Abstract

Let \(\varphi_N : \tilde{X} \rightarrow X\) be a regular covering projection of connected graphs with the group of covering transformations isomorphic to \(N\). If \(N\) is an elementary abelian \(p\)-group, then the projection \(\varphi_N\) is called \(p\)-elementary abelian. The projection \(\varphi_N\) is vertex-transitive (edge-transitive) if some vertex-transitive (edge-transitive) subgroup of the automorphism group of \(X\) lifts along \(\varphi_N\), and semisymmetric if it is edge- but not vertex-transitive. The projection \(\varphi_N\) is minimal semisymmetric if it cannot be written as a composition \(\varphi_N = \varphi \circ \varphi_M\) of two (nontrivial) regular covering projections, where \(\varphi_M\) is semisymmetric.

Malnič et al. [Semisymmetric elementary abelian covers of the Möbius–Kantor graph, Discrete Math. 307 (2007) 2156–2175] determined all pairwise nonisomorphic minimal semisymmetric elementary abelian regular covering projections of the Möbius–Kantor graph, the Generalized Petersen graph \(GP(8, 3)\), by explicitly giving the corresponding voltage rules generating the covering projections. It was remarked at the end of the above paper that the covering graphs arising from these covering projections need not themselves be semisymmetric (a graph with regular valency is said to be semisymmetric if its automorphism group is edge- but not vertex-transitive). In this paper it is shown that all these covering graphs are indeed semisymmetric.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05C25; 20B25

Keywords: Möbius–Kantor graph; Semisymmetric graph; Arc-transitive graph

1. Introduction

Throughout this paper a graph means a finite, connected, simple and undirected graph. For a graph \(X\), denote by \(V(X)\), \(E(X)\) and \(\text{Aut}(X)\) the vertex set, the edge set and the automorphism group of \(X\), respectively. For \(u, v \in V(X)\), \(u \sim v\) means that \(u\) and \(v\) are adjacent and denote by \(\{u, v\}\) the edge incident to \(u\) and \(v\) in \(X\). An \(s\)-arc in a graph \(X\) is an ordered \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_{s-1}, v_s)\) of vertices of \(X\) such that \(v_{i-1}\) is adjacent to \(v_i\) for \(1 \leq i \leq s\) and \(v_{i-1} \neq v_{i+1}\) for \(1 \leq i \leq s - 1\). A graph \(X\) is said to be \(s\)-arc-transitive if \(\text{Aut}(X)\) is transitive on the set of \(s\)-arcs in \(X\). In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A subgroup of \(\text{Aut}(X)\) is \(s\)-regular if the subgroup acts regularly on the set of \(s\)-arcs in \(X\), and \(X\) is said to be \(s\)-regular if \(\text{Aut}(X)\) is \(s\)-regular. A graph \(X\) is edge-transitive if \(\text{Aut}(X)\) acts transitively on \(E(X)\), and semisymmetric provided that it has regular valency and is edge-transitive but not vertex-transitive.

For convenience, in the following we introduce some concepts used in this paper, of which most can be extracted from [17,18,23]. An epimorphism \(\varphi : \tilde{X} \rightarrow X\) of graphs is called a regular covering projection if there is a semiregular subgroup \(\text{CT}(\varphi)\) of the automorphism group \(\text{Aut}(X)\) of \(X\) whose orbits in \(V(\tilde{X})\) coincide with the vertex fibres \(\varphi^{-1}(v)\),

E-mail address: yqfeng@bjtu.edu.cn (Y.-Q. Feng).

