]} \tau_n(x) + E_n^e \rho_n(h) \tau_n(x).
\]

From this and from (5.5) we obtain that

\[
\rho_{n-1}(h) E_n^u = E_n^u \rho_n(h) + E_n^u \rho_n(h) \tau_n(x)
\]

for \( h \in \mathcal{L}_0 \) and \( u \in \mathcal{L}/\mathcal{L}_0 \);

\[
\bigcap \{ \text{Ker} E_n^u : u \in \mathcal{L}/\mathcal{L}_0 \} = \{0\} \quad \text{for all } n \in \mathbb{N}.
\]

**Lemma 5.1.**

(i) \( Z(\mathcal{L}) = \mathcal{L}_1 \cap \text{Ker}(\rho_{-1}) \subseteq \text{Ker}(\rho_n) \) and \( \| \rho_n(h) \| \leq \| h \| \) for all \( n \in \mathbb{N} \) and \( h \in \mathcal{L}_0 \).

(ii) \( E_n^u \in \mathcal{B}(\hat{X}_n, \hat{X}_{n-1}) \) and \( \| E_n^u \| \leq \| u \| \) if \( u = \phi(e) \).

(iii) \( E_n^u E_{n+1}^v = E_n^u E_{n+1}^v \) for all \( n \in \mathbb{N} \) and \( u, v \in \mathcal{L}/\mathcal{L}_0 \).

**Proof.** Part (i) follows from Lemma 4.2(i) and the fact that, for all \( x \in X_n \),

\[
\| \rho_n(h) \tau_n(x) \|_{\hat{X}_n} = \| \tau_n(h x) \|_{\hat{X}_n} = \inf_{y \in X_{n+1}} \| h x + y \|_X \leq \inf_{y \in X_{n+1}} \| h(y + x) \|_X
\]

\[
\leq \inf_{y \in X_{n+1}} \| h \| \| x + y \|_X = \| h \| \| \tau_n(x) \|_{\hat{X}_n}.
\]

We have that \( \| E_n^u \| \leq \| u \| \), where \( u = \phi(e) \) and \( e \in \mathcal{L} \), since for \( x \in X_n \),

\[
\| E_n^{\phi(e)} \tau_n(x) \|_{\hat{X}_{n-1}} = \| \tau_{n-1}(e x) \|_{\hat{X}_{n-1}} = \inf_{y \in X_n} \| e x + y \|_X \leq \inf_{z \in X_{n+1}} \| e(x + z) \|_X
\]

\[
\leq \inf_{z \in X_{n+1}} \| e \| \| x + z \|_X = \| e \| \| \tau_n(x) \|_{\hat{X}_n}.
\]

Let \( u = \phi(e) \) and \( v = \phi(e') \), for some \( e, e' \in \mathcal{L} \). By (4.2), \( \tau_{n-1}(a x) = 0 \) for all \( x \in X_{n+1} \), \( a \in \mathcal{L} \) and \( n \in \mathbb{N} \). As \( \mathcal{L} \) is a Lie algebra, \( [e, e'] \in \mathcal{L} \), so

\[
(E_n^e E_{n+1}^e - E_{n+1}^e E_n^e) \tau_{n+1}(x) = \tau_{n-1}([e, e'] x) = 0.
\]

This proves that \( E_n^u E_{n+1}^v = E_n^v E_{n+1}^u \).

Note that the condition that \( \mathcal{L} \) is a *Lie algebra* and not just a module over a Lie algebra \( \mathcal{L}_0 \) is important and used in the proof of part (iii) of the above lemma.

Let \( M \neq \{0\} \) be a subspace of \( \mathcal{L}/\mathcal{L}_0 \). For all \( n \in \mathbb{N} \), set

\[
\text{Ker}_n(\{0\}) = \hat{X}_n \quad \text{and} \quad \text{Ker}_n(M) = \bigcap \{ \text{Ker}(E_n^u) : u \in M \}.
\]

Then \( \text{Ker}_n(M) \subseteq \hat{X}_n \). If \( M \subseteq M' \) then \( \text{Ker}_n(M') \subseteq \text{Ker}_n(M) \) for all \( n \).
Lemma 5.2. Let $M$ be a subspace of $L/L_0$ and $h \in L_0$.

(i) If $M$ is invariant for $\theta(h)$ then $\text{Ker}_n(M)$ is a closed subspace of $\hat{X}_n$ invariant for $\rho_n(h)$.
(ii) Each $E^u_n$, $v \in L/L_0$, maps $\text{Ker}_n(M)$ into $\text{Ker}_{n-1}(M)$.
(iii) If $\text{Ker}_n(M) = \{0\}$, for some $n \geq 0$, then $\text{Ker}_{n+1}(M) = \{0\}$.

Proof. As each $E^u_n$, $u \in L/L_0$, is a bounded operator on $\hat{X}_n$, $\text{Ker}(E^u_n)$ is a closed subspace of $\hat{X}_n$. Hence $\text{Ker}_n(M)$ is a closed subspace of $\hat{X}_n$. Let $u \in M$. Since $M$ is invariant for $\theta(h)$, $\theta(h)u \in M$. If $\xi \in \text{Ker}_n(M)$, then $E^u_n \xi = 0$ and $E^{\theta(h)u}_n \xi = 0$. Hence it follows from (5.7) that

$$E^u_n \rho_n(h) \xi = \rho_{n-1}(h) E^u_n \xi - E^{\theta(h)u}_n \xi = 0.$$ 

Therefore $\rho_n(h) \xi \in \text{Ker}_n(M)$, so $\text{Ker}_n(M)$ is invariant for $\rho_n(h)$. Part (i) is proved.

By Lemma 5.1(iii), $E^u_{n-1} E^v_n \xi = E^u_{n-1} E^v \xi = 0$, for $\xi \in \text{Ker}_n(M)$, $u \in M$ and $v \in L/L_0$. Hence $E^v_n$ maps $\text{Ker}_n(M)$ into $\text{Ker}_{n-1}(M)$.

By (5.8), for each $0 \neq \xi \in \hat{X}_{n+1}$, there is $v \in L/L_0$ such that $E^v_{n+1} \xi \neq 0$. If $\{0\} \neq \xi \in \text{Ker}_{n+1}(M)$, then, by (ii), $0 \neq E^u_{n+1} \xi \in \text{Ker}_n(M)$. 

Proposition 5.3. Let $M \subset K$ be subspaces of $L/L_0$ invariant for $\theta$, $q = \dim(K/M) < \infty$. Let the quotient representation $\theta|_{K/M}$ of $L_0$ on $K/M$ be an eigen-representation. If $\text{Ker}_m(K) = \{0\}$ for some $m$, then there is $r \in \mathbb{N}$ such that $\text{Ker}_n(M) = \{0\}$ for $n \geq r$.

Proof. As $\theta|_{K/M}$ is an eigen-representation, there are $\{u_i\}^q_{i=1}$ in $K \setminus M$ such that together with $M$ they span $K$, and $f_i = \theta(h_i)u_i - \mu_i u_i \in M$, for some $\{h_i\}^q_{i=1}$ in $L_0$ and $0 \neq \mu_i \in \mathbb{C}$. Set $U_n = \text{Ker}_n(M)$. By Lemma 5.2, $U_n$ are closed subspaces of $\hat{X}_n$ invariant for $\rho_n$.

Set $r_i = [2||h_i||/||\mu_i||]$ and $r = m + \sum_{i=1}^q r_i$. Assume that $U_r \neq \{0\}$. Then $E^f_i \xi = 0$ for $\xi \in U_n$.

Thus $E^f_i \xi = \mu_i E^f_i \xi + E^f_i \xi = \mu_i E^f_i \xi$. By (5.7),

$$\rho_{r-1}(h_i) E^f_i \xi = E^f_i \rho_r(h_i) \xi + E^{\theta(h)u}_i \xi = E^f_i \rho_r(h_i) \xi + \mu_i E^f_i \xi.$$ 

By Lemma 5.2(ii), $E^f_i \xi \in U_{r-1}$. Using (5.7), we obtain similarly that $E^{f_1}_r \ldots E^{f_i}_r \xi \in U_{r-k}$ and

$$\rho_{r-k-1}(h_i) E^{f_1}_r \ldots E^{f_i}_r \xi = E^{f_1}_r \ldots E^{f_i}_r \rho_r(h_i) \xi + (k+1) \mu_i E^{f_1}_r \ldots E^{f_i}_r \xi.$$ 

(5.10)

Fix $i$. Set $A = \rho_{r-k-1}(h_i)|_{U_{r-k-1}}$ and $B = \rho_r(h_i)|_{U_r}$. Consider the map $T : C \mapsto AC - CB$ on the Banach space of all bounded operators $C$ from $U_r$ to $U_{r-k-1}$. By Rosenblum’s theorem (see [15, Theorem 0.12]), the spectrum $\text{Sp}(T)$ of $T$ is contained in the set $\{\alpha - \beta : \alpha \in \text{Sp}(A), \beta \in \text{Sp}(B)\}$. By Lemma 5.1(i), $\max(||A||, ||B||) \leq ||h_i||$. Thus $\text{Sp}(T)$ lies in the circle of radius $2||h_i||$.

From this and from (5.10) it follows that, if $E^{f_1}_r \ldots E^{f_i}_r \xi \neq 0$, then $(k+1) \mu_i \in \text{Sp}(T)$, so that $(k+1) ||\mu_i|| \leq 2||h_i||$. Hence

$$E^{f_1}_r \ldots E^{f_i}_r \xi = 0,$$ 

(5.11)
Let $k_1$ be the smallest number such that $E_{r-k_1}^{u_1} \cdots E_r^{u_1} U_r = \{0\}$. By (5.11), $0 \leq k_1 \leq r_1$. Set $Y_1 = U_r$, if $k_1 = 0$, and $Y_1 = E_{r-k_1+1}^{u_1} \cdots E_r^{u_1} U_r$, otherwise. Then $\{0\} \neq Y_1 \subseteq U_{r-k_1}$ and $E_{r-k_1}^{u_1} Y_1 = \{0\}$. By Lemma 5.1(iii),

$$E_{r-k_1-k}^{u_2} \cdots E_{r-k_1}^{u_2} Y_1 = E_{r-k}^{u_2} \cdots E_{r-k_1}^{u_2} E_{r-k_1+1}^{u_1} \cdots E_r^{u_1} U_r = E_{r-k_1-k}^{u_2} \cdots E_{r-k}^{u_2} U_r.$$

By (5.11), there is the smallest $k_2$, $0 \leq k_2 \leq r_2$, such that $E_{r-k_1-k_2}^{u_2} \cdots E_{r-k_1}^{u_2} Y_1 = \{0\}$. Set $Y_2 = Y_1$, if $k_2 = 0$, and $Y_2 = E_{r-k_1-k_2+1}^{u_2} \cdots E_{r-k_1}^{u_2} Y_1$, otherwise. Then $\{0\} \neq Y_2 \subseteq U_{r-k_1-k_2}$ and $E_{r-k_1-k_2}^{u_2} Y_2 = \{0\}$. Moreover, by Lemma 5.1(iii),

$$E_{r-k_1-k_2}^{u_1} E_{r-k_1-k_2-k_2}^{u_2} \cdots E_{r-k_1-k_2+1}^{u_2} \cdots E_{r-k_1}^{u_2} Y_2 = E_{r-k_1}^{u_2} \cdots E_{r-k_1-k_2-k_2}^{u_2} Y_1 = E_{r-k_1}^{u_2} \cdots E_{r-k_1-k_2}^{u_2} Y_1 = \{0\}.$$

Repeating this procedure, we obtain integers $k_i$, $0 \leq k_i \leq r_i$, for $1 \leq i \leq q$, and the subspace $\{0\} \neq Y_q \subseteq U_{r-k_1-k_2-\cdots-k_q}$ such that $E_{r-k_1-k_2-\cdots-k_q}^{u_q} Y_q = \{0\}$ for all $i$. Set $p = r - k_1 - k_2 - \cdots - k_q$. Then $p \geq m$ and

$$\{0\} \neq Y_q \subseteq U_p \cap \{\ker(E_{p}^{u_i}): 1 \leq i \leq q\} = \ker_{p} (K).$$

On the other hand, as $\ker_{m} (K) = \{0\}$ and $p \geq m$, we have from Lemma 5.2(iii) that $\ker_{p} (K) = \{0\}$. This contradiction shows that $U_r = \{0\}$. □

We shall now prove the main result of this section.

