Existence of Multidimensional Travelling Wave Solutions of an Initial-Boundary Value Problem

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0. INTRODUCTION

A. This paper is concerned with the existence of travelling wave solutions of an initial-boundary value problem in two space variables, namely

\[ u_t = u_{xx} + u_{yy} + f(u) \]

\[ u(x, 0, t) = u(x, L, t) = 0 \]  \hspace{1cm} (0.1)

\[ u(x, y, 0) = u_0(x, y), \]

where \( x \in \mathbb{R}^1 \), \( 0 < y < L \), and \( f(u) = u(u - 1)(\alpha - u) \) with \( 0 < \alpha < \frac{1}{2} \). We shall locate solutions of the form

\[ u(x, y, t) = u(\xi, y) \quad (\xi = x - \theta t), \]  \hspace{1cm} (0.2)

where the wave velocity \( \theta \) is to be determined. The solutions will also be required to satisfy the limiting conditions

\[ \lim_{\xi \to -\infty} u(\xi, y) = u_0, \quad \lim_{\xi \to +\infty} u(\xi, y) = u_1(y), \]  \hspace{1cm} (0.3)

where the limiting states are required to be solutions of

\[ 0 = u_{yy} + f(u) \]

\[ u(0) = u(L) = 0. \]  \hspace{1cm} (0.4)

With regard to the multiplicity of solutions of (0.4) it is well known that if \( 0 < \alpha < \frac{1}{2} \) and \( L > L_0 \) for some \( L_0 > 0 \) then (0.4) admits exactly three

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(linearly) nondegenerate solutions: $0 = u_0 < u_s(y) < u_1(y)$; (see Smoller and Wasserman [18]). For more general nonlinearities, e.g., where $f(u)$ is a qualitative cubic, the existence theorem proved here remains valid provided that the above result concerning the multiplicity of solutions of (0.4) is postulated as $f$ is deformed to an exact cubic. This will be the case for any such $f$ provided that $L$ is sufficiently large.

The main result is the following.

**Theorem 0.1.** Suppose that $0 < x < t$ and $L > L_0$. Then there exists a solution $u$ of (0.1) which satisfies (0.2) and (0.3). For $0 < y < L$ and $\xi \in \mathbb{R}$, $u(\xi, y) > 0$.

If the first root of $f$ at $u = 0$ is perturbed to $u = \pm \epsilon$, the solution of (0.4) perturbs to a nonconstant function $u_\epsilon(y)$. The proof of Theorem 0.1 remains valid with only minor modifications.

It is frequently useful to have estimates for $\theta$. The following estimates are obtained in Section 4.

**Theorem 0.2.** (a) Suppose that $L = L_0 + \epsilon$. Then for sufficiently small $\epsilon > 0$, $\theta > 0$. (b) If $L \gg L_0$ then $\theta < 0$.

The "bistable" diffusion equation (with cubic nonlinearity) is well established as an interesting and yet analytically tractable problem in nonlinear diffusion. As such it has been a useful guide for predicting behavior of more complicated systems, both on a heuristic level and, more precisely, in the construction and asymptotic expansions. This has proved to be a particularly fruitful point of view in regard to travelling wave problems for systems in one space variable where the scalar travelling wave problem is well understood (see [8–11]). The formation of propagating fronts in multidimensional media has also been investigated by several authors [3, 13]. However, the problem is more difficult and the results are less complete than in space variable since the equations no longer necessarily admit the symmetry of a translation invariant wave (see, e.g., Jones [13]). It is therefore of interest to isolate a class of multidimensional problems which admit travelling wave solutions; the bistable diffusion equation is a natural place to begin. We remark that the existence of both small and large amplitude stationary waves in an infinite strip has been investigated by a number of authors, (see, e.g., [2, 15]).

In a forthcoming paper we shall investigate the existence of stationary solutions of (0.1) in infinite channels with slowly varying cross-section. Such solutions have applications to both population genetics and to steady rotational flows of an ideal incompressible fluid, wherein the solution $u$ is interpreted as a stream function and $\Delta u = -f(u)$ is the vorticity. In order for the equations to admit "heteroclinic" solutions the channel must have a
certain admissible shape; the travelling waves obtained here on the infinite strip play a crucial role in determining appropriate conditions on the shape of the channel.

C. Method of Proof. Most existence proofs of travelling waves of large amplitude in one space variable have employed a topological method of some sort. In this regard the Conley index of isolated invariant sets and a related invariant called the connection index have proved to be of particular importance (see [4-7, 9-11]). (The index is a homotopy invariant associated with certain invariant sets S of a flow on a locally compact metric space, wherein S is the maximal invariant set in some compact neighborhood N of itself. S is called an isolated invariant set and N is called an isolating neighborhood.) The index theory and its application to some specific travelling wave problems has been described elsewhere (see [4-6]). A brief outline is provided in the appendix; readers who are unfamiliar with these methods can obtain a more complete discussion in the references cited above. The main concern here is the location of a suitable isolating neighborhood N and the computation of the index.

The application of these methods to the multidimensional travelling wave problems presents some novel features compared with similar investigations in one space variable. In particular, the travelling wave solution satisfies an elliptic system of p.d.e.'s of the form

\[ \begin{align*}
  u_z &= w, \\
  w_z &= -\theta w - [u_{xx} + f(u)].
\end{align*} \tag{0.5} \]

The initial value problem for (0.5) is ill-posed, and so, the equations do not generate a flow on any reasonable space. The index theory cannot be applied directly.

Our approach is to discretize the y-interval into a net of N + 1 points, \( y_i = ih, \ 0 \leq i \leq N \), where Nh = L. The y-derivatives are then replaced with a difference operator. Thus if \( u_i(\xi) = u(\xi, y_i) \), then an approximate system for (0.5) is

\[ \begin{align*}
  u'_i &= w_i, \\
  w'_i &= -\theta w_i - [(u_{i+1} - 2u_i + u_{i-1})h^{-2} + f(u_i)],
\end{align*} \tag{0.6} \]

1 \leq i \leq N - 1, where \( u_0 = u_N = 0 \). In the sequel, \( U \) (resp. \( W \)) will denote the vector with components \( u_i \) (resp. \( w_i \)), 1 \leq i \leq N - 1. Equation (4) is replaced with the algebraic system

\[ \begin{align*}
  0 &= (u_{i+1} - 2u_i + u_{i-1})h^{-2} + f(u_i) \\
  u_0 = u_N = 0.
\end{align*} \tag{0.7} \]
It is not hard to show that the multiplicity results for (0.4) are inherited by (0.7) for sufficiently small $h$ (see Section 1). If $U_i$, $i = 0, \alpha, 1$, are the solutions of (0.7) then $(U_i, 0)$, $i = 0, \alpha, 1$, are the rest points of (0.5).

We show that for all small $h$ there exists a solution $(U, W)(t)$ of (0.6) running from $(U_0, 0)$ at $t = -\infty$ to $(U_1, 0)$ at $t = +\infty$. This is accomplished by locating an isolating neighborhood $\mathcal{N} \subset \mathbb{R}^{2N-2}$ for (0.6) which contains the connecting solution (see Section 2), and by computing the connection index associated with it (see Section 3). The "nontriviality" of the index, together with a natural gradient-like structure for solutions in the maximal invariant set isolated by $\mathcal{N}$, forces the existence of the connecting solution. (We remark that if more stringent hypotheses are imposed on $f$ and $L$ the existence of (approximate) connecting solutions follows from a general theorem of C. Conley; see [4, Sect. 4.3]. However, the construction given here has the advantage of providing more detailed qualitative information (e.g., that $W > 0$) about the connecting solution in the absence of the more restrictive conditions on the parameters. It is anticipated that this additional information will be useful in subsequent investigations.)

The system (0.6) is singularly perturbed in the parameter $h$, and so the construction of $\mathcal{N}$ (which depends on $h$) must be such that the set of approximate solutions is in some appropriate sense compact. This enables us to obtain an exact solution of (0.5) as a limit of a subsequence of approximate solutions. This program is made possible precisely because the limiting equations are elliptic and a priori estimates of their solutions are easily available.

The index is computed by deforming the boundary conditions from the Dirichlet problem to the Neumann problem, at which point the front coincides with a plane wave. This homotopy must be performed in such a manner that solutions of (0.4) remain nondegenerate for all relevant boundary value problems. Since $L$ is originally assumed to be large, we must evidently change $L$ as the boundary conditions are deformed. The following theorem on the global continuation of nondegenerate solutions is proved in [10].

**Theorem 0.3.** There exists a homotopy

$$
0 = u'' + f_{a(\lambda)}(u) \quad (0 < y < L_\lambda)
$$

$$
0 = l_\lambda(u(0), u'(0)) = r_{\lambda}(u(L_\lambda), u'(L_\lambda)),
$$

$\lambda \in [0, 1]$, where $f_{a(\lambda)}(u) = u(u-1)(x(\lambda) - u)$, such that $\lambda = 0$ is the Dirichlet problem, $x(0) = \alpha$, $L_0 = L$, and $\lambda = 1$ is the Neumann problem. For each $\lambda \in [0, 1]$, the boundary value problem admits exactly three solutions. The solutions are linearly nondegenerate; i.e., the spectra of the linearizations
about \( u_0 \) and \( u_1 \) are negative while that of \( u_2 \) has exactly one positive eigenvalue.

At \( \lambda = 1 \) a change of variables is introduced which decouple most of the components of (0.6) from one another. In fact, at the end of the homotopy the \((u_1, w_1)\) components satisfy the travelling wave equations for the bistable diffusion equation in one space variable, while the remaining components are linear with a saddle point at the origin. The connection index is easy to compute for this system. The continuation of (0.6) is described in Section 3.

D. Notation. We shall follow the following conventions with regard to spaces and norms. If \( \Omega \) is a bounded domain in \( \mathbb{R}^m \), \( H^{k,p}(\Omega) \) and \( H_0^{k,p}(\Omega) \) are the usual Sobolev spaces of order \( k \) based on \( L^p(\Omega) \) with norm denoted by \( \|u\|_{k,p} \). Also \( H^{k,2}(\Omega) \) and \( H_0^{k,2}(\Omega) \) are denoted by \( H^2(\Omega) \) and \( H_0^2(\Omega) \), respectively.

If \( U = (u_1, ..., u_{N-1}) \in \mathbb{R}^{N-1} \), we shall use the norms

\[
|U| = \max_{1 \leq i \leq N-1} |u_i|
\]

\[
\|U\| = \left( \sum_{i=1}^{N-1} u_i^2 h \right)^{1/2}
\]

\[
\|U\|_1 = (\|U\|^2 + \|\delta U\|^2)^{1/2}
\]

\[
\|U\|_2 = (\|U\|^2 + \|\delta^2 U\|^2)^{1/2},
\]

where \( \delta U \) is the vector with components \((u_{i+1} - u_i)/h, 0 \leq i \leq N-1 \), and \( \delta^2 U \) is the vector with components \((u_{i+1} - 2u_i + u_{i-1})h^{-2}, 1 \leq i \leq N-1 \). Here, \( u_0 \) and \( u_N \) are obtained from \( U \) by using the relevant boundary conditions (see Section 1).

1. Approximation of the Rest States

In this section it is shown that (0.7) provides a good approximation to (0.4) in the following sense. Suppose that (0.4) admits precisely three non-degenerate solutions, \( u_i(y), i = 0, \alpha, 1 \); then for sufficiently small \( h \), say \( h < h_0 \), there exist exactly three solutions of (0.7) which approximate the solutions of (0.4). Moreover, if \( \sigma_\rho, \Sigma_i \) is the spectrum of the linearized equations about \( u_i, U_i \), then \( \Sigma_i \) uniformly approximates \( \sigma_i \) on any compact subset of the complex plane.

As mentioned earlier we shall actually need to prove this result for all of the boundary value problems in Theorem 0.3. For boundary value problems other than the Dirichlet problem the approximation scheme has to
be chosen carefully, since we shall need to find an $h_0$ which is uniform in the homotopy parameter. In particular, we shall find a difference scheme which is uniformly consistent with all of the continuous boundary value problems. Since we shall need to obtain spectral estimates, the approximate equations should also be symmetric.

A. Formulation of the Problem

We consider the boundary value problems

\[ 0 = u'' + f(u), \quad 0 < y < L \]  

\[ 0 = l_a(u, u')|_{y=0} = r_a(u, u')|_{y=L}, \]  

where

\[ l_a(u, w) = aw - (1-a)u \]
\[ r_a(u, w) = aw + (1-a)u \quad (0 \leq a \leq 1). \]

Thus $a = 0$ is the Dirichlet problem and $a = 1$ is the Neumann problem.

The approximation scheme is defined as follows. Consider a net of the form $y_i = ih - x$, $0 \leq i \leq N$, where $h = h(a)$ and $x = x(a, h)$ are to be determined. $N$ is regarded as a large, fixed integer. We require that $x$ and $h$ satisfy the conditions

\[ 0 \leq x \leq h/2, \quad L = Nh - 2x; \]

thus $0 \in [y_0, y_0 + h/2]$ and $L \in [y_N - h/2, y_N]$.

The approximate equations are

\[ 0 = (u_{i+1} - 2u_i + u_{i-1}) h^{-2} + f(u_i), \quad 1 \leq i \leq N - 1 \]  

\[ 0 = l_a^h(u_0, u_1) = a(u_1 - u_0) h^{-1} - (1-a) \left[ \frac{h-x}{h} u_0 + \frac{x}{h} u_1 \right] \]  

\[ 0 = r_a^h(u_{N-1}, u_N) = a(u_N - u_{N-1}) h^{-1} + (1-a) \left[ \frac{x}{h} u_{N-1} + \frac{h-x}{h} u_N \right]. \]

Before stating the consistency theorem it will be convenient to express (1.1a) and (1.3a) in operator form. Thus let

\[ n[u] = u'' + f(u) \]
\[ n_h[u] = \mathcal{L}_h U + \mathcal{F}(U), \]
where \( \mathcal{F}(U) = (f(u_1), \ldots, f(u_{N-1}))' \), and \( \mathcal{L}_h \) is the \( (N-1) \times (N-1) \) tridiagonal symmetric matrix

\[
\mathcal{L}_h = h^{-2} \begin{bmatrix}
-2 + \gamma_a & 1 & & \\
1 & -2 & 1 & & \\
& & \ddots & \ddots & \ddots \\
& & & -2 & 1 \\
& & & & -2 + \gamma_a
\end{bmatrix}.
\]

The constant \( \gamma_a \) is determined by solving \( l_a^h(u_0, u_1) = 0 \) for \( u_0 \) in terms of \( u_1 \), namely, \( u_0 = \gamma_a u_1 \), where

\[
\gamma_a = \frac{a - (1 - a)x}{a + (1 - a)(h - x)}.
\]

It then follows that if \( r_a^h(u_{N-1}, u_N) = 0 \) then \( u_N = \gamma_a u_{N-1} \).

**Lemma 1.1.** Let \( u \) be a \( C^4 \) function which satisfies (1.1b), let \( u_i = u(y_i) \), \( 0 \leq i \leq N \), and let \( U \) be the vector with components \( u_i \), \( 1 \leq i \leq N - 1 \). Let \( \tilde{U} \) be the vector with components \( n[u](y_i) \), \( 1 \leq i \leq N - 1 \). Then there exists \( K > 0 \) independent of \( h \) and \( a \), and a unique \( h = h(a) \) and \( x = x(a, h) \) which satisfy (1.2) such that

\[
|n_a^h[U] - \tilde{U}| \leq Kh \tag{1.5a}
\]

\[
|l_a^h(u_0, u_1)|, |r_a^h(u_{N-1}, u_N)| \leq K[h^2 + (1 - a)h^3]. \tag{1.5b}
\]

If \( h_* = L/N \), then \( |h(a) - h_*| = \mathcal{O}(h_*^2) \). Finally, \( x(0, h) = 0 \), \( h(0) = h_* \), and \( 0 \leq \gamma_a \leq 1 \) with \( \gamma_i = i \), \( i = 0, 1 \).

**Proof.** The main problem is to determine \( x \) and \( h \) such that (1.5b) holds. To this end expand \( u(y) \) in a Taylor series about \( y = 0 \) to obtain

\[
u_a = u(0) - xu'(0) + x^2u''(0)/2 + \mathcal{O}(x^3)
\]

\[
u_1 = u(0) + (h - x)u'(0) + (h - x)^2u''(0)/2 + \mathcal{O}(h - x)^3.
\]

Substitute these expansions into the first equation in (1.3b). Since \( u \) satisfies (1.1b) the leading order terms are the quadratics. If \( x \) and \( h \) are chosen so that (1.5b) holds we obtain a quadratic equation in \( x \) and \( h \), namely

\[
(1 - a)x^2 - (2a + (1 - a)h)x + ah = 0.
\]

A similar computation at \( y = L \) yields the same equation for \( x \) and \( h \).
Regarding $h$ as fixed for the moment, the first condition in (1.2) allows us to solve uniquely for $x$, i.e.,

$$x(a, h) = \frac{2a + (1-a)h - [4a^2 + (1-a)^2 h^2]^{1/2}}{2(1-a)}, \quad 0 \leq a < 1$$

$$= h/2, \quad a = 1.$$

A simple computation shows that $x$ is continuous and $x(0, h) = x(a, 0) = 0$, and $x_a(a, h) \geq 0$ for $a \in [0, 1]$. It follows that $0 \leq x(a, h) \leq h/2$ for $h \geq 0$.