0012-365X/S - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.07.096
$v \in V(X)$, and the arc and edge orbits of $CT(\phi)$ coincide with the arc fibres $\phi^{-1}(u, v)$, $u \sim v$, and the edge fibres $\phi^{-1}[u, v]$, $u \sim v$, respectively. The semiregular group $CT(\phi)$ is called the covering transformation group. If $CT(\phi)$ is isomorphic to an abstract group $N$ then we speak of $\phi$ (or $\phi_N$ instead of $\phi$ for emphasizing) a regular $N$-covering projection. The projection $\phi_N$ is $p$-elementary abelian if $N$ is an elementary abelian $p$-group. Let $\phi : \tilde{X} \to X$ be a regular covering projection. An automorphism of $\tilde{X}$ is said to be fibre-preserving if it maps a vertex fibre to a vertex fibre, and all such fibre-preserving automorphisms form a group called the fibre-preserving group. Two regular covering projections $\phi : \tilde{X} \to X$ and $\phi' : \tilde{X}' \to X$ of a graph $X$ are isomorphic if there exists an automorphism $z \in \text{Aut}(X)$ and an isomorphism $\tilde{z} : \tilde{X} \to \tilde{X}'$ such that $\phi' = \phi \tilde{z}$. In particular, if $z = \text{id}$ then $\phi$ and $\phi'$ are equivalent.

If, in the above setting, $\tilde{X} = \tilde{X}'$ and $\phi = \phi'$, then we call $\tilde{z}$ a lift of $z$, and $\tilde{z}$ a projection of $\tilde{z}$ along $\phi$. In this case, concepts such as a lift of a subgroup of $\text{Aut}(X)$ and the projection of a subgroup of $\text{Aut}(\tilde{X})$ are self-explanatory. A regular covering projection $\phi : \tilde{X} \to X$ is vertex-transitive (edge-transitive) if some vertex-transitive (edge-transitive) subgroup of $\text{Aut}(X)$ lifts along $\phi$, and semisymmetric if it is edge- but not vertex-transitive. The projection $\phi$ is minimal vertex-transitive (minimal edge-transitive or minimal semisymmetric) if $\phi$ cannot be written as a composition $\phi = \phi_1 \phi_2$ of two “smaller” regular covering projections $\phi_1$ and $\phi_2$, where $\phi_2$ is vertex-transitive (edge-transitive or semisymmetric).

Covering techniques have been used as a powerful tool in topology and graph theory. Recently, several results pertaining to constructions and classifications for various classes of symmetric and semisymmetric graphs have been obtained using such techniques. For example, elementary abelian coverings of complete graphs were first dealt with in [6] and later on a slightly more general context was considered in [5]. Feng and Kwak [9] classified the regular cyclic coverings of the bipartite graph $K_{2,3}$ for each $1 \leq s \leq 5$ when the fibre-preserving group contains an arc-transitive subgroup. The $s$-regular cyclic or elementary abelian coverings of the hypercube $Q_3$ were classified in [10,11] for each $1 \leq s \leq 5$ when the fibre-preserving group is arc-transitive. In [20], it was shown that all cubic graphs admitting a solvable edge-transitive group of automorphisms arise as regular coverings of one of the following graphs: the complete graph $K_4$ of order 4, the dipole $\text{Dip}_3$ with two vertices and three parallel edges, the complete bipartite graph $K_{3,3}$, the Pappus graph, and the Gray graph (the smallest semisymmetric cubic graph). Moreover, it was shown that each such graph can be obtained from these “basic” graphs by a sequence of edge-transitive elementary abelian regular coverings. A more detailed study of edge- and/or vertex-transitive elementary abelian coverings of “small” cubic graphs $\text{Dip}_3$, $K_4$, $Q_3$, and $K_{3,3}$ can be found in [5,8–10,21]. Furthermore, Malnić and Potočnik [24] classified vertex-transitive elementary abelian coverings of the Petersen graph when the fibre-preserving group is vertex-transitive. The study of semisymmetric graphs was initiated by Folkman [7] who, among others, gave constructions of several infinite families of such graphs and posed a number of open problems which spurred the interest in this topic (see for example [1,2,7,12–15,26,31]). In particular, Marušič [25] constructed the first infinite family of cubic semisymmetric graphs and as one of the first applications of covering techniques, Malnić et al. [22] classified cubic semisymmetric graphs of order 2$p^3$. A beautiful recent result on the automorphism groups of cubic semisymmetric graphs of twice odd order was obtained by Parker [27]. Malnić et al. [21] classified cubic semisymmetric cyclic coverings of the bipartite graph $K_{3,3}$ when the fibre-preserving group contains an edge- but not vertex-transitive subgroup. Semisymmetric elementary abelian coverings of the Heawood graph were considered in [4,20]. Using the method developed in [19,20], Malnić et al. [18] determined all pairwise nonisomorphic minimal semisymmetric elementary abelian regular covering projections of the Möbius–Kantor graph, the Generalized Petersen graph $\text{GP}(8, 3)$ and at the end of that paper, they remarked that the respective covering graphs related to these covering projections need not themselves be semisymmetric. The purpose of this paper is to prove that all such respective covering graphs are semisymmetric.