**Theorem 5.4.** Let $X$ be an infinite-dimensional Banach space and $\mathcal{L}$ be a finite- or infinite-dimensional Lie subalgebra of $\mathcal{B}(X)$. Let $X_0$ be a closed subspace of $X$ invariant for a Lie subalgebra $\mathcal{L}_0$ of $\mathcal{L}$ and let $(\mathcal{L}, X_0)$-filtration $\{X_n\}$ of $X$ be non-trivial. If $\mathcal{L}_0$ is non-degenerate in $\mathcal{L}$ then, for some $m$, $\{0\} \neq X_m \subseteq X_0$ is a closed subspace of $X$ invariant for $\mathcal{L}$.

**Proof.** By (5.2), $\theta$ is non-degenerate. Theorem 2.6 implies that there are subspaces $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = \mathcal{L}/\mathcal{L}_0$ invariant for $\theta$ such that all quotient representations $\theta_j$ of $\mathcal{L}_0$ on $V_j/V_{j-1}$ are eigen-representations. By (5.8), $\ker_{0} (V_k) = \bigcap \{\ker(E_{0}^{u_i}): u \in \mathcal{L}/\mathcal{L}_0\} = \{0\}$. As the representation $\theta$ of $\mathcal{L}_0$ on $V_k/V_{k-1}$ is an eigen-representation, Proposition 5.3 implies that there is $n_{k-1} \geq 0$ such that $\ker_{n_{k-1}} (V_{k-1}) = \{0\}$. Repeating this process, we obtain $n_0$ such that $\ker_{n_0} (V_0) = \{0\}$. As $V_0 = \{0\}$, we have from (5.9) that $\ker_{n_0} (V_0) = \hat{X}_{n_0}$. Hence $\hat{X}_{n_0} = \{0\}$. Thus $X_{n_0}$ is invariant for $\mathcal{L}$. □

**Corollary 5.5.** Let $X$ and $\mathcal{L}$ be as in Theorem 5.4. Let $X_0$ be a closed subspace of $X$ invariant for a Lie subalgebra $\mathcal{L}_0$ of $\mathcal{L} \subseteq \mathcal{B}(X)$ and let $\mathcal{L}_0$ be related to $\mathcal{L}$.

(i) If $(\mathcal{L}, X_0)$-filtration $\{X_n\}$ of $X$ is non-trivial, then $\mathcal{L}$ has a closed non-trivial invariant subspace contained in $X_0$.

(ii) If $\text{codim}(X_0) < \infty$ then $\mathcal{L}$ has a closed non-trivial invariant subspace $W$ contained in $X_0$ and $\text{codim}(W) < \infty$.

**Proof.** Consider Lie subalgebras $\mathcal{L}_0 = \mathcal{L}_0^0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{L}_p = \mathcal{L}$ of $\mathcal{L}$ such that each $\mathcal{L}^i$ is non-degenerate in $\mathcal{L}^{i+1}$. Set $Y_0 = X_0$. As $\mathcal{L}_1 \subseteq \mathcal{L}$, we have from (4.2) that $(\mathcal{L}_1, Y_0)$-filtration $\{Y_n\}$
of \( X \) satisfies \( X_n \subseteq Y_n \subseteq X_0 \), so it is non-trivial. Hence, by Theorem 5.4, for some \( m_1 \), the closed subspace \( Y_{m_1} \subseteq X_0 \) is invariant for \( \mathcal{L}^1 \). Set \( Z_0 = Y_{m_1} \). Then \( X_{m_1} \subseteq Z_0 \). It follows from (4.2) that \( (\mathcal{L}_2, Z_0) \)-filtration \( \{Z_n\} \) of \( X \) satisfies \( X_{n+m_1} \subseteq Z_n \subseteq Z_0 \), for \( n \in \mathbb{N} \), so that it is non-trivial. If \( p = 2 \) then \( \mathcal{L} = \mathcal{L}_2 \) and, by Theorem 5.4, for some \( m_2 \), the closed subspace \( Z_{m_2} \subseteq X_0 \) is invariant for \( \mathcal{L} \). If \( p > 2 \), repeating this, we obtain that \( \mathcal{L} \) has a closed non-trivial invariant subspace contained in \( X_0 \). Part (i) is proved.

If \( X_0 \) has finite codimension, then \( (\mathcal{L}, X_0) \)-filtration \( \{X_n\} \) of \( X \) is non-trivial and \( Y_{m_1} \) has finite codimension, so that \( Z_{m_2} \) has finite codimension. Repeating this process, we complete the proof. \( \square \)

**Remark 5.6.** (1) Let \( \mathcal{L} \subseteq \mathcal{B}(X) \), let \( \mathcal{L}_0 \) be a Lie subalgebra of \( \mathcal{L} \) with \( \text{codim}(\mathcal{L}_0) < \infty \), let \( X_0 \) be a non-trivial closed subspace of \( X \) invariant for \( \mathcal{L}_0 \) and let \( (\mathcal{L}, X_0) \)-filtration of \( X \) be non-trivial. Suppose that a one-dimensional subspace \( \mathcal{C} \phi(e), e \in \mathcal{L} \), of \( \mathcal{L}/\mathcal{L}_0 \) is invariant for all \( \theta(h), h \in \mathcal{L}_0 \). Then \( \mathcal{L} = \mathcal{C} e + \mathcal{L}_0 \) is a Lie subalgebra of \( \mathcal{L} \).

If \( \mathcal{L}_0 \) is not a Lie ideal of \( \mathcal{L} \) and \( \mathcal{L} \) is related to \( \mathcal{L}_0 \), then \( \mathcal{L}_0 \) is related to \( \mathcal{L} \). Hence \( \mathcal{L}_0 \) is related to \( \mathcal{L} \) and, by Corollary 5.5, \( \mathcal{L} \) is reducible. The case when \( \mathcal{L}_0 < \mathcal{L} \) and \( \mathcal{L} \) is related to \( \mathcal{L}_0 \) will be considered in Theorem 8.1(ii).

**Corollary 5.7.** Let \( \mathcal{L}_0 \) be a Lie subalgebra of \( \mathcal{L} \subseteq \mathcal{B}(X) \) and let \( \mathcal{L}_0 \) be related to \( \mathcal{L} \). If \( \mathcal{L}_0 \) is operator-nilpotent (\( \mathcal{L}_0^n = \{0\} \) for some \( n \)), then \( \mathcal{L} \) is reducible.

**Proof.** Any operator-nilpotent Lie subalgebra \( \mathcal{L}_0 \) of \( \mathcal{B}(X) \) has a closed invariant subspace of codimension 1. Indeed, \( \overline{\mathcal{L}_0 X} \) is a proper subspace of \( X \) and each subspace containing it is invariant for \( \mathcal{L}_0 \). Applying now Corollary 5.5, we complete the proof. \( \square \)

In the above corollary one can consider more general condition \( \overline{\mathcal{L}_0 X} \neq X \) instead of the condition that \( \mathcal{L}_0 \) is operator-nilpotent. It should be noted that if a Lie subalgebra \( \mathcal{L}_0 \) of \( \mathcal{L} \) is operator-nilpotent and has finite codimension but not related to \( \mathcal{L} \), then \( \mathcal{L} \) may be irreducible as the following example shows.

**Example 5.8.** Let \( Y \) be a Banach space, \( \dim(Y) = \infty \). Then \( X = Y \oplus Y \) is a Banach space with norm \( \|x \oplus y\| = \sup(\|x\|, \|y\|) \). Let a bounded operator \( S \) on \( Y \) have no non-trivial closed invariant subspaces. Set \( \hat{S} = S \oplus S \). For \( a = (a_{11}, a_{12}) \in M_2(\mathbb{C}) \), let \( A_a \) be the operator on \( X \) acting by the formula \( A_a(x \oplus y) = (a_{11}x + a_{12}y) \oplus (a_{21}x + a_{22}y) \) and let \( \mathcal{A} = \{A_a; a \in M_2(\mathbb{C})\} \). Then \( \mathcal{L} = \mathcal{C} \hat{S} + \mathcal{A} \) is an irreducible Lie subalgebra of \( \mathcal{B}(X) \). Set \( \mathcal{L}_0 = \{A_a; a \in M_2(\mathbb{C}), a_{11} = a_{21} = a_{22} = 0\} \). Then \( \mathcal{L}_0 \) is a Lie subalgebra of \( \mathcal{L} \), it is operator-nilpotent and codim(\( \mathcal{L}_0 \)) = 4. However, since \( \mathcal{A} \) is a Lie ideal of \( \mathcal{L} \), \( \mathcal{L}_0 \) is not related to \( \mathcal{L} \).

