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A note on the theorems of M.G. Krein and L.A. Sakhnovich on continuous analogs of orthogonal polynomials on the circle☆

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Abstract

Continuous analogs of orthogonal polynomials on the circle are solutions of a canonical system of differential equations, introduced and studied by Krein and recently generalized to matrix systems by Sakhnovich. We prove that the continuous analogs of the adjoint polynomials converge in the upper half-plane in the case of L^2 coefficients, but in general the limit can be defined only up to a constant multiple even when the coefficients are in L^p for any $p > 2$, the spectral measure is absolutely continuous and the Szegö–Kolmogorov–Krein condition is satisfied. Thus, we point out that Krein's and Sakhnovich's papers contain an inaccuracy, which does not undermine known implications from these results.

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1. Introduction

Orthogonal polynomials on the unit circle have interesting features that relate properties of their spectral measure to the properties of coefficients of generating recur-

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sive formulas (see Section 2 for more details). The present paper deals with continuous analogs of such polynomials.

The one-dimensional analogs were introduced by Krein in [K]. They provide, in a sense, a generalization of the Fourier transform from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \tau)$. Here τ is a Borel spectral measure on $\mathbb{R}$. In this generalization of the Fourier transform, the usual exponentials $e^{ir\lambda}$ are replaced with $p(r, \lambda)$, the continuous analog of orthogonal polynomials. We consider only "one sided" situation, that is, r is nonnegative and the Fourier transform is from a half-line to the whole line (see Section 3).

Note that the Fourier transform itself is a continuous analog of the expansion into the Fourier series, insofar as

$$\{e^{ir\lambda} \mid r \in \mathbb{R}_+, \lambda \in \mathbb{R}\}$$

are analogous to

$$\{z^n \mid n \in \mathbb{Z}_+, |z| = 1\}.$$

Similarly,

$$\{p(r, \lambda) \mid r \in \mathbb{R}_+, \lambda \in \mathbb{R}\}$$

are analogous to

$$\{\varphi_n(z) \mid n \in \mathbb{Z}_+, |z| = 1\},$$

orthonormal polynomials of degree n on the unit circle with respect to an arbitrary probability Borel spectral measure τ. To add one more analogy, note that $\varphi_n(z) = z^n$ are the orthogonal polynomials with the normalized Lebesgue measure as the spectral measure.

In [S1,S2,S3,S4,S5] Sakhnovich defined and studied matrix valued continuous analogs of orthogonal polynomials on the unit circle, and generalized Krein's results for this case (see Section 4).

The functions $p(r, \lambda)$, together with the continuous analog $p^*(r, \lambda)$ of the adjoint polynomials, are solutions of a canonical system of differential equations (3.1). The spectral measure τ is uniquely determined by these differential equations. The Krein differential equations are related to the study of the one-dimensional continuous Schrödinger equation [D1,D5,DK2,K]. Also they can be used to solve an important factorization problem in the theory of analytic functions [A,DK1,G,Sz,Si].

As an expository remark, we note that another way to define $p(r, \lambda)$ and $p^*(r, \lambda)$ is by the formulas

$$p(r, \lambda) = e^{ir\lambda} \left(1 - \int_0^r \Gamma_r(s, 0) e^{-is\lambda}\right) ds,$$

$$p^*(r, \lambda) = 1 - \int_0^r \Gamma_r(0, s) e^{is\lambda}\, ds.$$

Here $\Gamma_r(s,t)=\overline{\Gamma_r(t,s)}$ is the resolvent of a positive integral operator S_r, that is

$$\Gamma_r(s,t)+\int_0^r H(s-u)\Gamma_r(u,t)\,\mathrm{d}u = H(s-t),$$

where $H(t)=\overline{H(-t)}$ and

$$S_r f(x) = f(x)+\int_0^r H(x-t)f(t)\,dt.$$

The coefficient $a(r)$ of Eq. (3.1) is $a(r)=\Gamma_r(0,r)$. Usually, the accelerant $H(t)$ is assumed to be continuous to construct the corresponding Krein system with continuous coefficient $a(r)$. In our work, we do not use such a construction, but define $p(r,\lambda)$ and $p^*(r,\lambda)$ as solutions of Krein's canonical system of differential equations (3.1).

If τ' is the density of the absolutely continuous component of the spectral measure, then the Szegö–Kolmogorov–Krein condition

$$\int_{\mathbb{R}} \frac{|\log \tau'(\lambda)|}{1+\lambda^2}\,d\lambda < \infty \tag{1.1}$$

is satisfied if and only if

$$\int_0^\infty |p(r,\lambda)|^2\,dr < \infty \tag{1.2}$$

for $\operatorname{Im}\lambda > 0$. Notice that no assumption on the singular part of τ is made except (3.2).

In the midst of our discussion is the existence of the limit

$$\Pi(\lambda) = \lim_{r\to\infty} p^*(r,\lambda), \tag{1.3}$$

where $\Pi(\lambda)$ is analytic for $\operatorname{Im}\lambda > 0$. Krein pointed out in [K] that if the coefficients are square integrable, then limit (1.3) converges. In Section 5 we prove that this so even in the matrix case, and therefore $\Pi(\lambda)$ is uniquely defined for square integrable coefficients. Section 5 also contains other results related to the convergence of limit (1.3) in the case of Sakhnovich differential equations.

An important relation, which follows from (3.1) and was noted by Krein in [K], is

$$|p^*(r,\lambda)|^2 - |p(r,\lambda)|^2 = 2\operatorname{Im}\lambda \int_0^r |p(s,\lambda)|^2\,ds. \tag{1.4}$$

This a particular case of Lagrange identity, which is an analog of the Christoffel–Darboux formula for orthogonal polynomials (see, for instance, [At]). Thus we must have

$$|\Pi(\lambda)|^2 = 2\,\mathrm{Im}\,\lambda \int_0^\infty |p(r,\lambda)|^2\,dr$$

if the integral converges and limit (1.3) exists.

The existence of limit (1.3) implies the convergence of integrals (1.1) and (1.2), but the converse is not true in general. In Section 6, we prove that there are situations when (1.1) and (1.2) hold, but $\Pi(\lambda)$ has to be defined as a limit of a convergent subsequence. We show that this situation is not “pathological", but can occur even if the spectral measure τ is absolutely continuous with positive continuous density (Theorem 2). In another example (Theorem 3), this happens even though

$$\lim_{r\to\infty} |p^*(r,\lambda)|^2 = |\Pi(\lambda)|^2$$

and the coefficients are in L^p for any $p > 2$. Moreover, the function $\Pi(\lambda)$ cannot be defined uniquely, but only up to a constant factor of absolute value one (up to left multiplication by a unitary matrix in the case of the Sakhnovich theorem).

Note that results of Section 5 apply to the Krein system, since it is a particular case of the Sakhnovich system. Two of the three results are new even for the Krein system. At the same time results of Section 6 are stated for the Krein system, but are applicable for the Sakhnovich system as well.

The fundamental paper [K] presents a number of important results, though it does not contain proofs due to the type of the journal it was published in. Later proofs of Krein’s results were given independently by the author in 1990 (partly published in [T1]) and Sakhnovich in 1998 ([S2–S4]). The main subject of [T1] was to prove that the spectral measure τ is absolutely continuous with probability one if the coefficient $a(r)$ is a random function satisfying certain conditions.

In [T1] the author noted and rectified an inaccuracy in the statement of Krein’s theorem, and gave a proof of the corrected main theorem (see Section 3 for more details). Theorems 2 and 3 in Section 6 prove, in particular, that a part of the statement of the Krein theorem in [K] needs to be revised.