2. Main results

For a subgroup $H$ of a group $G$, denote by $C_G(H)$ the centralizer of $H$ in $G$ and by $N_G(H)$ the normalizer of $H$ in $G$. Then, $C_G(H)$ is normal in $N_G(H)$.

**Proposition 2.1** (Suzuki [28, I. Theorem 6.11]). The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(H)$ of $H$.

Let $X$ be a cubic symmetric graph and let $G$ be an $s$-regular subgroup of $\text{Aut}(X)$ for some $s \geq 1$. Let $N$ be a normal subgroup of $G$. The quotient graph $X_N$ of $X$ relative to $N$ is defined as the graph whose vertices are the orbits of $N$
Then \(N\) has more than two orbits then \(N\) is semiregular on Proposition 2.2.

have the following proposition.

Furthermore, \(N\) is the kernel of \(G\) acting on \(V(X)\) and \(G/N\) is an s-regular subgroup of \(\text{Aut}(X)\).

Let \(X\) be a graph and \(N\) a finite group. Assign to each arc of \(X\) a voltage \(\zeta(u, v) \in N\) such that \(\zeta(v, u) = (\zeta(u, v))^{-1}\), where \(\zeta : X \to N\) is called a voltage assignment of \(X\). Let \(\text{Cov}(X; \zeta)\) be the derived graph from \(\zeta\) which has vertex set \(V(X) \times N\) and adjacency relation defined by \((u, a) \sim (v, a\zeta(u, v))\), where \(a \in N\) and \(u \sim v\) in \(X\). Then the projection onto the first coordinate \(\varphi : \text{Cov}(X; \zeta) \to X\) is a regular \(N\)-covering projection, where the group \(N\), viewed as \(\text{CT}(\varphi)\), acts via left multiplication on itself. Moreover, it can be shown that each regular \(N\)-covering projection \(\varphi' : \tilde{X} \to X\) is equivalent to \(\varphi : \text{Cov}(X; \zeta) \to X\) for some voltage assignment \(\zeta : X \to N\). Furthermore, one can assume that \(\zeta = id\) on the arcs of an arbitrary spanning tree \(T\). For an extensive treatment of graph coverings see [17,23].

For convenience, in the remainder of the paper we always assume that \(X\) is the Möbius–Kantor graph \(G(8, 3)\) which is depicted in Fig. 1.

Let
\[
\rho = (1 2 3 4 5 6 7 8)(9 10 11 12 13 14 15 16), \\
\sigma = (1 14 7 12 5 10 3 16)(2 11 8 9 6 15 4 13), \\
\omega = (2 8 9)(3 16 14)(4 13 6)(7 12 10).
\]

