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Journal of Algebra 323 (2010) 3039–3053

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Ample filters and Frobenius amplitude ☆

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ARTICLE INFO

Article history:
Received 22 December 2009
Available online 17 March 2010
Communicated by Michel Van den Bergh

Keywords:
Vanishing theorems
Ampleness
Coherent sheaves
Vector bundles
Frobenius morphism

ABSTRACT

Let X be a projective scheme over a field. We show that the vanishing cohomology of any sequence of coherent sheaves is closely related to vanishing under pullbacks by the Frobenius morphism. We also compare various definitions of ample locally free sheaf and show that the vanishing given by the Frobenius morphism is, in a certain sense, the strongest possible. Our work can be viewed as various generalizations of the Serre Vanishing Theorem.

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1. Introduction

Throughout this paper, X will be a projective scheme over a field k.

Vanishing theorems are of great importance in algebraic geometry. They often allow one to control the behavior of a sheaf by the behavior of its global sections. This then allows certain algebra versus geometry connections, such as the Serre Correspondence Theorem.

Arguably, the most important vanishing theorem is the Serre Vanishing Theorem, which could be stated thusly: For any coherent sheaf $\mathcal{F}$ on X, there exists n_0 such that

$$H^q(X, \mathcal{F} \otimes \mathcal{L}_n) = 0 \tag{1.1}$$

for $q > 0$, $n \geqslant n_0$, and $\mathcal{L}_n = \mathcal{L}^{\otimes n}$ for an ample invertible sheaf $\mathcal{L}$. In noncommutative algebraic geometry, it has been extremely useful to allow the sequence $\mathcal{L}_n$ to be something other than $\mathcal{L}^n$, possibly even a sequence of non-invertible sheaves, so that one may construct noncommutative rings from geometric data. See [AV,Ke1,Ke3,KRS,R,RS].

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doi:10.1016/j.jalgebra.2010.02.028

Let $\mathcal{E}$ be a fixed vector bundle and let F be the absolute Frobenius morphism on X if $\operatorname{char} k = p > 0$. The vanishing (1.1) has also been studied for the sequences of locally free sheaves, $\mathcal{L}_n = S^n(\mathcal{E})$ (in which case $\mathcal{E}$ is called ample) [H1] and $\mathcal{L}_n = \mathcal{E}^{(p^n)} = F^{*n}\mathcal{E}$ (in which case $\mathcal{E}$ is called cohomologically p-ample, Frobenius ample, or F-ample) [A1,B,G].

Arapura recently extended this definition to characteristic 0 in [A1, Definition 1.3], allowing him to generalize several vanishing theorems, such as those of Kodaira–Nakano and Kawamata–Viehweg [A1, §8]. He also recovers Le Potier's Vanishing Theorem by studying the "Frobenius amplitude" of a coherent sheaf. As our main theorem depends on this definition, we give it now.

Definition 1.2. Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf. If $\operatorname{char} k = p > 0$ and F is the absolute Frobenius morphism, then define $\mathcal{F}^{(p^n)} = F^{*n}\mathcal{F}$. The *Frobenius amplitude* or *F-amplitude* of $\mathcal{F}$, $\phi(\mathcal{F})$, is the smallest integer such that for any locally free coherent sheaf $\mathcal{E}$, there exists n_0 such that

$$H^i\big(X, \mathcal{E} \otimes \mathcal{F}^{(p^n)}\big) = 0, \quad i > \phi(\mathcal{F}),\ n \geqslant n_0.$$

If $\operatorname{char} k = 0$, then $\phi(\mathcal{F}) \leqslant t$ if and only if $\phi(\mathcal{F}_q) \leqslant t$ for all closed fibers on some arithmetic thickening. If $\phi(\mathcal{F}) = 0$, then $\mathcal{F}$ is *F-ample*.

In this paper we will study the vanishing (1.1) for any sequence of coherent sheaves $\mathcal{L}_n$. We show that the vanishing from such a sequence is closely related to F-amplitude. Our main theorem is

Theorem 1.3. *Let X be a projective integral scheme, smooth over a perfect field k, let $\{\mathcal{G}_n\}$ be a sequence of coherent sheaves, and let $t \geqslant 0$. Then these two statements are equivalent*:

(1) *For any locally free coherent sheaf $\mathcal{E}$, there exists n_0 such that*

$$H^q(X, \mathcal{E} \otimes \mathcal{G}_n) = 0, \quad q > t,\ n \geqslant n_0.$$

(2) *For any invertible sheaf $\mathcal{H}$, there exists n_1 such that the F-amplitude $\phi(\mathcal{G}_n \otimes \mathcal{H}) \leqslant t$ for $n \geqslant n_1$.*

(More generally, we allow the $\mathcal{L}_n$ to be indexed by a filter instead of just a sequence, because it has been useful to filter by $\mathbb{N}^n$ [Chn,Ke2]. See Section 3.)