We shall now develop a new approach to the problem of the existence of invariant subspaces for Lie subalgebras of \( \mathcal{B}(X) \). It will allow us to give a different proof of Theorem 5.4 and to treat in the next section the case when \( \mathcal{L}_0 \) is a Lie ideal of \( \mathcal{L} \). Let \( \{X_n\} \) be \( (\mathcal{L}, X_0) \)-filtration of \( X \). Then \( \hat{X}_n = X_n / X_{n+1} \) are Banach spaces with norms \( \|\cdot\|_n \). Consider the graded Banach space

\[
\hat{X} = \bigoplus_{n=-1}^{\infty} \hat{X}_n = \{\hat{x} = \hat{x}_{-1} \oplus \cdots \oplus \hat{x}_n \oplus \cdots; \hat{x}_n \in \hat{X}_n \text{ with } \|\hat{x}\| = \sup\|\hat{x}_n\|_n < \infty\}.
\]
Set $V = L/L_0$. Let $u \mapsto E_n^u$, $n \in \mathbb{N}$, be the linear maps from $V$ into the Banach spaces of all bounded operators from $\hat{X}_n$ into $\hat{X}_{n-1}$ defined in (5.6). Denote by $E^u$ the operators on $\hat{X}$ that act by

$$E^u \hat{x} = E_0^u \hat{x}_0 + E_1^u \hat{x}_1 + \cdots + E_n^u \hat{x}_n + \cdots.$$  

(5.12)

It follows from Lemma 5.1 that $u \mapsto E^u$ is a linear map from $V$ into $\mathcal{B}(\hat{X})$, $\|E^u\| \leq \|e\|$, for each $e \in L$ satisfying $u = \phi(e)$, and

$$E^v E^u = E^u E^v \text{ for all } u, v \in V.$$  

(5.13)

This map extends to a linear bounded map $\Psi$ from $L$ into $\mathcal{B}(\hat{X})$ by $\Psi(e) = E^{\phi(e)}$. Note that $\Psi$ is not, generally speaking, a Lie homomorphism.

Set $V^\text{nil} = \{u \in V : E^u$ is nilpotent\} (see (3.4)). Then $M = \{E^u : u \in V\}$ is a linear subspace of $\mathcal{B}(\hat{X})$ of mutually commuting bounded operators and $N_M = \{E^u : u \in V^\text{nil}\}$. Set

$$Y_n = \left(\bigcap \{\text{Ker}(E^u) : u \in V^\text{nil}\}\right) \cap \hat{X}_n.$$  

(5.14)

Let $V^\text{nil} \neq \{0\}$, $\dim(V^\text{nil}) = p < \infty$ and let $\{u_i\}_{i=1}^p$ be a basis in $V^\text{nil}$. Then $(E^{u_i})^{k_i} = 0$ for some $k_i$. Set $E_i = E^{u_i}$ and $k = \max(k_i)$. Then $E_i \hat{X}_n \subseteq \hat{X}_{n-1}$ and $E_i^{k_i} = 0$ for all $i$ and $n$.

**Proposition 5.9.** If $\hat{X}_n \neq \{0\}$ for some $n \geq pk$, then $Y_m \neq \{0\}$ for some $m$ satisfying $n - pk \leq m < n$.

**Proof.** Take the largest $m_1$ such that $K_1 = E_1^{m_1} \hat{X}_n \neq \{0\}$. Then $0 \leq m_1 < k$ and $E_1 K_1 = \{0\}$. Take the largest $m_2$ such that $K_2 = E_2^{m_2} K_1 \neq \{0\}$. Then $0 \leq m_2 < k$ and $E_2 K_2 = \{0\}$. As $E_1$ and $E_2$ commute (see (5.13)), $E_1 K_2 = E_1 E_2^{m_2} K_1 = E_2^{m_2} E_1 K_1 = \{0\}$. Repeating this process, we obtain

$$\{0\} \neq K_p = E_p^{m_p} K_{p-1} = E_p^{m_p} E_{p-1}^{m_{p-1}} \cdots E_2^{m_2} E_1^{m_1} \hat{X}_n \subseteq \hat{X}_{n-m_1 - \cdots - m_p},$$

for some $0 \leq m_p < k$, such that $E_1 K_p = E_2 K_p = \cdots = E_p K_p = \{0\}$. As $N_M$ is the linear span of all $E_i$, $\{0\} \neq K_p \subseteq Y_{n-m_1 - \cdots - m_p}$. Setting $m = n - m_1 - \cdots - m_p$, we complete the proof. \[\square\]

Let $\rho_n$, $n \in \mathbb{N} \cup \{-1\}$, be the representations of $\mathcal{L}_0$ on $\hat{X}_n$ defined in (5.3), satisfying $\|\rho_n(h)\| \leq \|h\|$ for $h \in L_0$. Let $\rho(h)$ be the operator on $\hat{X}$ that acts by the formula

$$\rho(h) \hat{x} = \rho_{-1}(h) \hat{x}_{-1} + \cdots + \rho_n(h) \hat{x}_n + \cdots.$$  

(5.15)

Then $\rho$ is a representation of $\mathcal{L}_0$ on $\hat{X}$ and $\|\rho(h)\| \leq \|h\|$.

Let $\theta$ be the representation of $\mathcal{L}_0$ on $V$ defined in (5.1). It follows from (5.7) that

$$\rho(h) E^u = E^u \rho(h) + E^{\theta(h)u} \text{ for } h \in \mathcal{L}_0 \text{ and } u \in V.$$  

(5.16)

Using this approach, we will now give another proof of Theorem 5.4.
6. Invariant subspaces of operator Lie algebras: the case when \( \mathcal{L}_0 \) is a Lie ideal of \( \mathcal{L} \)

In this section we will turn to the case when \( \mathcal{L}_0 \) is a Lie ideal of \( \mathcal{L} \). The restriction \( \operatorname{codim}(\mathcal{L}_0) < \infty \) will be dropped; instead we will only assume that \( \dim(\mathcal{L}_0) = \infty \). We will show that if \( \mathcal{L}_0 \) has a closed invariant subspace \( X_0 \) of finite codimension, then \( \mathcal{L} \) has a non-trivial closed invariant subspace.

Recall (see (4.3)) that \( Z(\mathcal{L}) \subseteq L_1 \) and \( Z(\mathcal{L})X \subseteq X_0 \).

**Proposition 6.1.** Let \( \mathcal{L}_0 \) be a Lie subalgebra of \( \mathcal{L} \subseteq B(X) \), let \( X_0 \) be a closed subspace of \( X \) invariant for \( \mathcal{L}_0 \) and \( \{X_n\} \) be \( (\mathcal{L}, X_0) \)-filtration of \( X \). Assume that \( Z(\mathcal{L}) \ll \mathcal{L} \). Set \( X = \bigcap_{n=0}^{\infty} X_n \). Then

(i) \( Z(\mathcal{L})X \subseteq X \).

(ii) If \( \dim(\mathcal{L}_0) = \infty \), \( \operatorname{codim}(\mathcal{L}_0) < \infty \) and \( \operatorname{codim}(X_0) < \infty \), then \( X \) is a non-trivial closed invariant subspace for \( \mathcal{L} \).

(iii) If \( \dim(\mathcal{L}_0) = \infty \), \( \mathcal{L}_0 \ll \mathcal{L} \) and \( \operatorname{codim}(X_0) < \infty \), then \( X \) is a non-trivial closed subspace invariant for \( \mathcal{L} \) and \( \dim(\mathcal{L}_0/Z(\mathcal{L})) < \infty \).

**Proof.** Set \( Z = Z(\mathcal{L}) \). We have \( ZX \subseteq X_0 \). If \( ZX \subseteq X_k \), for some \( k \geq 0 \), then, as \( Z \ll \mathcal{L} \),

\[
LZX \subseteq [\mathcal{L}, Z]X + ZLX \subseteq ZX + ZX \subseteq X_k.
\]

Hence it follows from (4.2) that \( ZX \subseteq X_{k+1} \). By induction, \( ZX \subseteq X_n \) for all \( n \in \mathbb{N} \). Part (i) is proved.

Clearly, \( X \) is invariant for \( \mathcal{L} \). By Lemma 4.2(iv), \( X \) is closed. Since \( Z \ll \mathcal{L} \), we have from (i) that \( ZX \subseteq X \). Let us show that \( X \neq \{0\} \).

(ii) Since \( \operatorname{codim}(\mathcal{L}_0) < \infty \) and \( \operatorname{codim}(X_0) < \infty \), it follows from Lemma 4.2(iii) that \( \dim(\mathcal{L}_0/Z) < \infty \). As \( \dim(\mathcal{L}_0) = \infty \), \( Z \neq \{0\} \). Hence \( X \neq \{0\} \).

(iii) Since \( \operatorname{codim}(X_0) < \infty \), the representation \( \rho_{-1} \) of \( \mathcal{L}_0 \) on \( X/X_0 \) is finite-dimensional. As \( \mathcal{L}_0 \ll \mathcal{L} \), \( \mathcal{L}_0 = L_1 \). Hence, by (4.3), \( \operatorname{Ker}(\rho_{-1}) = Z \). Thus \( \dim(\mathcal{L}_0/Z) < \infty \). If \( X = \{0\} \) then \( Z = \{0\} \), so that \( \dim(\mathcal{L}_0) < \infty \) which contradicts our assumption. \( \Box \)

We will now show that the conditions \( \operatorname{codim}(X_0) < \infty \) and \( \mathcal{L}_0 \ll \mathcal{L} \) automatically imply \( Z(\mathcal{L}) \ll \mathcal{L} \). By Proposition 6.1(iii), this will guarantee \( \bigcap_{n=0}^{\infty} X_n \neq \{0\} \).

By Lemma 4.2(ii), \( z X_n \subseteq X_{n+1} \) for \( z \in Z(\mathcal{L}) \). Hence, for all \( n \in \mathbb{N} \), \( z \) defines linear operators \( F_n^z \) from \( \hat{X}_{n-1} \) into \( \hat{X}_n \) by the formula

\[
F_n^z\tau_{n-1}(x) = \tau_n(zx), \quad \text{for } x \in X_{n-1}.
\]  

(6.1)

Denote by \( F^z \) the operator on \( \hat{X} \) that acts on \( \hat{x} = \hat{x}_{-1} \oplus \cdots \oplus \hat{x}_n \oplus \cdots \in \hat{X} \) by the formula

\[
F^z\hat{x} = 0 \oplus F_0^z\hat{x}_{-1} \oplus F_1^z\hat{x}_0 \oplus \cdots \oplus F_{n+1}^z\hat{x}_n \oplus \cdots.
\]  

(6.2)
Proposition 6.2. If $Z(L) \subseteq L_2$, the operators $\rho([z, e])$ are quasinilpotent, for all $e \in L$, $z \in Z(L)$.

Proof. Let $e \in L$. For $u = \phi(e)$ and $z \in Z(L)$, we have from (5.12) and (6.2) that
\[
[F^z, E^u] = (-E^u F^z_0) \hat{\tau}_{-1} \oplus (F^z_0 E^u_0 - E^u_1 F^z_1) \hat{\tau}_0 \oplus \cdots \oplus (F^z E^u_n - E^u_{n+1} F^z_{n+1}) \hat{\tau}_n \oplus \cdots.
\]
As $zX \subseteq X_0$, $\tau_{-1}(zX) = 0$. Since $[z, e] \in L_0$, it follows from (5.4) and (6.1) that
\[
-E^u_0 F^z_0 \tau_{-1}(x) = -E^u_0 \tau_0 (zx) = \tau_{-1}(-ezx) = \tau_{-1}((ze - ez)x) = \rho_{-1}([z, e]) \tau_{-1}(x), \quad (6.3)
\]
for $x \in X$. For $x \in X_n$, we have
\[
(F^z E^u_n - E^u_{n+1} F^z_{n+1}) \tau_n(x) = F^z \tau_{n-1}(ex) - E^u_{n+1} \tau_{n+1}(zx) = \tau_n(zex) - \tau_n(ezx)
\]
Therefore $[F^z, E^u] \hat{\tau}_{-1} = \rho([z, e]) \hat{\tau}_{-1}$, so $[F^z, E^u] = \rho([z, e])$.