We finally show that with $x(a, h)$ as above there exists a unique choice of $h$ such that the second equation in (1.2) holds for sufficiently large $N$. In fact, $h = h_\ast + O(h^2)$, where $h_\ast = L/N$. The equation we must solve is

$$g(a, h) = h - 2x(a, h)/N - h = O. \quad (1.7)$$

A simple computation shows that $|x_a(a, 0)|$ and $|x_h(a, 0)|$ are uniformly bounded for all $a \in [0, 1]$, and so, these derivatives are uniformly bounded for all $a$ and sufficiently small $h$, say $0 \leq h \leq h_1$. Thus

$$\frac{dh}{da} = \frac{2x_a(a, h)/N}{1 - 2x_h(a, h)/N}.$$

Since $x(0, h) = 0$ for all $h$ we must choose $h(0) = h_\ast$. For sufficiently large $N$ it follows from the above remark that there exists a unique solution $h = h(a)$ of this initial value problem on $0 \leq a \leq 1$ and that the solution remains of order $1/N$. Finally, note that since $x(a, 0) = 0$, $|h(a) - h_\ast| = O(h/N) = O(h^2)$.

The proof of (1.5a) easily follows from (1.5b). For $i \neq 1$, $N - 1$ it is routine to verify (1.5a) for the $i$th component of $n_h(U) - \bar{u}$; in fact these differences are of order $Kh^2$. For $i = 1$, $N - 1$ it follows from (1.5) and (1.4) that

$$u_0 = \gamma_a u_1 + A,$$

$$u_N = \gamma_a u_N, \quad A = O\left(\frac{ah^3 + (1-a)h^4}{a + (1-a)(h-x)}\right).$$

It is easily seen that $A$ is uniformly of order $h^3$ for $a \in [0, 1]$. Thus we obtain order $h$ accuracy in the extreme components.

B. Spectral Estimates

Suppose that $\bar{u}$ is a solution of (1.1) and let $\sigma$ be the spectrum of $\Gamma = dn$ at $\bar{u}$ relative to the subspace of $H^2[0, L]$ functions which satisfy (1.1b). Then $\sigma$ consists of real, simple eigenvalues, $\lambda_0 > \lambda_1 > \lambda_2 \cdots$. Let $\bar{U}$ be the vector with components $\bar{u}(y_i)$, $1 \leq i \leq N - 1$, and let $\bar{\Sigma}$ be the spectrum of $\bar{F}_h = dn_h$ at $\bar{U}$.
LEMMA 1.2. (A) With notation as above $\Sigma_h$ consists of real eigenvalues $\lambda_0^h \geq \lambda_1^h \geq \cdots \geq \lambda_{N-2}^h$, and if $U_0$ is an eigenvector of $\hat{\Gamma}_h$ for the eigenvalue $\lambda_0^h$, then the components of $U_0$ are of one sign. (B) For each $K \geq 0$ there exists $R_K > 0$ such that for all sufficiently small $h$,

$$|\lambda_k^h - \lambda_k| < R_k h, \quad 0 \leq k \leq K.$$  

Proof. (A) Define a bilinear form $B_h: \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \to \mathbb{R}$ by

$$B_h(U, V) = \hat{\Gamma}_h U \cdot V h.$$  

Since $\hat{\Gamma}_h$ is symmetric, the spectrum $\Sigma_h$ of $\hat{\Gamma}_h$ is real, and it can be characterized by the minimax theorem with respect to the functional

$$A_h(U) = B_h(U, U) / \|U\|^2;$$

(see Section 0.D. for notation). A simple calculation shows that for $\|U\| = 1$,

$$A_h(U) = h^{-1}(\gamma - 1)(u_i^2 + u_{i-1}^2) - \sum_{i=1}^{N-2} \frac{(u_{i+1} - u_i)^2}{h} + \sum a_i u_i^2 h,$$  

where $a_i = f'(\bar{u}(y_i))$. Clearly $\lambda_0$ is the maximum of $A_h$ over the unit sphere. It can be seen from the equation satisfied by $U_0$ that two adjacent components of $U_0$ cannot both be zero; otherwise, $U_0$ would vanish identically. If $U_0$ had components with different signs, it then follows that $A_h(U_+) > \lambda_0$, where the components of $U_+$ are the absolute values of those of $U_0$. This is a contradiction and the proof of (A) is complete.

(B) Fix $\lambda_k \in \sigma$ and let $u_k$ be a corresponding eigenfunction of $\Gamma$ with unit $L_2$ norm. Let $\hat{U}_k$ be the vector with components $u_k(y_i)$, $1 \leq i \leq N-1$. From Lemma 1.1 it follows that $|\langle (\hat{\Gamma}_h - \lambda_k) \hat{U}_k \rangle| \leq M_k h$, where $M_k$ depends only on the $C^4$ norm of $u_k$, so that

$$\|\langle (\hat{\Gamma}_h - \lambda_k) \hat{U}_k \rangle\| \leq MN^{1/2} h^{3/2} = o_k(h).$$

Moreover, $\|\hat{U}_k\|$ tends to $\|u_k\|_{L_2} = 1$ as $h$ tends to zero, so that $\|\hat{U}_k\| = o(1)$. From a standard theorem on the approximation of eigenvalues of symmetric matrices it follows that

$$\min_i |\lambda_i^h - \lambda_i^h| \leq \frac{\|\langle (\hat{\Gamma}_h - \lambda_k) \hat{U}_k \rangle\|}{\|\hat{U}_k\|} = o_k(h)$$  

(see Issacson and Keller [12, Chap. 4, Theorem 1.5]).

We finally show that for $0 \leq k \leq K$ and $h \leq h(K)$ sufficiently small, $\lambda_k^h = \lambda_k^h(h_k)$, where $\lambda_k^h(h_k)$ minimizes the left side of (1.9). Since $\sigma$ consists of simple eigenvalues, $\gamma = \inf_{i \geq 0} (\lambda_i - \lambda_{i-1})$ is strictly positive. Let $B_k(\delta)$ be an
interval of length $\delta$ about $\lambda_k$, where $\delta < \gamma / 2$. From (A) and (1.9) it follows that for sufficiently small $h$ that $\lambda_k^h \in B(I, \delta)$ and $\hat{X}_k \cap B_k(\delta) \neq \emptyset$ for $0 \leq k \leq K$ and $K$ fixed.

We claim that for $h < h(K)$, where $h(K)$ depends on each of the $C_k(h)$, $0 \leq k \leq K$, $\lambda_k^h \in B_k(\delta)$. If this were not the case, for some sequence of $h$’s tending to zero there would exist $J(h) \geq K$ such that $\lambda_{J(h)}^h \in B_{J(h)}(\delta)$ for some $i < J(h)$. Let $J_m(h)$ be the minimal index for which this occurs. From (A) we may take $i = J_m(h) + 1$. Since $J_m(h)$ is integer valued it can be assumed by passing to a suitable subsequence that $J_m(h) \equiv J$ as $h \to 0$, and so, $\lambda_J^h, \lambda_{J+1}^h \in B_J(\delta)$.

Let $U_k$ be an eigenvector of $\hat{X}_h$ corresponding to $\lambda_k^h$ with $\|U_k\| = 1$. Since $\hat{X}_h$ is symmetric, $U_k \cdot U_j = 0$ for $k \neq j$. Let $u_k^h(y)$ be a piecewise linear interpolant of $U_k$, i.e.,

$$u_k^h(y) = (u_{i+1}^h - u_i^h)(y - y_i)h^{-1} + u_i^h \quad (y_i \leq y \leq y_{i+1}),$$

for $0 \leq i \leq N - 1$, where $u_0 = \gamma_a u_1$ and $u_N = \gamma_a u_{N-1}$ (see (1.4)). We will show that $u_k^h(y), u_{k+1}^h(y)$ converge to orthogonal eigenfunctions $u_J, u_{J+1}$ of $\hat{X}$ which correspond to the same eigenvalue $\lambda_J$. Since $\sigma$ is simple this will provide the desired contradiction.

To this end we obtain uniform bounds on $\|\delta U_k\|, k = J, J + 1$ (see Section 0.D)). A simple computation shows that

$$\|u_k^h\|_{L^2[0, L]}^2 = \|\delta U_k\|^2 - \frac{x}{h}[(u_1 - u_0)^2 + (u_N - u_{N-1})^2],$$

so that if $\|\delta U_k\|$ is uniformly bounded, $\{u_k^h\}$ is uniformly bounded in $H^1[0, L]$.

First note that for $k = J, J + 1$, $A_k^h(U_k) = \lambda_k^h \in B_J(\delta)$; thus $A_k^h(U_k)$ is uniformly bounded as $h \to 0$. It follows from (1.8) and $\|U_k\| = 1$ that

$$\|\delta U_k\|^2 \leq -A_h(U_k) + h^{-1}(\gamma_a - 1)(u_1^2 + u_{N-1}^2) + M,$$

where $M = \max|a|$ is uniformly bounded. Since $0 \leq \gamma_a \leq 1$ for all $a \in [0, 1]$ it follows that

$$\|\delta U_k\|^2 \leq \max(-\lambda_J + \delta, -\lambda_J - \delta) + M.$$

Thus $\{u_k^h\}$ is uniformly bounded in $H^1[0, L], k = J, J + 1$.

It follows that some subsequences converge strongly in $L^2[0, L]$ and weakly in $H^1[0, L]$ to limits, say $u_J(y), u_{J+1}(y)$. Moreover, since $U_J, U_{J+1} = 0$ it follows from the uniform bounds on $\|\delta U_k\|$ that $(u_J^h, u_{J+1}^h)_{L^2} = O(h)$ as $h \to 0$. Thus $(u_J, u_{J+2})_{L^2} = 0$. In addition, since $\|\delta U_k\|$ is uniformly bounded, $\|U_k\| = 1$ implies that $\|u_k^h\|_{L^2} = 1 + O(h)$, and so $\|u_k\|_{L^2} = 1$ for $k = J, J + 1$. 
It only remains to show that \( u \) satisfies
\[
(\Gamma - \lambda_j) u_k = 0, \quad k = J, J + 1,
\]
and that \( u_k \) satisfies (1.1b). To this end let \( \psi \) be a \( C^\infty \) function which satisfies (1.1b), and consider the bilinear form
\[
b(u, \psi) = \int_0^1 [u\psi'' + f'(\bar{u}(y)) u\psi - \lambda_j u\psi] \, dy.
\]
Let \( \psi \) (resp. \( \bar{U} \)) be the vector with components \( \psi(y_i) \) (resp. \( u_k(y_k) \)), \( 1 \leq i \leq N - 1 \). Since \( u_k \in H^1[0, L] \) and satisfies (1.1b), it follows Lemma 1.1 that
\[
b(u_k, \psi) = \left[ L \psi \cdot \bar{U} + D_j \psi \cdot \bar{U} \right] h + O(h),
\]
where \( D_j \) is the diagonal matrix with entries \( f'(u(y_k)) - \lambda_j \). Since \( u_k \) tends to \( u_k \) in \( L^2[0, L] \) it easily follows that
\[
\lim_{h \to 0} \| U_k - \bar{U} \| = 0.
\]
Moreover, since \( \delta \) (in the definition of \( B_k(\delta) \)) is arbitrary, we have that \( \lambda_k^h \to \lambda_j \) as \( h \to 0 \), for \( k = J, J + 1 \). Thus for all small \( h \), we have that
\[
b(u_k, \psi) = (\hat{f}^h - \lambda_j) \Psi \cdot u_k h + \hat{f}^h \Psi \cdot (U_k - \bar{U}) h + (\lambda_j - \lambda_k^h) U_k \cdot \Psi h + O(h)
\]
\[
= \hat{f}^h \Psi \cdot (U_k - \bar{U}) h + (\lambda_j - \lambda_k^h) U_k \cdot \Psi + O(h),
\]
since \( \hat{f}^h \) is symmetric and \( (\hat{f}^h - \lambda_k^h) U_k = 0 \). Since \( \hat{f}^h \Psi \) is uniformly bounded it follows from the above remarks that \( b(u_k, \psi) = 0 \) for all smooth \( \psi \) satisfying (1.1b). It follows that \( u_k \) satisfies (1.10) in the sense of distributions. Moreover, since \( u_k \in H^1[0, L] \) it follows from the differential equation (1.10) that \( u_k \in H^2[0, L] \), and in fact, \( u_k \in C^\infty[0, L] \).

Now take \( \psi \) to be a smooth function with \( \psi(L) = \psi'(L) = 0 \) and \( \psi(0) = a, \psi'(0) = 1 - a \). Then
\[
0 = b(u_k, \psi) = \int_0^L \psi(u'' + (f'(a) - \lambda_j) u) \, dy + (\phi' u - \phi u') \bigg|_0^L
\]
\[
= [(1 - a) u_k(0) - au_k'(0)].
\]
A similar argument applied at \( y = L \) shows that \( u_k \) satisfies (1.1b).

Remark. Since \( \sigma \) has at most a finite number of positive elements, it follows from Lemma 1.2 that if \( 0 \notin \sigma \), then \( 0 \notin \hat{\Sigma}_h \) for all sufficiently small \( h \), and that \( \hat{\Sigma}_h \) has exactly the same number of positive eigenvalues as \( \sigma \).
C. The Nonlinear Problem

The exact number of solutions of (1.3) can now be described provided that such information is available for the solution set of (1.1). We begin with a local result. The notation will be as in B, above.

**Lemma 1.3.** Suppose that \( 0 \not\in \sigma \). Then for all sufficiently small \( h \) there exists a neighborhood \( B_{\varepsilon} \) of \( 0 \) of the form (see Section 0.D)

\[
B_{\varepsilon} = \{ U : \| U - \hat{U} \| \leq \varepsilon \}
\]

such that there exists a unique solution \( \hat{U} \) of \( n_h(U) = 0 \) in \( B_{\varepsilon} \). Moreover, \( \varepsilon \) can be chosen independently of \( h \). If \( \bar{F}_h \) is the linearization of \( n_h \) at \( \hat{U} \), then the spectrum \( \Sigma_h \) of \( \bar{F}_h \) has the same number of positive eigenvalues as \( \sigma \), and \( 0 \not\in \Sigma_h \).

Let \( \hat{U} \) be the vector with components \( \hat{u}(y_i) \), \( 1 \leq i \leq N - 1 \). If \( V = U - \hat{U} \), we have that

\[
n_h(U) = \hat{F}_h V + q(V) + n_h[\hat{U}],
\]

where \( q \) is a vector with quadratic components of the form

\[
q(V) = \int_a^b \int_a^b f''(\tau) \, d\tau \, d\sigma.
\]

Note that \( q \) is independent of \( h \). Let \( Q = \hat{F}_h^{-1} \) and let \( \eta(V) = Qq(V) \). The equation \( n_h(U) = 0 \) is equivalent to

\[
\phi(V) = V + \eta(V) = -Qn_h[\hat{U}].
\]

If it can be shown for some \( \alpha \in (0, 1) \) that

\[
\| \eta(V) - \eta(W) \| \leq \alpha \| V - W \| \tag{1.11}
\]

for \( \| V \|, \| W \| \leq \varepsilon \), it follows from the contradiction mapping theorem that if \( A_\varepsilon \) is a ball of \( (L^2) \) radius \( \varepsilon \) about the origin, then \( \phi(A_\varepsilon) \) univalently covers a ball of \( (L^2) \) radius \( \varepsilon(1 - \alpha) \) about the origin (see Schwartz [16, Lemma 1.18]). Assuming (1.11) for the moment, we can prove the first assertion of the lemma. It follows from lemma 1.2 that \( \| Q \| \) is uniformly bounded as \( h \to 0 \), where \( \| Q \| \) is the operator norm induced by the discrete \( L^2 \) norm. From Lemma 1.1, it follows that \( |n_h(\hat{U})| \leq Kh \), and so,

\[
\| n_h(\hat{U}) \| \leq K^{1/2}h^{3/2} = O(h).
\]

Thus, if \( \alpha \) and \( \varepsilon \) are independent of \( h \), it follows that \( -Qn_h(\hat{U}) \) lies in a ball of radius \( \varepsilon(1 - \alpha) \) for all sufficiently small \( h \). This provides both the existence and local uniqueness of the solution \( \bar{V} \) of \( \phi(V) = 0 \). Finally, we define \( B_\varepsilon = \{ \bar{U} \} + QA_\varepsilon \). It follows that there exists a unique solution \( \bar{U} \) of \( n_h(U) = 0 \) in \( B_\varepsilon \) for all small \( h \).
We now prove (1.11). It follows from the definition of \( q(V) \) that

\[
\| \eta(V) - \eta(W) \| \leq \| Q \| \| V - W \| \left| \int_0^1 q'(tV + (1-t)W) \, dt \right|
\]

\[
\leq K \| Q \| (\| V \| + \| W \|) \| V - W \|,
\]

where \( K \) is constant depending only on \( \dot{U} \) and \( f' \). Thus if \( \| V \|, \| W \| \leq \varepsilon \), we must choose \( \alpha = 2K\| Q \| \varepsilon < 1 \). It is now clear that both \( \alpha \) and \( \varepsilon \) can be chosen so that (1.11) holds, where the choice of \( \alpha \) and \( \varepsilon \) is independent of \( h \).