Sections 2 and 3 contain preliminaries, though the Castelnuovo–Mumford regularity bounds in Section 2 may be of independent interest. The main theorem is proven in Section 4. For general projective schemes, partial generalizations are given in Section 5. Along the way, we obtain a generalization of Fujita's Vanishing Theorem [F2, Theorem 1], which is in turn a generalization of Serre Vanishing.

Theorem 1.4. *Let X be a projective scheme over a field k with ample invertible sheaf $\mathcal{L}$. Let $\mathcal{F}$ be a coherent sheaf. Then there exists m_0 such that*

$$H^q\big(X, \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{G}\big) = 0, \quad q > t,\ m \geqslant m_0,$$

for any locally free coherent $\mathcal{G}$ with $\phi(\mathcal{G}) \leqslant t$. (Note that m_0 does not depend on $\mathcal{G}$.)

(See Theorems 4.5 and 5.5 for statements in which $\mathcal{G}$ is only "nearly" locally free.)

We then turn our attention to other ampleness properties of a locally free sheaf. Theorem 6.2 shows that if $\mathcal{E}$ is an ample vector bundle (or even just ample on a complete intersection curve), then $\phi(\mathcal{E}) < \dim X$. Finally, in Theorem 7.2, we compare 5 possible ways to define ampleness of a sequence of locally free sheaves.

2. Subadditive regularity bounds on schemes

Let X be a projective scheme over a field k and let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on X. In this section we will explore how the Castelnuovo–Mumford regularities $\operatorname{reg}\mathcal{F}$, $\operatorname{reg}\mathcal{G}$, and $\operatorname{reg}(\mathcal{F}\otimes\mathcal{G})$ are related. More specifically, we generalize [S, Proposition 1.5], which states that if $X=\mathbb{P}^n$, then $\operatorname{reg}(\mathcal{F}\otimes\mathcal{G})\leqslant \operatorname{reg}\mathcal{F}+\operatorname{reg}\mathcal{G}$ when $\mathcal{F}$ and $\mathcal{G}$ are "locally free in codimension 2."

First, we must clarify the concept of regularity on X.

Definition 2.1. Let X be a projective scheme with an ample, globally generated invertible sheaf $\mathcal{O}_X(1)$ and let $t\in\mathbb{N}$. Then a coherent sheaf $\mathcal{F}$ on X is (m,t)-*regular* with respect to $\mathcal{O}_X(1)$ if

$$H^i\big(X,\mathcal{F}\otimes\mathcal{O}_X(m-i)\big)=0,\quad i>t.$$

The minimum m such that $\mathcal{F}$ is (m,t)-regular with respect to $\mathcal{O}_X(1)$ is denoted $\operatorname{reg}^t_{\mathcal{O}_X(1)}(\mathcal{F})$. Note that if $f:X\to\mathbb{P}^n$ is the finite morphism defined by $\mathcal{O}_X(1)$, so that $\mathcal{O}_X(1)=f^*\mathcal{O}_{\mathbb{P}^n}(1)$, then $\operatorname{reg}^t_{\mathcal{O}_X(1)}(\mathcal{F})=\operatorname{reg}^t_{\mathcal{O}_{\mathbb{P}^n}(1)}(f_*\mathcal{F})$. If $\mathcal{F}$ is $(m,0)$-regular, we simply say $\mathcal{F}$ is m-regular.

We study (m,t)-regularity instead of just $(m,0)$-regularity because it is necessary for the study of F-amplitude as defined in Definition 1.2. This concept of (m,t)-regularity shares a basic property with $(m,0)$-regularity.