Since $Z(L) \subseteq L_2$, we have $[z, e] \in L_1 = \text{Ker}(\theta)$. Hence, by (5.16), $[\rho([z, e]), E^u] = E^\theta ([z, e]) u = 0$. Therefore it follows from the Kleinecke–Shirokov theorem (see [6]) that $\rho([z, e])$ is quasinilpotent. □

Let $X_0$ be a closed subspace of $X$ invariant for $L_0$. We will say that $X_0$ is maximal, if it is not contained in a larger closed proper subspace of $X$ invariant for $L_0$. If $X_0$ is not maximal and $\text{codim}(X_0) < \infty$, it can always be extended to a maximal invariant subspace.

In the proof of our next result we use the following extension of Engel Theorem proved in [8]: if $K$ is a set of nilpotent operators on a finite-dimensional space such that $[k_1, k_2] \in K$, for all $k_1, k_2 \in K$, then $\text{Ker}(K) \neq 0$. This is why we have to impose the condition $\text{codim}(X_0) < \infty$.

Proposition 6.3. Let $X_0$ be a maximal closed subspace of $X$ invariant for $L_0$ and let $(X_n)$ be $(L, X_0)$-filtration of $X$. If $\text{codim}(X_0) < \infty$ and $L_0 \triangleleft L$ then $Z(L) \triangleleft L$.

Proof. Set $Z = Z(L)$. As $L_0 \triangleleft L$, $L_0 = L_1$. Hence, by Lemma 5.1(i),
\[
Z = \text{Ker}(\rho_{-1}). \quad (6.4)
\]
Since $X_0$ is maximal, the representation $\rho_{-1}$ of $L_0$ on $\hat{X}_{-1}$ is irreducible. Hence the Lie algebra $L = \rho_{-1}(L_0)$ of operators on $\hat{X}_{-1}$ has no non-trivial invariant subspaces. Its subset $K = \{\rho_{-1}([z, e]): e \in L, z \in Z\}$ consists of nilpotent operators. Indeed, since $Z \subseteq L_2$, we have from Proposition 6.2 that $\rho([z, e])$ is quasinilpotent in $\hat{X}$ for $e \in L$ and $z \in Z$. As $\hat{X}_{-1}$ is finite-dimensional, each $\rho_{-1}([z, e])$ is nilpotent.

Let us show that all $\rho_{-1}([z, e]) = 0$. As $\rho_{-1}$ is a Lie homomorphism, we have for $h \in L_0$ that
\[
[\rho_{-1}(h), \rho_{-1}([z, e])] = \rho_{-1}([h, [z, e]]) = \rho_{-1}([[h, z], e]) + \rho_{-1}([z, [h, e]]).
\]
As \( L_0 \triangleleft \mathcal{L} \), we have \([h,e] \in L_0\). By Lemma 4.2(ii), \( Z \triangleleft L_0 \). Therefore \([z, [h, e]] \in Z \) and \([h, z] \in Z \). Hence, by (6.4), \( \rho^{-1}([z, [h, e]]) = 0 \), so that \( \rho^{-1}(h), \rho^{-1}([z, e]) = \rho^{-1}([h, z], e)) \in K \). Thus

\[
[a, k] \in K \quad \text{for all } a \in L \text{ and } k \in K.
\]

(6.5)

In particular, \([k_1, k_2] \in K \) for all \( k_1, k_2 \in K \). As \( K \) consists of nilpotent operators, it follows from [8, Theorem II.2.1’] that there is \( 0 \neq k_0 \in \mathcal{X}_{-1} \) such that \( k_0 \xi_0 = 0 \) for all \( k \in K \).

Let \( R = \{ \xi \in \mathcal{X}_{-1}: K \xi = \{0\} \} \). Then \( R \neq \{0\} \) and, by (6.5), \( R \) is invariant for \( L \). As \( L \) has no non-trivial invariant subspaces, \( R = \mathcal{X}_{-1} \). Hence \( K \mathcal{X}_{-1} = \{0\} \), so \( K = \{0\} \). Thus \( \rho^{-1}([z, e]) = 0 \) for all \( e \in \mathcal{L} \) and \( z \in Z \). Therefore, by (6.4), \( [Z, \mathcal{L}] \subseteq Z \), so that \( Z \triangleleft \mathcal{L} \). \( \square \)

The following theorem is the central result of this section.

**Theorem 6.4.** Let \( L_0 \triangleleft \mathcal{L} \subseteq B(X) \), let \( \dim(L_0) = \infty \) and let \( L_0 \) have a closed invariant subspace \( X_0 \) of finite codimension. Then

(i) \( \mathcal{L} \) has a non-trivial closed invariant subspace \( Y \) such that \( Y + X_0 \neq X \).

(ii) \( \mathcal{L} \) has a Lie ideal \( \mathcal{C} \subseteq L_0 \) such that \( \mathcal{C}X \subseteq Y \) and \( \dim(L_0/\mathcal{C}) < \infty \).

(iii) If \( X_0 \) is maximal, \( Y \) can be taken as \( \bigcap X_n \) (so \( Y \subseteq X_0 \)) and \( \mathcal{C} = Z(\mathcal{L}, \mathcal{L}_0, X_0) \).

**Proof.** If \( X_0 \) is not maximal, replace it by a larger maximal closed subspace \( Y_0 \) invariant for \( L_0 \). Let \( \{Y_n\} \) be \((\mathcal{L}, Y_0)\)-filtration of \( X \). The closed subspace \( Y = \bigcap Y_n \) is invariant for \( \mathcal{L} \) and \( Y + X_0 \subseteq Y_0 \neq X \). Set \( \mathcal{C} = Z(\mathcal{L}, \mathcal{L}_0, Y_0) \). By Proposition 6.3, \( \mathcal{C} \triangleleft \mathcal{L} \). Applying Proposition 6.1(iii), we complete the proof. \( \square \)

**Remark 6.5.** The closed subspace \( Y \) in Theorem 6.4 invariant for \( \mathcal{L} \) is not, generally speaking, finite codimensional. Indeed, let \( \mathcal{X} \) be an infinite-dimensional Banach space such that there exists a bounded operator \( e \) on \( \mathcal{X} \) that has no non-trivial closed invariant subspaces (for example, \( \mathcal{X} = l_1 \) (see [17])). Let \( Z \) be a closed subspace of codimension 1 in \( \mathcal{X} \). Set \( \mathcal{X} = \mathcal{X} \oplus \mathcal{X}, \ X_0 = Z \oplus \mathcal{X} \). Set \( \mathcal{L} = \mathcal{C}e \oplus B(\mathcal{X}) \) and \( \mathcal{L}_0 = \{0\} \oplus B(\mathcal{X}) \). Then \( \mathcal{L}_0 \triangleleft \mathcal{L} \) and \( \dim(L_0) = 1 \), the subspace \( X_0 \) is invariant for \( L_0 \) and \( \dim(X_0) = 1 \), while the Lie algebra \( \mathcal{L} \) has only two non-trivial closed invariant subspaces \( \{0\} \oplus \mathcal{X} \) and \( \mathcal{X} \oplus \{0\} \); both of them have infinite codimension.

As the following example shows, the conditions \( \dim(L_0) = \infty \) and \( \dim(X_0) < \infty \) in Theorem 6.4 are crucial.

**Example 6.6.** Let \( L_0 \triangleleft \mathcal{L} \subseteq B(X) \) and let \( L_0 \) have a closed invariant subspace \( X_0 \). If either \( \dim(L_0) < \infty \), or \( \dim(X_0) = \infty \), then \( \mathcal{L} \) may be irreducible.

Indeed, let \( H \) and \( K \) be Hilbert spaces, \( \dim(H) = \infty \) and \( \dim(K) < \infty \). Let \( X = H \otimes K \) and \( \mathcal{L} = B(H) \otimes \mathbb{C}1_K + \mathbb{C}1_H \otimes B(K) \). Let \( \mathcal{H} \) be a subspace of \( H \) of codimension 1. Consider the Lie ideals \( B(H) \otimes \mathbb{C}1_K \) and \( \mathbb{C}1_H \otimes B(K) \) of \( \mathcal{L} \).

(i) Let \( L_0 = B(H) \otimes \mathbb{C}1_K \). Then \( \dim(L_0) = \infty \), the closed subspace \( X_0 = H \otimes \mathbb{C}k \), for each \( k \in K \), is invariant for \( L_0 \) and \( \dim(X_0) = \infty \);

(ii) Let \( L_0 = \mathbb{C}1_H \otimes B(K) \). Then \( \dim(L_0) < \infty \), the closed subspace \( X_0 = \mathbb{C}h \otimes K \), for each \( h \in H \), is invariant for \( L_0 \) and \( \dim(X_0) = \infty \);
(iii) Let $L_0 = C_1 \mathcal{H} \otimes B(K)$. Then $\dim(L_0) < \infty$, the closed subspace $X_0 = \mathcal{H} \otimes K$ is invariant for $L_0$ and $\text{codim}(X_0) < \infty$.

However, as the commutant of $L$ coincides with $C_1^X$, $L$ has no non-trivial closed invariant subspaces.

It follows from Proposition 6.3 that if a Lie ideal of an irreducible Lie algebra has an invariant subspace of finite codimension then it has a special simple structure.

**Proposition 6.7.** Let $L_0 \triangleleft L \subseteq B(X)$. Suppose that $L_0$ has a maximal closed invariant subspace $X_0$ such that $k = \text{codim}(X_0) < \infty$ and $\bigcap X_n = \{0\}$, where $\{X_n\}$ is $(L, X_0)$-filtration of $X$. (In particular, this holds if $L$ is irreducible.) Then

(i) $L_0$ is isomorphic to an irreducible Lie algebra of operators on $k$-dimensional space.

(ii) If $k = 1$ then $L_0 = C_1$.

**Proof.** By Proposition 6.3, $Z(L) \triangleleft L$. If $\bigcap X_n = \{0\}$ then, by Proposition 6.1(i), $Z(L) = \{0\}$. By (6.4), $Z(L) = \text{Ker}(\rho_{-1})$. Hence $\rho_{-1}$ is a faithful representation of $L_0$ on the $k$-dimensional space $\tilde{X}_{-1}$. Thus $L_0$ is isomorphic to the Lie algebra $\rho_{-1}(L_0)$ and, since $X_0$ is maximal, $\rho_{-1}(L_0)$ has no invariant subspaces, part (i) is proved.