In [S1–S5] Sakhnovich defined and studied matrix valued continuous analogs of orthogonal polynomials on the unit circle, and proved matrix generalizations of Krein’s results. Unfortunately, these works contain the same kind of inaccuracy as [K]. In Section 4, we present the corrected statement, and the corrected part of the proof.

We emphasize that the inaccuracy in the statement of Krein’s and Sakhnovich’s theorems is not significant, and does not undermine known implications from these important results. For instance, if (1.1) and (1.2) hold, then there is the function $\Pi(\lambda)$ which is analytic and has no zeros for $\mathrm{Im}\,\lambda > 0$, and

$$\tau'(\lambda) = \frac{1}{2\pi|\Pi(\lambda)|^2}$$

for Lebesgue almost all $\lambda \in \mathbb{R}$ (there is an analogous matrix version proved by Sakhnovich in [S4]). This result remains unchanged even if limit (1.3) diverges, and the nonuniqueness of $\Pi(\lambda)$ mentioned above takes place.

2. Orthogonal polynomials on the circle

If $\{\varphi_n(z)\}_{n=0}^{\infty}$ are polynomials of degree n, orthonormal on the unit circle with respect to a probability Borel measure τ, then there exists a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ such that the following recurrent relations hold:

$$\begin{aligned} \varphi_{n+1}(z) &= (1-|a_n|^2)^{-1/2}\big(z\varphi_n(z) - \bar{a}_n\varphi_n^*(z)\big),\\ \varphi_{n+1}^*(z) &= (1-|a_n|^2)^{-1/2}\big(\varphi_n^*(z) - a_n z\varphi_n(z)\big), \end{aligned} \tag{2.1}$$

with initial conditions

$$\varphi_0(z) = \varphi_0^*(z) = 1.$$

The auxiliary polynomials $\varphi_n^*(z)$ are adjoint to the orthogonal polynomials $\varphi_n(z)$ in the sense that $\varphi_n^*(z) = \bar{c}_0 z^n + \cdots + \bar{c}_j z^{n-j} + \cdots + \bar{c}_n$ if $\varphi_n(z) = c_0 + \cdots + c_j z^j + \cdots + c_n z^n$.

The so-called circular (reflection, Shur's) parameters $\{a_n\}_{n=0}^{\infty}$ satisfy

$$|a_n| < 1 \tag{2.2}$$

for all n if and only if the measure τ is not concentrated in a finite number of atoms. Conversely, if conditions (2.2) are satisfied, then there exists a unique Borel probability measure τ on the unit circle such that polynomials $\{\varphi_n(z)\}_{n=0}^{\infty}$, defined by (2.1), are orthonormal with respect to τ.

The theory of orthogonal polynomials on the circle was developed by Szegö, Akhiezer, Geronimus et al. ([A,G,Sz]). The following theorem is a combination of results of Szegö, Kolmogorov, Krein and Geronimus (see [G,Si]).

Theorem. *The linear span of* $\{\varphi_n(z)\}_{n=0}^{\infty}$ *is not dense in* L^2_τ *if and only if any of the following five equivalent statements hold*:

(I)

$$\int_0^{2\pi} \log \tau'(e^{i\theta})\, \mathrm{d}\theta > -\infty, \tag{2.3}$$

where τ' *is the density of the absolutely continuous component of* τ *with respect to the Lebesgue measure on the unit circle.*

(II) *There exists at least one* z *in the unit disk* $D = \{z : |z| < 1\}$ *such that*

$$\sum_{n=0}^{\infty} |\varphi_n(z)|^2 < \infty. \tag{2.4}$$

(III) *There exists at least one* $z \in D$ *such that*

$$\liminf_{n\to\infty} |\varphi_n^*(z)| < \infty.$$

(IV) *Series* (2.4) *converges uniformly on compact subsets of* D.

(V) *There exists a function* $\Pi(z)$, *analytic in* D, *such that the limit*

$$\Pi(z) = \lim_{n\to\infty} \varphi_n^*(z) \tag{2.5}$$

is uniformly convergent on compact subsets of D.

Moreover, *statements* (I)–(V) *are equivalent to the condition*

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Note that in (I) the integral is always less than $+\infty$, and that there is no restrictions on the singular part of τ.

3. Krein theorem

In [K] Krein studied the following canonical system of ordinary differential equations:

$$\begin{aligned} \tfrac{d}{dr}\, p\,(r,\lambda) &= \boldsymbol{i}\lambda p(r,\lambda) - \overline{a(r)}\; p^*(r,\lambda), \\ \tfrac{d}{dr}\, p^*(r,\lambda) &= -a(r)\; p(r,\lambda), \end{aligned} \tag{3.1}$$

with the initial conditions

$$p(0,\lambda) = p^*(0,\lambda) = 1.$$

In our paper, we consider only the case when $a(\cdot)$ is continuous on $[0,\infty)$.

There is a Borel measure τ on $\mathbb{R}$, which is called the spectral measure, such that

$$\int_{\mathbb{R}} \frac{1}{1+\lambda^2}\, d\tau(\lambda) < \infty \tag{3.2}$$

and the map $\mathcal{U} : L^2_{[0,\infty)} \to L^2_\tau$ defined by

$$\mathcal{U} f(\lambda) = \int_0^\infty f(r) p(r,\lambda)\, dr \tag{3.3}$$

is an isometry.

A simple example is the situation when $a(r) \equiv 0$ and $\mathcal{U}$ is the usual Fourier transform. In this case τ is the Lebesgue measure normalized by 2π. For a more detailed study see [AR,R,D2–D5,DK2].

Theorem. *The isometry $\mathcal{U}$ is not onto if and only if any of the following five equivalent statements hold*:

(I)
$$\int_{\mathbb{R}} \frac{\log \tau'(\lambda)}{1+\lambda^2}\, d\lambda > -\infty, \tag{3.4}$$

where τ' is the density of the absolutely continuous component of τ with respect to the Lebesgue measure on $\mathbb{R}$.

(II) *There exists at least one λ in the domain $\mathbb{C}^+ = \{\lambda : \operatorname{Im}\lambda > 0\}$ such that*

$$\int_0^\infty |p(r,\lambda)|^2\, dr < \infty. \tag{3.5}$$

(III) *There exists at least one $\lambda \in \mathbb{C}^+$ such that*

$$\liminf_{r\to\infty} |p^*(r,\lambda)| < \infty. \tag{3.6}$$

(IV) *Integral* (3.5) *converges uniformly on compact subsets of* $\mathbb{C}^+$.

(V) *There exists an analytic in $\mathbb{C}^+$ function $\Pi(\lambda)$ and a sequence $r_n \to \infty$ such that the limit*

$$\Pi(\lambda) = \lim_{n\to\infty} p^*(r_n,\lambda) \tag{3.7}$$

converges uniformly on compact subsets of $\mathbb{C}^+$.

Note that in (I) the integral is always less than $+\infty$, and that there is no restrictions on the singular part of τ.

Remark 3.1. This theorem was stated by Krein in [K] without a proof because of the type of the journal it was published in. Parts (III) and (V) of this theorem were not stated in [K] correctly. Namely, it was written as if (I), (II) and (IV) were equivalent to:

(III′) There exists at least one $\lambda \in \mathbb{C}^+$ such that $\sup_{r\geqslant 0} |p^*(r,\lambda)| < \infty$.

(V′) The limit $\Pi(\lambda) = \lim_{r\to\infty} p^*(r,\lambda)$ converges uniformly on compact subsets of $\mathbb{C}^+$.