Lemma 2.2. *Let X be a projective scheme with ample, globally generated sheaf $\mathcal{O}_X(1)$ and coherent sheaf $\mathcal{F}$. If $\mathcal{F}$ is (m,t)-regular, then $\mathcal{F}$ is (n,t)-regular for all $n\geqslant m$.*

Proof. The proof is just as in [Kle, p. 307, Proposition 1(i)]. □

In [A2, §1], the related concept of the *level* $\lambda(\mathcal{F})$ of a sheaf was introduced. The level is the smallest natural number for which $q>\lambda(\mathcal{F})$ implies

$$H^q\big(\mathcal{F}(-1)\big)=H^{q+1}\big(\mathcal{F}(-2)\big)=\cdots=0.$$

From this we immediately have

Lemma 2.3. *Let X be a projective scheme with ample, globally generated sheaf $\mathcal{O}_X(1)$ and coherent sheaf $\mathcal{F}$. Then $\lambda(\mathcal{F})\leqslant t$ if and only if $\mathcal{F}$ is (t,t)-regular.*

We can now merge, and thus slightly strengthen, [S, Lemma 1.4] and [A1, Lemma 3.7]. The reader may also confer [Ca, Section 3] for similar proofs for the regularity of modules over a commutative ring.

Lemma 2.4. *Let X be a projective scheme of dimension d with ample, globally generated invertible sheaf $\mathcal{O}_X(1)$, and let*

$$\cdots\xrightarrow{\phi_3}\mathcal{E}_2\xrightarrow{\phi_2}\mathcal{E}_1\xrightarrow{\phi_1}\mathcal{E}_0\to 0$$

be a complex of sheaves on X with homology sheaves $\mathcal{H}_j$, for $j\geqslant 0$. Let $t\in\mathbb{N}$. Suppose that the dimension of the support of the higher homology of the complex satisfies

$$\dim\operatorname{Supp}\mathcal{H}_j-j-t<2,\quad 1\leqslant j\leqslant d-t-2.$$

Then

$$\operatorname{reg}^t_{\mathcal{O}_X(1)}(\mathcal{H}_0) \leqslant \max\bigl\{\operatorname{reg}^{t+j}_{\mathcal{O}_X(1)}(\mathcal{E}_j) - j\colon\ 0 \leqslant j \leqslant d-t-1\bigr\}.$$

Proof. For this proof, we assume that all regularity values are with respect to $\mathcal{O}_X(1)$. Let $m = \max\{\operatorname{reg}^{t+j}(\mathcal{E}_j) - j\colon\ 0 \leqslant j \leqslant d-t-1\}$. Fix $i > t$. We wish to show that $H^i(X, \mathcal{H}_0(m-i)) = 0$.

Let $\mathcal{Z}_j = \operatorname{Ker}(\phi_j)$ and $\mathcal{B}_j = \operatorname{Im}(\phi_{j+1})$, so that $\mathcal{H}_j = \mathcal{Z}_j/\mathcal{B}_j$. The hypotheses on the $\mathcal{H}_j$ give us that $H^{i+j+1}(X, \mathcal{H}_j(m-i)) = 0$ for $j \geqslant 1$ because $i+j+1 > \dim \operatorname{Supp} \mathcal{H}_j$. Thus

$$\dim H^{i+j+1}\bigl(X, \mathcal{B}_j(m-i)\bigr) \leqslant \dim H^{i+j+1}\bigl(X, \mathcal{Z}_j(m-i)\bigr), \quad j \geqslant 1. \tag{2.5}$$

For $0 \leqslant j \leqslant d-t-1$, we have that $\mathcal{E}_j$ is $(m+j, t+j)$-regular by Lemma 2.2 because $m+j \geqslant \operatorname{reg}^{t+j}\mathcal{E}_j$. Thus for all $j \geqslant 0$, we have $H^{i+j}(X, \mathcal{E}_j(m-i)) = 0$ (if $j \geqslant d-t$, then $H^{i+j}(X, \mathcal{E}_j(m-i)) = 0$ because $i+j > \dim X$). Thus

$$\dim H^{i+j}\bigl(X, \mathcal{B}_{j-1}(m-i)\bigr) = \dim H^{i+j+1}\bigl(\mathcal{Z}_j(m-i)\bigr), \quad j \geqslant 1. \tag{2.6}$$

Now $H^{i+j+1}(\mathcal{Z}_j(m-i)) = 0$ for $j \geqslant d-t-1$ for dimensional reasons. Then by descending induction on j and Eqs. (2.5) and (2.6), we have that

$$H^{i+j+1}\bigl(X, \mathcal{Z}_j(m-i)\bigr) = H^{i+j}\bigl(X, \mathcal{B}_{j-1}(m-i)\bigr) = 0, \quad j \geqslant 1.$$

We also have $H^i(X, \mathcal{Z}_0(m-i)) = 0$ because $\mathcal{Z}_0 = \mathcal{E}_0$ is (m,t)-regular by hypothesis. Thus $H^i(X, \mathcal{H}_0(m-i)) = 0$, as desired. □

The following lemma will allow us to build an appropriate complex on which to apply Lemma 2.4.