If $k = 1$, $\dim(L_0) = 1$. Hence $L_0 = \mathbb{C}a$ for some $0 \neq a \in L$. Suppose that $a \neq \lambda 1$ and consider the Lie algebra $C = L + C_1$. Then $C_0 = L_0 + C_1 < C$. $X_0$ is invariant for $C_0$ and $\{X_n\}$ is also $(C, X_0)$-filtration of $X$. By the above argument, $\dim(C_0) = 1$. This contradiction shows that $L_0 = C_1$. $\square$

7. **Finite-dimensional invariant subspaces**

In this section we consider the case when $L_0$ has an invariant subspace $X_0$ of finite dimension.

Let $X^*$ be the dual space of $X$ and $X_0^\perp = \{ f \in X^* : f(x) = 0 \text{ for all } x \in X_0 \}$ be the annihilator of $X_0$ in $X^*$. Then $X_0^\perp$ is a closed subspace of $X^*$ and $\dim(X_0^\perp) = \dim(X_0) < \infty$. For $a \in B(X)$, denote by $a^*$ the conjugate operator on $X^*$ defined by $(a^* f)(x) = f(ax)$ for $f \in X^*$ and $x \in X$. Set $L^* = \{a^* : a \in L\}$ and $L_0^* = \{a^* : a \in L_0\}$. As

$$[a, b]^* = -[a^*, b^*], \quad (7.1)$$

$L^*$ is a Lie subalgebra of $B(X^*)$ and $L_0^*$ is its Lie subalgebra. The subspace $X_0^\perp$ is invariant for $L_0^*$.

**Theorem 7.1.** Let $L$ be a finite- or infinite-dimensional Lie subalgebra of $B(X)$. Let a Lie subalgebra $L_0$ of $L$ have a non-trivial finite-dimensional invariant subspace $X_0$.

(i) If $\text{codim}(L_0) < \infty$ and the representation $\theta$ of $L_0$ on $L/L_0$ is non-degenerate, then $L$ has a finite-dimensional invariant subspace that contains $X_0$.

(ii) If $\dim(L_0) = \infty$ and $L_0 \triangleleft L$, then $L$ has a non-trivial closed invariant subspace $W$. If $X_0$ is a minimal subspace invariant for $L_0$, then there is a non-trivial closed invariant subspace $W$ such that $X_0 \subseteq W$. 

Proof. Since codim($X^\perp_0$) < ∞, ($L^*$, $X^\perp_0$)-filtration \{X^\perp_n\} of $X^*$ is non-trivial.

(i) Let codim($L_0$) < ∞ and $\theta$ be non-degenerate. By (5.2), $L = L_0 + [L_0, L]$. Hence it follows from (7.1) that $L^* = L^*_0 + [L^*_0, L^*]$. Therefore, by (5.2), $\theta^*$ is non-degenerate.

Since codim($L^*_0$) < ∞, it follows from Theorem 5.4 that, for some $m$, $X^*_m$ is a closed subspace of $X^*$ invariant for $L^*$ and codim($X^*_m$) < ∞.

Let $X^{**}$ be the dual space of $X^*$ and $Y$ be the annihilator of $X^*_m$ in $X^{**}$. Then $Y$ is a subspace of $X^{**}$ invariant for the Lie subalgebra $L^{**} = \{a^{**}: a \in L\}$ of $\mathcal{B}(X^{**})$ and dim($Y$) < ∞. The space $X$ can be considered as a closed subspace of $X^{**}$ invariant for $L^{**}$ and $a^{**}|_X = a$ for $a \in L$. Hence $X_0 \subseteq Y$, as $X^*_m \subseteq X^*_0 = X^\perp_0$. Therefore the finite-dimensional subspace $W = Y \cap X$ of $X$ is invariant for $L$ and $X_0 \subseteq W$. Part (i) is proved.

(ii) Let dim($L_0$) = ∞ and $L_0 < L$. Then dim($L^*_0$) = ∞ and, by (7.1), $L^*_0 < L^*$. We may assume that $X_0$ is the minimal subspace invariant for $L_0$. Then $X^\perp_0$ is a maximal subspace invariant for $L^*_0$ and codim($X^\perp_0$) < ∞. Hence it follows from Theorem 6.4 that $L^*$ has a non-trivial closed invariant subspace $Y = \bigcap X^*_n$. Then the annihilator $Y^\perp$ of $Y$ in $X^{**}$ is a closed subspace invariant for the Lie algebra $L^{**}$. Consider $X$ as a closed subspace of $X^{**}$ invariant for $L^{**}$. Then $a^{**}|_X = a$, for $a \in L$, and $X_0 \subseteq Y^\perp$, as $Y \subseteq X^*_0 = X^\perp_0$. Hence the subspace $W = Y^\perp \cap X$ of $X$ is invariant for $L$ and $X_0 \subseteq W$. Thus $W \neq \{0\}$.

We have that $X$ is dense in $X^{**}$ in the $\sigma(X^{**}, X^*)$ topology. On the other hand, $Y^\perp$ is closed in $X^{**}$ in the $\sigma(X^{**}, X^*)$ topology and $\{0\} \neq Y^\perp \neq X^{**}$. Therefore $X \not\subseteq Y^\perp$. Hence $W \neq X$. □

Remark 7.2. As Example 6.6(ii) shows, if dim($L_0$) < ∞ in Theorem 7.1(ii), then $L$ can be irreducible.

Corollary 7.3. Let $L$ be a finite- or infinite-dimensional Lie subalgebra of $\mathcal{B}(X)$. Let a Lie subalgebra $L_0$ of $L$ have a non-trivial finite-dimensional invariant subspace $X_0$. If $L_0$ is related to $L$, then $L$ has a finite-dimensional invariant subspace that contains $X_0$.

Proof. As $L_0$ is related to $L$, there are Lie subalgebras $L_0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_p = L$ of $L$ such that the representations $\theta_i$, $0 \leq i \leq p - 1$, of $L_i$ on $L_{i+1}/L_i$ are non-degenerate. Apply now Theorem 7.1(i) to each pair ($L^i, L^{i+1}$). □

8. Combined cases. Invariant subspaces of associative operator algebras and semigroups

8.1. Combined cases

In this subsection we shall combine the results of Sections 5–7.

Theorem 8.1. Let $L_0 \subseteq L \subseteq L$ be Lie subalgebras of $\mathcal{B}(X)$ and let dim($L_0$) = ∞. Suppose that $L_0$ has a closed invariant subspace $X_0$.

(i) Let $L_0$ be related to $L$ and $L \not\subseteq L$. If codim($X_0$) < ∞ or dim($X_0$) < ∞ then $L$ is reducible.
(ii) Let $L_0 \not\subseteq L$, let codim($L_0$) < ∞ and $L$ be related to $L$. If codim($X_0$) < ∞ and $L_0$ has no operator-nilpotent Lie ideals of finite codimension (see Section 4), then $L$ is reducible.

Proof. (i) As $L_0$ and $L$ are related, it follows from Corollaries 5.5 and 7.3 that $L$ has a non-trivial closed invariant subspace of finite codimension (respectively, of finite dimension). As $L \not\subseteq L$ and dim($L$) = ∞, we have from Theorems 6.4 and 7.1(ii) that $L$ is reducible.
(ii) As $L_0 \vartriangleleft L$, $\dim(L_0) = \infty$ and $\text{codim}(X_0) < \infty$, it follows from Theorem 6.4(i) and (ii) that $L$ has a non-trivial closed invariant subspace $Y_0$ and a Lie ideal $C$ contained in $L_0$ such that $CX \subseteq Y_0$ and $\dim(L_0/C) < \infty$.

If $\dim(Y_0) < \infty$ then, as $L$ is related to $L$, it follows from Corollary 7.3 that $L$ has a finite-dimensional invariant subspace.

Let $\dim(Y_0) = \infty$. As $\dim(L_0) < \infty$ and $\dim(L_0/C) < \infty$, we have $\text{codim}(C) < \infty$. Set $C_0 = C$ and consider $C_n$-filtration $\{C_n\}$ of $L$. By Corollary 4.4(iii), $\text{codim}(C_1) < \infty$, so that $C_1 \neq \emptyset$. As $C_1 X \subseteq CX \subseteq Y_0$, we have $C_1 \subseteq [L, C_0, Y_0]$. We also have $[C_1, L_0] \subseteq [C_0, L_0] \subseteq C_0$ and

$$\text{[}[C_1, L_0], L] \subseteq [C_1, [L_0, L]] + [C_1, L, L_0] \subseteq [C_1, L] + [C_0, L_0] \subseteq C_0.$$

Hence $[C_1, L_0] \subseteq C_1$, so that $C_1 \vartriangleleft L_0$. As the Lie ideal $C_1$ is not operator-nilpotent on $X$, we have that, for any $n \geq 1$, $\{0\} \neq C_1^{n+1} X = C_1^n(C_1 X) \subseteq C_1^n Y_0$. Therefore, by Corollary 4.5(ii), $(L, Y_0)$-filtration $\{Y_n\}$ of $X$ is non-trivial. As $Y_0$ is invariant for $L$ and $L$ is related to $L$, we obtain from Corollary 5.5 that $L$ is reducible. □

**Problem 1.** Let $L_0$ be a Lie subalgebra of $L \subseteq B(X)$ of finite codimension and $\dim(L_0) = \infty$. Assume that there is $L$ such that $L_0 \vartriangleleft L$ and $L \vartriangleleft L$. If $L_0$ has a closed invariant subspace of finite codimension or dimension, is $L$ always reducible?

Note that it follows from Theorems 6.4 and 7.1(ii) that in the conditions of Problem 1 $L$ has a closed non-trivial invariant subspace $W$. However, $W$ may have infinite codimension and dimension.

Let $L_0$ be a Lie subalgebra of $B(X)$. Set $\text{Nor}(L_0) = \{A \in B(X): [A, L_0] \subseteq L_0\}$. Then $\text{Nor}(L_0)$ is a Lie subalgebra of $B(X)$ and $L_0 \vartriangleleft \text{Nor}(L_0)$. A closed subspace $Y$ of $X$ is **superinvariant for $L_0$** if it is invariant for $\text{Nor}(L_0)$. Corollary 6.4 and Theorem 7.1(ii) yield

**Corollary 8.2.** Let $L_0$ be a Lie subalgebra of $B(X)$ and $\dim(L_0) = \infty$. If $L_0$ has a closed invariant subspace $X_0$ of finite codimension (respectively, finite dimension), then it has a non-trivial superinvariant subspace $W$ such that $W \subseteq X_0$ (respectively, $W \supseteq X_0$).

The example below shows that the conditions $\text{codim}(X_0) < \infty$ and, respectively, $\dim(X_0) < \infty$ in Corollary 8.2 cannot be dropped.