In Section 6, we present two counterexamples. We refer to this theorem as the Krein theorem because most of the results were stated correctly by Krein in [K], and the rest is correct in spirit despite of a relatively minor mistake. The corrected statement appeared first in [T1].

In [K] Krein noted that if $a \in L^1_{[0,\infty)}$ then (I)–(V) hold and τ is absolutely continuous with positive continuous density. Also, he noted that if $a \in L^2_{[0,\infty)}$ then (I)–(V) as well as (III′) and (V′) hold. The converse of this fact is not true, unlike the case of orthogonal polynomials in Section 2.

In Section 5, we give a proof that if $a(r) \in L^2[0,\infty)$ then (I)–(V) hold, but the result is sharp in the sense of Theorem 2 and Remark 5.1. We also prove two more results related to convergence in (I)–(V). In Section 6, we prove that, in general, $\Pi(\lambda)$ cannot be defined uniquely, but only up to a factor of absolute value one.

4. Sakhnovich theorem

In [S1–S5] Sakhnovich introduced and studied matrix analogs of the Krein system. He considered a system of canonical differential equations

$$\tfrac{d}{dr} Y(r,\lambda) = \boldsymbol{i}\lambda J\mathcal{H}(r)Y(r,\lambda), \quad r \geqslant 0,$$

that can be transformed by a change of variables into a system

$$\begin{aligned} \tfrac{d}{dr} P_1(r,\lambda) &= \boldsymbol{i}\lambda D P_1(r,\lambda) + A_1(r)\, P_1(r,\lambda) + A_2^*(r)\, P_2(r,\lambda), \\ \tfrac{d}{dr} P_2(r,\lambda) &= A_2(r)\, P_1(r,\lambda), \end{aligned} \tag{4.1}$$

with the initial conditions

$$P_1(0,\lambda) = P_2(0,\lambda) = I_m,$$

where $r \in [0,\infty)$, $\lambda \in \mathbb{C}$, and I_m is the $m \times m$ identity matrix. Here D, $P_1(r,\lambda)$, $P_2(r,\lambda)$, $A_1(r)$, $A_2(r)$ are $m \times m$ matrices. It is assumed that $A_1(r) = -A_1^*(r)$, and D is a constant diagonal matrix with positive diagonal entries. Functions $A_1(\cdot)$ and $A_2(\cdot)$ are assumed to be continuous on $[0,\infty)$.

There is a Borel matrix valued measure τ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{1+\lambda^2}\, d\tau(\lambda) < \infty \tag{4.2}$$

and the map $\mathcal{U} : L^2_{[0,\infty)} \to L^2_\tau$ defined by

$$\mathcal{U} f(\lambda) = \int_0^\infty f(r) P_1(r,\lambda)\, dr \tag{4.3}$$

is an isometry.

Theorem. *The following five statements are equivalent*:

(I)

$$\int_{\mathbb{R}} \frac{\log \det \tau'(\lambda)}{1+\lambda^2}\, d\lambda > -\infty. \tag{4.4}$$

where τ' *is the density of the absolutely continuous component of* τ *with respect to the Lebesgue measure on* $\mathbb{R}$,

(II) *There exists at least one* λ *in the domain* $\mathbb{C}^+ = \{\lambda : \operatorname{Im} \lambda > 0\}$ *such that*

$$\int_0^\infty \|P_1(r,\lambda)\|^2\, dr < \infty, \tag{4.5}$$

where $\|\cdot\|$ *is a matrix norm.*

(III) *There exists at least one* $\lambda \in \mathbb{C}^+$ *such that*

$$\liminf_{r\to\infty} \|P_2(r,\lambda)\| < \infty. \tag{4.6}$$

(IV) *Integral* (4.5) *converges uniformly on compact subsets of* $\mathbb{C}^+$.

(V) *There exists an analytic in* $\mathbb{C}^+$ *matrix valued function* $\Pi(\lambda)$ *and a sequence* $r_n \to \infty$ *such that the limit*

$$\Pi(\lambda) = \lim_{n\to\infty} P_2(r_n,\lambda) \tag{4.7}$$

converges uniformly on compact subsets of $\mathbb{C}^+$.

Remark 4.1. This important result was proved by Sakhnovich in [S2,S3,S4]. Unfortunately, parts (III) and (V) of this theorem were not stated in [S2,S3,S4] correctly in that it was written as if (I), (II) and (IV) implied the existence of the limit

$$\Pi(\lambda) = \lim_{r\to\infty} P_2(r,\lambda). \tag{4.8}$$

Despite that, we refer to this theorem as the Sakhnovich theorem because most of the results were stated correctly by Sakhnovich, and the rest is correct in spirit except for a relatively minor mistake.

The precise location of the gap in Sakhnovich's papers is after the proof of the fact that $\lim_{n\to\infty} P_1(t_n,\lambda) = 0$ for a sequence $t_n \to \infty$ (see [S2, formula (1.35)] and [S4, formula (2.10)]). The cited formulas do not imply (4.8). What may seem more surprising is that it does not even imply $\lim_{n\to\infty} P_2(t_n,\lambda) = \Pi(\lambda)$ but only $\lim_{n\to\infty} \|P_2(t_n,\lambda)\| = \|\Pi(\lambda)\|$, as shown in Theorem 3.

Since the Krein system is a particular case of the Sakhnovich system, the counterexamples of Section 6 apply to this situation as well. Also it is easy to construct "true" matrix-valued counterexamples along the lines of Section 6.

In Section 5, show that if $A_2(r) \in L^2[0,\infty)$, then the finite limit (4.8) exists, and so $\Pi(\lambda)$ is unique. In Section 6 we prove that, in general, $\Pi(\lambda)$ cannot be defined uniquely.

Below we give a corrected part of the proof of the Sakhnovich theorem. Following the lines of [S2–S4], we will show that statements (II)–(V) are equivalent. An alternative approach can be found in [T1].

The following is a Lagrange identity, which is an analog of the Christoffel-Darboux formula for orthogonal polynomials (see, for instance, [At]).

Lemma 4.2.

$$P_2^*(r,\lambda_0)P_2(r,\lambda) - P_1^*(r,\lambda_0)P_1(r,\lambda) = \boldsymbol{i}(\overline{\lambda_0} - \lambda)\int_0^r P_1^*(s,\lambda_0)DP_1(s,\lambda)\,ds. \tag{4.9}$$

Proof. Note that the relation is clearly true for $r = 0$. Also, the derivatives with respect to r of both sides of (4.9) coincide because of (4.1).

Proof of a part of the Sakhnovich theorem. Statements (II) and (III) are equivalent because of the relation

$$P_2^*(r,\lambda)P_2(r,\lambda) - P_1^*(r,\lambda)P_1(r,\lambda) = 2\,\mathrm{Im}\,\lambda\int_0^r P_1^*(s,\lambda)DP_1(s,\lambda)\,ds, \tag{4.10}$$

which is a particular case of (4.9).

Clearly, (IV)–(V) imply (II) and (III) because of (4.10). So we have to show that (II) and (III) imply (IV) and (V).

Now assume that (II) and (III) hold for some $\lambda = \lambda_0 \in \mathbb{C}^+$. By (4.1) and (4.10), the family $\{\|P_2(r,\lambda)\| : r \geqslant 0, \lambda \in S\}$ is uniformly bounded from below for any compact $S \subset \mathbb{C}^+$. By (4.6) and Montel's theorem, there exists a sequence $r_n \to \infty$ such that limit (4.7) converges uniformly on compact subsets of $\mathbb{C}^+$. Thus (V) holds, and so does (IV) because of (4.10). $\square$

5. Some convergence results

All the results in this section apply to the Krein system if we set $m = 1$, $D = 1$, $A_1(r) = 0$, $a(r) = -A_2(r)$, $p(r,\lambda) = P_1(r,\lambda)$ and $p^*(r,\lambda) = P_2(r,\lambda)$.

