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Note

Exact Coverings of 2-Paths by Hamilton Cycles

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We construct a $C(2m, 2m, 2)$ design which is a family of Hamilton cycles in K_{2m} so that each 2-path of K_{2m} lies in exactly two of the cycles.

1. INTRODUCTION

Let K_n be the complete graph on n vertices. A $C(n, k, \lambda)$ design is a family of k-cycles in K_n in which each 2-path (path of length two) of K_n occurs in exactly λ of the cycles. Necessary and sufficient conditions for the existence of $C(n, 4, \lambda)$ designs are known [6]. A $C(n, n, 1)$ design is a solution of Dudeney's round table problem which is [2, Problem 273]:

> Seat the same n persons at a round table on $(n-1)(n-2)/2$ occasions so that no person shall ever have the same two neighbours twice. This is, of course, equivalent to saying that every person must sit once, and only once, between every possible pair.

A $C(n, n, 1)$ design is known to exist when $n = p + 1$, $n = 2p$ (p is prime), $n = p^k + 1$ (p is prime and k is a natural number), $n = p + 2$ (p is prime and 2 is a generator of the multiplicative group of $GF(p)$), $n = pq + 1$ (p and q are odd primes), $n = p^e q^f + 1$ (p and q are odd primes satisfying $p \geqslant 5$ and $q \geqslant 11$, and e and f are natural numbers), and some sporadic cases: $n = 11$, 19, 23, 43 [1, 4, 5, 7, 9, 10]. It is also known that if there is a $C(n+1, n+1, 1)$ design then there is a $C(2n, 2n, 1)$ design, where $n \geqslant 2$ [4].

In this paper we will show

1.1. THEOREM. *There is a $C(n, n, 2)$ design when n is even and $n \geqslant 4$.*

After we submitted this paper, we have found the method of construction

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for $C(2m, 2m, 1)$ design in which some subtle graphical properties are used [8]. The present paper for $C(2m, 2m, 2)$ design requires quite simple characteristics comparing with the case $\lambda = 1$.

2. Preliminaries

Let $K_n = (V_n, E_n)$ be the complete graph on n vertices, where V_n is the vertex-set and E_n is the edge-set. The following lemma is well known (see, for example, [3]).

2.1. Lemma. *There is a 1-factorization in K_n when n is even.*

A family of Hamilton cycles $\mathscr{C}$ in K_n is called a faithful (double) Hamilton cycle cover of K_n if each edge of K_n belongs to exactly one (two) cycle(s) in $\mathscr{C}$. The following lemma is easy to prove.

2.2. Lemma. (i) *When m is odd and $m \geqslant 3$, K_m has a faithful Hamilton cycle cover.*

(ii) *When m is even and $m \geqslant 4$, K_m has a double Hamilton cycle cover.*

Let $\mathscr{F}$ be a 1-factorization in K_n, $n = 2m$, and let $F = \{e_1, e_2, ..., e_m\}$ be a 1-factor in $\mathscr{F}$, where $e_i = \{a_i, b_i\} \in E_n$ and $a_i, b_i \in V_n$. We direct each edge e_i $(1 \leqslant i \leqslant m)$ and so introduce the directed edge $f_i = (a_i, b_i)$.

Using this notation we now construct a specific double Hamilton cycle cover of $K_m = (V_m, E_m)$, where $V_m = \{f_1, f_2, ..., f_m\}$, m is even, and $m \geqslant 4$, which will be used in Section 4. Let C be the Hamilton cycle

$$(f_1, f_2, f_3, f_m, f_4, f_{m-1}, f_5, f_{m-2}, ..., f_{m/2+1}, f_{m/2+2}).$$

Let σ be the permutation $\sigma = (f_1)(f_2 f_3 f_4 \cdots f_m)$. Then $\{C, \sigma(C), \sigma^2(C), ..., \sigma^{m-2}(C)\}$ is a double Hamilton cycle cover of K_m.