**Example 8.3.** (Theorems 4.2 and 4.3(ii) of [9].) Let $H$ be an infinite-dimensional Hilbert space and $S$ be a closed unbounded self-adjoint operator on $H$ with domain $D(S)$ dense in $H$. Let $A(S)$ be the associative subalgebra of $B(H)$ of all operators $A$ such that $AD(S) \subseteq D(S)$ and that the operator $[S, A]$ on $D(S)$ extends to a bounded operator $[\hat{S}, \hat{A}]$ on $H$. Set $X = H \oplus H$,

$$E = \begin{pmatrix} 0 & 1_H \\ 0 & 0 \end{pmatrix}, \quad L_0 = \left\{ \begin{pmatrix} A & i[S, A] \\ 0 & A \end{pmatrix} : A \in A(S) \right\}.$$

$$X_t = \left\{ \begin{pmatrix} iSx + tx \\ x \end{pmatrix} : x \in D(S) \right\}$$

for $t \in \mathbb{C}$. Then $L_0$ is an associative subalgebra of $B(X)$, $\text{Nor}(L_0) = L_0 + \mathbb{C} E$ and $L_0$ is a Lie ideal of codimension 1 in $\text{Nor}(L_0)$. Non-trivial invariant subspaces of $L_0$ are $H \oplus \{0\}$ and all $X_t$, for $t \neq 0$. However, $X_t$ is reducible for $t = 0$. □
for \( t \in \mathbb{C} \). On the other hand, \( \text{Nor}(\mathcal{L}_0) \) has a unique non-trivial invariant subspace \( H \oplus \{0\} \). It neither contains, nor is contained in any subspace \( X_t \).

### 8.2. Operator Lie algebras with Lie subalgebras of small codimension

In order to illustrate the situation we will consider Lie algebras of operators with Lie subalgebras \( \mathcal{L}_0 \) satisfying \( \text{codim}(\mathcal{L}_0) \leq 3 \).

**Corollary 8.4.** Let \( \mathcal{L} \) be a finite- or infinite-dimensional Lie subalgebra of \( \mathcal{B}(X) \). Let \( \mathcal{L}_0 \) be a Lie subalgebra of codimension 1 in \( \mathcal{L} \) and let \( \mathcal{L}_0 \) have a non-trivial closed invariant subspace of finite codimension (respectively, dimension).

(i) If \( \mathcal{L}_0 \not\triangleleft \mathcal{L} \), then \( \mathcal{L} \) has a non-trivial closed invariant subspace of finite codimension (respectively, dimension).

(ii) If \( \mathcal{L}_0 \triangleleft \mathcal{L} \) and \( \dim(\mathcal{L}_0) = \infty \), then \( \mathcal{L} \) is reducible.

**Proof.** If \( \mathcal{L}_0 \not\triangleleft \mathcal{L} \), the representation \( \theta \) of \( \mathcal{L}_0 \) on \( \mathcal{L}/\mathcal{L}_0 \) is non-degenerate and (i) follows from Theorems 5.4 and 7.1(i). Part (ii) follows from Theorems 6.4 and 7.1(ii). \( \square \)

Set \( \mathcal{L} = \mathcal{L}_0 + [\mathcal{L}_0, \mathcal{L}] \). Then \( \mathcal{L}_0 \subseteq \mathcal{L} \subseteq \mathcal{L} \). If \( \dim(\mathcal{L}/\mathcal{L}_0) = 1 \) then \( \mathcal{L} \) is a Lie subalgebra of \( \mathcal{L} \).

**Corollary 8.5.** Let \( \text{codim}(\mathcal{L}_0) = 2 \) and let \( \mathcal{L}_0 \) have a non-trivial closed invariant subspace of finite codimension (respectively, dimension).

(i) If \( \mathcal{L}_0 = \mathcal{L} \) and \( \dim(\mathcal{L}_0) = \infty \), then \( \mathcal{L} \) is reducible.

(ii) Let \( \dim(\mathcal{L}/\mathcal{L}_0) = 1 \).

(1) If \( \mathcal{L}_0 \not\triangleleft \mathcal{L} \) and \( \mathcal{L} \not\triangleleft \mathcal{L} \), then \( \mathcal{L} \) has a non-trivial closed invariant subspace of finite codimension (respectively, dimension).

(2) If \( \mathcal{L}_0 \not\triangleleft \mathcal{L} \), \( \mathcal{L} \triangleleft \mathcal{L} \) and \( \dim(\mathcal{L}_0) = \infty \), then \( \mathcal{L} \) is reducible.

(3) If \( \mathcal{L}_0 \triangleleft \mathcal{L} \), \( \mathcal{L} \not\triangleleft \mathcal{L} \), \( \dim(\mathcal{L}_0) = \infty \) and \( \mathcal{L}_0 \) has no operator-nilpotent Lie ideals of finite codimension (see Section 4), then \( \mathcal{L} \) is reducible.

(iii) If \( \mathcal{L} = \mathcal{L} \) then \( \mathcal{L} \) has a non-trivial closed invariant subspace of finite codimension (respectively, dimension).

**Proof.** Part (i) follows from Theorems 6.4 and 7.1(ii). In (ii)(1) and (iii) \( \mathcal{L}_0 \) is related to \( \mathcal{L} \), so that their proofs follow from Corollaries 5.5 and 7.3. In (ii)(2) \( \mathcal{L}_0 \) is related to \( \mathcal{L} \) and in (ii)(3) \( \mathcal{L} \) is related to \( \mathcal{L} \). Hence their proofs follow from Theorem 8.1. \( \square \)

If \( \dim(\mathcal{L}_0) < \infty \) in Corollary 8.5(ii)(2), then \( \mathcal{L} \) may be irreducible. Indeed, if in Example 5.8 \( \mathcal{L} = \mathcal{A} \) and \( \mathcal{L}_0 = \{ \mathcal{A}_a; a \in M_2(\mathbb{C}), a_{21} = 0 \} \), then \( \text{codim}(\mathcal{L}_0) = 2 \). \( \mathcal{L}_0 \) has a non-trivial closed invariant subspace of finite codimension, \( \mathcal{L}_0 \) is related to \( \mathcal{L} \) and \( \mathcal{L} \triangleleft \mathcal{L} \). However, \( \dim(\mathcal{L}_0) = 3 < \infty \) and \( \mathcal{L} \) is irreducible.

We will now briefly consider the case when \( \text{codim}(\mathcal{L}_0) = 3 \) and \( \mathcal{L}_0 \) has a non-trivial closed invariant subspace of finite codimension or dimension.

1. Let \( \mathcal{L}_0 = \mathcal{L} \). Then \( \mathcal{L}_0 \triangleleft \mathcal{L} \). If \( \dim(\mathcal{L}_0) = \infty \), it follows from Theorems 6.4 and 7.1(ii) that \( \mathcal{L} \) is reducible.
2. Let \( \dim(L/L_0) = 1 \). Then \( L \) is a Lie subalgebra of \( L \) of codimension 1.

(i) If \( L_0 \not\triangleleft L \), then, by Corollary 8.4(i), \( L \) has a non-trivial closed invariant subspace of finite codimension or dimension. As \( \dim(L) = 2 \), the problem is reduced to the case considered in Corollary 8.5.

(ii) If \( L_0 \triangleleft L \) and \( \dim(L_0) = \infty \) then, by Corollary 8.4(ii), \( L \) has a closed non-trivial invariant subspace \( Y \). If \( Y \) has finite codimension or dimension, the problem is reduced to the case considered in Corollary 8.5. Otherwise, the problem is open.

3. Let \( \dim(L/L_0) = 2 \). If \( L \) is a Lie subalgebra of \( L \), then in Corollary 8.5 we have various conditions when \( L \) has a closed non-trivial invariant subspace \( Y \). If \( Y \) has finite codimension or dimension then, by Corollary 8.4, \( L \) is reducible.

4. If \( L = L_0 \) then \( L_0 \) and \( L \) are related, so that \( L \) has a non-trivial closed invariant subspace of finite codimension or dimension.

8.3. Invariant subspaces of associative algebras and semigroups of operators

The situation turns out to be much simpler if instead of a Lie algebra with a reducible Lie subalgebra of finite codimension one considers an associative algebra \( A \) of operators with a reducible subalgebra \( A_0 \) of finite codimension or such that \( A \) is finitely generated as a module over \( A_0 \). In this subsection we will obtain several (possibly known) results about invariant subspaces of \( A \).

If \( I \neq \{0\} \) is a reducible two-sided ideal of \( A \), then (see [22, Lemma 5]) \( A \) is reducible. Indeed, if \( Y \) is a non-trivial subspace invariant for \( I \), the closed spaces \( IY \) and \( \operatorname{Ker}(I) \) are invariant for \( A \). At least one of them is non-trivial, since \( \operatorname{Ker}(I) \neq X \) and, if \( \operatorname{Ker}(I) = \{0\} \) then \( \{0\} \neq IY \subseteq Y \neq X \).

**Proposition 8.6.** Let \( A \) be an associative subalgebra of \( B(X) \) and let \( A_0 \) be a reducible subalgebra of \( A \). If \( \dim(A/A_0) < \infty \) then \( A \) is reducible.

**Proof.** Let \( \dim(A) < \infty \). For each \( x \notin \operatorname{Ker}(A) \), \( Ax \) is a non-trivial, finite-dimensional invariant subspace of \( A \).

Let \( \dim(A) = \infty \). As \( \dim(A/A_0) < \infty \), it follows from [12, Lemma 2.1] that \( A_0 \) contains a two-sided ideal \( I \) of \( A \) such that \( \dim(A/I) < \infty \). Thus \( I \neq \{0\} \). Since \( A_0 \) is reducible, \( I \) is reducible. By the argument before the proposition, \( A \) is reducible. \( \square \)

We will now consider the case when an associative algebra \( A \) is finitely generated as a module over a reducible subalgebra.

**Proposition 8.7.** Let \( A_0 \) be a subalgebra of an associative algebra \( A \subseteq B(X) \) and let \( X_0 \) be a non-trivial closed subspace invariant for \( A_0 \).

(i) Suppose that there are \( e_1, \ldots, e_n \) in \( B(X) \) such that \( A \subseteq \sum_{i=1}^n A_0 e_i \). If \( \dim(X_0) < \infty \), then \( A \) has a closed invariant subspace \( X_1 \subseteq X_0 \) with \( \dim(X_1) < \infty \).

(ii) Suppose that there are \( e_1, \ldots, e_n \) in \( B(X) \) such that \( A \subseteq \sum_{i=1}^n e_i A_0 \). If \( \dim(X_0) < \infty \), then \( A \) has a closed invariant subspace \( X_1 \) with \( \dim(X_1) < \infty \) and \( X_0 \subseteq X_1 \).
Proof. (i) As in (4.1), set $X_1 = \{x \in X_0: ax \in X_0, a \in A\}$. By Lemma 4.2(iv), $X_1$ is a closed subspace of $X_0$. For $x \in X_1$ and $b \in A$, $a(bx) = (ab)x \in X_0$ for all $a \in A$. Hence $bx \in X_1$, so that $X_1$ is invariant for $A$. The closed subspaces $L_i = \{x \in X_0: ei \in X_0\}$ have finite codimension in $X_0$ and, hence in $X$. Therefore $Y = \bigcap L_i$ has finite codimension in $X$. As $eiY \subseteq X_0$, we have $AY \subseteq \sum_i A_0 e_i Y \subseteq A_0 X_0 \subseteq X_0$ whence $Y \subseteq X_1$. Hence $\text{codim}(X_1) < \infty$.