The last remark concerning the spectrum \( \Sigma_h \) of \( \Gamma_h \) follows from Lemma 1.2 and standard continuous dependence theorems; we omit the details.

We conclude this section with a global result.

**Lemma 1.4.** Suppose that (1.1) admits exactly three solutions \( u_i, i = 0, \alpha, 1 \). Let \( \sigma_i \) be the spectrum of \( \sigma_n \) at \( u_i \) relative to the subspace of \( H^2[0, L] \) of functions satisfying (1.1b), and suppose that \( 0 \notin \sigma_i, i = 0, \alpha, 1 \).

Then for each sufficiently large \( M > 0 \), there exist exactly three solutions \( U_i, i = 0, \alpha, 1 \), of (1.3) in the ball \( |U| \leq M \), where \( U_i \) approximates \( u_i \) in the sense of Lemma 1.3.

**Proof:** Lemma 1.3 provides nondegenerate solutions \( U_i \) of (1.3) which lie in a ball \( B_i \) of \( \| \cdot \|_2 \) radius \( \varepsilon \) about \( \dot{U}_i \); here \( \dot{U}_i \) is obtained from \( u_i \) as in Lemma 1.3. Moreover, \( U_i \) is the unique solution of (1.3) in \( B_i \).

Suppose that the lemma is false. Then for a sequence \( h \to 0 \) there exist solutions \( \bar{U} \) of (1.3) such that \( \bar{U} \notin B_0 \cup B_{\alpha} \cup B_1 \) and \( |\bar{U}| \leq M \). We choose \( M \) so large that \( |U_i| < M \) for \( i = 0, \alpha, 1 \). (Since \( \| U_i \|_2 \) is uniformly bounded for all \( h \), it follows from the discrete Sobolev inequalities that \( |U_i| \) is uniformly bounded.) Thus \( \| U_i - \bar{U} \|_2 \geq \varepsilon \) for \( h \to 0 \).

Let \( \Gamma_i^h \) be the differential of \( \sigma_n \) at \( U_i \). If \( V_i = \bar{U} - U_i \), then

\[
\Gamma_i^h V = -q(V),
\]

where \( q \) is an \( h \)-independent quadratic term. It follows that

\[
\| \delta^2 V_i \| \leq L \| V_i \|
\]

for some constant \( L \) independent of \( h \) (see Section 0.D for notation). From the discrete Sobolev inequalities it follows that

\[
\| V_i \|_2 \leq L \| V_i \|
\]

for some \( L > 0 \) independent of \( h \). Thus for some constant \( F > 0 \) depending only on \( f \) and \( M \),

\[
\varepsilon \leq \| V_i \|_2 \leq F \left( \| \Gamma_i^h V_i \| + \| U_i \| \right) \| V_i \| \leq 2LFM \| V_i \|,
\]

so that \( \| V_i \| \geq \varepsilon F \) for some \( \varepsilon > 0 \).

Let \( \tilde{u}_i^h(y) \) be a piecewise linear interpolant of \( \bar{U} \); it follows as in
Lemma 1.3 that $\tilde{u}^h$ converges to a smooth solution $\tilde{u}$ of (1.1). However, 
$\|\tilde{u} - u_i\|_{L^2} \geq c\epsilon$ for $i = 0, \alpha, 1$, which is a contradiction. 

2. AN ISOLATING NEIGHBORHOOD FOR THE APPROXIMATE TRAVELLING WAVE EQUATIONS

A. Construction of an Isolating Neighborhood

We now construct an isolating neighborhood $\mathcal{N} \subset \mathbb{R}^{2N-2}$ for (0.6). $\mathcal{N}$ will then be used to define the connection index.

The relevant equations are

$$U' = W$$

$$W' = -\theta W - [\mathcal{L}U + \mathcal{F}(U)],$$

where $U = (u_1, \ldots, u_{N-1})'$, $W = (w_1, \ldots, w_{N-1})'$, $\mathcal{F}(U) = (f(u_1), \ldots, f(u_{N-1}))'$, and $\mathcal{L}$ is the $(N-1) \times (N-1)$ matrix

$$
\frac{1}{h^2} \begin{bmatrix}
-2 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & -2
\end{bmatrix}
$$

With regard to the notation of Section 1 we have that $u_0 = u_N = 0$, $h = L/N$, and $x = 0$.

As a first approximation to $\mathcal{N}$ let $\mathcal{N}_0$ be the region

$$\mathcal{N}_0 = \{(U, W); -\varepsilon \leq u_i \leq u_i^l + \varepsilon, 0 \leq w_i, 1 \leq i \leq N - 1, W^2h \leq D\},$$

where $u_i^l$ are the components of $U_1$, $\varepsilon$ is small and positive, and $D$ is to be determined; also, $W^2 = W \cdot W$.

For any set $M \subset \mathbb{R}^{2N-2}$ let $S(M)$ denote the maximal invariant set contained in $M$ of the flow generated by (2.1).

We remark that $g = \sum u_i$ is a Liapunov function for the flow in $S(\mathcal{N}_0)$. Thus $S(\mathcal{N}_0)$ consists of the set of critical points $\mathcal{C} = \{(U_i, 0); i = 0, \alpha, 1\}$ and solutions which connect rest points of lower index to rest points of higher index.

The final isolating region $\mathcal{N}$ will be of the form

$$\mathcal{N} = \mathcal{N}_0 \cup B_0 \cup B_1 \backslash B_2,$$

where $B_i$ is a neighborhood of $(U_i, 0)$. The ultimate goal is therefore to
show that \( S(\mathcal{N}) \cap \partial \mathcal{N} = \phi \). We begin by identifying the points in \( S(\mathcal{N}_0) \cap \partial \mathcal{N}_0 \). To this end let

\[
B_D = \{(U, W) \in \mathcal{N}_0 : W^2 h = D\}.
\]

**Lemma 2.1.** \( S(\mathcal{N}_0) \cap \partial \mathcal{N}_0 \subset \mathcal{C} \cup B_D \).

**Proof.** Let \((\bar{U}, \bar{W}) \in \partial \mathcal{N}_0 \setminus \mathcal{C} \cup B_D\) and let \((U, W)(\xi)\) be the solution curve of (2.1) passing through \((\bar{U}, \bar{W})\) at \(\xi = 0\). It must be shown that \((U, W)(\xi) \notin \mathcal{N}_0\) for some \(\xi\); there are several cases to be considered.

(i) Suppose \(\bar{u}_i = -\varepsilon\) for some \(i\). Recall that solutions in \(S(\mathcal{N}_0)\) consist of the critical set \(\mathcal{C}\) together with solutions connecting rest points whose \(U\)-components are monotone increasing. Since the \(U\)-components of all rest points are nonnegative it follows that \(w_j(\xi) = u_j(\xi)\) must be negative for some \(\xi < 0\). Such solutions must therefore leave \(\mathcal{N}_0\) in backward time.

(ii) Suppose that \(\bar{w}_i = 0\) for some \(i\). If \(w_i(0) \neq 0\) then \(w_i\) assumes negative values along either the forward or backward flow and so the solution leaves \(\mathcal{N}_0\). Suppose then that \(w_i(0) = 0\). Then

\[
w_i(0) = - (w_{i+1} + w_{i-1})h^{-2},
\]

where, if \(i = 1, N-1\) we take \(w_0, w_N\) equal to zero. If either of \(w_{i+1}\) or \(w_{i-1}\) are positive then \(w_i(0) < 0\) and \(w_j(\xi)\) therefore assumes negative values for \(\xi \neq 0\). Such solutions leave \(\mathcal{N}_0\). If \(w_{i+1} = w_{i-1} = 0\) we apply the above argument to each of these components. Proceeding inductively it follows that either \(w_j\) assumes negative values for some \(j\) or \(\bar{W} = 0\) and \(\mathcal{W}''(0) = 0\). In the latter case it then follows that \(\mathcal{L} \bar{U} + \mathcal{F}(\bar{U}) = 0\), and so \((\bar{U}, 0) \in \mathcal{C}\) contrary to our hypothesis.

**B. Derivative Estimates and Convergence Theorems**

The next task is to determine a suitable choice of the constant \(D\) in the definition of \(\mathcal{N}_0\). The main goal is to show that \(D\) can be chosen independently of \(h\). This yields sufficient compactness so that a convergent subsequence can be obtained as the mesh tends to zero. In the following we restrict to an interval \(|\theta| \leq \theta^*\); \(\theta^*\) appears later (see Lemma 2.12) as an a priori bound for the wave velocity of the continuous problem.

**Lemma 2.2.** Suppose that \(|\theta| \leq \theta^*\); then there exists \(D = D(\mathcal{F}, L, \theta^*)\) such that \(S(\mathcal{N}_0) \cap B_D = \phi\) for all sufficiently small \(h > 0\).

**Proof.** Suppose \((U, W)(\xi) \in S(\mathcal{N}_0)\); then \(U(\xi)\) satisfies

\[
U'' + \mathcal{L} U + \theta U' + \mathcal{F}(U) = 0.
\]
Let $I_j = \{ |\xi| < j \}$ for $j = 1, 2, 3$, and let $\phi(\xi)$ be a smooth function with $\phi \equiv 1$ on $I_2$ and $\phi \equiv 0$ on $\mathbb{R}^l \setminus I_3$. Then $\phi U$ satisfies

$$(\phi U)^'' + \phi L U = - \phi (\theta U' + \mathcal{F}(U)) + 2\phi U' + \phi'' U.$$ 

Now multiply this equation by $\phi U$ and integrate by parts to obtain (see Section 0.D)

$$\int_{I_3} [ -|(\phi U)'|^2 - \|\delta U\|^2 \phi^2 ] d\xi$$

$$= \int_{I_3} [(\theta - 2)(\phi^2)' + \phi''') \| U \|^2 + \phi'^2 \cdot U \cdot \mathcal{F}(U) \cdot h \] d\xi.$$ 

Since the solution lies in $S(A_0), |U| \leq 1$, and from the above, we obtain

$$\int_{I_3} [ \| U \|^2 + \| \delta U \|^2 ] d\xi \leq D_1 = D_1(\theta^*, \phi, f, L), \quad (2.3)$$

where $D_1$ is independent of $h$.

Now let $\psi(\xi)$ be a smooth function satisfying $\psi \equiv 1$ on $I_1$ and $\psi \equiv 0$ on $\mathbb{R}^l \setminus I_2$. We now apply an argument identical to the above to the equation satisfied by $\psi W(\xi)$, where $W = U'$ to obtain

$$\int_{I_2} [ \| (\psi W)' \|^2 + \psi^2 \| \delta W \|^2 ] d\xi \leq D_2 = D_2(\theta^*, \psi, f, L, D_1). \quad (2.4)$$

Let $D_3 = \int_{I_2} \psi^2 \| W' \|^2 d\xi$ and expand $(\psi W)'$ in (2.4); from (2.4) we obtain

$$D_3 \leq D_2 - 2 \int_{I_2} \psi \psi' W' \cdot W' h d\xi$$

$$\leq D_2 + 2KD_1^{1/2} D_3^{1/2},$$

where $K$ is a bound for $|\psi'|$. From the above it follows that

$$D_3 \leq [KD_1^{1/2} + (K^2 D_1 + D_2)^{1/2}]^{1/2}.$$ 

We can now prove the lemma. Let $\Gamma(\xi) = \psi(\xi) W^2(\xi) h$; from (2.3) and the above it follows that

$$\int_{I_2} |\Gamma'(\xi)| d\xi \leq KD_1 + 2D_1^{1/2} D_3^{1/2} = D_4,$$

so that $\Gamma(\xi) \leq 2D_4$. Thus if $D \geq 2D_4$, $W^2 h < D$ for $|\xi| \leq 1$. Since the center of the intervals $I_j$ was arbitrary, the same estimate holds for all $\xi \in \mathbb{R}^l$. \[}
Remark. The estimate (2.3) also yields an estimate on difference quotients of the solution, namely

\[ \sum_{i=0}^{N-1} \frac{(u_{i+1} - u_i)^2}{h} = -U \cdot \mathcal{L} U h \leq D_1. \] (2.5)

We can now prove a convergence theorem. To this end, we suppose for all small \( h \) that there exists a solution \((U^h, W^h)(\xi)\) of (2.1) in \( S(\mathcal{N}_0) \). Let \( u^h(\xi, y) \) be a piecewise linear interpolant of \( U^h \), i.e.,

\[ u^h(\xi, y) = (u_{i+1}(\xi) - u_i(\xi)) h^{-1}(y - y_i) + u_i(\xi), \quad y_i \leq y \leq y_{i+1}, \]

when \( u_0 = u_N = 0 \). For \( \gamma > 0 \) let

\[ S_\gamma = \{ (\xi, y) : |\xi| \leq \gamma, 0 \leq y \leq L \}. \]

**Lemma 2.3.** With notation as above there exists a sequence \( h_k \to 0 \) such that \( u^h \) converges to an exact solution \( u(\xi, y) \) of

\[ Au + \theta u_y + f(u) = 0, \quad u|_{\xi \leq \gamma} = 0. \] (2.6)

The convergence is in \( L^2(S_\gamma) \) for each \( \gamma > 0 \), and the limit satisfies the boundary conditions in the sense that \( \phi u \in H^1_0(S_\infty) \) for all smooth functions \( \phi \) compactly supported in \( \xi \).

**Proof.** A simple calculation shows that

\[ \iint_{S_\gamma} (u^h)^2 \, d\xi \, dy = \gamma \| \partial U^h(\xi) \|^2 \, d\xi \leq D_1, \]

by (2.5). Also,

\[ |u^h| \leq |w_{i+1} - w_i| |y - y_i| h^{-1} + w_i \leq w_{i+1} + 2w_i \]

on \( y_i \leq y \leq y_{i+1} \). Thus we have that

\[ \iint_{S_\gamma} |u^h|^2 \, d\xi \, dy \leq 6 \| W(\xi) \|^2 \, d\xi < 12\gamma D. \]

Hence

\[ \iint_{S_\gamma} [(u^h)^2 + |\nabla u^h|^2] \, d\xi \, dy \leq 2\gamma L(1 + D_1 + 12\gamma D) = D_\ast^2, \]

where \( D_\ast \) is independent of \( h \). Thus \( \{ u^h \} \) is uniformly bounded in \( H^1(S_\gamma) \) for all \( \gamma, h > 0 \). So that for each \( \gamma > 0 \) there exists \( h_\gamma(\gamma) \) such that \( u^h(\gamma) \) converges in \( L^2(S_\gamma) \) to a limit \( u_\gamma \). For \( \gamma \in \mathbb{Z}_+ \), we can choose \( h_\gamma(\gamma + 1) \) to be a
subsequence $h_k(\gamma)$. By a diagonal procedure we obtain a sequence $h_k = h_k(k)$ which converges in $L^2(S_\gamma)$ to a limit $u(\xi, u)$ for each $\gamma > 0$.

We now show that $u$ satisfies (2.6) weakly. Let $\psi \in C_0^\infty(S_\gamma)$. Then

$$I \int_{S_\gamma} \int [u \Delta \psi - \theta \psi \xi + \psi f(u)] \, d\xi \, dy = \lim_{k \to \infty} \int_{S_\gamma} \int [u_k \Delta \psi - \theta \psi \xi \, u_k + \psi \, f(u_k)] \, d\xi \, dy$$

$$= \lim_{k \to \infty} I_k$$

(note that as $u_k$ is uniformly bounded, $f(u_k)$ tends to $f(u)$ in $L^2(S_\gamma)$).

For any continuous $\rho \in H^1(S_\gamma)$ a simple computation shows that

$$\int_{S_\gamma} \int \rho(\xi, y) \, dy \, d\xi = \int_{-\gamma}^{\gamma} \rho(\xi, y) \, h \, d\xi + O(h),$$

where $O(h)$ depends only on the $H^1$ norm of $\rho$ and $h$. Let $\Psi(\xi)$ be the vector with components $\psi(\xi, y_i)$ and let $U_k(\xi) = U^{k}(\xi)$. It follows from Lemmas 1.1, 2.2, and the above that

$$I_k = O(h_k) + \int_{-\gamma}^{\gamma} \left[ \Psi \xi \cdot U_k + \mathcal{L} \Psi \cdot U_k - \theta \Psi \xi \cdot U_k + \psi \cdot \mathcal{F}(U_k) \right] \, d\xi = O(h_k)$$

since $(U_k, W_k)$ is a solution of (2.1). Thus $\lim I_k = 0$.