Then \(\rho, \sigma, \omega \in \text{Aut}(X)\). By [18, Section 3], \(\text{Aut}(X)\) has a unique minimal edge- but not vertex-transitive subgroup \(H = \langle \rho^2, \omega \rangle \cong Q_8 \times \mathbb{Z}_3\) where \(Q_8\) is the quaternion group, and a unique maximal edge- but not vertex-transitive subgroup \(M = \langle H, \sigma \rangle = H \rtimes \mathbb{Z}_2\). Note that \(H\) has no normal Sylow 3-subgroups. The edge set of \(X\) is the union of the outer edges \(\{(i, 1 + (i \text{ mod } 8)) | i \in \{1, \ldots , 8\}\}\), the inner edges \(\{(8 + i, 9 + ((i + 2) \text{ mod } 8)) | i \in \{1, \ldots , 8\}\}\), and the spokes \(\{(i, 8 + i) | i \in \{1, \ldots , 8\}\}\). Let \(T\) be the spanning tree of \(X\) containing all spokes and inner edges except for the edge \(\{16, 11\}\). Let \(x\) denote the arc \(\{16, 11\}\) and let \(x_i\) denote the arc \(\{i, 1 + (i \text{ mod } 8)\}, i \in \{1, 2, \ldots , 8\}\). For a prime \(p\), denote by \(\mathbb{Z}_p^d\) the elementary abelian group of order \(p^d\). Let \(\varphi : \tilde{X} \to X\) be regular \(\mathbb{Z}_p^d\)-covering projection. Then one may assume that \(\tilde{X} = \text{Cov}(X; \zeta)\), where \(\zeta = 0\) on the spanning tree \(T\). In view of [18, Section 7], we have

Proposition 2.3. Let \(\varphi : \tilde{X} = \text{Cov}(X; \zeta) \to X\) be a minimal semisymmetric regular \(\mathbb{Z}_p^d\)-covering projection. Then \(p \geq 3\) and \(d = 2\) or \(3\). Let \(n\) be the number of all pairwise nonisomorphic minimal semisymmetric regular \(\mathbb{Z}_p^d\)-covering projections of \(X\). Then \(n = 1 + (p - 1)/4\) if \(p \equiv 1\) (mod 24), \(n = (p - 1)/4\) if \(p \equiv 1\) (mod 4) and \(p \equiv 1\) (mod 24), \(n = 1 + (p + 1)/4\) if \(p \equiv 19\) (mod 24), and \(n = (p + 1)/4\) if \(p \equiv 3\) (mod 4) and \(p \equiv 19\) (mod 24). The voltages \(\zeta(x_1), \ldots , \zeta(x_8)\) of the covering graphs related to these covering projections are explicitly given in [18, Section 7] and the largest groups that lift along these covering projections are \(H\) or \(M\). Furthermore,

(1) If \(\varphi\) is minimal edge-transitive then \(d = 2\). The largest group that lifts is \(M\) if \(p = 3\) and \(H\) if \(p = 5\) or \(7\) (mod 8).

(2) If \(\varphi\) is not minimal edge-transitive then \(d = 3\), \(p \equiv 1\) or \(19\) (mod 24), and the largest group that lifts is \(H\).

The following is the main result of this paper.
Theorem 2.4. Let \( \varphi : \widehat{X} = \text{Cov}(X; \zeta) \to X \) be a minimal semisymmetric regular \( \mathbb{Z}_p^d \)-covering projection as given in Proposition 2.3. Then the covering graph \( \widehat{X} \) is semisymmetric.

Proof. Suppose on the contrary that \( \widehat{X} \) is not semisymmetric. Since \( \varphi \) is a minimal semisymmetric regular \( \mathbb{Z}_p^d \)-covering projection, \( \widehat{X} \) is a cubic symmetric graph of order \( 2^d \cdot p^d \). Set \( K = \mathbb{Z}_p^d \) and \( A = \text{Aut}(\widehat{X}) \). Then \( K \leq A \). If \( K \leq A \) then the projection of \( A \) along \( \varphi \) is an arc-transitive automorphism group of \( X \). However, by Proposition 2.3, the largest group that lifts is either \( H \) or \( M \), but neither is arc-transitive, a contradiction. It follows that \( K \) is not normal in \( A \). By Tutte [29,30], the stabilizer \( A_v \) of a vertex \( v \in V(\widehat{X}) \) in \( A \) has order dividing 48, and hence \( |A||2^8 \cdot 3 \cdot p^d| \).