In what follows the matrix norm $\|\cdot\|$ is defined by $\|M\| = \sqrt{\mathrm{Tr}\,M^*M}$.

Note that, even under conditions (1) and (2) of the following theorem, the limit $\lim_{n\to\infty} P_2(r_n,\lambda)$ may not exist by Remark 5.1.

Theorem 1. (i) (1) *Suppose that the equivalent conditions* (I)–(V) *of the Sakhnovich theorem hold, and*

$$\lim_{n\to\infty} P_1(t_n, \lambda_0) = 0$$

for some $t_n \to \infty$ *and* λ_0 *in a nonempty open subset* S *of* $\mathbb{C}^+$. *Then the limits*

$$\begin{aligned} &\lim_{n\to\infty} P_2^*(t_n, \xi) P_2(t_n, \lambda) = \Pi^*(\xi)\Pi(\lambda),\\ &\lim_{n\to\infty} \|P_2(t_n, \lambda)\| = \|\Pi(\lambda)\|,\\ &\lim_{n\to\infty} P_1(t_n, \lambda) = 0, \end{aligned} \tag{5.1}$$

converge uniformly on compact subsets of $\mathbb{C}^+ \times \mathbb{C}^+$ *and* $\mathbb{C}^+$, *respectively*. *Here* $\Pi(\lambda)$ *is an analytic function on* $\mathbb{C}^+$.

(2) *Suppose that the equivalent conditions* (I)–(V) *of the Sakhnovich theorem hold, and*

$$\inf_{\varepsilon>0}\left(\sup_{r\geqslant 0}\int_r^{r+\varepsilon} \|A_2(r)\|\, dr\right) = 0. \tag{5.2}$$

Then the limits

$$\begin{aligned} &\lim_{r\to\infty} P_2^*(r, \xi) P_2(r, \lambda) = \Pi^*(\xi)\Pi(\lambda),\\ &\lim_{r\to\infty} \|P_2(r, \lambda)\| = \|\Pi(\lambda)\|,\\ &\lim_{r\to\infty} P_1(r, \lambda) = 0, \end{aligned} \tag{5.3}$$

converge uniformly on compact subsets of $\mathbb{C}^+ \times \mathbb{C}^+$ *and* $\mathbb{C}^+$, *respectively*.

(3) *Suppose that* $A_2(r) \in L^2[0, \infty)$. *Then conditions* (I)–(V) *of the Sakhnovich theorem hold and, moreover, the limits*

$$\begin{aligned} &\lim_{r\to\infty} P_2(r, \lambda) = \Pi(\lambda),\\ &\lim_{r\to\infty} P_1(r, \lambda) = 0, \end{aligned} \tag{5.4}$$

converge uniformly on compact subsets of $\mathbb{C}^+$.

Remark 5.1. This result is sharp in the sense that there is a real C^∞ coefficient $A_2(r)$, which is in L^p for any $p > 2$, such that statements (I)–(V) of the Sakhnovich theorem do not hold.

Also this result is sharp in a more delicate sense: by Theorem 3 there exists a coefficient $A_2(r)$, which is again in L^p for any $p > 2$, such that $\lim_{r\to\infty} P_1(r,\lambda) = 0$, statements (I)–(V) of the Sakhnovich theorem hold, but the limit $\lim_{r\to\infty} P_2(r,\lambda)$ does not exist. In fact, we show that $\Pi(\lambda)$ cannot be defined uniquely, but only up to a constant factor, even though the limit $\lim_{r\to\infty} \|P_2(r,\lambda)\| = \|\Pi(\lambda)\|$ exists by part 2 of Theorem 1.

Note that in this theorem there is no restriction on the skew-symmetric coefficient $A_1(r)$, except for the usual assumption of continuity.

It was proved in [S2,S3] that if $A_2(r) \in L^1[0,\infty)$, then conditions (I)–(V) of the Sakhnovich theorem hold, and the limits (5.4) converge uniformly on compact subsets of $\mathbb{C}^+ \cup \mathbb{R}$ and $\mathbb{C}^+$, respectively. This fact and statement 3 of Theorem 1 were formulated in [K] for the Krein system. Also, for the Krein system statements 2 and 3 of Theorem 1 are related to the results of [D2].

Proof of 1. Identity (4.9) implies that if (I)–(V) hold and

$$\lim_{n\to\infty} P_1^*(r_n,\lambda_0)P_1(r_n,\lambda) = 0, \tag{5.5}$$

then $\Pi(\lambda)$ satisfies

$$\Pi(\lambda) = \boldsymbol{i}(\overline{\lambda_0} - \lambda)\left(\Pi^*(\lambda_0)\right)^{-1} \int_0^\infty P_1^*(s,\lambda_0)DP_1(s,\lambda)\,ds. \tag{5.6}$$

Let $\lambda_0 \in S$ and $\lambda \in \mathbb{C}^+$. Then using (4.10) at λ_0 and at λ we obtain

$$P_1(t_n,\lambda_0) = o\left(P_2(t_n,\lambda_0)\right)_{n\to\infty},$$

$$P_1(t_n,\lambda) = O(P_2(t_n,\lambda))_{n\to\infty}$$

and therefore

$$P_1^*(t_n,\lambda_0)P_1(t_n,\lambda) = o\left(P_2^*(t_n,\lambda_0)P_2(t_n,\lambda)\right)_{n\to\infty}.$$

Hence we have (5.5) and

$$\lim_{n\to\infty} P_2^*(t_n,\lambda_0)P_2(t_n,\lambda) = \boldsymbol{i}(\overline{\lambda_0} - \lambda)\int_0^\infty P_1^*(s,\lambda_0)DP_1(s,\lambda)\,ds \tag{5.7}$$

by (4.9).

By (4.10) and (5.7), the family of analytic functions $\{P_1(t_n,\lambda)\}_{n\geqslant 1}$ is locally uniformly bounded and so is relatively compact. Thus, any its subsequence has a convergent subsubsequence, and our assumptions imply that its limit has to be zero on

$\mathbb{C}^+$ because it is an analytic function which is zero on an nonempty open set S. Hence $\lim_{n\to\infty} P_1(t_n, \lambda) = 0$ uniformly on compact subsets of $\mathbb{C}^+$.

Therefore, the sequence $\{P_2(t_n, \lambda)\}_{n\geqslant 1}$ is bounded by (4.10) and (5.5), and so has a convergent subsequence. Then we define $\Pi(\lambda)$ as the limit of this subsequence. The right-hand side of (5.7) does not depend on the choice of the subsequence, and so (5.7) extends to $\lambda_0, \lambda \in \mathbb{C}^+$ by analyticity. This implies the first and second limits in (5.1). □

Proof of 2. From (4.1) we have that

$$\begin{aligned}\tfrac{d}{dr}\|P_1(r,\lambda)\|^2 &= Tr\ \tfrac{d}{dr} P_1^* P_1\\ &= Tr(-2\,\mathrm{Im}\,\lambda P_1^* D P_1 + P_1^* A_2 P_2 + P_2^* A_2 P_1)\\ &\geqslant 2\left(\mathrm{Im}\,\lambda \|D\|\,\|P_1(r,\lambda)\|^2 + \|A_2(r)\|\,\|P_1(r,\lambda)\|\,\|P_2(r,\lambda)\|\right) \qquad (5.8)\end{aligned}$$

and also

$$\begin{aligned}\tfrac{d}{dr}\log\left(\|P_1(r,\lambda)\|^2 + \|P_2(r,\lambda)\|^2\right) &= Tr\,\frac{\tfrac{d}{dr}\left(P_1^*P_1 + P_2^*P_2\right)}{\|P_1\|^2 + \|P_2\|^2}\\ = 2Tr\,\frac{-\mathrm{Im}\,\lambda P_1^* D P_1 + P_1^* A_2 P_2 + P_2^* A_2 P_1}{\|P_1\|^2+\|P_2\|^2} &\leqslant 4\|A_2(r)\| \qquad (5.9)\end{aligned}$$

since $\mathrm{Im}\,\lambda > 0$.