For a sign $s \in \{+, -\}$ and a directed edge (a, b), we define

$$s(a, b) = \begin{cases} (a, b) & \text{if } s = +, \\ (b, a) & \text{if } s = -. \end{cases}$$

Further, for a sign sequence $S = (s_1, s_2, ..., s_m)$, where $s_i \in \{+, -\}$, $1 \leqslant i \leqslant m$ and a Hamilton cycle $C = (f_{i_1}, f_{i_2}, ..., f_{i_m})$ in K_m we define

$$SC = (s_1 f_{i_1}, s_2 f_{i_2}, ..., s_m f_{i_m}).$$

Using SC we define a Hamilton cycle in K_n. If $SC = ((c_1, d_1), (c_2, d_2), ..., (c_m, d_m))$, then $SC^* = (c_1, d_1, c_2, d_2, ..., c_m, d_m)$ is a Hamilton cycle in K_n.

The proof of Theorem 1.1 is divided into two cases depending on the parity of m.

3. The Case $n=2m=4k+2$

Let m be odd and put $n=2m=4k+2$, where $m\geqslant 3$ (because of our assumption that $n\geqslant 4$). We define four sign sequences

$$S_i\colon (s_{i_1}, s_{i_2}, ..., s_{i_m}), \qquad 1\leqslant i\leqslant 4,$$

where $s_{ij}\in\{+,-\}$, $1\leqslant i\leqslant 4$, $1\leqslant j\leqslant m$, satisfying

$s_{1j}=+\ (1\leqslant j\leqslant m)$

$s_{2j}=+\ (1\leqslant j\leqslant m-1,\ j$ is odd), $s_{2j}=-\ (1\leqslant j\leqslant m-1,\ j$ is even), $s_{2j}=-\ (j=m)$

$s_{3j}=-\ (1\leqslant j\leqslant m,\ j$ is odd), $s_{3j}=+\ (1\leqslant j\leqslant m,\ j$ is even)

$s_{4j}=-\ (1\leqslant j\leqslant m-1)$, $s_{4j}=+\ (j=m)$,

that is,

$$S_1\colon (+\,+\,+\,+\quad\cdots\quad +\,+\,+\,+\,+)$$
$$S_2\colon (+\,-\,+\,-\quad\cdots\quad +\,-\,+\,-\,-)$$
$$S_3\colon (-\,+\,-\,+\quad\cdots\quad -\,+\,-\,+\,-)$$
$$S_4\colon (-\,-\,-\,-\quad\cdots\quad -\,-\,-\,-\,+).$$

3.1. *Observation.* For any j, $1\leqslant j\leqslant m$,

$$\{(s_{1j}, s_{1,j+1}), (s_{2j}, s_{2,j+1}), (s_{3j}, s_{3,j+1}), (s_{4j}, s_{4,j+1})\}$$
$$=\{(+,+), (+,-), (-,+), (-,-)\},$$

where the second subscripts of s_{ij} are calculated modulo m.

Now let $\mathscr{F}$ be any 1-factorization of K_n. For each 1-factor $F=\{e_1, e_2, ..., e_m\}\in\mathscr{F}$, we introduce directed edges $\{f_1, f_2, ..., f_m\}$ and consider the complete graph $K_m=(V_m, E_m)$, where $V_m=\{f_1, f_2, ..., f_m\}$. Let $\mathscr{C}_F$ be a faithful Hamilton cycle cover of K_m. For each Hamilton cycle $C=(f_{i_1}, f_{i_2}, ..., f_{i_m})\in\mathscr{C}_F$, we define

$$\mathscr{H}_F(C)=\{S_1C^*, S_2C^*, S_3C^*, S_4C^*\}.$$

So $\mathscr{H}_F(C)$ is a set of four Hamilton cycles in K_n.