Assume \( p = 3 \). Then, by Proposition 2.3, \( d = 2 \) and the largest subgroup that lifts is \( M \). Thus, \( K = \mathbb{Z}_3^d \) and \( \widehat{X} \) is a cubic symmetric graph of order \( 144 = 2^4 \cdot 3^2 \). By checking the list of cubic symmetric graphs up to 768 vertices in Conder and Dobcsányi [3], \( \widehat{X} \) is at most 2-regular, implying \( |A||2^5 \cdot 3 \cdot 7| \). Let \( T \) be the lift of \( M \). Then, \( T/K \cong M \). Recall that \( M = H \times \mathbb{Z}_2 \cong (\mathbb{Q}_8 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \). It follows that \( |T| = |M|/|K| = 2^4 \cdot 3^3 \) and hence \( |A : T| \leq 2 \), forcing \( T \leq \text{Syl}_3 \). Since \( T/K \cong M = (\mathbb{Q}_8 \times \mathbb{Z}_3) \times \mathbb{Z}_2 \) has no normal Sylow 3-subgroups, one has \( O_3(T) = T \), the largest normal 3-subgroup of \( T \). It follows that \( K \leq A \) because \( O_3(T) \) is characteristic in \( T \), a contradiction.

Assume \( p > 3 \). In this case, \( K \) is a Sylow \( p \)-subgroup of \( A \) because \( |A||2^8 \cdot 3 \cdot p^d| \). Note that \( H \leq M \) and by Proposition 2.3, \( H \) lifts. Let \( B \) be the lift of \( H \). Then \( B/K \cong H \) and \( |B| = |K|/|H| = 2^3 \cdot 3 \cdot p^d \) because \( H \cong \mathbb{Q}_8 \times \mathbb{Z}_3 \), implying \( |A : B||2^5| \). Since \( M = N \cdot \text{Aut}(\widehat{X}) \), \( \text{Aut}(\widehat{X}) \) is a cubic symmetric graph of order 400. By Conder and Dobcsányi [3], \( \widehat{X} \) is at most 2-regular, forcing \( |A||2^5 \cdot 3 \cdot 5^2| \). However, \( |A| = |A : N||N|/|A : N||B| = 2^3 \cdot 3 \cdot 5^2 \geq 2^3 \cdot 3 \cdot 5^2 \), a contradiction.

Case 1: \( k = 3 \) and \( p = 5 \).

In this case, \( |A : N| = 2^5 \). Since \( |A : N||A : B| = |A : N||B| \), one has \( |A : B| = |A : B| \), forcing \( N = B \). Thus, \( |A| = 32 \cdot 2^8 \cdot 3 \cdot 5^2 \) and hence \( \widehat{X} \) is a 5-regular cubic symmetric graph of order \( 2^4 \cdot 3^2 \). Consider the action of \( A \) on the right cosets of \( N \) in \( A \) by right multiplication and denote by \( N_A \) the kernel of this action. Then, \( A/N_A \leq S_{32} \) and \( N_A \) is the largest normal subgroup of \( A \) contained in \( N \). Since \( 31 \cdot 32! \), one has \( |N_A| = |A : B| \). Let \( P \) be a Sylow 3-subgroup of \( N_A \). Since \( N = B \) has normal Sylow 3-subgroup \( K \), \( P \) is normal in \( N \) and hence characteristic in \( N \). It follows that \( P \trianglelefteq A \) because \( N_A \trianglelefteq A \) and \( P \) has order \( 31^2 \) then \( K = P \trianglelefteq A \), a contradiction. Thus, \( |P| = 31 \). By Proposition 2.2, the quotient graph \( \widehat{X}_P \) of \( \widehat{X} \) relative to \( P \) is a cubic 5-regular graph of order 16 \cdot 31 = 496, which is impossible by Conder and Dobcsányi [3].

Case 2: \( k = 1 \) and \( p = 7 \).