Let us assume that $\limsup_{r\to\infty}\|P_1(r,\lambda)\| > 0$ for some $\lambda \in \mathbb{C}^+$. Then there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \|P_1(t_n,\lambda)\| = \delta > 0.$$

Relation (4.10) implies that

$$\lim_{n\to\infty} \|P_2(t_n,\lambda)\| = \gamma > \delta.$$

Then (4.9) and (5.9) implies that for any $\delta_0, \varepsilon_0 > 0$ there exist $C > 0$ such that

$$\|P_1(r,\lambda)\|^2 + \|P_2(r,\lambda)\|^2 \leqslant (\delta^2+\gamma^2)\exp\left(\int_{t_n}^{t_n+\varepsilon_0} 4\|A_2(r)\|\,dr\right) + \delta_0 < C$$

for all large enough n and any $r \in [t_n, t_n+\varepsilon_0]$. Therefore, we can conclude from (5.8) that there are $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that

$$\|P_1(r,\lambda)\| > \delta_1$$

for all large enough n and any $r \in [t_n, t_n + \varepsilon_1]$. This is a contradiction with (4.5), and so $\lim_{r\to\infty} P_1(r,\lambda) = 0$ for any $\lambda \in \mathbb{C}^+$.

Then the proof of (5.3) follows from statement (1) of this theorem. $\square$

Proof of 3. Our first aim is to show that integral (4.5) converges for any $\lambda \in \mathbb{C}^+$. Let us assume that $\int_0^\infty \|P_1(r,\lambda)\|^2\,dr = \infty$ for some $\lambda \in \mathbb{C}^+$. Then one can see that

$$\|P_2(r,\lambda)\|^2 \leqslant \left(\int_0^r \|A_2(s)\|\|P_1(s,\lambda)\|\,ds\right)^2 = o\left(\int_0^r \|P_1(s,\lambda)\|^2\,ds\right)_{r\to\infty},$$

which contradicts to (4.10). Thus integral (4.5) converges for any $\lambda \in \mathbb{C}^+$ and so

$$\Pi(\lambda) = \lim_{r\to\infty} P_2(r,\lambda) = I_m + \int_0^\infty A_2(r)P_1(r,\lambda)\,dr$$

holds, since $A_2(r) \in L^2[0,\infty)$. The rest of the proof follows from (4.10) and (4.9). $\square$

6. Two results on nonconvergence

Theorem 2. *There exists a real-valued continuous function $a(r)$ such that the spectral measure τ is absolutely continuous with positive continuous density, statements* (I)–(V) *of the Krein theorem hold, but*

$$\liminf_{r\to\infty} |p^*(r,\lambda)| < \limsup_{r\to\infty} |p^*(r,\lambda)| \tag{6.1}$$

for any $\lambda \in \mathbb{C}^+$. In addition, the lim sup *in* (6.1) *can be either finite or identically $+\infty$ on $\mathbb{C}^+$.*

Remark 6.1. In this theorem, by construction, $a(r)$ can be chosen to be a C^∞ function.

Before giving a detailed proof of Theorem 2, we describe a simple construction of a function $a(r)$ such that (6.1) holds for a fixed $\lambda \in \mathbb{C}^+$.

A sketch of the proof of Theorem 2. We choose positive constants ε_n and r_n such that $\varepsilon_n \to 0$ and $r_n - r_{n-1} \to \infty$ as $n \to \infty$, and then define

$$a(r) = \begin{cases} -\frac{1}{\varepsilon_n}, & r \in [r_n, r_n+\varepsilon_n), \\ \frac{1}{\varepsilon_n}, & r \in [r_n+\varepsilon_n, r_n+2\varepsilon_n), \\ 0, & r \in [r_n+2\varepsilon_n, r_{n+1}), \end{cases}$$

assuming the intervals involved do not intersect each other and $r_0 = 0$. Note that $p^*(r, \lambda)$ is constant and $|p(r, \lambda)|$ decreases exponentially when $r \in [r_n+2\varepsilon_n, r_{n+1})$. So we can assume $|p(r_n, \lambda)|$ are arbitrarily small if $r_n - r_{n-1}$ are large enough. Then it is easy to see that, if ε_n are small enough, $p^*(r_n+\varepsilon_n, \lambda)$ are arbitrarily close to $\cosh(1)\, p^*(r_n, \lambda)$ and $p^*(r_n+2\varepsilon_n, \lambda)$ are arbitrarily close to $p^*(r_n, \lambda)$. To justify it formally, see (6.14) and consider the change of variable $s = r/\varepsilon_n$. Thus, if $r_n - r_{n-1}$ are large enough and ε_n are small enough, then $\liminf_{r\to\infty} |p^*(r, \lambda)|$ is arbitrarily close to 1 and $\limsup_{r\to\infty} |p^*(r, \lambda)|$ is arbitrarily close to $\cosh(1)$. □

Before the proof of Theorem 2, we need the following lemma.

Lemma 6.2. *Let $b(r)$ be any real continuous function such that*

$$\int_0^1 b(r)\, dr = 0.$$

For $0 < \varepsilon < 1$ let $p_\varepsilon(r, \lambda)$ and $p^*_\varepsilon(r, \lambda)$ be the solutions of (3.1) *with*

$$a(r) = a_\varepsilon(r) = -\tfrac{\log|\log\varepsilon|}{\varepsilon} b(\tfrac{r}{\varepsilon})$$

and initial conditions $p^*_\varepsilon(0, \lambda) = c$, $p^*_\varepsilon(0, \lambda) = c^*$.

Then

$$p_\varepsilon(\varepsilon, \lambda) = c + o(\sqrt{\varepsilon})_{\varepsilon\to 0},$$
$$p^*_\varepsilon(\varepsilon, \lambda) = c^* + o(\sqrt{\varepsilon})_{\varepsilon\to 0}, \tag{6.2}$$

where the limits are uniform for λ, c, c^* in any compact subset of $\mathbb{C}$. In addition, if $c \neq -c^*$ and

$$\int_0^{\frac{1}{2}} b(r)\, dr > 0, \tag{6.3}$$

then

$$\lim_{\varepsilon\to 0} |p_\varepsilon(\tfrac{\varepsilon}{2}, \lambda)| = \lim_{\varepsilon\to 0} |p^*_\varepsilon(\tfrac{\varepsilon}{2}, \lambda)| = \infty. \tag{6.4}$$

Proof. First, we consider differential equations

$$\tfrac{d}{dr}\, q_\varepsilon(r) = -a_\varepsilon(r)\, q^*_\varepsilon(r),$$
$$\tfrac{d}{dr}\, q^*_\varepsilon(r) = -a_\varepsilon(r)\, q_\varepsilon(r), \tag{6.5}$$

with initial conditions $q_\varepsilon(0) = c$, $q^*_\varepsilon(0) = c^*$. Then we have

$$q_\varepsilon(r) - q^*_\varepsilon(r) = (c - c^*) \exp\left\{\int_0^r a_\varepsilon(r)\,dr\right\},$$

$$q_\varepsilon(r) + q^*_\varepsilon(r) = (c + c^*) \exp\left\{-\int_0^r a_\varepsilon(r)\,dr\right\}. \tag{6.6}$$