Let $f_{i_t} = (a, b)$ be any vertex in V_m $(1 \leqslant t \leqslant m)$. Put $f_{i_{t-1}} = (c, d)$ and $f_{i_{t+1}} = (g, h)$, where subscripts of i are calculated modulo m. Then

$$(+f_{i_{t-1}}, +f_{i_t}) \text{ yields the path } (c, d, a, b)$$
$$(+f_{i_{t-1}}, -f_{i_t}) \text{ yields the path } (c, d, b, a)$$
$$(-f_{i_{t-1}}, +f_{i_t}) \text{ yields the path } (d, c, a, b)$$
$$(-f_{i_{t-1}}, -f_{i_t}) \text{ yields the path } (d, c, b, a).$$

Similarly,

$$(+f_{i_t}, +f_{i_{t+1}}) \text{ yields the path } (a, b, g, h)$$
$$(+f_{i_t}, -f_{i_{t+1}}) \text{ yields the path } (a, b, h, g)$$
$$(-f_{i_t}, +f_{i_{t+1}}) \text{ yields the path } (b, a, g, h)$$
$$(-f_{i_t}, -f_{i_{t+1}}) \text{ yields the path } (b, a, h, g).$$

Clearly in all four Hamilton cycles in $\mathscr{H}_F(C)$, a has all four vertices of $f_{i_{t-1}}$ and $f_{i_{t+1}}$ as neighbours. We consider $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$ which is a collection of $4k$ Hamilton cycles in K_n.

3.2. *Observation.* Let $e = \{a, b\}$ be any edge in F.

(i) The vertex a always has b as one of its two neighbours in any Hamilton cycle of $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$.

(ii) For any vertex c $(\neq a, b)$ of K_n, the vertex a has c as a neighbour in exactly one Hamilton cycle of $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$.

This observation implies that if $e = \{a, b\} \in F$, then the 2-path (a, b, c) lies in one of the cycles of $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$.

Let $D = \bigcup_{F \in \mathscr{F}} \bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$, where multiplicities are retained if they occur and so clearly $|D| = (n-1)\, 4k = (n-1)(n-2)$.

Let (a, b, c) be any 2-path in K_n. There exists a 1-factor $F \in \mathscr{F}$ such that $\{a, b\} \in F$. The 2-path (a, b, c) belongs to $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$ by Observation 3.2. On the other hand, there exists a 1-factor $F' \in \mathscr{F}$ such that $\{b, c\} \in F'$ and the 2-path (a, b, c) belongs to $\bigcup_{C \in \mathscr{C}_{F'}} \mathscr{H}_{F'}(C)$. Thus the 2-path (a, b, c) belongs to at least two cycles in D. The number of 2-paths belonging to cycles in D is $n(n-1)(n-2)$ and the number of 2-paths in K_n is $n(n-1)(n-2)/2$. Therefore any 2-path of K_n belongs to D exactly twice. Hence D is a $C(n, n, 2)$ design and thus Theorem 1.1 is proved when $n = 2m = 4k + 2$.

4. The Case $n=2m=4k$

When $n=4$, Theorem 1.1 is obvious, so we assume $n\geqslant 8$. We define two sign sequences

$$S_i\colon (s_{i1}, s_{i2}, \dots, s_{im}), \qquad 1\leqslant i\leqslant 2,$$

where $s_{ij}\in\{+,-\}$, $1\leqslant i\leqslant 2$, $1\leqslant j\leqslant m$, satisfying

$$s_{11}=-, \quad s_{1j}=+ \quad (2\leqslant j\leqslant k+1)$$
$$s_{2j}=- \quad (1\leqslant j\leqslant k+1)$$

and if k is odd,

$$s_{1j}=- \quad (k+2\leqslant j\leqslant m,\ j\text{: odd})$$
$$s_{1j}=+ \quad (k+2\leqslant j\leqslant m,\ j\text{: even})$$
$$s_{2j}=+ \quad (k+2\leqslant j\leqslant m,\ j\text{: odd})$$
$$s_{2j}=- \quad (k+2\leqslant j\leqslant m,\ j\text{: even});$$

if k is even,

$$s_{1j}=- \quad (k+2\leqslant j\leqslant m,\ j\text{: even})$$
$$s_{1j}=+ \quad (k+2\leqslant j\leqslant m,\ j\text{: odd})$$
$$s_{2j}=+ \quad (k+2\leqslant j\leqslant m,\ j\text{: even})$$
$$s_{2j}=- \quad (k+2\leqslant j\leqslant m,\ j\text{: odd}).$$

Then S_1 and S_2 are as follows: If k is odd,

$$S_1\colon (-+++\cdots+++-+-+\cdots-+)$$
$$S_2\colon (----\cdots---+-+-\cdots+-);$$

if k is even,

$$S_1\colon (-+++\cdots+++-+-+\cdots-+-)$$
$$S_2\colon (----\cdots---+-+-\cdots+-+).$$

Then the sign sequences S_1, S_2 have the following property.