In this case, \( |A : N| = 8 \). Consider the action of \( A \) on the right cosets of \( N \) in \( A \) by right multiplication and denote by \( N_A \) the kernel of this action. Then, \( A/N_A \leq S_8 \) and \( N_A \) is the largest normal subgroup of \( A \) contained in \( N \). Since \( 7^2 \cdot 8! \), one has \( |N_A| = 7! \). Let \( P \) be a Sylow 7-subgroup of \( N_A \). Then a similar argument to Case 2 implies that \( P \trianglelefteq A \) and \( |P| = 7 \). By Proposition 2.2, the quotient graph \( \widehat{X}_P \) of \( \widehat{X} \) relative to \( P \) is a cubic symmetric graph of order \( 16 \cdot 7 = 112 \) with \( A/P \) as an arc-transitive group of automorphisms of \( \widehat{X}_P \). By Conder and Dobcsányi [3], \( \widehat{X}_P \) is at most 3-regular, forcing \( |A/P||2^6 \cdot 3 \cdot 7| \). It follows that \( |A||2^6 \cdot 3 \cdot 7^2 \). Since \( |A| = |A : N||N|/|A : N||B| = 2^6 \cdot 3 \cdot 7^2 \), one has \( |A||2^6 \cdot 3 \cdot 7^2 \), implying that \( \widehat{X} \) is 3-regular of order \( 2^4 \cdot 7^2 \). Since \( 2^7||N_A| \), \( N_A \) has more than two orbits on \( V(\widehat{X}_P) \), and by Proposition 2.2, \( N_A \) is semiregular on \( V(\widehat{X}) \). The quotient graph \( \widehat{X}_{N_A} \) of \( \widehat{X} \) relative to \( N_A \) is a cubic symmetric graph and hence has even order. It follows that \( |N_A| = 7, 14, 28 \) or 56. For each subgroup \( T \leq \text{Syl}_2 \), one has \( |N| \geq |B| \geq 2^2 \cdot 3 \cdot 7 \).

By Proposition 2.1, \( \overline{N}/\overline{C} \subseteq \text{Aut}(\overline{K}) \subseteq \text{Z}_6 \), forcing \( |\overline{C}| \geq |N|/6 \geq 14 \). Thus, \( \overline{C} \) is a proper subgroup of \( \overline{C} \), contrary to the fact that \( \overline{C} = \overline{K} \).

Let \( |N| = 28 \). Set \( C = C_{N_A}(P) \) and \( O = O_2(N_A) \), the largest normal 2-subgroup of \( N_A \). Then, by Proposition 2.1, \( N_A/C \subseteq \text{Aut}(P) \subseteq \text{Z}_6 \), implying \( |C| \geq 14 \). Let \( Q \) be a Sylow 2-subgroup of \( C \). Then, \( C = Q \times P \) because \( N_A \) is \( [2, 7]\)-group and hence \( Q \) is characteristic in \( C \). It follows that \( Q \trianglelefteq N_A \), implying \( Q \trianglelefteq O \). Thus, \( |O| = 2 \) or 4. Clearly, \( O \) is characteristic in \( N_A \) and hence normal in \( A \). Consider the quotient graph \( \widehat{X}_O \) of \( \widehat{X} \) relative to \( O \). By Proposition 2.2, \( \widehat{X}_O \) is a cubic symmetric graph of order \( 392 \) or \( 196 \) with \( A/O \) as a 3-regular group of automorphisms of \( \widehat{X}_O \). This is impossible by Conder and Dobcsányi [3].
Let $|N_A| = 56$. Consider the quotient graph $\tilde{X}_{N_A}$ of $\tilde{X}$ relative to $N_A$. Then, by Proposition 2.2, $\tilde{X}_{N_A}$ is a cubic symmetric graph of order 14 with $A/N_A$ as a 3-regular subgroup of $\text{Aut}(\tilde{X}_{N_A})$. By Conder and Dobcsányi [3], there is a unique cubic symmetric graph of order 14, that is, the 4-regular Heawood graph. It is well-known that the Heawood graph has $\text{PGL}(2, 7)$ as its 4-regular automorphism group, which has no 3-regular subgroup ($\text{PGL}(2, 7)$ has a unique index 2 subgroup isomorphic to $\text{PSL}(2, 7)$ and this subgroup fixes the bipartite sets of the Heawood graph that is not transitive), a contradiction. □

Acknowledgements

Supported by the National Natural Science Foundation of China (10571013), the Key Project of Chinese Ministry of Education (106029), and the Specialized Research Fund for the Doctoral Program of Higher Education in China (20060004026).

References