Hence $q_\varepsilon(\varepsilon) = c$ and $q^*_\varepsilon(\varepsilon) = c^*$. Thus our aim is to show that for $0 \leqslant r \leqslant \varepsilon$ we have

$$|p_\varepsilon(r,\lambda) - q_\varepsilon(r)| = o(\sqrt{\varepsilon})_{\varepsilon\to 0} \quad \text{and} \quad |p^*_\varepsilon(r,\lambda) - q^*_\varepsilon(r)| = o(\sqrt{\varepsilon})_{\varepsilon\to 0}.$$

To show this, we use Gronwall's lemma: if $\alpha(r)$ is a nonnegative integrable function such that

$$\alpha(r) \leqslant c_1 \int_0^r \alpha(s)\,ds + c_2 \tag{6.7}$$

for some constants $c_1, c_2 \geqslant 0$, then

$$\alpha(r) \leqslant c_2 e^{c_1 r}.$$

First, we use Gronwall's lemma with

$$c_1 = M_\varepsilon = |\lambda| + \tfrac{\log|\log\varepsilon|}{\varepsilon} \max_{0\leqslant s\leqslant 1} |b(s)|$$

and $c_2 = |c| + |c^*|$ to estimate $\alpha(r) = |p_\varepsilon(r,\lambda)| + |p^*_\varepsilon(r,\lambda)|$. Thus, by (3.1) and the definition of $p_\varepsilon(r,\lambda)$ and $p^*_\varepsilon(r,\lambda)$ we have

$$|p_\varepsilon(r,\lambda)| + |p^*_\varepsilon(r,\lambda)| \leqslant (|c| + |c^*|)e^{M_\varepsilon r}. \tag{6.8}$$

Then we use Gronwall's lemma once more to estimate

$$\alpha(r) = |p_\varepsilon(r,\lambda) - q_\varepsilon(r)| + |p^*_\varepsilon(r,\lambda) - q^*_\varepsilon(r)|.$$

Using the previous estimate, (3.1) and (6.6) we obtain (6.7) with $c_1 = M_\varepsilon$ and

$$c_2 = \varepsilon|\lambda|(|c| + |c^*|)e^{M_\varepsilon \varepsilon} \geqslant r|\lambda\, p_\varepsilon(s,\lambda)|$$

for any $0 \leqslant r \leqslant \varepsilon$. Then by estimate (6.8) we have

$$|p_\varepsilon(r,\lambda) - q_\varepsilon(r)| + |p^*_\varepsilon(r,\lambda) - q^*_\varepsilon(r)| \leqslant \varepsilon|\lambda|(|c| + |c^*|)e^{2M_\varepsilon \varepsilon} = o(\sqrt{\varepsilon})_{\varepsilon\to 0}$$

for any $0 \leqslant r \leqslant \varepsilon$.

Moreover, by (6.3) and (6.6)

$$q_\varepsilon(\tfrac{\varepsilon}{2}) = q^*_\varepsilon(\tfrac{\varepsilon}{2}) + o(1)_{\varepsilon\to 0} = \frac{1}{2}(c+c^*)\exp\left\{\log|\log\varepsilon|\cdot\int_0^{\frac{1}{2}} b(r)\,dr\right\} + o(1)_{\varepsilon\to 0},$$

which completes the proof. □

Proof of Theorem 2. In this proof $n\to\infty$ means that the limit is taken over positive integers, and $r\to\infty$ means that the limit is taken over positive reals.

We fix a function $b(r)$ which satisfies all the conditions of Lemma 6.2. Also we assume that $b(r)=0$ if $r\notin[0,1]$. Let $a(r)$ be defined by

$$a(r) = -\sum_{n=1}^{\infty}(2^n\log n)b(2^n r - n2^n) = \sum_{n=1}^{\infty} a_{\varepsilon_n}(r-n),$$

where $a_\varepsilon(\cdot)$ is defined as in Lemma 6.2, and $\varepsilon_n = 2^{-n}$. This sum is a continuous function since for any r the sum contains at most one nonzero term. Then by Lemma 6.2 we have

$$|p^*(n,\lambda) - p^*(n+2^{-n},\lambda)| = o(2^{-n/2})_{n\to\infty}.$$

Note that $p^*(r,\lambda)$ does not change when r is in an interval $[n+2^{-n}, n+1]$ since $a(r)=0$ on such intervals. Therefore by (3.1) we have

$$|p^*(n,\lambda) - p^*(n+1,\lambda)| = o(2^{-n/2})_{n\to\infty}. \tag{6.9}$$

Hence a finite limit $\lim_{n\to\infty} p^*(n,\lambda)$ exists for any $\lambda\in\mathbb{C}$. Note that $\lim\limits_{n\to\infty} p^*(n,\lambda)\neq 0$ for $\operatorname{Im}\lambda\geqslant 0$ since, by (3.1),

$$\frac{d}{dr}\left(|p^*(r,\lambda)|^2 - |p(r,\lambda)|^2\right) = 2\operatorname{Im}\lambda|p(r,\lambda)|^2\geqslant 0. \tag{6.10}$$

By the same argument, for any $r>0$ and $\operatorname{Im}\lambda>0$ we have $p(r,\lambda)\neq -p^*(r,\lambda)$. Then Lemma 6.2 implies that

$$\lim_{n\to\infty}|p(n+2^{-n-1},\lambda)| = \lim_{n\to\infty}|p^*(n+2^{-n-1},\lambda)| = \infty.$$

Note that if in Lemma 6.2 we define $a_\varepsilon(r) = -\frac{M}{\varepsilon}b(\frac{r}{\varepsilon})$, then

$$\liminf_{r\to\infty}|p^*(r,\lambda)| < \limsup_{r\to\infty}|p^*(r,\lambda)| < \infty$$

for any large enough M.

In order to complete the proof we need to show that the spectral measure τ is absolutely continuous with positive continuous density. Estimates (6.9) and Lemma 6.2 shows that the limit $\Pi(\lambda) = \lim_{n\to\infty} p^*(n,\lambda)$ converges uniformly on compact sets of $\lambda \in \mathbb{C}$. As a byproduct we have proved that $\Pi(\lambda)$ is continuous for $\lambda \in \mathbb{C}$ and has no zeros in the closed half-plane $\operatorname{Im}\lambda \geqslant 0$. In particular, this is so for real λ.

For the rest of the proof we assume $\lambda \in \mathbb{R}$. Let τ_r be the measure absolutely continuous with respect to the Lebesgue measure with the density

$$\frac{d\tau_r(\lambda)}{d\lambda} = \frac{1}{2\pi|p^*(r,\lambda)|^2}.$$

Then τ_r converges weakly to τ as $r \to \infty$ (see, for instance, [T1]). From the previous paragraph,

$$\frac{d\tau(\lambda)}{d\lambda} = \lim_{n\to\infty} \frac{1}{2\pi|p^*(n,\lambda)|^2} = \frac{1}{2\pi|\Pi(\lambda)|^2}$$

is a positive continuous function on $\mathbb{R}$, which completes the proof. □

Theorem 3. *There exists a continuous function $a(r)$ such that* (I)–(V) *of the Krein theorem hold, but the function $\Pi(\lambda)$, which is analytic in $\mathbb{C}^+ = \{\lambda : \operatorname{Im}\lambda > 0\}$, is not unique in the following sense: for any complex θ of absolute value one there is a sequence $t_n \to \infty$ such that*

$$\lim_{n\to\infty} p^*(t_n,\lambda) = \theta\Pi(\lambda). \tag{6.11}$$

In addition, we can have the following conditions satisfied: $a(r) \in L^p[0,\infty)$ for any $p > 2$, $\lim_{r\to\infty} a(r) = 0$, and for any $\lambda \in \mathbb{C}^+$:

$$\lim_{r\to\infty} p(r,\lambda) = 0,$$
$$\lim_{r\to\infty} |p^*(r,\lambda)| = |\Pi(\lambda)|. \tag{6.12}$$

Remark 6.3. In this theorem, by construction, $a(r)$ can be chosen to be a C^∞ function.