(ii) The subspace $X_1 = X_0 + AX_0$ is invariant for $A$ and contains $X_0$. Since $X_1 \subseteq X_0 + \sum_{i=1}^n (e_i X_0)$, we have that $X_1$ is finite-dimensional. □

We will apply now Proposition 8.7 to the Invariant Subspace problem for semigroups of bounded operators (see the excellent discussion of this subject in [16]). For semigroups containing compact operators, we refer the reader to [18,21].

Corollary 8.8. Let $H$ be a subsemigroup of a semigroup $G \subseteq B(X)$. Let $H$ have a non-trivial closed invariant subspace $X_0$.

(i) If $\text{codim}(X_0) < \infty$ and there are $e_1, \ldots, e_n \in B(X)$ such that $G \subseteq H \cup He_1 \cup \cdots \cup He_n$, then $G$ has a closed invariant subspace $X_1 \subseteq X_0$ with $\text{codim}(X_1) < \infty$.

(ii) If $\text{dim}(X_0) < \infty$ and there are $e_1, \ldots, e_n \in B(X)$ such that $G \subseteq H \cup e_1 H \cup \cdots \cup e_n H$, then $G$ has a closed invariant subspace $X_1$ with $\text{dim}(X_1) < \infty$ and $X_0 \subseteq X_1$.

Proof. Let $A = \text{span}(G)$, $A_0 = \text{span}(H)$. Then $A \subseteq \sum_{i=1}^n A_0 e_i$ in (i) and $A \subseteq \sum_{i=1}^n e_i A_0$ in (ii). It only remains to apply Proposition 8.7. □

9. Lie ideals in Banach Lie algebras

In this section we consider infinite-dimensional complex Banach Lie algebras. Recall that a complex Lie algebra $L$ is called a Banach Lie algebra, if it is a Banach space in some norm $\| \cdot \|$ that satisfies

$$\| [a, b] \| \leq D \| a \| \| b \|$$

for some $D > 0$ and all $a, b \in L$. (9.1)

For example, closed Lie subalgebras of $B(X)$ are Banach Lie algebras.

Throughout this section we assume that $L$ has a proper closed Lie subalgebra $L_0$ of finite codimension and study the question when $L$ has Lie ideals. We will write $L_0 \triangleleft L$ when $L_0$ is a closed Lie ideal of $L$. Set

$$M = \text{ad}(L) \quad \text{and} \quad M_0 = \text{ad}(L_0).$$

Then $M_0 \subseteq M \subseteq B(L)$ and $\dim(M/M_0) \leq \dim(L/L_0)$; whence either $M_0 = M$, or $M_0$ has finite non-zero codimension in $M$. If the representation $\theta$ of $L_0$ on $L/L_0$ is non-degenerate then, by (5.2), $L = L_0 + [L_0, L]$. As $\text{ad}([a, b]) = [\text{ad}(a), \text{ad}(b)]$, we have

$$M = M_0 + [M_0, M].$$

Hence, if $M_0 \neq M$ then the representation of $M_0$ on the quotient space $M/M_0$ is non-degenerate. It implies that if $L_0$ is related to $L$, then $M_0$ is related to $M$.

The next result gives a partial answer to the question raised in [10].
Corollary 9.1. Let \( \mathcal{L}_0 \) be related to \( \mathcal{L} \). Then \( \mathcal{L} \) has a closed Lie ideal of finite codimension contained in \( \mathcal{L}_0 \).

**Proof.** Set \( X = \mathcal{L} \) and \( X_0 = \mathcal{L}_0 \). If \( \mathcal{M}_0 = \mathcal{M} \), \( \mathcal{L}_0 \) is the required Lie ideal. If \( \mathcal{M}_0 \neq \mathcal{M} \), then \( \mathcal{M}_0 \) is related to \( \mathcal{M} \subseteq B(X) \), \( X_0 \) is invariant for \( \mathcal{M}_0 \) and \( \text{codim}(X_0) < \infty \). By Corollary 5.5(ii), there is a non-trivial closed subspace \( W \) of \( X_0 \) invariant for \( \mathcal{M} \) and \( \text{codim}(W) < \infty \). Hence \( W \) is the required Lie ideal. \( \square \)

Amayo showed in Lemma 2.2(c) of [1] that, if \( \mathcal{L} \) is a finite-dimensional Lie algebra over field of characteristic 0 and \( \text{codim}(\mathcal{L}_0) = 1 \), then \( \mathcal{L}_0 \) contains a Lie ideal \( \mathcal{K} \) of \( \mathcal{L} \) such that \( \text{dim}(\mathcal{L}/\mathcal{K}) \leq 3 \). The extension of this result to Banach Lie algebras obtained in [10] is an easy consequence of Corollary 9.1.

Corollary 9.2. Let \( \mathcal{L}_0 \) be a closed Lie subalgebra of \( \mathcal{L} \) and let \( \text{codim}(\mathcal{L}_0) = 1 \). Then \( \mathcal{L}_0 \) contains a closed Lie ideal \( \mathcal{K} \) of \( \mathcal{L} \) such that \( \text{codim}(\mathcal{K}) \leq 3 \).

**Proof.** If \( \mathcal{L}_0 \) is not a Lie ideal of \( \mathcal{L} \) then \( \mathcal{L}_0 \) is non-degenerate in \( \mathcal{L} \). By Corollary 9.1, \( \mathcal{L} \) has a closed Lie ideal \( \mathcal{W} \) of finite codimension contained in \( \mathcal{L}_0 \). Applying the result of Amayo stated above to \( \mathcal{L}/\mathcal{W} \), we obtain that \( \mathcal{L}_0 \) contains a closed Lie ideal \( \mathcal{K} \) of \( \mathcal{L} \) such that \( \text{codim}(\mathcal{K}) \leq 3 \). \( \square \)

Let us show now that the above result extends to the case when \( \text{codim}(\mathcal{L}_0) = 2 \).

Corollary 9.3. Let \( \mathcal{L}_0 \) be a closed Lie subalgebra of \( \mathcal{L} \) and let \( \text{codim}(\mathcal{L}_0) = 2 \). Then \( \mathcal{L} \) has a non-trivial closed Lie ideal \( \mathcal{K} \) of finite codimension. Moreover, if \( \mathcal{L}_0 + [\mathcal{L}_0, \mathcal{L}] \neq \mathcal{L} \) then \( \text{codim}(\mathcal{K}) \leq 3 \).

**Proof.** If \( \mathcal{L}_0 \triangleleft \mathcal{L} \), the result holds. Let \( \mathcal{L}_0 \not\triangleleft \mathcal{L} \). Set \( L = \mathcal{L}_0 + [\mathcal{L}_0, \mathcal{L}] \). Then \( L \neq \mathcal{L}_0 \).

If \( L = \mathcal{L} \) then \( \mathcal{L}_0 \) and \( \mathcal{L} \) are related. By Corollary 9.1, \( \mathcal{L} \) has a closed Lie ideal of finite codimension contained in \( \mathcal{L}_0 \).

Let \( L \neq \mathcal{L} \). Since codimension of \( \mathcal{L}_0 \) in \( L \) is 1, \([L, L] = [\mathcal{L}_0, L] \subseteq [\mathcal{L}_0, \mathcal{L}] \subseteq L \). Thus \( L \) is a Lie subalgebra of \( \mathcal{L} \). Since \( \text{codim}(L) = 1 \), the result follows from Corollary 9.2. \( \square \)

Let \( \text{codim}(\mathcal{L}_0) = 3 \) and set \( L = \mathcal{L}_0 + [\mathcal{L}_0, \mathcal{L}] \). If \( \mathcal{L}_0 \) is not a Lie ideal of \( \mathcal{L} \), then either \( L = \mathcal{L} \), or \( \text{codim}(L) = 2 \), or \( \text{codim}(L) = 1 \).

If \( L = \mathcal{L} \) then \( \mathcal{L}_0 \) is related to \( \mathcal{L} \) and, by Corollary 9.1, \( \mathcal{L}_0 \) contains a Lie ideal of \( \mathcal{L} \) of finite codimension. Let \( \text{codim}(L) = 2 \). As \( \mathcal{L}_0 \) is closed and has codimension 1 in \( L \), \( L \) is a closed Lie subalgebra. By Corollary 9.3, \( \mathcal{L} \) has a non-trivial closed Lie ideal of finite codimension.

Let \( \text{codim}(L) = 1 \). If \( L \) is a Lie algebra, then, by Corollary 9.2, \( \mathcal{L} \) has a non-trivial closed Lie ideal of finite codimension. However, if \( L \) is not a Lie algebra, the question is open.

**Problem 2.** Let \( 3 \leq \text{codim}(\mathcal{L}_0) < \infty \). Does \( \mathcal{L} \) always have a Lie ideal of finite codimension?

We will consider now examples of Banach Lie algebras \( \mathcal{L} \) with closed Lie subalgebras \( \mathcal{L}_0 \) of finite codimension such that \( \mathcal{L} \) has non-trivial closed Lie ideals but none of them lies in \( \mathcal{L}_0 \). Let \( L \) be a closed Lie subalgebra of the algebra \( B(X) \) of all bounded operators on a Banach space \( X \). Denote by \( \mathcal{L} \) the direct sum \( \mathcal{L} = L \oplus X \) and endow it with the following binary operation:

\[
[(a; x), (b; y)] = ([a, b]; ay - bx), \quad \text{for} \ a, b \in L \text{ and } x, y \in X. \quad (9.2)
\]
Direct calculations (see [4, Chapter I, 1, 8, Example 2]) show that \( L \) is a Lie algebra. It is a Banach space with respect to the norm \( \| (a; x) \| = \max \{ |a|, \|x\| \} \). Note that \( L \) is a Banach Lie algebra, as

\[
\| [(a; x), (b; y)] \| = \| [(a, b); ay - bx] \| \leq \max \{ 2 \|a\| \|b\|, \|a\| \|y\| + \|b\| \|x\| \} \\
\leq 2 \max \{ \|a\|, \|x\| \} \max \{ \|b\|, \|y\| \} = 2 \| (a; x) \| \| (b; y) \|.
\]

**Lemma 9.4.** Let a closed Lie subalgebra \( L \subseteq B(\mathfrak{X}) \) be irreducible. Then each closed non-zero Lie ideal of the Banach Lie algebra \( L = L \oplus \mathfrak{X} \) is of the form \( K = L_0 \oplus \mathfrak{X} \), where \( L_0 \) is a Lie ideal of \( L \).