Finally let $\phi$ be as in the statement of the lemma and let $\gamma > 0$ be such that $\text{supp } \phi \subset S_\gamma$. Then $\phi u_k$ is a uniformly bounded sequence in $H^1_0(S_\gamma)$ since $\phi u_k$ is continuous and vanishes in $\partial S_\gamma$. It follows from the Banach–Alaoglu theorem that some subsequence of $\{\phi u_k\}$ converges weakly to an $H^1_0(S_\gamma)$ limit, say $u$. It easily follows that $\phi u = u$ (a.e.) and so $\phi u$ actually lies in $H^1_0(S_\gamma)$.

We remark that although $H^1$ bounds on $S_\gamma$ obtained above on $S_\gamma$ blow up as $\gamma \to \infty$, we can clearly perform estimates similar to those of Lemma 2.2 on any compact subset $K$ of $S_\infty$. Thus the solution $u$ obtained above is uniformly bounded in $H^1(K)$ for any translate $K_j = S_1 + \{j\}, j \in \mathbb{Z}$. Standard bootstrapping arguments then yield the following (see [1, Theorem 5.3]).

**Lemma 2.4.** Let $u$ be the solution of (2.8) obtained in Lemma 2.3. For each $R \geq 0$ there exists a constant $d_k$ depending only on $k, \theta^*$, $L$ and $f$ such that

$$\sup_{S_\infty} \sum_{|a| \leq k} |D^a u| \leq d_k.$$
Remark. Let \(w^h(\xi, y)\) be a piecewise linear interpolant of \(W^h(\xi)\), where 
\((U^h, W^h)(\xi)\) is a solution of (2.1) in \(S(\mathcal{N}_0)\). It follows from (2.4) that 
\[||\nabla w^h||_{L^2(K)}\] 
is uniformly bounded for any compact \(K \subset S_x\). It follows that 
some subsequence converges to a limit \(w(\xi, y)\), and that \(u_\xi = w\).

C. Increasing \(\mathcal{N}_0\)

Since \((U_0, 0)\) and \((U_1, 0)\) lie in \(\delta \mathcal{N}_0 \cap S(\mathcal{N}_0)\) small neighborhoods \(B_i\) of 
\((U_i, 0), i = 0, 1\), must be included so that these critical points are interior to 
the new region. However, this must be performed in such a manner that 
the maximal invariant set is not increased.

To this end we will need some information about the linearization of 
(2.1) about the critical set. Let \(X = (U, W)\) and let \(r(X)\) be the vector field 
on the right side of (2.1). If \(X_i = (U_i, 0)\) let \(R_i\) be the differential of \(r\) at \(X_i\). 
Thus if \(n(U) = \mathcal{L}U + \mathcal{F}(U)\) and \(M_i = dn\) at \(U_i\), then
\[R_i = \begin{bmatrix} 0 & I \\ -M_i & -\theta I \end{bmatrix}.
\]

In the following we assume that \(i = 0\) or \(1\) and we shall drop the subscript \(i\).
From Lemma 1.4 we have that the eigenvalues \(\lambda_{N-1} \leq \ldots \leq \lambda_1 < 0\) of \(M\) are 
negative and that \(\lambda_1\) is simple for small \(h\). It follows that the eigenvectors \(\gamma_j\) 
of \(R\) are of the form \((U_j, \mu_j U_j)\), \(1 \leq |j| \leq N-1\), where \(U_j = U_{-j}\) is an eigenvector 
of \(M\) corresponding to \(\lambda_j\) and
\[
\mu_j = \theta/2 + \frac{1}{2}((\theta^2 - 4\lambda_j)^{1/2}'}, \quad j \geq 1
\]
\[
= \theta/2 - \frac{1}{2}((\theta^2 - 4\lambda_j)^{1/2}', \quad j \leq -1.
\]

Since \(M\) is symmetric, \(U_i \cdot U_j = 0\) for \(i \neq j\). It follows that \(\gamma_i \cdot \gamma_j = 0\) for 
\(i \neq -j, j\). Since \(\lambda_j < 0\) for all \(j\) we have that \(\mu_j\) is real and nonzero for all \(j\).
We summarize this in the following lemma.

**Lemma 2.5.** \(X_i\) is a hyperbolic critical point of \(R_i, i = 0, 1\). The eigenvalues of \(R_i\) are 
real and the principal positive and negative eigenvalues \(\mu_{\pm 1}\) are simple.

Now let \(B_i\) be a small sup norm neighborhood of \(X_i, i = 0, 1\), say of radius \(\delta < \varepsilon\) and 
let \(\mathcal{N}_\varepsilon = \mathcal{N}_0 \cup B_0 \cup B_1\), where \(\varepsilon\) is as in Section 2.A.

**Lemma 2.6.** \(S(\mathcal{N}_\varepsilon)\) consists of the critical set \(\mathcal{C}\) together with orbits running 
from \(X_i\) to \(X_j\) with \(i < j\).

**Proof:** Let \(X(\xi)\) be a nonconstant solution in \(S(\mathcal{N}_\varepsilon)\). If \(W(\xi) > 0\) there 
is nothing to prove. Suppose then that some \(w_i < 0\) at some point along the
orbit, say at $\xi = 0$. At this point $X \in B_i$ for $i = 0$ or $1$. Assume that $i = 1$, the case $i = 0$ is handled in a similar manner.

We claim that $|U(\xi) - U_i| \leq \epsilon$ for all $\xi \geq 0$. Since $X$ lies in $S(\mathcal{N}_*)$ and $\delta < \epsilon$ it is immediate $U \leq U_i + \epsilon$ componentwise for all $\xi$, where $\epsilon$ is the vector whose components are all equal $\epsilon$. We also have that $U(\xi) > U_i - \epsilon$ for all $\epsilon > 0$ since if $u_i(\xi) = u_i' - \epsilon$ for some $\xi > 0$ and some $i$, it would follow that $w_i = u_i' < 0$ for some $\xi \in [0, \xi]$ for which $X(\xi) \notin B_1$, since $\delta < \epsilon$. Such solutions do not lie in $S(\mathcal{N}_*)$.

It also follows that $|W(\xi)| < A\sqrt{\epsilon}$ for all $\xi \geq 0$ and for some constant $A$ depending on $h$. For example, suppose that $w_i(0) > \delta$ for $\delta > 0$. Then since $w_i'$ is uniformly bounded, $w_i(\xi) > \delta/2$ for $D < \xi < K\delta$ for some $K$ depending on $h$. Thus

$$u_i(K\delta) \geq u_i(0) + \int_0^K w_i(\tau) \, d\tau \geq u_i' - \epsilon + K\delta^2/2.$$ 

Thus if $\delta = \sqrt{4\epsilon/K}$ we obtain a contradiction.

Since $X_i$ is hyperbolic it follows that $X(\xi)$ tends to $X_1$ in positive time since the solution remains in a small neighborhood of $X_1$ for $\xi > 0$.

It also follows that $X(\xi)$ must leave $B_1$ in backward time, since if this were not the case the solution would remain uniformly near $X_1$ for $\xi < 0$ and so, it would have to tend to $X_1$ in this direction. The solution $X(\xi)$ would then be a homoclinic orbit which is uniformly near $X_1$. If $\epsilon$ is small enough this is impossible since $X_1$ is a hyperbolic point.

Since $X$ lies in $S(\mathcal{N}_*)$ and $W \geq 0$, where $X \notin B_0 \cup B_1$, it easily follows that $X(\xi)$ tends either to $X_0$ or it enters $B_0$ in backward time; in the latter case we apply an argument similar to the above to the solution once it enters $B_0$.

\textbf{Lemma 2.1.} Let $\epsilon$ be so small that Lemma 2.6 is valid and that $f(-\epsilon) > 0$ and $f'(u) < 0$ for $|u| \leq \epsilon$. Then $S(\mathcal{N}_0 \cup B_0) = S(\mathcal{N}_o)$.

\textit{Proof.} We need only consider solutions in $S(\mathcal{N}_0 \cup B_0)$ along which either $u_i$ or $w_i$ assumes negative values.

Suppose that $u_i < 0$ at some point along the solution. Then by Lemma 2.6,

$$n = \inf \{u_i(\xi); 1 \leq i \leq N - 1, \xi \in \mathbb{R}^1\} < 0$$

is assumed at some finite $\xi$, say $\xi = 0$, for some particular $i$. Since $n$ is minimal, $w_i(0) = u_i'(0) = 0$ so that

$$u_i''(0) = w_i''(0) = -[(u_{i+1} - 2n + u_{i-1}) + f(n)].$$
Since $f(n) > 0$ and $n$ is minimal it follows that $u''(0) < 0$, which contradicts the minimality of $n$. Thus $U > 0$ along solutions in $S(M_0 \cup B_0)$.

Now suppose that $w_i < 0$ at some point. Then

$$n = \inf \{ w_i(\xi) : 1 \leq i \leq N - 1, \xi \in \mathbb{R}^1 \} < 0$$

is assumed at some finite $\xi$, say $\xi = 0$, for some particular $i$. It follows that $w'_i(0) = 0$ and that

$$w''_i(0) = -[(w_{i+1} - 2n + w_{i-1})h^{-2} + f'(u_i)n] < 0,$$

since $n$ is minimal and $f'(u_i)n > 0$. This again leads to a contradiction.

The inclusion of a neighborhood of $X_1$ is more difficult since both $f$ and $f'$ change sign along $u_i(y)$. In view of Lemmas 2.8 and 2.9, solutions in $S(M_0)$ along which $w_i < 0$ for some $i$ lie in the stable manifold $M_0$ of $X_1$ and by Lemma 2.7; this can occur only when the solution lies near $X_0$. We will show that this is impossible through a detailed analysis of the linearized flow

$$X' = R_1 X$$

in the $(N - 1)$-dimensional stable subspace $\mathcal{S}$ spanned by the eigenvectors $\gamma_j$ of $R_1$, $j \leq -1$. This will be accomplished through analysis of the flow induced by (2.7) on the projective space $\mathbb{R}P^{N-2}$. Recall that $\mathbb{R}P^{N-2}$ is defined to be $\mathbb{R}^{N-1}\{0\}/\sim$, where $x \sim y$ if $x = Ay$ for some $A \in \mathbb{R}\{0\}$. This method is borrowed in part from Jones [14].

We first introduce some notation. Let $\pi: \mathcal{S} \to \mathbb{R}P^{N-2}$ be the projection map and for a set $S \subset \mathcal{S}$ let $\pi(S)$ be denoted by $\hat{S}$. Let $\mathcal{S}_j$ be the “fast” stable subspace spanned by $\gamma_j$, $j \leq -2$. $\hat{S}_j$ is therefore a copy of $\mathbb{R}P^{N-j-3}$ embedded in $\mathbb{R}P^{N-2}$.

Recall that $U_k$ are the eigenvectors of $M_1$ and that $\gamma_j = (U_k, \mu_k U_k)$ are the eigenvectors of $R_1$. For $U \in \mathbb{R}^{N-1}$ let $\|U\| = U \cdot U$ and assume that $\|U_k\| = 1$ for all $k$. For $X \in \mathcal{S}$ define $\|X\|^2 = \|U\|^2 + \|W\|^2$. Thus if $X = \sum_{j \leq -1} c_j \gamma_j$ then $\|X\|^2 = \sum_{j \leq -1} (1 + \mu_j^2) c_j^2$. Also, for $X \in \mathcal{S}$ we denote the $j$th Fourier coefficient by $c_j(X)$.

We put local coordinates on $\hat{S} \setminus \mathcal{S}_j$ as follows. For $X \in \hat{S} \setminus \mathcal{S}_j$, $c_{-1}(\pi^{-1}(X)) \neq 0$. Let $\hat{X}$ denote the unique element of the fiber $\pi^{-1}(X)$ of unit length whose $c_{-1}$ coefficient is negative.

Let the flow generated by (2.7) be denoted by $X \cdot \xi$ and for a set $S \subset \mathcal{S}$ let

$$S \cdot \xi = \{ X \cdot \xi : X \in S \}.$$
Lemma 2.8. There exist positively invariant neighborhoods $\hat{V}_{-1}$ of $\hat{\gamma}_{-1}$, a negatively invariant neighborhood $\hat{V}_f$ of $\hat{\Sigma}$ such that $\hat{C} \cap \hat{V}_f = \phi$, and $\xi_0 < 0$ depending on $\hat{V}_{-1}$ and $\hat{V}_f$ such that for all $X \in \mathcal{C} \setminus \pi^{-1}(\hat{V}_{-1})$, $X \cdot \xi_0 \in \hat{V}_f$.

If for such $X$, $\|X\| = \alpha$ then there exists $K > 0$ depending only on $\xi_0$ such that $\|X \cdot \xi_0\| \leq \beta$, where $\beta = K\alpha$.

Proof. From (A) of Lemma 1.2 we have that $U_1$ has positive components; let $u_1 > 0$ be the minimum of the components of $U_1$.

Since $\mu_{-(N-1)} \leq \ldots \leq \mu_{-1} < 0$ it follows that $\hat{\gamma}_{-1}$ is an attracting critical point for the induced flow on $\mathbb{R}P^{N-2}$ and that $\hat{\Sigma}$ is a repeller for this flow (see, e.g., Jones [14]). Let $\hat{V}_{-1}$ be an attracting, positively invariant neighborhood of $\hat{\gamma}_{-1}$ and let $\hat{V}_f$ be a repelling, negatively invariant neighborhood of $\hat{\Sigma}$. (The existence of such neighborhoods is guaranteed by the abstract result that every attractor is contained in an attractor block; see Conley [4].) We choose these neighborhoods to be disjoint and $\hat{V}_{-1}$ to be so small that for $(U, W) \in \pi^{-1}(\hat{V}_{-1})$, $W$ has components of one sign. We will put more stringent conditions on $\hat{V}_{-1}$ and $\hat{V}_f$ later.

Since $\mathbb{R}P^{N-2}$ is compact there exists $\xi_0 < 0$ depending only on $\hat{V}_{-1}$ and $\hat{V}_f$ such that

$$\left(\mathbb{R}P^{N-2} \setminus \hat{V}_{-1}\right) \cdot \xi_0 \subset \hat{V}_f.$$  

(2.8)

Suppose that $X \in C$ with $\|X\| = \alpha > 0$. We will show that there exists $L > 0$ such that

$$c_{-1}(X) < -La.$$  

(2.9)

Here $L$ depends on the norm on $\mathcal{C}$ and hence, on $h$. However, this is not important here.

First note that for such $X = (U, W)$,

$$\alpha^2 = \sum_{j \leq -1} c_j^2(1 + \mu_j^2) \leq L_1 \sum_{j \leq -1} \mu_j^2 c_j^2 = L_1 \|W\|^2,$$

where $L_1$ is the maximum of $(\mu_j^2 + 1)/\mu_j^2$. Since all norms on $\mathcal{C}$ are equivalent and $W \geq 0$, there exists $L_2 > 0$ such that

$$\alpha L_2 \leq \max_{1 \leq j \leq N-1} \omega_j.$$

Finally, $W - \sum_{j \leq -1} c_j \mu_j U_j$ and so it follows that

$$u_1 L_2 \alpha \leq U_1 \cdot W = c_{-1} \mu_{-1}.$$

Thus (2.9) holds with $L = -\mu_{-1}^{-1} L_2 u_1$. 


The neighborhood $\hat{V}_{-1}$ of $\hat{y}_{-1}$ is chosen so small that $[\hat{V}_{-1}]^\sim$ lies interior to $C$; this is possible since $-\gamma_{-1}$ lies in the interior of $C$. From (2.9) it then follows that

$$C_{-1}(\hat{X}) < -L$$

for all $\hat{X} \in \hat{V}_{-1}$. We now define $\hat{V}_f$ to consist of all points in $\mathcal{G}_f$ together with

$$\{ \hat{X} : C_{-1}(\hat{X}) \geq -L/2 \},$$

since $\xi_0$ is chosen such that if $\hat{X} \notin \hat{V}_{-1}$ then $\hat{X} \cdot \xi_0 \in \hat{V}_f$, it follows from (2.9) that $\hat{X} \cdot \xi_0 \notin C$.

The last statement clearly holds with $K = \exp (\mu_{(N-1)} \xi_0)$.

We are now prepared to prove the main result of this section. Recall that

$$\mathcal{N}_* = \mathcal{N}_0 \cup B_0 \cup B_1.$$

**Lemma 2.11.** Let $B_1(\delta)$ be a ball of radius $\delta$ about $X_1$. Then for all sufficiently small $\delta$, $S(\mathcal{N}_*) = S(\mathcal{N}_0)$, i.e., the $W$-components of solutions in $S(\mathcal{N}_*)$ remain nonnegative.