Proof. We will construct a function $a(r)$ which is piecewise constant, and then can be approximated by continuous functions that still have the desired properties.

First, note that the system of differential equations:

$$\frac{d}{dr}q(r) = -\overline{a(r)}\,q^*(r),$$
$$\frac{d}{dr}q^*(r) = -a(r)\,q(r), \tag{6.13}$$

with constant coefficient $a(r) = -C$ has a matrix solution

$$Q(r) = \begin{pmatrix} \cosh|Cr| & \overline{D}\sinh|Cr| \\ D\sinh|Cr| & \cosh|Cr| \end{pmatrix}, \tag{6.14}$$

where $D = \frac{C}{|C|}$.

Now let b be positive real and

$$a_{b,\xi,\varepsilon}(r) = \begin{cases} -b & \text{for } 0 \leqslant r \leqslant \varepsilon, \\ \overline{\xi}b, & \text{for } \varepsilon \leqslant r \leqslant 2\varepsilon, \\ 0, & \text{for } r \geqslant 2\varepsilon, \end{cases} \tag{6.15}$$

where the constant $\xi \in \mathbb{C}$ is such that $|\xi| = 1$. Let $q(r) = q_{b,\xi,\varepsilon}(r)$ and $q^*(r) = q^*_{b,\xi,\varepsilon}(r)$ be the solutions of the system of Eqs. (6.13) with $a(r) = a_{b,\xi,\varepsilon}(r)$, and initial conditions $q(0) = 0$, $q^*(0) = 1$. Then

$$\begin{aligned} q_{b,\xi,\varepsilon}(\varepsilon) &= \sinh b\varepsilon, \quad q_{b,\xi,\varepsilon}(2\varepsilon) = \tfrac{1}{2}(1-\xi)\sinh 2b\varepsilon, \\ q^*_{b,\xi,\varepsilon}(\varepsilon) &= \cosh b\varepsilon, \quad q^*_{b,\xi,\varepsilon}(2\varepsilon) = 1 + (1-\overline{\xi})\sinh^2 b\varepsilon. \end{aligned} \tag{6.16}$$

Let $p_{b,\xi,\varepsilon}(r,\lambda)$ and $p^*_{b,\xi,\varepsilon}(r,\lambda)$ be the solutions of the system of Eqs. (3.1) with $a(r) = a_{b,\xi,\varepsilon}(r)$, and initial conditions $p_{b,\xi,\varepsilon}(0,\lambda) = 0$ and $p^*_{b,\xi,\varepsilon}(0,\lambda) = 1$.

To estimate these solutions we use the following form of Gronwall's lemma: if $\alpha(r)$ is a nonnegative integrable function such that

$$\alpha(r) \leqslant c\int_0^r \alpha(s)\,ds + \beta(r) \tag{6.17}$$

for some c and $\beta(r) \geqslant 0$, then

$$\alpha(r) \leqslant c\int_0^r e^{c(r-s)}\beta(s)\,ds + \beta(r). \tag{6.18}$$

In the following estimates we assume that $\lambda \in \mathbb{C}$ is fixed. We write "*const*" for a constant, different in different inequalities, which depends on λ, but is independent of ε, r and b provided $0 < \varepsilon, r, b < 1$.

First, we use Gronwall's lemma with $\alpha(r) = |p_{b,\xi,\varepsilon}(r,\lambda)| + |p^*_{b,\xi,\varepsilon}(r,\lambda)|$. Then (3.1) implies (6.17) with $\beta(r) = 1$ and $c = |\lambda| + b$ and so (6.18) implies

$$|p_{b,\xi,\varepsilon}(r,\lambda)| + |p^*_{b,\xi,\varepsilon}(r,\lambda)| \leqslant e^{(|\lambda|+b)r} < const.$$

Second, we apply this form of Gronwall's lemma with $\alpha(r) = |p_{b,\xi,\varepsilon}(r,\lambda)|$. Then (3.1) and the previous estimate imply (6.17) with $c = |\lambda|$ and

$$\beta(r) = const \cdot br > \int_0^r |b\, p^*_{b,\xi,\varepsilon}(s,\lambda)|\, ds.$$

Therefore (6.18) implies

$$|p_{b,\xi,\varepsilon}(r,\lambda)| < const \cdot br.$$

Using the same form of Gronwall's lemma the third time with $c = |\lambda| + b$,

$$\beta(r) = const \cdot br^2 > \int_0^r |\lambda\, p_{b,\xi,\varepsilon}(s,\lambda)|\, ds$$

and

$$\alpha(r) = |p_{b,\xi,\varepsilon}(r,\lambda) - q_{b,\xi,\varepsilon}(r)| + |p^*_{b,\xi,\varepsilon}(r,\lambda) - q^*_{b,\xi,\varepsilon}(r)|,$$

we obtain

$$|p_{b,\xi,\varepsilon}(r,\lambda) - q_{b,\xi,\varepsilon}(r)| + |p^*_{b,\xi,\varepsilon}(r,\lambda) - q^*_{b,\xi,\varepsilon}(r)| < const \cdot br^2 \tag{6.19}$$

by (3.1), (6.13) and the previous estimates. This implies

$$|p^*_{b,\xi,\varepsilon}(r,\lambda) - q^*_{b,\xi,\varepsilon}(r)| < const \cdot b^2 r^3 \tag{6.20}$$

by (3.1) and (6.13).

We define

$$\varepsilon_n = \frac{1}{\log^2 n}, \quad b_n = \frac{\log^2 n}{\sqrt{n}}$$

for $n \geqslant 3$. Also we define ξ_n as a unique complex number such that

$$|\xi_n| = 1, \quad |1 - \xi_n| = \frac{1}{\log n} \quad \text{and} \quad \operatorname{Im} \xi_n > 0.$$

Note that

$$\xi_n = 1 + \frac{\boldsymbol{i}}{\log n} + O\left(\frac{1}{\log^2 n}\right)_{n\to\infty}. \tag{6.21}$$

Let $a(r)$ be defined by

$$a(r) = \sum_{n=3}^{\infty} a_{b_n,\xi_n,\varepsilon_n}(r - r_n),$$

where $a_{b,\xi,\varepsilon}(\cdot)$ is defined by (6.15), and r_n are as follows. We fix any $\lambda_0 \in \mathbb{C}^+$. Then we choose $r_2 = 0$ and each $r_n - r_{n-1}$ to be large enough so that

$$\frac{p^*(r_n + 2\varepsilon_n, \lambda_0)}{p^*(r_n, \lambda_0)} = 1 + \frac{\boldsymbol{i}}{n \log n} + O\left(\frac{1}{n \log^2 n}\right)_{n\to\infty}. \tag{6.22}$$

This is possible since $|p(r_n, \lambda_0)| \to 0$ exponentially as r_{n-1} is fixed and $(r_n - r_{n-1}) \to \infty$. Therefore, we can use (6.20), (6.21), and the fact that

$$q^*_{b,\xi,\varepsilon}(2\varepsilon, \lambda) = 1 + (1 - \overline{\xi})\left(b^2\varepsilon^2 + O(b^4\varepsilon^4)_{b\varepsilon\to 0}\right)$$

by (6.16).