4.1. *Observation.* For any j, $2\leqslant j\leqslant k$,

$$\{(s_{1j}, s_{1,j+1}), (s_{2j}, s_{2,j+1})\}=\{(+,+),(-,-)\}.$$

For any j, $k+1 \leqslant j \leqslant 2k-1$,

$$\{(s_{1j}, s_{1,j+1}), (s_{2j}, s_{2,j+1})\} = \{(+, -), (-, +)\}.$$

Let $\mathscr{F}$ be any 1-factorization in K_n. For any 1-factor $F = \{e_1, e_2, ..., e_m\} \in \mathscr{F}$, introduce directed edges $\{f_1, f_2, ..., f_m\}$ as before and let $K_m = (V_m, E_m)$, where $V_m = \{f_1, f_2, ..., f_m\}$.

Let $\mathscr{C}_F$ be the double Hamilton cycle cover of K_m defined in Section 2. For any cycle C in $\mathscr{C}_F$, we define $\mathscr{H}_F(C) = \{S_1 C^*, S_2 C^*\}$ which is a set of two Hamilton cycles in K_n. Then $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$ is a set of $2(m-1)$ Hamilton cycles in K_n.

4.2. *Observation.* Let $f = (a, b)$ and $g = (c, d)$ be two vertices in V_m. Then each of the 2-paths (a, b, c), (b, a, c), (a, b, d), and (b, a, d) occurs in exactly one of the Hamilton cycles of $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$.

Proof. Since $\mathscr{C}_F$ is a double Hamilton cycle cover, there exist exactly two cycles $C, C' \in \mathscr{C}_F$ containing the edge $\{f, g\}$. Put $C = (f_{i_1}, f_{i_2}, ..., f_{i_m})$, $C' = (f'_{i_1}, f'_{i_2}, ..., f'_{i_m})$, where $f_{i_1} = f'_{i_1} = f_1$ and think of the cycles as directed in this way.

Suppose $f \neq f_1$ and $g \neq f_1$. If (f, g) belongs to C, then (f, g) (not (g, f)) belongs to C'. Moreover, if $(f, g) = (f_{i_l}, f_{i_{l+1}})$, for some l, $2 \leqslant l \leqslant k$, then $(f, g) = (f'_{i_t}, f'_{i_{t+1}})$, for some t, $k+1 \leqslant t \leqslant 2k-1$. From Observation 4.1 we obtain in $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$ the 3-paths (a, b, c, d), (b, a, d, c), (a, b, d, c), and (b, a, c, d) and the result follows.

Suppose, without loss of generality, that $f = f_1$ and $(f_1, g) \in C$. Then we have $(g, f_1) \in C'$, and obtain the 3-paths (b, a, c, d), (b, a, d, c), (c, d, b, a), and (d, c, b, a) and again the claim follows.

From this observation the next follows immediately.

4.3. *Observation.* Let $e = \{a, b\}$ be any edge in F.

(i) The vertex a always has b as one of its two neighbours in any Hamilton cycle of $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$.

(ii) For any vertex c ($\neq a, b$) of K_n, the vertex a has c as a neighbour in exactly one Hamilton cycle of $\bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$.

Again, we denote by D the family of all Hamilton cycles of K_n in $\bigcup_{F \in \mathscr{F}} \bigcup_{C \in \mathscr{C}_F} \mathscr{H}_F(C)$ (multiplicities being permitted if they arise). Then, arguing as in the previous section, D is a $C(n, n, 2)$ design by Observation 4.3. Thus Theorem 1.1 has proved when $n = 2m = 4k$.

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