**Proof.** Let $X(\xi)$ be a solution in $S(\mathcal{N}_*)$ and suppose that $w_i(\xi)$ assumes negative values at some point for some $i$. By previous remarks $X(\xi)$ lies in $\mathcal{M}_1$ and $W(\xi)$ can leave the positive cone only when $X(\xi)$ lies in $B_1(\delta)$. Let $\xi_*$ be the first time beyond which $W(\xi)$ leaves the positive cone, so that $W(\xi) > 0$ for all $\xi < \xi_*$. We parametrize the solution so that $\xi_* = 0$.

For simplicity we translate $X_1$ to the origin, so that $\mathcal{S}$ is the tangent space to $\mathcal{M}_1$ at the origin.

The idea is to locate a solution $X_1(\xi)$ of (2.7) in $\mathcal{S}$ which approximates $X(\xi)$ on the interval $\xi_0 \leq \xi \leq 0$ and which satisfies the hypotheses of Lemma 2.8. This should provide the desired contradiction, since $X(\xi_0) \in C_0$, where $C_0$ are the elements of $\mathcal{M}_1$ with nonnegative $W$-components.

To this end let $p : \mathcal{M}_1 \to \mathcal{S}$ be the projection map

$$p \left( \sum_{|i| \leq N-1} c_i \gamma_i \right) = \sum_{j \leq -1} c_i \gamma_i.$$

If $\mathcal{M}_1(\delta) = \mathcal{M}_1 \cap B_1(\delta)$ then $p$ is a diffeomorphism of $\mathcal{M}_1(\delta)$ onto its image for sufficiently small $\delta$, and for some constant $K > 0$,

$$\| p(X) - X \| \leq K\delta^2$$

for $X \in \mathcal{M}_1(\delta)$. 

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Let \( \alpha = \|X(0)\| \leq \delta \), and let \( X(0) = p(X(0)) \). It easily follows from Gronwall's inequality for some \( K_1 \) of order \( \exp (2\xi_0 \mu_{-(N-1)}) \) that

\[
\|X(\xi) - X(\xi)\| \leq K_1 \alpha^2 \tag{2.10}
\]

for \( \xi_0 \leq \xi \leq 0 \).

Since \( X(0) \in \partial C^n \) and \( X(0) = X(0) + \mathcal{O}(\alpha^2) \) it follows that some \( W \)-component of \( X(0) \) is of order \( \alpha^2 \). Recalling that \( \hat{V}_{-1} \) was defined so that \([\hat{V}_{-1}]^c \) is interior to \( C \), it follows that

\[
0 < m = \min \left\{ \tilde{w}_i : 1 \leq i \leq N - 1, (\hat{U}, \tilde{W}) \in \hat{V}_{-1} \right\}.
\]

However, \( \tilde{X}(0) = \mathcal{O}(\alpha^{-1})X(0) \) and it follows that \( \tilde{X}(0) \) has a \( W \)-component of order \( \alpha \). Thus for sufficiently small \( \alpha \), \( \tilde{X}(0) \notin \hat{V}_{-1} \). From Lemma 2.8 it follows that \( \tilde{X}(\xi_0) \in \hat{V}_{f_1} \), and so,

\[
0 > C_{-1}(X(\xi_0)) \geq -L\|X(\xi_0)\|/2
\]

(since \( X(\xi_0) \in C^n \), the left-hand inequality is easy to check for small \( \alpha \)).

Finally, since \( X(\xi_0) \in C^n \), \( X(\xi_0) \) can be expressed as \( X_+ + X_- \) where if some entry of \( X_+ \) is nonzero the corresponding entry of \( X_- \) is zero, and vice versa, and where the only entries of \( X_- \) which are nonzero are the negative \( W \)-components of \( X(0) \). It follows from (2.10) that \( \|X_-\| \leq K\alpha^2 \) for some constant \( K \) depending on \( K_1 \). Since \( X_+ \in C \) it follows from (2.9) and the above that

\[
C_{-1}(X(\xi_0)) < -L\|X(\xi_0)\| + 2K\alpha^2.
\]

Thus we will have obtained a contradiction if

\[
-L\|X(\xi_0)\| + 2K\alpha^2 < -L\|X(\xi_0)\|/2, \tag{2.11}
\]

i.e.,

\[
4K\alpha^2/L < \|X(\xi_0)\|.
\]

A simple computation shows that there exists a constant \( \tilde{K} \) depending only on \( \xi_0 \) such that

\[
\|X(\xi_0)\| < \tilde{K}\alpha.
\]

Since \( \alpha \leq \delta \), (2.11) will hold provided that, in addition to previous conditions imposed on \( \delta \), we choose \( \delta \leq \tilde{K}L/(4K) \).

D. The excision of \((U_*, 0)\)

In order to complete the construction it will be necessary to excise a
neighborhood $B_2(r)$ of $L_2$-radius $r$ from the region $\mathcal{N}_*$. Here, it will be important that the radius of $B_2(r)$ be chosen independently of $h$. Thus let

$$B_2(r) = \{(U, W): \|U - U_s\|^2 + \|W\|^2 \leq r^2\};$$

the final isolating region $\mathcal{N}$ is defined to be

$$\mathcal{N} = \mathcal{N}_* \setminus B_2(r) = \mathcal{N}_0 \cup B_0 \cup B_1(\delta) \setminus B_2(r).$$

From our previous results it follows that $S(\mathcal{N}) \cap \partial \mathcal{N} \subset \partial B_2(r) \cap \mathcal{N}$. The main result of this section is the following.

**Proposition 2.10.** (a) There exists $r > 0$ such that $S(\mathcal{N}) \cap \partial \mathcal{N} = \emptyset$ for all $|\theta| \leq \theta^*$ and for all sufficiently small $h > 0$. (b) If for each $h > 0$ there exists a nonconstant solution $(U^h, W^h) \in S(\mathcal{N})$ at some $|t| < 0$ then (some subsequence of) $(U^h, W^h, \theta^h)$ converges to an exact solution $(u, w, \theta)$ of (0.5), (0.3) as $h$ tends to zero.

The proof of Proposition 2.10 will be established with the aid of the following lemma.

**Lemma 2.11.** Let $u(\xi, y)$ be a smooth solution of (2.2) with $u(\xi, 0) = u(\xi, L) = 0$ and such that $u_\xi \geq 0$ in the infinite strip and that $u_\xi > 0$ at some point. Then

$$\lim_{\xi \to -\infty} u(\xi, y) = u_i(y), \quad \lim_{\xi \to \infty} u(\xi, y) = u_j(y),$$

(2.12)

where $u_i$ and $u_j$ are solutions of (0.4) with $i < j$. The above limits are uniform in $y$.

**Proof.** Since $u$ is monotone in $\xi$ and $u_\xi \not\equiv 0$ it follows that

$$\lim_{\xi \to -\infty} u(\xi, y) = u_-(y), \quad \lim_{\xi \to +\infty} u(\xi, y) = u_+(y),$$

where $u_-(y) \neq u_+(y)$.

Suppose first that $u_\xi$ and $|u_{\xi\xi}|$ tend uniformly to zero for large $|\xi|$ and that $u$ tends uniformly to $u_\pm(y)$. Then, from (2.6) $u_{\xi y}$ tends uniformly to $u_{\xi\pm}(y)$. It follows that $u_{\xi\pm}$ are solutions of (0.4).

Now let $\varepsilon = u_+(y) - u(\xi, y) \geq 0$. If the convergence of $u$ to $u_+$ were nonuniform there would exist $\delta > 0$ and a sequence $(\xi_k, y_k)$ with $\xi_k \to +\infty$ such that $\varepsilon(\xi_k, y_k) > \delta$. Let $\bar{y}$ be a limit point of $\{y_k\}$. From Lemma 2.4, $u_{\xi\bar{y}}$ is uniformly bounded and so, $\varepsilon(\xi_k, \bar{y}) > \delta/2$ for sufficiently large $k$. This yields a contradiction.

The uniform decay of $u_\xi$ and $|u_{\xi\xi}|$ is proved in a similar manner and we omit the details.
Proof of Proposition 2.10. We assume that the proposition is false and obtain a contradiction. With this assumption we make the following claim: there exist two solutions \( u'(\xi, y) \) of (2.12), \( i = 1, 2 \), such that \( u^1 \) satisfies (2.12) with \( u_i = u_0, \ u_j = u_a, \) and \( u^2 \) satisfies (2.12) with \( u_i = u_a, \ u_j = u_1. \) Moreover, these solutions exist at the same value of \( \theta. \)

We first assume that the claim is valid and derive a contradiction. To this end, consider the functional \( Q_0(u) \)

\[
Q_0(u) = \int_0^L \left[ -u^2/2 + F(u) \right] dy,
\]

where \( F(u) = f(u) \) and \( F(0) = 0. \) We will show that

\[
Q_0(u_x) < Q_0(u_i), \quad i = 0, 1. \tag{2.13}
\]

Assume (2.13) holds for the moment. Let \( u = u', \ i = 1 \) or \( 2 \) and define

\[
J_0(\xi) = \int_0^L u_i^2/2 \ dy + Q_0(u(\xi, y)).
\]

A simple computation shows that

\[
J_0(\xi) = -\theta \int_0^L u_i^2 \ dy \quad \tag{2.14}
\]

and so, \( J_0 \) converges to limits at \( \xi = \pm \infty. \) By Lemma 2.11, \( u_\xi \) tends to zero in \( L^2[0, L] \) for large \( |\xi|, \) and so

\[
\text{sgn}(Q_0(u_x)) = \text{sgn}(-\theta)
\]

\[
\text{sgn}(Q_0(u_1) - Q_0(u_2)) = \text{sgn}(\theta),
\]

where the first (resp. second) equation is obtained by integrating (2.14) over the real line. From (2.13) it can be seen that these equations are inconsistent for all real \( \theta. \)

We now prove (2.13). To this end consider the equation

\[
v_t = v_{yy} + f(v) \quad \tag{0 < y < L, \ t > 0}
\]

\[
v(t, 0) = v(t, L) = 0.
\]

Let \( Q(t) = Q_0(v(t, y)) \) where \( Q_0 \) is as above. Then

\[
Q'(t) = \int_0^L v_t^2(t, y) \ dy > 0
\]

if \( v \) is not an equilibrium. Note that the equilibria of (2.15) are \( u_i, \)
It is known that there exist bounded solutions \( v_1 \) and \( v_2 \) defined for all \( t \in \mathbb{R}^1 \) such that

\[
\lim_{t \to -\infty} v_1 = u_x, \quad \lim_{t \to +\infty} v_1 = u_0
\]

\[
\lim_{t \to -\infty} v_2 = u_x, \quad \lim_{t \to +\infty} v_2 = u_1
\]

(see Smoller [17, Lemma 24.12]). Let \( Q_1(t) = Q_0(v_1(t, y)) \). It easily follows that

\[
Q_1(-\infty) = Q_0(u_x) < Q_0(u_0) = Q_1(+\infty)
\]

\[
Q_1(-\infty) = Q_0(u_x) < Q_0(u_1) - Q_2(+\infty).
\]

This establishes (2.13).

We finally prove the claim made at the beginning of the proof. Since we are assuming that the proposition is false there exists \((r_k, h_k) \to (0, 0)\) such that \( S(\mathcal{N}) \cap \partial B_2(r_k) \neq \emptyset \) when \( h = h_k \). Let \((U^k, W^k)\) be a solution in \( S(\mathcal{N})\) which hits such a point. Let \( B_0 \) be a ball of fixed \( L_2\)-radius \( \rho \) about \((U_0, 0)\) and parametrize \((U^k, W^k)\) so that \((U^k, W^k)(0) \in \partial B_0\); (it follows from previous remarks that any nonconstant solution in \( S(\mathcal{N})\) connects \((U_0, 0)\) to \((U_1, 0)\)). Let \( \xi_k \) be the smallest \( \xi > 0 \) such that \((U^k, W^k)(\xi) \in \partial B_2(r_k)\).

We remark that by Lemma 2.3 and the remark after Lemma 2.4, piecewise linear interpolants of \( U^k, W^k \) converge to limits \( u, w \) respectively, with \( u_\xi = w \). Thus we need only show that \( u \) satisfies (2.12) with \( u_\xi = u_0 \) and \( u_j = u_x \).

To this end, consider the function

\[
\rho_k(\xi) = \|U^k(\xi)\|^2 + \|W^k(\xi)\|^2.
\]

It follows from the proof of Lemma 2.3 that (some subsequence of) \( \rho_k(\xi) \) converges uniformly to a continuous limit, namely

\[
\rho(\xi) = \int_0^\xi u^2 \, dy + \int_0^\xi u_x^2 \, dy.
\]

Moreover, \( \rho_k(0) = \rho^2 \) for all \( k \), so that \( \rho(0) = \rho^2 \). Thus \( u(\xi, y) \) is a nonconstant solution of (2.2) which is monotone in \( \xi \). Since \( u(0, y) \) is near zero, it follows from Lemma 2.13 that \( u \) tends to \( u_0 \) as \( \xi \to -\infty \).

We now show that \( u \) approaches \( u_x(y) \) as \( \xi \to +\infty \). To this end we first show that \( \xi_k \to +\infty \) for large \( k \). If this were not the case \( \xi_k \) would tend to a finite limit \( \xi > 0 \). Let \( r > 0 \) be fixed. For sufficiently large \( k \) \((U^k, W^k)(\xi) \in B_\alpha(r)\), and so, by an argument similar to that of the previous paragraph

\[
\int_0^\xi (u(\xi, y) - u_x(y))^2 \, dy + \int_0^\xi w(\xi, y)^2 \, dy < r.
\]
Since $r$ was arbitrary it follows that $u(x, y, t) \equiv u_x(y)$ and that $w(x, y, t) \equiv 0$.

Note that if $w(x, y, t) = w(x - \theta t, y)$ and $z = \exp(\gamma t)w$, then $z$ is a non-negative solution of

$$z_t = \Delta z + (f'(u) + \gamma)z.$$ 

Suppose that $\gamma$ is chosen so large that $f'(u) + \gamma > 0$. Moreover, $z(x, y, 0)$ does not vanish identically. It follows from the strong maximum principle for parabolic equations that $z(x, y, 1)$ and hence, $w$, is positive for all $x$ and $0 < y < L$. This yields a contradiction, so that $\lim_{t_k} = +\infty$.

Now let $\xi > 0$ be fixed. For sufficiently large $k$, $\xi < \xi_k$, and as $U^k$ is monotone in $\xi$, it follows that $U^k(\xi)$ is uniformly bounded away from $U^1$ as $k \to \infty$. Thus $u(x, y, t)$ is uniformly bounded away from $u(y)$ for all $\xi > 0$. From Lemma 2.11 it follows that $u(x, y, t)$ tends to $u_x(y)$ for large $\xi$.

A similar argument provides the solution $u^2$ in the statement of the claim.

We finally prove that, given the bound $\theta^*$, the assertion in part (b) of Proposition (2.10) easily follows from an argument similar to the one used in the proof of Lemma 2.3, since it can be assumed (by passing to a suitable subsequence) that $\theta^n$ converges to a limit $\theta$. We omit the details.

E. Bounds on the Wave Velocity

In order to define the connection index it will be necessary to determine $h$-independent bounds for $\theta$.

**Lemma 2.12.** There exists $\theta^* > 0$ such that $S(\mathcal{N}) = \{(U_0, 0), (U_1, 0)\}$ when $|\theta| \geq \theta^*$ for all sufficiently small $h$.

**Proof.** We obtain $\theta > 0$ such that there exist no solutions of the continuous problem (0.5), (0.3) whenever $|\theta| \geq \theta$. In view of part (b) of Proposition 2.10 it suffices to choose $\theta^* > 0$.

Let $(u, \theta)$ be a solution of (0.5), (0.3). We shall need bounds of derivatives of $u$ up to fourth order which are independent of $\theta$. Since the bounds in Lemma 2.3 depend on $\theta$, a different argument must be provided. To this end, we view $u(x, y, t) = u(x - \theta t, y)$ as a solution of the parabolic equation (0.1); since $u_x = u_x$ and $u_x = u_x$, it suffices to obtain $\theta -$ independent bounds on the derivatives of $\dot{u}$. Let $D$ be a bounded subdomain of $S$ and let $\Omega_i = D \times [t_i, T], i = 0, 1, 2$, where $0 = t_0 < t_1 < t_2 < T$ are fixed. It follows from the interior Schauder estimates that

$$|\dot{u}|_{1, \Omega_1} \leq K(|\dot{u}|_{0, \Omega_0} + |f(\dot{u})|_{0, \Omega_0})$$

$$|\dot{u}|_{2, \Omega_2} \leq K(|f(\dot{u})|_{0, \Omega_0} + |\dot{u}|_{0, \Omega_0})$$

$$< K(|f(\dot{u})|_{0, \Omega_0} + |\dot{u}|_{0, \Omega_0}),$$
where $K$ is a generic constant which is independent of $u$. Since $u = 0$ on $\partial S_\infty$ it follows from the Schauder boundary estimates that $K$ is also independent of the distance of $D$ from $\partial S_\infty$, and so, these estimates remain valid if $D$ coincides with a portion of $\partial S_\infty$. Estimates on higher derivatives are obtained in a similar manner. Finally, since $D$ was arbitrary, the estimates hold on all of $S_\infty$. Thus the derivatives of $\hat{u}$, and hence, of $u$ are uniformly bounded independently of $\theta$. (In fact, since $\hat{u}_\xi = \theta u_\xi$, the $\xi$-derivatives of $u$ decay as $|\theta| \to \infty$).