We have that $p^*(r, \lambda)$ is constant for $r \in [r_n + 2\varepsilon_n, r_{n+1}]$, in particular,

$$p^*(r_n + 2\varepsilon_n, \lambda) = p^*(r_{n+1}, \lambda).$$

Hence (6.22) imply that

$$\left|\frac{p^*(r_{n+1}, \lambda_0)}{p^*(r_n, \lambda_0)}\right| - 1 = O\left(\frac{1}{n \log^2 n}\right)_{n\to\infty}$$

and so the limit $\lim\limits_{n\to\infty} |p^*(r_n, \lambda_0)| = |\Pi(\lambda_0)|$ converges, since

$$\sum_{n=3}^{\infty} \frac{1}{n \log^2 n} < \infty.$$

Thus statements (I)–(V) of the Krein theorem hold by (1.4) and (3.6).

If each $r_n - r_{n-1}$ is large enough, then the sum that defines $a(r)$ is a sum of the functions with disjoint support. Therefore

$$\|a(r)\|^p_{L^p} = 2 \sum_{n=3}^{\infty} n^{-p/2} \log^{2p-2} n$$

and so $a(r) \in L^p[0, \infty)$ if and only if $p > 2$. In particular, this means that part 2 of Theorem 1 implies (6.12).

To complete the proof note that the limit $\lim_{n\to\infty} p^*(r_n, \lambda_0)$ does not exists because

$$\frac{p^*(r_n + 2\varepsilon_n, \lambda_0)}{p^*(r_n, \lambda_0)} = \exp\left\{\frac{\boldsymbol{i}}{n\log n} + O\left(\tfrac{1}{n\log^2 n}\right)_{n\to\infty}\right\} \tag{6.23}$$

by (6.22), and the series $\sum_{n=3}^{\infty} \frac{1}{n\log n}$ diverges, while $\sum_{n=3}^{\infty} \frac{1}{n\log^2 n} < \infty$. At the same time $\lim_{n\to\infty} \frac{1}{n\log n} = 0$ and so for any complex θ of absolute value one there is a sequence $t_{\theta,n} \to \infty$, which is a subsequence of r_n, such that

$$\lim_{n\to\infty} p^*(t_{\theta,n}, \lambda_0) = \theta|\Pi(\lambda_0)|.$$

Note that $|\Pi(\lambda)|$ is well defined for any $\lambda \in \mathbb{C}^+$ since $\lim_{r\to\infty} |p^*(r, \lambda)| = |\Pi(\lambda)|$ converges by (5.3). Also using (5.3) we can define a function $\Pi(\lambda)$, which is analytic in $\mathbb{C}^+$, by

$$\Pi(\lambda) = |\Pi(\lambda_0)|^{-1} \lim_{n\to\infty} p^*(t_{1,n}, \lambda)\overline{p^*(t_{1,n}, \lambda_0)} = \lim_{n\to\infty} p^*(t_{1,n}, \lambda).$$

Then $\lim_{n\to\infty} p^*(t_{\theta,n}, \lambda) = \theta\Pi(\lambda)$ for any $\lambda \in \mathbb{C}^+$ because of (5.3). $\square$

Proposition 6.4. *If $r_n - r_{n-1}$ are large enough in the proof of Theorem* 3, *then for all $\lambda \in \mathbb{C}^+$ we have* (6.22) *as well as estimates*

$$\left|\frac{p(r, \lambda)}{p^*(r_n, \lambda)}\right| < \frac{const}{\sqrt{n}\log n} \tag{6.24}$$

for $r_n + 2\varepsilon_n \leqslant r \leqslant r_{n+1}$, and

$$\left|\frac{p^*(r, \lambda)}{p^*(r_n, \lambda)} - 1\right| < \frac{const}{n}, \quad \left|\frac{p(r, \lambda)}{p^*(r_n, \lambda)}\right| < \frac{const}{\sqrt{n}} \tag{6.25}$$

for $r_n \leqslant r \leqslant r_n + 2\varepsilon_n$. This gives, in particular, a constructive proof of (3.5) *and* (6.12).

Proof. We can demonstrate (6.24) and (6.25) for $\lambda = \lambda_0$ using estimates

$$\left|q^*_{b,\xi,\varepsilon}(r, 2\varepsilon)\right| < const \cdot b\varepsilon|1 - \xi|$$

and, for $0 \leqslant r \leqslant 2\varepsilon$,

$$\left|q_{b,\xi,\varepsilon}(r, \lambda)\right| < const \cdot b\varepsilon, \quad \left|q^*_{b,\xi,\varepsilon}(r, \lambda) - 1\right| < const \cdot b^2\varepsilon^2,$$

which follows from (6.14) and (6.16).

We also can obtain (6.25) and (6.22) for all $\lambda \in \mathbb{C}^+$ if the sequence r_n is chosen as follows. It is easy to see that estimates like (6.19) and (6.20) can be established uniformly in λ in a compact subsets of $\mathbb{C}$. Also $|p(r_n, \lambda)| \to 0$ uniformly in λ in a compact subsets of $\mathbb{C}^+$ as r_{n-1} is fixed and $(r_n - r_{n-1}) \to \infty$. Thus for any compact subset H of $\mathbb{C}^+$ there is a sequence r_n^H such that (6.22), (6.24) and (6.25) hold for $r_n = r_n^H$, and also for r_n that is any subsequence of r_n^H. We can represent $\mathbb{C}^+$ as an increasing union of compact subsets H_k. Without loss of generality we can assume that $r_n^{H_{k+1}}$ is a subsequence of $r_n^{H_k}$ for each k. Then we define r_n by the "diagonal process" $r_n = r_n^{H_n}$. □

Conjecture 6.5. *We conjecture that if* $a(r)$ *is a* real-valued *function, and conditions* (I)–(V) *of the Krein theorem hold, then* $\Pi(\lambda)$ *is unique in the following sense*: *if* $t_n \to \infty$ *and* $\lim_{n\to\infty} p(t_n, \lambda) = 0$, *then the limit* $\lim_{n\to\infty} p^*(t_n, \lambda) = \Pi(\lambda)$ *converges uniformly on compact subsets of* $\mathbb{C}^+$. *If true, this conjecture implies that the original form of Krein's theorem holds if* $a(r)$ *is real and "locally uniformly integrable" in the sense of part* (2) *of Theorem* 1.

Conjecture 6.6. *We conjecture that if* $a(r) \in L^1_{\rm loc}$ *is real, and conditions* (I)–(V) *of the Krein theorem hold, then* $\Pi(\lambda)$ *is the limit in average of* $p^*(t_n, \lambda)$, *that is,*

$$\Pi(\lambda) = \lim_{r\to\infty} \frac{1}{r} \int_0^r p^*(s, \lambda)\, ds$$

uniformly on compact subsets of $\mathbb{C}^+$. *Here* $a(r) \in L^1_{\rm loc}$ *if*

$$\sup_{r \geqslant 0} \int_r^{r+1} |a(s)|\, ds < \infty.$$

If true, this conjecture also implies the uniqueness of $\Pi(\lambda)$. Note that in the situation of Theorem 3 the limit in average of $p^*(t_n, \lambda)$ does not exists if $r_{n+1} - r_n$ are large enough.

These two conjectures may be related to the results of [D2].

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