**Proof.** Set \( Y = \{ y \in \mathfrak{X}: (0; y) \in K \} \). Then \( Y \) is a linear subspace of \( \mathfrak{X} \). Let \( 0 \neq (a; x) \in K \). If \( a = 0 \) then \( 0 \neq x \in Y \). If \( a \neq 0 \) then, by (9.2), \( [(a; x), (0; z)] = (0; az) \in K \) for all \( z \in \mathfrak{X} \). Thus \( Y \neq \{0\} \). As \( K \) is closed, \( Y \) is closed. For all \( y \in Y \) and \( a \in L \), \( [(a; 0), (0; y)] = (0; ay) \in K \). Hence \( ay \in Y \), so \( Y \) is invariant for \( L \). Thus \( Y = \mathfrak{X} \). Hence \( \{0\} \oplus \mathfrak{X} \subseteq K \) which concludes the proof. \( \square \)

**Corollary 9.5.** Let a closed Lie subalgebra \( L \) of \( B(\mathfrak{X}) \) be irreducible (for example, let \( \mathfrak{X} = l_1 \) and \( L = C \mathbb{C} \), where \( e \) is a bounded operator on \( \mathfrak{X} \) that has no non-trivial closed invariant subspaces (see [17])). Then, for each closed subspace \( Y \subseteq \mathfrak{X} \), \( L_0 = \{ (0; y): y \in Y \} \) is a closed Lie subalgebra of the Banach Lie algebra \( L = L \oplus \mathfrak{X} \) and \( 2 \leq \text{codim}(L_0) = \text{codim}(Y) + \dim(L) \). However, \( L_0 \) contains no non-trivial closed Lie ideals of \( L \).

Suppose that \( L_0 \subseteq L \subseteq B(\mathfrak{X}) \), \( L_0 \) has an invariant subspace \( X_0 \) of finite codimension and \( \text{codim}(L_0) < \infty \), but \( L_0 \) is not related to \( L \). In Example 5.8 we considered the case when \( \dim(L_0) < \infty \) and \( L \) is irreducible. Below we consider an example when \( \dim(L_0) = \infty \) and \( L \) is reducible, but its invariant subspaces do not lie in \( X_0 \) (cf. Corollary 5.5).

**Example 9.6.** Let \( \mathfrak{X} = l_1 \), let \( e \) be a bounded irreducible operator on \( \mathfrak{X} \) and let \( Y \) be a proper closed subspace of \( \mathfrak{X} \) of finite codimension. As above, consider the Banach Lie algebra \( L = C \mathbb{C} e \oplus \mathfrak{X} \). Then \( L_0 = \{ (0) \oplus Y \} \) is a closed Lie subalgebra of \( L \) of finite codimension and \( \dim(L_0) = \infty \). Since \( L_0 \) is contained in the Lie ideal \( \{0\} \oplus \mathfrak{X} \) of \( L \), it is not related to \( L \).

As \( \text{Ker}(\text{ad}) = \{0\} \), \( L \) is isomorphic to the Lie subalgebra \( \text{ad}(L) \) of the algebra of all bounded operators on the Banach space \( \mathfrak{X} = L \). Then the Lie subalgebra \( \text{ad}(L_0) \) of finite codimension in \( \text{ad}(L) \) has a closed invariant subspace \( X_0 = L_0 \) of finite codimension in \( \mathfrak{X} \). By Lemma 9.4, \( \text{ad}(L) \) has only one non-trivial closed invariant subspace \( \{0\} \oplus \mathfrak{X} \) and it does not lie in \( X_0 \).

We will now consider various cases when \( L_0 \) contains a closed Lie subalgebra \( K_0: K_0 \subseteq L_0 \subseteq L \).

**Corollary 9.7.** Let \( L_0 \) be related to \( L \) and let \( K_0 \subset L_0 \).

(i) If \( \dim(L_0/K_0) < \infty \), then \( L \) has a closed Lie ideal of finite codimension in \( K_0 \).

(ii) If \( \dim(K_0) < \infty \), then \( L \) has a finite-dimensional Lie ideal that contains \( K_0 \).

**Proof.** Set \( X = L \) and \( X_0 = K_0 \). Then the Lie subalgebra \( M_0 = \text{ad}(L_0) \) is related to \( M = \text{ad}(L) \) or \( M_0 = M \); and \( X_0 \) is invariant for \( M_0 \). The case \( M_0 = M \) is evident.
Theorem 9.9. Let \( \mathcal{L}_0 \) be a non-trivial closed Lie subalgebra of \( \mathcal{L} \). A Lie ideal of \( \mathcal{L} \) is a closed Lie ideal of \( \mathcal{L} \).

Proof. If \( \dim(\mathcal{L}/K_0) < \infty \) then \( \text{codim}(X_0) < \infty \). It follows from Corollary 5.5(ii) that there is a closed subspace \( W \) of \( X_0 \) invariant for \( M \) and \( \text{codim}(W) < \infty \). Hence \( W \) is the required Lie ideal.

(ii) If \( \dim(K_0) < \infty \) then \( \dim(X_0) < \infty \). We have from Corollary 7.3 that \( M \) has an invariant subspace \( W, X_0 \subseteq W \) and \( \text{dim}(W) < \infty \). Hence \( W \) is the required Lie ideal. \( \square \)

Corollary 9.8. Let \( \mathcal{L}_0 \) be a closed, non-commutative Lie ideal of \( \mathcal{L} \) and \( \dim(\mathcal{L}_0) = \infty \). Then \( \mathcal{L} \) has a non-trivial closed Lie ideal \( \mathcal{K} \leq \mathcal{L}_0 \) in the following cases:

(i) \( \mathcal{L}_0 \) contains a proper closed Lie subalgebra \( K_0 \) related to \( \mathcal{L}_0 \);
(ii) \( \mathcal{L}_0 \) contains a proper closed Lie ideal \( K_0 \) and \( \dim(\mathcal{L}_0/K_0) < \infty \);
(iii) \( \mathcal{L}_0 \) contains a proper closed Lie ideal \( K_0 \) and \( \dim(\mathcal{L}_0) < \infty \).

Proof. Let \( C \) be the center of \( \mathcal{L}_0 \). As \( \mathcal{L}_0 \) is non-commutative, \( C \neq \mathcal{L}_0 \). If \( C \neq \{0\} \), then \( \mathcal{K} = C \) is the required Lie ideal of \( \mathcal{L} \). Indeed, as \( \mathcal{L}_0 \) is a Lie ideal of \( \mathcal{L} \),

\[
[[C, \mathcal{L}], \mathcal{L}_0] \subseteq [C, [\mathcal{L}, \mathcal{L}_0]] + [[C, \mathcal{L}_0], \mathcal{L}] = \{0\}.
\]

Assume now that \( C = \{0\} \). Set \( X = \mathcal{L}_0 \), \( N = \text{ad}(K_0) \mid X \), \( N_0 = \text{ad}(\mathcal{L}_0) \mid X \) and \( \mathcal{N} = \text{ad}(\mathcal{L}) \mid X \subseteq B(X) \). Then \( \dim(X) = \dim(N_0) = \infty \), \( N_0 \triangleleft \mathcal{N} \) and \( X_0 = K_0 \neq X \) is a closed subspace.

(i) The subspace \( X_0 \) is invariant for \( \mathcal{N} \) and \( \text{codim}(X_0) < \infty \). As \( \mathcal{N} \) is related to \( N_0 \), it follows from Corollary 5.5(ii) that there is a closed subspace \( W \) of \( X_0 \) invariant for \( N_0 \) and \( \text{codim}(W) < \infty \). Applying now Theorem 6.4(i), we obtain that \( \mathcal{N} \) has a non-trivial closed invariant subspace \( \mathcal{K} \). Hence \( \mathcal{K} \) is a Lie ideal of \( \mathcal{L} \) and \( \mathcal{K} \leq \mathcal{L}_0 \).

(ii) The subspace \( X_0 \) is invariant for \( \mathcal{N}_0 \) and \( \text{codim}(X_0) < \infty \). As \( \mathcal{N}_0 \triangleleft \mathcal{N} \) and \( \dim(N_0) = \infty \), it follows from Theorem 6.4(i) that \( \mathcal{N} \) has a non-trivial closed invariant subspace \( \mathcal{K} \) of \( X_0 \). Hence \( \mathcal{K} \) is a closed Lie ideal of \( \mathcal{L} \) and \( \{0\} \neq \mathcal{K} \leq \mathcal{L}_0 \).

(iii) The subspace \( X_0 \) is invariant for \( \mathcal{N}_0 \) and \( \dim(X_0) < \infty \). Replacing the pair \( \mathcal{L}_0 \triangleleft \mathcal{L} \) in Theorem 7.1(ii) by \( \mathcal{N}_0 \triangleleft \mathcal{N} \), we obtain that there is a non-trivial closed subspace \( \mathcal{K} \) invariant for \( \mathcal{N} \). Hence \( \mathcal{K} \) is a Lie ideal of \( \mathcal{L} \). \( \square \)

A linear map \( \delta \) on \( \mathcal{L} \) is a bounded derivation if there is \( C > 0 \) such that \( \|\delta(a)\| \leq C\|a\| \), for all \( a \in \mathcal{L} \), and \( \delta([a, b]) = [\delta(a), b] + [a, \delta(b)] \), for \( a, b \in \mathcal{L} \). Denote by \( \mathcal{D}(\mathcal{L}) \) the set of all bounded derivations on \( \mathcal{L} \). A Lie ideal of \( \mathcal{L} \) is called characteristic if it is invariant for all \( \delta \in \mathcal{D}(\mathcal{L}) \).

Theorem 9.9. Let \( \mathcal{L} \) be an infinite-dimensional, non-commutative Banach Lie algebra and \( \mathcal{L}_0 \) be a non-trivial closed Lie subalgebra of \( \mathcal{L} \). Then \( \mathcal{L} \) has a non-trivial closed characteristic Lie ideal \( W \) if one of the following conditions holds:

(i) \( \mathcal{L}_0 \) is related to \( \mathcal{L} \).
(ii) \( \mathcal{L}_0 \triangleleft \mathcal{L} \) and \( \text{codim}(\mathcal{L}_0) < \infty \).
(iii) \( \mathcal{L}_0 \triangleleft \mathcal{L} \) and \( \dim(\mathcal{L}_0) < \infty \).

Proof. Replacing \( (\mathcal{K}_0, \mathcal{L}_0, \mathcal{L}) \) in Corollary 9.8 by \( (\mathcal{L}_0, \mathcal{L}, \mathcal{D}(\mathcal{L})) \), we obtain the proof of the theorem. \( \square \)
In many cases a Banach Lie algebra can have a big variety of closed Lie ideals of finite codimension but no characteristic ideals. For example, if $\mathcal{L}$ is commutative then each closed subspace of $\mathcal{L}$ is a Lie ideal and each bounded operator on $\mathcal{L}$ is a derivation.

References