We now introduce a change of dependent variables in (0.5), namely $w \to \gamma$, where

$$\gamma(\xi, y) = w_\xi = -\theta w - [u_{yy} + f(u)].$$

It follows that (0.5) is equivalent to

$$u_\xi = -\frac{1}{\theta}(u_{yy} + f(u) + \gamma)$$
$$\gamma_\xi = -\frac{\theta}{\gamma} + \left[\gamma_{yy} + u_{yy} + f(u)_{yy} + f'(u)(\gamma + u_{yy} + f(u))\right].$$

(2.16)

Since $\gamma = w_\xi = u_\xi$, it follows from the previous paragraph that $\gamma$ and $\gamma_{yy}$ are bounded independently of $\theta$. Thus the entire term in square brackets in (2.16) remains uniformly bounded as $|\theta| \to \infty$. Let $\zeta = -\frac{1}{\theta} \xi$ and define another change of variables

$$\nu(\zeta, y) = u(-\theta \zeta, y)$$
$$\delta(\zeta, y) = \gamma(-\theta \zeta, y).$$

Then $(\nu, \delta)$ is uniformly bounded for all $\theta$ and satisfies

$$\nu_\zeta = \nu_{yy} + f(\nu) + \delta(\zeta, y)$$
$$\delta_\zeta = \theta^2 \delta + b(\zeta, y),$$

(2.17)

where $b(\zeta, y)$ is the term in square brackets in (2.16) with $(\nu, \delta)$ substituted in for $(u, \gamma)$. Thus $b$ is uniformly bounded independently of $\theta$. From the second equation in (2.17) it easily follows that $|\delta| \leq \sup |b|/\theta^2$ for all $(\zeta, y)$.

We first assume that $\theta \leq 0$. If $u$ solves (2.2), (0.3) then $u$ must be uniformly near zero for $\xi$ sufficiently negative, say for $\xi \leq \xi_0$. Thus $v$ is uniformly near zero for $\zeta_0 = -\theta \xi_0 < 0$. Let $\rho \in (0, \alpha)$ and choose $\xi_0$ so that $0 \leq v(\zeta_0, y) < \rho$ for all $y$. Choose $-\theta > 0$ so large that $f(\rho) + \delta(\xi_0, y) < 0$ for $\theta \leq \theta_-$ and all $(\zeta, y)$. We now view the first equation in (2.17) as an initial value problem for $v$ with initial data prescribed at $\zeta = \zeta_0$ where $\delta$ is regarded as a given function of $(\zeta, y)$. It follows from the maximum principle for the heat equation that $v(\zeta, y) < \rho$ for all $\zeta \geq \zeta_0$; this contradicts (0.3). More precisely, the lemma is valid when $\theta \leq \theta_-$ where

$$\theta_- > \frac{\sup |b|}{\sqrt{-f(\rho)}},$$
If \( \theta > 0 \) then for \( \xi \) sufficiently positive, say \( \xi \geq \xi_0, u(\xi, y) \) is near \( u_1(y) \) and so for \( \xi = \xi_0 = -\theta \xi_0 < 0, v(\xi_0, y) \) will be near \( u_1(y) \).

Now consider the equation

\[
v_\xi = v_{yy} + f(v).
\]

(2.18)

It is well known (see Smoller and Wasserman \([18]\)) that \( u_1(y) \) is a stable equilibrium of (2.18). Let \( W \) be a positively invariant, attractor block for \( u_1(y) \), say in \( L^2[0, L] \). It follows that for sufficiently large \( \theta \), say \( \theta \geq \theta_+ \), \( W \) is also positively invariant for the first equation in (2.17), and a contradiction is obtained as before. (We shall not provide explicit estimates for \( \theta_+ \); however, it appears that this would not be difficult, since precise information about the spectrum of the linearization of (0.4) about \( u_1(y) \) is available; see [18].)

Finally, we choose \( \theta \geq \max (\theta_-, \theta_+) \).

F. The Connection Index

It is now possible to define the connection index (see the Appendix). To this end augment (2.1) with the additional equation, \( \theta' = 0 \). The augmented system now generates a flow on \( \mathbb{R}^{2N-1} \). Let \( X_i = (U_i, 0), i = 0, 1 \) and define

\[
S = S(N \times [-\theta^*, \theta^*])
\]

\[
S_0 = \{X_0\} \times [-\theta^*, \theta^*]
\]

(2.19)

\[
S_1 = \{X_1\} \times [-\theta^*, \theta^*].
\]

It follows from Proposition 2.10 and Lemma 2.12 that \((S, S_0, S_1)\) determines a connection triple. Thus the connection index, \( \tilde{h}(S, S_0, S_1) \) is well defined for all sufficiently small \( h > 0 \).

3. Computation of the Connection Index

We now exploit the homotopy invariance of \( \tilde{h} \) by deforming (2.1) to a simpler set of equations. The main part of the homotopy consists of deforming the boundary conditions from the Dirichlet problem to the Neumann problem (see Section 0.C, Theorem 0.3). To this end we shall locate an isolating neighborhood \( \mathcal{N}_\lambda \), \( 0 \leq \lambda \leq 1 \), where \( \lambda \) is the homotopy parameter, analogous to the region \( \mathcal{N} \) constructed in Section 2. The information essential to this construction is provided by Theorem 0.3, namely, the multiplicity and spectra of solutions of the boundary value problems (0.8)\( \lambda \). We remark that each of these boundary value problems (0.8)\( \lambda \) are of the form (1.1), where \( a = a(\lambda) \). Since \( a(\lambda) \), \( L(\lambda) \) and \( a(\lambda) \) are uniformly bounded, we can apply the results of Section 1 to each boundary value problem.
to obtain exact information about the multiplicity and spectra of the solutions of the approximate equations (1.3). In particular, Lemma 1.4 holds for $h \leq h_0$, for all $\lambda \in [0, 1]$, where $h_0$ is uniform in $\lambda$. The details of the proof of Theorem 0.3 are not important here, and are provided in a separate paper (see [10]).

A. A Homotopy of the equations

We first describe the deformation of the equations (2.1). The continuous equations take the form

$$u_\xi = w \quad (0 < \eta < L_\lambda)$$
$$w_\xi = -\theta w - \left[u_{\eta\eta} + f_{\alpha(\lambda)}(u)\right]$$

$$l_{\alpha(\lambda)}(u(0), u_\eta(0)) = r_{\alpha(\lambda)}(u(L_\lambda), u_\eta(L_\lambda)) = 0,$$

or equivalently,

$$Au + \theta u_\eta + f_{\alpha(\lambda)}(u) = 0$$
$$l_{\alpha(\lambda)}(u(0), u_\eta(0)) = r_{\alpha(\lambda)}(u(L_\lambda), u_\eta(L_\lambda)) = 0.$$

Here $l_{\alpha}, r_{\alpha}$ are as in (1.1b) and $\alpha(\lambda), L_\lambda, a(\lambda)$ are as in Theorem 0.3 for $\lambda \in [0, 1]$. Thus $a(0) = 0, L_0 = L$, and $\alpha(0) = a$; also $a(1) = 1, \alpha(1) \in (0, 1)$ is arbitrary, and $L_1 = L_1(\alpha(1))$ is small enough that the Neumann problem (0.8) admits only the constant solutions $\{0, a, 1\}$.

The approximate equations take the form

$$U' = W$$
$$W' = -\theta W - \left[\mathcal{L}_h U + \mathcal{F}_h(U)\right],$$

where $U \in \mathbb{R}^{N-1}$, $\mathcal{F}_h(U)$ is the vector field with components $f_{\alpha(\lambda)}(u_i)$, $1 \leq i \leq N-1$, and $\mathcal{L}_h$ is the matrix $\mathcal{L}_h$ (see Section 1.A) with $\gamma_a = \gamma_{\alpha(\lambda)}$ (see (1.4)). From Lemma 1.1 we see that the mesh parameter $h$ also depends on $\lambda$, where $h(\lambda) = \mathcal{O}(\mathcal{L}/N)$. Thus for $\lambda \in [0, 1]$ we must choose $N$ so large that $h(\lambda)$ is uniformly small.

The equations (3.3) are equivalent to the second order system

$$U'' + \theta U' + \mathcal{L}_h U + \mathcal{F}_h(U) = 0.$$

B. An Isolating Neighborhood for (3.3)

An isolating region $\mathcal{N}_\lambda$ analogous to the region $\mathcal{N}$ of Section 2 will now be constructed for (3.3), $\lambda \in [0, 1]$. More precisely, the family $\{\mathcal{N}_\lambda\}$ will have the following properties: (i) $\mathcal{N}_0 = \mathcal{N}$, and (ii) if $S_{\lambda, \gamma} = S_{\lambda}(\mathcal{N}_\gamma)$, where $S_\gamma(\mathcal{N}_\lambda)$ is the maximal invariant set in $\mathcal{N}_\gamma$ relative to the flow induced by (3.3), then $S_{\lambda, \gamma} = S_{\lambda, \lambda}$ for $|\gamma - \lambda|$ sufficiently small. This ensures that the connection index $h_\lambda$ is well defined and independent of $\lambda$ for $0 \leq \lambda \leq 1$. 

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The construction is similar to that of $\mathcal{N}$ in Section 2. Let $U_i^\lambda$, $i = 0, \alpha, 1$, be the three solutions of (0.8), so that $(U_i^\lambda, 0)$ are the rest points of (3.3). Define

$$\mathcal{N}_0^{\lambda} = \{(U, W): -\varepsilon \leq u_i \leq u_i^\lambda + \varepsilon, 0 \leq w_i, 1 \leq i \leq N - 1, \text{ and } \|W\|^2 \leq D\};$$

here, $U_i^\lambda$ are the components of $U_i^\lambda$. Let $B_i$ be a small neighborhood about $(U_i^\lambda, 0)$; the region $\mathcal{N}_0^{\lambda}$ is then

$$\mathcal{N}_0^{\lambda} = \mathcal{N}_0^{\lambda} \cup B_0^\lambda \cup B_1^\lambda \setminus B_x^\lambda. \quad (3.5)$$

The details of the proof that $\mathcal{N}_0^{\lambda}$ is an isolating region for sufficiently small $h$ and $\lambda > 0$ are virtually identical to the case $\lambda = 0$ with the exception of a few minor points which we now describe. More precisely, the proof of Lemma 2.3 and Proposition 2.10 must be modified slightly to accommodate the more general boundary conditions.

With regard to Lemma 2.3 it must be shown that the convergence results established there with $\lambda = 0$ hold for all $\lambda \in [0, 1]$. The main point is that the boundary conditions are always coercive; i.e., the boundary terms in the weak formulation of (3.2) always have the "correct" sign. Thus the compactness result, Lemma 2.2, follows in exactly the same manner as before. This yields a convergent subsequence of the piecewise linear interpolants $u^h(\xi, y)$ of $U^h(\xi)$, where $(U^h, W^h)(\xi)$ is a solution in $S(\mathcal{N}_0^{\lambda})$. Here, we define

$$u_0 = \gamma_{a_i^1}u_1, \quad u_N = \gamma_{a_i^{N-1}}u_{N-1}.$$ 

It also follows as before that the limit, $u$, is a weak $(H_{1, loc}^1)$ solution of

$$\Delta u + \theta u + f(u) = 0.$$ 

The main job is to show that $u$ satisfies the boundary conditions. To this end, we note that for any compact $K \subset S_\infty$, $u$ lies in $H^1(K)$ since our subsequence can be taken to be weakly convergent in $H^1(K)$. Also, by the remark following Lemma 2.4, we have that the interpolants $w^h$ of $W^h$ converge weakly in $H^1(K)$ to a limit $w$, where $w = u$. Thus $u_\xi, u_{\xi\xi}, u_{\xi y} \in L^2(K)$. It follows from the differential equation that $u_{xy} \in L^2(K)$, and so, the limit actually lies in $H^2(K)$. Thus the traces of both $u$ and $u_{y}$ are well defined on $\partial S_\infty$ and lie in $H^{3/2}_{1, loc}(\partial S_\infty)$ and $H^{1/2}_{1, loc}(\partial S_\infty)$, respectively. Moreover, the trace of $u^h$ converges weakly in $H^{1/2}_{1, loc}(\partial S_\infty)$ to the trace of $u$.

We finally show that the trace of $u^h$ converges weakly in $H^{1/2}_{1, loc}(\partial S_\infty)$ to $u_{y}$. A simple computation shows that for $a > 0$.

$$u_{y}^h(\xi, 0) = \pm \frac{1-a}{a} u^h(\xi, 0).$$
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(Note that although \( u^h \notin H^2 \), \( u^h_0 \) still has a well-defined trace.) Thus \( u^h_0 \) converges weakly to \( \pm (1 - a) u^h(\xi, 0)/a \), and therefore \( u \) satisfies the boundary conditions.

It is now possible to apply the boundary estimates of Agmon, Douglis, and Nirenberg [1, Theorem 15.2] to obtain bounds on derivatives of arbitrary order.

The proof of Proposition 2.10 employed the results of Smoller [17] for the equation 2.15 to obtain estimates (2.13) on the functional \( Q_0(u_i) \), where \( u_i \) is a solution (0.4). The main facts required in Smoller's proof concern the exact multiplicity and spectral properties of solutions of (0.4). This is precisely the information provided by Theorem 0.3 for all boundary value problems in the homotopy, namely in each \( \lambda \), there exist exactly three solutions \( u_i, i = 0, 1, 2 \), of (8). Moreover, \( u_0 \) and \( u_1 \) always have negative spectra, while \( u_2 \) has exactly one positive eigenvalue. Smoller's proof therefore applies to all boundary value problems appearing in the homotopy and Proposition 2.12 is valid for all \( \lambda \in [0, 1] \); the details will be omitted.

We summarize this in the following lemma.

**Lemma 3.1.** Let \( \mathcal{N}^{\lambda} \) be defined as in (3.5). Then for all \( \lambda \in [0, 1] \) there exist \( h(\lambda) \) and \( \theta^* \) such that \( \mathcal{N}^{\lambda} \) is an isolating neighborhood for (3.3) for all \( \theta \leq \theta^* \) where \( \theta^* \) is independent of \( \lambda \), and for all \( h < h(\lambda) \) (see 3.8). If \( S(\mathcal{N}^{\lambda}) \setminus \emptyset \neq \emptyset \) for a sequence \( (\theta(h), h) \) with \( h \to 0 \), a subsequence of the approximate solutions converge to a classical solution of (3.2) for some \( |\theta| < \theta^* \).

Now append the equation \( \theta' = 0 \) to (3.3), and let

\[
S^i = S(\mathcal{N}^{\lambda}) \times [-\theta^*, \theta^*] \\
S^i_0 = \{(U^i_0, 0)\} \times [-\theta^*, \theta^*], \quad i = 0, 1.
\]

Then \((S^0, S^0_0, S^1)\) is a connection triple and \( h_\lambda = h(S^0, S^0_0, S^1) \) is well defined for \( h \leq h(\lambda) \) and \( 0 \leq \lambda \leq 1 \).

**Lemma 3.2.** (a) There exists \( h_* > 0 \) such that \( h(\lambda) > h_* \) for all \( \lambda \in [0, 1] \). (b) For \( h < h_* \), \( h_0 = h_1 \).

The proof of (a) follows in a manner similar to the proof of the uniformity of the parameters of \( \mathcal{N}^{\lambda} \) relative to \( \theta \), and we omit the details. Part (b) immediately follows from (a) and the continuation property of the connection index.

C. Decoupling (3.3)

At \( \lambda = 1 \), \( L_1 \) can be chosen arbitrarily small. We now show that if \( L_1 \) is
small enough then \( u_1 = u_2 = \cdots = u_{N-1} \) along solutions in \( S(N') \) for all sufficiently small \( h \). After a change of variables, we will see that most of the components of (3.3), decouple from one another.

**Lemma 3.3.** There exists \( \sigma > 0 \) such that for all \( L_1 < \sigma \) and for all sufficiently small \( h \), the components of solutions \((U, W)\) in \( S(N') \) are equal, i.e., \( u_i = u_i \) and \( w_i = w_i \) for \( 1 \leq i \leq N - 1 \).

**Proof.** Let \( g(\xi) = -U \mathcal{L}^1 U h \); it is easily verified that

\[
g(\xi) = \sum_{i=1}^{N-2} (u_{i+1}(\xi) - u_i(\xi))^2/h.
\]

We will show that if \( L_1 \) is sufficiently small, then \( g(\xi) = 0 \) for all \( \xi \), where \((U(\xi), W(\xi))\) is a solution in \( S(N') \).

To this end, we show that \( g \) satisfies a certain differential inequality. First note that since \( \mathcal{L}^1 \) is symmetric,

\[
g_\xi = -2U_\xi \mathcal{L}^1 Uh
\]

\[
g_{\xi\xi} = -2U_\xi \mathcal{L}^1 U_\xi h - 2U_\xi \mathcal{L}^1 Uh.
\]

Since \( U \) is a solution of (3.4), we have that

\[
g_{\xi\xi} = -2U_\xi \mathcal{L}^1 U_\xi h - 2(-\theta U_\xi - \mathcal{L}^1 U - \mathcal{F}_0(U)) \mathcal{L}^1 U h
\]

\[
= 2p(\xi) + \theta g_\xi + 2\|\mathcal{L}^1 U\|^2 + 2\mathcal{F}_0(U) \mathcal{L}^1 U h,
\]

where \( p(\xi) = -U_\xi \mathcal{L}^1 U_\xi h \) is nonnegative.

Let the spectrum of \( \mathcal{L}^1 \) be \( \mu_0(h) > \mu_1(h) \geq \cdots \geq \mu_{N-2}(h) \). If \( 1 \) is the vector whose entries all equal 1, then \( \mathcal{L}^1 1 = 0 \), so that \( \mu_0(h) = 0 \) and \( \mu_1(h) < 0 \). Let \( 0 = \mu_0 > \mu_1 > \cdots \) be the spectrum of \( d^2/dy^2 \) on \([0, L_1]\) with homogeneous Neumann conditions. Then \( \mu_1 = -\pi^2/L_1^2 \), and by Lemma 1.2, \( \mu_1(h) \to \mu_1 \) as \( h \to 0 \).

We claim that

\[
\|\mathcal{L}^1 U\|^2 \geq -\mu_1(h) g(\xi).
\]

Since \( \mathcal{L}^1 \) is symmetric there is an orthonormal basis of eigenvectors of \( \mathcal{L}^1 \), say \( U_i, 0 \leq i \leq N - 2 \). Then if \( U = \sum_{i=1}^{N-2} c_i U_i \), it follows that

\[
\|\mathcal{L}^1 U\|^2 = \sum_{i=1}^{N-2} \mu_i^2(h) c_i^2 \geq -\mu_1(h) \sum_{i=1}^{N-2} c_i^2 = -\mu_1(h) g(\xi).
\]
We now consider the last term in $g_{\xi \xi}$. In particular

$$|\mathcal{F}_1(U) \mathcal{L}^1 U h| \leq \sum_{i=1}^{N-2} \left( \frac{f_i(u_{i+1}) - f_i(u_i)}{h} (u_{i+1} - u_i) \right)$$

$$\leq K \sum_{i=2}^{N-2} \frac{(u_{i+1} - u_i)^2}{h} = Kg,$$  \hspace{1cm} (3.8)

where $K$ depends only on $|f'_1(u)|$, $0 \leq u \leq 1$.

Using (3.7) and (3.8) in (3.6) we obtain the desired differential inequality, namely

$$g_{\xi \xi} - 2\theta g_{\xi} + 2(\mu_1(h) + K) g \geq 2p(\xi) \geq 0.$$  \hspace{1cm} (3.9)

If $(U(\xi), W(\xi))$ is in $S(M^{-1})$, then the solution connects $(U_0^1, 0)$ at $\xi = -\infty$ to $(U_1^1, 0)$ at $\xi = +\infty$. However, the components of $U_1^1$ are all equal to $i$, for $i = 0, 1$. Thus $g(\pm \infty) = 0$. Suppose that $L_i$ is so small that $-\pi^2/L_i^2 + K < 0$. Then if $g$ is not identically zero, $g$ must have a positive maximum at some point; this contradicts (3.9). Thus $g \equiv 0$, and $u_i = u_i$, $2 \leq i \leq N-1$. Since the components of $U$ are equal, it follows that $\mathcal{L}^1 U(\xi) = 0$ for all $\xi$, and since $g_{\xi \xi}$ is identically zero, we have that $-W \mathcal{L}^1 W = U_1^1 \mathcal{L}^1 U_1 = 0$ for all $\xi$. Thus the components of $W$ are equal to each other.

We now introduce a change of variables $(u_i, w_i) \rightarrow (\gamma_i, \delta_i)$, for $2 \leq i \leq N-1$. In particular, let

$$\gamma_i = u_i - u_{i-1}, \quad \delta_i = w_i - w_{i-1}, \quad 2 \leq i \leq N-1.$$

The equation (3.3) transforms to

$$u'_i = w_i$$

$$w'_i = -\theta w_i - [\gamma_1 h^{-2} + f(u_1)]$$

$$\gamma'_i = \delta_i$$

$$\delta'_i = -\theta \delta_i - [(-2\gamma_2 + \gamma_3) h^{-2} + f(u_1 + \gamma_2) - f(u_1)]$$

$$\gamma'_2 = \delta_2$$

$$\delta'_2 = -\theta \delta_2 - [(\gamma_2 - 2\gamma_3 + \gamma_4) h^{-2} + f(u_1 + \gamma_2 + \gamma_3) - f(u_1 + \gamma_2)]$$

$$\vdots$$

$$\gamma'_{N-1} = \delta_{N-1}$$

$$\delta'_{N-1} = -\theta \delta_N - [(\gamma_{N-2} - 2\gamma_{N-1}) h^{-2} + f(u_1 + \sum_{i=2}^{N-1} \gamma_i) - f(u_1 + \sum_{i=2}^{N-2} \gamma_i)].$$

(3.10)
If $\gamma$ (resp. $\delta$) is the vector with components $\gamma_i$ (resp. $\delta_i$), (3.10) can be expressed as

$$u'_i = w_i, \quad \gamma' = \delta$$

$$w'_i = -\theta w_i - [\gamma_i h^{-2} + f(u_i)], \quad \delta' = -\theta \delta - [L_D \gamma + K(\gamma) \gamma],$$

where $L_D$ is the $(N-2) \times (N-2)$ matrix

$$L_D = h^{-2} \begin{bmatrix} -2 & 1 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & -2 \end{bmatrix}$$

and $K(u_i, \gamma)$ is the diagonal matrix with entries

$$k_{ij}(u_i, \gamma) = f(u_i + \sum_{i=2}^{j} \gamma_i) - f(u_i + \sum_{i=2}^{j-1} \gamma_i), \quad \gamma_j \neq 0$$

$$= f'(u_i + \sum_{i=1}^{j-1} \gamma_i), \quad \gamma_j = 0.$$

Note that $K$ is uniformly bounded along solutions in $S(\mathcal{N}^{-1})$ and that $L_D$ approximates $d^2/dy^2$ on $[0, L_1]$ with homogeneous Dirichlet conditions.

We shall need an isolating neighborhood for (3.11). If $u = (u_1, w_1, \gamma, \delta)$, then $u$ is a diffeomorphism of $\mathbb{R}^{2N-2}$ onto itself. Thus if $\hat{\gamma} = \pi(u)$, then $\hat{\gamma}$ is an isolating neighborhood for (3.11) and $h(\hat{\gamma})$ is a connection triple for (3.11); moreover,

$$h(S^1, S^0, S^1) = h(\hat{S}, \hat{S_0}, \hat{S_1}).$$

Suppose that $L_1 < \sigma$, so that by Lemma 3.2, $\gamma(\xi) = \delta(\xi) = 0$ along solutions in $S(\hat{N})$. Let $(\tilde{u}_1, \tilde{w}_1, 0, 0) \in \hat{N}$ and define

$$\rho(\tilde{u}_1, \tilde{w}_1) = \{(\gamma, \delta); (\tilde{u}_1, \tilde{w}_1, \gamma, \delta) \in \hat{N}\}.$$

Since $\hat{N}$ is an isolating neighborhood it follows that for each point $(\tilde{u}_1, \tilde{w}_1, 0, 0) \in S(\hat{N})$, $\rho(\tilde{u}_1, \tilde{w}_1)$ contains a neighborhood $\rho$ of the origin in the $(\gamma, \delta)$-plane, and as $S(\hat{N})$ is compact $\rho$ can be chosen uniformly for all $(\tilde{u}_1, \tilde{w}_1, 0, 0) \in S(\hat{N})$.

We define another isolating region for (3.11). Let

$$A_0 = \{(u_1, w_1); -\epsilon \leq u_1 \leq 1 + \epsilon, 0 \leq w_1 \leq D\}$$

and let $B_i$ be a small neighborhood of $(i, 0)$, $i = 0, \alpha, 1$. Define

$$A = A_0 \cup B_0 \cup B_1 \setminus B_\alpha;$$
if $D$ is sufficiently large, then $A$ is an isolating region for

$$u'_1 = w_1, \quad w'_1 = -\theta w_1 - f(u_1)$$  \hspace{1cm} (3.12)

for all $\theta$ (see [4]). Let $\mathcal{N}^* = A \times \rho$. By Lemma 3.2, $\mathcal{N}^*$ isolates the same invariant set as $\mathcal{N}$ with respect to the equations (3.11). It follows that the connection triple obtained from $\mathcal{N}^*$ is precisely $(\mathcal{S}, \mathcal{S}_0, \mathcal{S}_1)$, and so $h(\mathcal{S}, \mathcal{S}_0, \mathcal{S}_1)$ can be computed using either $\mathcal{N}$ or $\mathcal{N}^*$, since $h$ depends only on the connection triple.

Next, consider the homotopy of (3.11),

$$u'_1 = w_1, \quad \gamma' = \delta$$

$$w'_1 = -\theta w_1 - [\varepsilon_1 h^{-2} + f(u_1)], \quad \delta' = -\varepsilon \delta - [L_D \gamma + \varepsilon K(u_1, \gamma) \gamma],$$

where $1 \geq \varepsilon \geq 0$.

**Lemma 3.4.** Suppose that $L_1$ is as in Lemma 3.3. Then for sufficiently small $h$, $\mathcal{N}^*$ is an isolating neighborhood for (3.13), for each $\varepsilon \in [0, 1]$ and $\gamma(\xi) = \delta(\xi) = 0$ along all solutions in $S(\mathcal{N}^*)$.

**Proof.** The matrix $L_D$ approximates the operator $L = d^2/dy^2$ on $[0, L]$ with homogeneous Dirichlet conditions. Let $\nu_2(h) > \nu_3(h) > \cdots > \nu_{N-1}(h)$ be the spectrum of $L_D$, and $\nu_2 > \nu_3 > \cdots$ be the spectrum of $L$. By Lemma 1.2, $\nu_2(h)$ approaches $-\pi^2/L^2$ as $h \to 0$; thus the spectrum of $L_D$ is of order $L^{-2}$.

Now let $(u_1, w_1, \gamma, \delta)(\xi) \in S(\mathcal{N}^*)$ and define $g(\xi) = \|\gamma(\xi)\|^2$. It follows that

$$g(\xi) = 2\gamma \cdot (-\varepsilon \delta - L_D \gamma - \varepsilon K(u_1, \gamma) \gamma) h + 2\gamma \cdot \gamma \cdot h.$$

Since the components of $\varepsilon K(\gamma)$ are uniformly bounded for solutions in $S(\mathcal{N}^*)$ we have that $|2\gamma \cdot \varepsilon K(u_1, \gamma)\gamma| \leq \varepsilon Kg$ for some constant $K > 0$. It also follows that $-\gamma \cdot L_D \gamma \geq |\nu_2(h)|^{-1} g$; this is verified by expanding $\gamma$ in the (orthonormal) eigenvectors of $L_D$. Thus

$$g(\xi) + \varepsilon \theta g + (v_2(h) - \varepsilon K) g \geq 2\gamma \cdot \gamma \cdot h \geq 0.$$  \hspace{1cm} (3.14)

For $L_1 < \sigma$ and $\epsilon \in [0, 1)$ the coefficient of $g$ in (3.14) is negative.

We claim that $g(\xi) \equiv 0$ for all $\xi$. Clearly, $g(\xi) \geq 0$ for all $\xi$. Suppose that $g(\xi_0) = 0$ at some $\xi_0$. If $g(\xi_0) > 0$ then, as $L_1 < \sigma$ it follows from (3.14) that $g(\xi_0) > 0$ so that $g$ has a strict local minimum at $\xi_0$. Suppose that $g(\xi_0) = 0$. Then if $\gamma_2(\xi_0) = \delta(\xi_0) \neq 0$ it again follows that $g(\xi_0) > 0$. If $\delta(\xi_0) = 0$, then $\gamma(\xi) \equiv \delta(\xi) \equiv 0$ for all $\xi$, since $\gamma = \delta = 0$ is an invariant manifold of (3.12). Thus, if $g(\xi)$ does not vanish for all $\xi$ then $g$ has exactly one critical point which must be a strict local minimum. In the lat-
ter case $g$ tends to positive limits $g_{\pm}$ as $\xi \to \pm \infty$. Let $(u_1, w_1, \gamma, \delta)_{\pm}(\xi)$ be solutions in the $\alpha$- and $\omega$-limit sets of $(u_1, w_1, \gamma, \delta)(\xi)$. These solutions lie in $S(N^*)$ and, in particular, they lie on the spheres $\|\gamma\|^2 = g_{\pm}$. Thus $g_{\pm}(\xi) = \|\gamma_{\pm}(\xi)\|^2$ is constant for all $\xi$. However, (3.14) would then imply that $g_{\pm} > 0$ for all $\xi$, which is impossible. Thus $g_+ = g_- = 0$. Since $g$ cannot have a local maximum it must be that $g(\xi) = 0$ for all $\xi$. Thus the $(u_1, w_1)$-components of solutions of (3.13) are solutions of (3.12). Since $(0,0)$ is an interior point of $\rho$ and $\Lambda$ is an isolating neighborhood for (3.12) it follows that $N^*$ is an isolating neighborhood (3.13) for each $\varepsilon \in [0,1]$.

D. Proof of Theorem 0.1

We can now complete the proof of Theorem 0.1. It follows from Lemma 3.4 that $N^*$ contains exactly two critical points of (3.13) for each $\varepsilon \in [0,1]$, namely $(i,0,0,0)$, $i = 0, 1$. Let

$$S^* = S(N^* \times [-\theta^*, \theta^*])$$

(relative to (3.13))

$$S_0^* = \{(0,0,0,0)\} \times [-\theta^*, \theta^*]$$

$$S_1^* = \{(1,0,0,0)\} \times [-\theta^*, \theta^*].$$

Then $(S^*, S_0^*, S_1^*)$ is a connection triple for (3.13) and $S^*$ lies in $\Lambda \times \{(0,0)\}$ for each $\varepsilon$. Thus $S^*$ is independent of $\varepsilon$ and $h(S^*, S_0^*, S_1^*)$ is the same for each of the flows (3.13).

We have finally continued the flow (2.1) in $N^*$ to

$$\begin{align*}
u_1' &= w_1, \\
\gamma' &= \delta \\
w_1' &= -\theta w_1 - f_1(u_1), \\
\delta' &= -L_D \gamma
\end{align*}$$

in $N^*$, i.e., the connection $h$ index of the triple (2.18) for (2.1) equals the index $h$ of (3.15) for the flow (3.16) for all sufficiently small $h$.

Note that $(\gamma, \delta)$ is completely decoupled from $(u_1, w_1)$ in (3.16) and that the $(\gamma, \delta)$-equations are linear with a saddle point at the origin; the latter remark easily follows from the fact that the spectrum of $-L_D$ is positive. In fact, the saddle point has $(n-2)$ positive and $(n-2)$ negative eigenvalues.

Now let

$$s = S(\Lambda \times [-\theta^*, \theta^*])$$

(relative to (3.12))

$$s_0 = \{(0,0)\} \times [-\theta^*, \theta^*]$$

$$s_1 = \{(1,0)\} \times [-\theta^*, \theta^*].$$

Then $(s, s_0, s_1)$ is a connection triple for (3.12), and the index $h$ of $(s, s_0, s_1)$
is $\bar{0}$, i.e., the homotopy type of a pointed point; the details of this com-
putation can be found in Refs. [5], [6] and will be omitted here.

It follows that $h(S, S_0, S_1)$ for the equations (2.1) is $\bar{0} \wedge \Sigma^{N-2} = \bar{0}$, where
$\Sigma^{N-2}$ is a pointed $N-2$ sphere and " $\wedge$ " is the smash product.

It can now be shown that $S \ni S_0 \cup S_1$. It must be shown that
$h(S, S_0, S_1) = \bar{0}$ is "nontrivial" in the sense of Theorem A.1 (see the Appen-
dix). To this end, note that the (Conley) indices of $s_0$ and $s_1$ with respect to
(3.12) are each equal to $\Sigma^1$, and so $h(S_0^*) = h(S_1^*) = \Sigma^1 \wedge \Sigma^{N-2} = \Sigma^{N-1}$.
Since $S_0^*$ and $S_1^*$ continue as isolated invariant sets throughout the
homotopy, it follows that $h(S, S_0, S_1) = \Sigma^{N-1}$, $i = 0, 1$, where $S_i$ is the critical
point $(U_i, 0)$ of (2.1). Thus
$$h(S, S_0, S_1) = \bar{0} \neq (\Sigma^1 \wedge h(S_0)) \wedge h(S_1) = \Sigma^N \vee \Sigma^{N-1},$$
so that by theorem A.1, $S(N)$ contains a nonconstant solution for some
$\theta^h \in (-\theta^*, \theta^*)$ for all sufficiently small $h$. By Proposition 2.10 it follows
that some subsequence converges to a solution $u$ of (0.5), (0.3).

Since the $W$-components of the approximate solutions are positive it
follows that $u_\xi \geq 0$ in the entire strip $S_x$. We finally show that $u_\xi > 0$
everywhere. If $w = u_\xi$, let $z(x, y, t) = e^{i\xi t}w(x - \theta t, y)$. Then $z$ is nonpositive
and satisfies the equation
$$\Delta z - z_t = (f'(u) + \gamma)z.$$  
Choose $\gamma$ so that $f'(u) + \gamma < 0$. Then $\Delta z - z_t \geq 0$; if $z$ were equal to zero at
an interior point then it would follow from the strong maximum principle
for the heat equation that $z \equiv 0$. This contradicts (0.3). Thus $u_\xi(\xi, y) > 0$ for
all $\xi$ and for $0 < y < L$.  

4. ESTIMATES ON THE WAVE VELOCITY (PROOF OF THEOREM 0.2)

We conclude with a few simple estimates for the wave velocity $\theta$. To this
end it will be convenient to rescale $y$ to $\tilde{y}$ and work on a fixed $\tilde{y}$ interval,
$0 < \tilde{y} < L_0$, where $L_0$ is as in Theorem 0.1. Thus let $y = D^{-1}\tilde{y}$, where
$D = L_0/L$, and define $\tilde{u}(\xi, \tilde{y}) = u(\xi, D\tilde{y})$. Dropping the bar, we see that (0.5)
is equivalent to
$$u_{\xi\xi} + D^2 u_{\theta \theta} + \theta u_\xi + f(u) = 0$$
$$u(\xi, 0) = u(\xi, L_0) = 0.$$  

The limiting states $u_0$ and $u_1$ are solutions of
$$0 = D^2 u_{\theta \theta} + f(u)$$
$$u(0) = u(L_0) = 0.$$  

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The multiplicity results in [18] can then be stated as follows

\[
(4.2) \begin{cases}
1 \text{ solution if } D > 1 \\
2 \text{ solutions if } D = 1 \\
3 \text{ solutions if } D < 1.
\end{cases}
\]

It is easily seen that solutions of (4.2) can be characterized as critical points of the functional

\[
Q_D(u) = \int_0^T \left[ -D^2 u_y^2 / 2 + F(u) \right] \, dy
\]

on the space \(H^1_0(0, L_0)\). Now suppose that \(u(\xi, y)\) is the connecting solution of (4.1) and define \(J_D(\xi) = \int_0^{L_0} u_y^2 / 2 \, dy + Q_D(u(\xi, \cdot))\). If (4.1) is multiplied by \(u_x\) and integrated with respect to \(y\), we obtain

\[
J'_{D}(\xi) = \theta \int_0^{L_0} u_x(\xi, y)^2 \, dy,
\]

and so,

\[
J_D(A) - J_D(-A) = -\theta \int_{-A}^A \int_0^{L_0} u_x^2 \, d\xi \, dy.
\]

From Lemma 2.13, it follows that \(u_x\) tends to zero uniformly as \(|\xi| \to +\infty\). Thus

\[
\text{sgn}(Q_D(u_1) - Q_D(u_0)) = \text{sgn}(-\theta),
\]

and since \(u_0 = 0\), it follows that \(\text{sgn}(\theta) = \text{sgn}(-Q(u_1))\).

We can provide estimates for \(Q_D(u_1)\) in two different regimes.

**Lemma 4.1.** Suppose that \(D = 1 - \epsilon\). Then for \(\epsilon\) small and positive, \(Q_D(u_1) < 0\) and so, \(\theta > 0\).

**Proof.** \(Q_D(u)\) is a continuous functional for \((D, u) \in \mathbb{R}_+ \times H^1_0(0, L_0)\). Let \(\bar{u}\) be the second, nonzero solution of (4.2) when \(D = 1\). We claim that \(Q_1(\bar{u}) < 0\). Assuming this for the moment, the lemma easily follows. In particular, for \(D = 1 - \epsilon\) with \(\epsilon > 0\), there are two branches of solutions \(u_x\) and \(u_1\) of (4.2) which are continuous \(H^1_0(0, L_0)\) valued functions of \(D\). Since \(u_x\) and \(u_1\) coalesce to \(\bar{u}\) as \(\epsilon \to 0\) it follows that \(Q_D(u_1) \to Q_1(\bar{u})\) as \(D \to 1^-\), so that for sufficiently small \(\epsilon\), \(Q_D(u_1) < 0\).

We now show that \(Q_1(\bar{u}) < 0\). To this end, consider the parabolic equation

\[
\begin{align*}
&u_t = D^2 u_{yy} + f(u) \\
&u(0, t) = u(L_0, t) = 0.
\end{align*}
\]

(4.3)
If \( u(y, t) \) is a solution of (4.3) then it is easily checked that

\[
\frac{d}{dt} Q_D(u(\cdot, t)) = \int_0^{L_0} u^2 \, dy,
\]

(4.4)

so that \( Q_D \) is a Liapunov function for the semiflow generated by (4.3), say on the space \( H^1_0(0, L_0) \). From general considerations it follows that all bounded solutions of (4.3) tend to critical points of \( Q_D \), i.e., solutions of (4.2), as \( t \to +\infty \).

It is easily shown that the zero solution \( u_0 = 0 \) of (4.2) is a strict local maximum of \( Q_D \) for all \( D > 0 \). In particular, the second variation of \( Q_D \) at \( u_0 \)

\[
\delta^2 Q_D(0), (\phi, \psi) = \int_0^{L_0} f''(0) \phi \psi \, dy,
\]

is negative definite since \( f'(0) < 0 \). Thus, given \( \delta > 0 \) there exists a neighborhood \( K_0 \) about \( u = 0 \) in \( H^1_0 \) such that \( Q_D(u) = -\delta \) for all \( u \in \partial K_0 \) and \( -\delta \leq Q_D(u) \leq 0 \) for all \( u \) in \( K_0 \).

If \( Q_D(\tilde{u}) > 0 \) then there exists \( \varepsilon > 0 \) such that \( Q_D(\tilde{u}) > -\delta/2 \) for \( 1 \leq D \leq 1 + \varepsilon \). We now use \( \tilde{u} \) as initial data for (4.3) with \( D = 1 + \varepsilon \); let the solution be denoted by \( \tilde{u}(y, t) \). It follows from the maximum principle that \( 0 \leq u \leq 1 \) is a positively invariant region for (4.3), and since \( 0 \leq \tilde{u}(y) < 1 \), \( \tilde{u}(y, t) \) remains bounded for all \( t > 0 \), and so, \( \tilde{u}(y, t) \) tends to a solution of (4.2) as \( t \to +\infty \). However, the only solution of (4.2) for \( D > 1 \) is \( u = 0 \), and since \( Q_D(u) \geq -\delta/2, Q_D(\partial K_0) < -\delta \), \( Q_D \) must decrease at some point along \( u(y, t) \). This contradicts (4.4).

**Lemma 4.2.** Suppose that \( 0 < D \leq 1 \); then \( Q_D(u_1) > 0 \) and \( \theta < 0 \).

**Proof.** The solution \( u_1 \) is positive with a (unique) maximum at \( y = L_0/2 \) so that \( \tilde{u}_1(L_0/2) = 0 \). Let \( p = u_1(L_0/2) \); it is easily seen that \( p \to 1 \) as \( D \to 0 \) (this follows from the results in [18]).

Since \( 0 < x < \frac{1}{2} \) it follows that there exists a unique \( y \in (x, 1) \) such that \( F(y) = 0 \). Also, \( F(u) \) is negative (resp. positive and monotone increasing) for \( u \in (0, y) \) (resp. \( u \in (y, 1) \)). Select \( \bar{p} \in (y, 1) \) such that

\[
2F(u) - F(1) > \rho > 0
\]

(4.5)

for all \( u \in [\bar{p}, 1] \) and for some \( \rho > 0 \). For \( D \) sufficiently small, \( p \in (\bar{p}, 1) \).

If \( u \) is any solution of (4.2) it is easily seen that

\[
D^2 \left( \frac{du}{dy} \right)^2 + F(u) = C.
\]
If \( u = u_1 \) we can take \( C = F(p) \), and so

\[
\frac{D^2}{2} \left( \frac{du_1}{dy} \right)^2 = F(p) - F(u_1). \tag{4.6}
\]

Now let \( \tilde{y} \in (0, L_0/2) \) be such that \( u_1(\tilde{y}) = \rho; \) it follows that \( u_1(L_0 - \tilde{y}) = \rho \) also. For \( 0 \leq y \leq \tilde{y} \) and \( L_0 - \tilde{y} \leq y \leq L_0 \) there exists a positive \( H \) such that

\[
F(p) - F(u_1(y)) > H;
\]

this follows from previous remarks about the sign and monotonicity of \( F \) on \((0, \gamma) \) and \((\gamma, 1)\). From (4.6) we see that

\[
\left| \frac{du_1}{dy} \right| \geq \sqrt{\frac{2H}{D}}, \quad 0 \leq y \leq \tilde{y}, \quad L_0 - \tilde{y} \leq y \leq L_0.
\]

It follows that \( \tilde{y} \leq \rho D/\sqrt{2H} \).

We can now estimate \( Q_D(u_1) \) for small \( D \). First express \( Q_D(u_1) \) as

\[
Q_D(u_1) = \int_{\tilde{y}}^{y} \left( -\frac{D^2}{2} \left[ \frac{du_1}{dy} \right]^2 + F(u_1) \right) dy + \int_{L_0 - \tilde{y}} L_0 \left( -\frac{D^2}{2} \left[ \frac{du_1}{dy} \right]^2 + F(u_1) \right) dy.
\]

From (4.6) we see that the integrands of \( I_1 \) and \( I_2 \) are uniformly bounded as \( D \to 0 \), and since \( \tilde{y} = O(D) \), we have that \( |I_1| + |I_2| = O(D) \) as \( D \to 0 \). From (4.6) we have that

\[
I_3 = \int_{\tilde{y}}^{L_0 - \tilde{y}} (2F(u_1(y)) - F(p)) dy.
\]

If \( D \) is sufficiently near zero then \( p \) will lie close enough to unity so that by (4.5),

\[
2F(u) - F(p) > \rho/2
\]

for all \( u \in [\tilde{\rho}, \rho] \). Thus the integrand of \( I_3 \) is uniformly positive for \( \tilde{y} \leq y \leq L_0 - \tilde{y} \), and as \( \tilde{y} \to 0 \) as \( D \to 0 \) it follows that \( Q_D(u_1) > 0 \) for sufficiently small \( D \).

It is plausible that the wave velocity \( \theta \) is bounded from below by the wave speed \( \bar{\theta} \) of the travelling wave solution of the bistable diffusion equation in one space variable, and that as \( D \to 0 \), \( \theta \) tends to \( \bar{\theta} \). We will not pursue these questions further.
Consider a local flow on $\mathbb{R}^n$ generated by the differential equations

$$\frac{dx}{d\xi} = f(x). \tag{A.1}$$

If $N$ is a compact neighborhood in $\mathbb{R}^n$, let $S(N)$ denote the points on solution curves which remain in $N$ for all $\xi$. $N$ is called an isolating neighborhood and $S(N)$ is called an isolated invariant set provided that $S(N) \cap \partial N = \emptyset$.

A homotopy invariant associated with isolated invariant sets $S = S(N)$ can be defined as follows. Suppose that $N^2$ is a closed subset of $N$ such that

1. $N^2$ is positively invariant relative to $N$;
2. $S(N) \subset N \setminus N^2$;
3. any solution of (A.1) with initial data in $N$ and which leaves $N$ for some $\xi > 0$ hits a point in $N_2$.

If $S = S(N)$ is isolated then an $N_2$ with these properties exists; $(N, N^2)$ is called an index pair. The Conley index of $S$ is defined to be $[N/N^2]$, i.e., the homotopy type of the space obtained by collapsing $N_2$ to a point. It can be shown that if $S$ is isolated, index pairs exist and that the index depends only on $S$; $[N/N^2]$ is therefore denoted by $h(S)$. Moreover, $h(S)$ is invariant under homotopies of the flow which preserve the isolation property. The proofs of these remarks can be found in Conley's monograph [4].

The following facts are also discussed in [4]:

1. If $S = \emptyset$ then $h(S)$ is the homotopy type of a pointed point; we denote this homotopy type by $0$.
2. If $S$ is a hyperbolic critical point with $k$ positive eigenvalues then $h(S) = \Sigma^k$, i.e., a pointed $k$-sphere.
3. If $S_1$ and $S_2$ are disjoint isolated invariant sets then $S_1 \cup S_2$ is isolated, and $h(S_1 \cup S_2) = h(S_1) \vee h(S_2)$, i.e., the space obtained by gluing $h(S_1)$ and $h(S_2)$ together at their distinguished point.
4. If $S_i$ is an isolated invariant set for $x_i = f_i(x_i), i = 1, 2$, then $S_1 \times S_2$ is an isolated invariant set for the product system, $(x_1, x_2) = (f_1(x_1), f_2(x_2))$, and $h(S_1 \times S_2) = h(S_1) \wedge h(S_2)$, where "$\wedge$" is the smash product.
B. The Connection Index

Suppose that there is given a parametrized system

\[ \frac{dx}{d\xi} = f(x, \theta), \quad (A.2) \]

where \( x \in \mathbb{R}^n \) and \( \theta_0 \leq \theta \leq \theta_1 \). If \((A.2)\) is augmented with the additional equation \( d\theta/d\xi = 0 \), the equations generate a flow on \( Y = \mathbb{R}^n \times [\theta_0, \theta_1] \) for which each slice \( \theta = \) constant is invariant. If \( S \) is a subset of \( Y \), let \( S_{\theta} \) denote the points in \( S \) whose last coordinate equals \( \theta \).

A homotopy invariant for the augmented equations is defined as follows. Suppose that \( S, S^0 \) and \( S^1 \) are subsets of \( Y \) with the following properties:

(i) \( S^0 \), \( S^1 \), and \( S^i \) are isolated invariant sets with respect to \((A.2)\) for each \( \theta \in [\theta_0, \theta_1] \);
(ii) \( S^0 \cup S^1 \subset S \);
(iii) \( S^0 \cup S^1 = S_{\theta} \) when \( \theta = \theta_i, i = 0, 1 \).

Then \((S, S^0, S^1)\) is called a connection triple. The connection index, denoted by \( h(S, S^0, S^1) \) can then be defined; \( h \) depends only on the connection triple and it is invariant under suitable homotopies (see \([5]\)). The construction of \( h \) entails modifying the \( \dot{\theta} \) equation near \( S_{\theta}, i, j = 0, 1 \). We shall not describe the details here except to mention that the modified equations admit an isolating neighborhood \( \hat{N} \) for the flow on the augmented space \( Y \).

The connection index \( h \) is then defined to be the Conley index \( h(S(N)) \), where \( S(N) \) is with respect to the augmented, modified flow.

Roughly speaking, \( h \) can be understood as follows. If \((\hat{N}, \hat{N}^2)\) is an index pair then for \( \theta_0 < \theta < \theta_1 \), \((\hat{N}_\theta, \hat{N}_\theta^2)\) is an index pair for \( S_{\theta} \). At \( \theta = \theta_i, i = 0, 1 \), \( \hat{N}_\theta \) also consists of points in the “unstable manifold” of \( S_{\theta}, i, j = 0, 1 \), which, by (iii), lie in the exit set. Thus \( h \) measures a change in the manner in which the unstable manifold of \( S^0 \) leaves \( \hat{N} \) at \( \theta = \theta_0 \) and \( \theta = \theta_1 \). (The modifications in the \( \dot{\theta} \) equation mentioned earlier are such as to make this picture correct.) If the index is “nontrivial” in a sense to be made precise below, the unstable manifold sweeps across \( S^1 \) as \( \theta \) varies from \( \theta_0 \) to \( \theta_1 \), and so, at some \( \theta^* \in (\theta_0, \theta_1) \) there is a connection from \( S_{\theta^*} \) to \( S_{\theta^*} \).

The following theorem is proved in \([6]\).

**Theorem A.1.** Suppose that \( h(S, S^0, S^1) \neq (h(S^0) \wedge S^1) \vee h(S^1) \). Then \( S \nsubseteq S^0 \cup S^1 \).

If the flow in \( S \) happens to admit a Liapunov function, Theorem A.1 provides a solution running from \( S^0 \) to \( S^1 \) at some \( \theta^* \in (\theta_0, \theta_1) \).

The “standard” example appearing at the end of the homotopy in Section 3 is discussed in detail in several references (see \([5, 6, 9]\)).
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