Lemma 2.7. *Let X be a projective scheme with ample, globally generated invertible sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be a coherent sheaf on X such that $\mathcal{F}$ is m-regular with respect to $\mathcal{O}_X(1)$. Then for any $N \geqslant 0$, there exist vector spaces V_i and a resolution*

$$V_N \otimes \mathcal{O}_X(-m-NR) \to \cdots \to V_1 \otimes \mathcal{O}_X(-m-R) \to V_0 \otimes \mathcal{O}_X(-m) \to \mathcal{F} \to 0$$

where $R = \max\{1, \operatorname{reg}^0_{\mathcal{O}_X(1)}(\mathcal{O}_X)\}$.

Proof. The proof is exactly the same as in [A1, Corollary 3.2], using [Kle, p. 307, Proposition 1] for the necessary facts regarding regularity for ample, globally generated invertible sheaves. □

We now generalize [S, Proposition 1.5].

Proposition 2.8. *Let X be a projective scheme of dimension d with ample, globally generated invertible sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on X, and let Y be the closed subscheme of X where both $\mathcal{F}$ and $\mathcal{G}$ are not locally free. If $\dim Y \leqslant t+2$, then*

$$\operatorname{reg}^t_{\mathcal{O}_X(1)}(\mathcal{F} \otimes \mathcal{G}) \leqslant \operatorname{reg}^0_{\mathcal{O}_X(1)}(\mathcal{F}) + \operatorname{reg}^t_{\mathcal{O}_X(1)}(\mathcal{G}) + (d-t-1)(R-1)$$

where $R = \max\{1, \operatorname{reg}^0_{\mathcal{O}_X(1)}(\mathcal{O}_X)\}$.

Proof. All regularities in this proof are with respect to $\mathcal{O}_X(1)$. Let $m = \operatorname{reg}^0(\mathcal{F})$ and $n = \operatorname{reg}^t(\mathcal{G})$. Form a resolution of $\mathcal{F}$ as in Lemma 2.7 with $N = d$. Tensoring this resolution with $\mathcal{G}$, we have a complex $\mathcal{E}_\bullet$ with $\mathcal{E}_i = V_i \otimes \mathcal{O}_X(-m - iR) \otimes \mathcal{G}$. This complex has homology $\mathcal{H}_i = \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and, as argued in [S, Proposition 1.5], the hypothesis on Y ensures $\dim \operatorname{Supp} \mathcal{H}_i \leqslant t + 2$ for $i \geqslant 1$.

Thus Lemma 2.4 applies to this complex and so

$$\begin{aligned}\operatorname{reg}^t(\mathcal{H}_0) &\leqslant \max\bigl\{\operatorname{reg}^t(\mathcal{E}_i) - i\colon\ 0 \leqslant i \leqslant d - t - 1\bigr\} \\ &= \max\bigl\{m + n + i(R-1)\colon\ 0 \leqslant i \leqslant d - t - 1\bigr\} \\ &= m + n + (d - t - 1)(R - 1).\end{aligned}$$

Now $\mathcal{H}_0 = \mathcal{F} \otimes \mathcal{G}$, so we have the desired result. □

Note that this result is used in [KRS, Proposition 4.12] and [R,RS].

One could make an even sharper results using $\operatorname{reg}^{t+i}(\mathcal{G})$ for $0 \leqslant i \leqslant d - t - 1$, since $\operatorname{reg}^t(\mathcal{G}) \geqslant \operatorname{reg}^{t+1}(\mathcal{G})$. But that becomes rather unsightly.

It seems that Proposition 2.8 cannot be pushed further to the case when $\dim Y > t + 2$. Chardin has an example [Chr, Example 13.6] where, put in geometric terms, $X = \mathbb{P}^n, \mathcal{O}_X(1)$ is the degree one very ample line bundle (so $R = 1$), $\mathcal{F}$, $\mathcal{G}$ are ideal sheaves, and $\dim Y > 2$. In this example, $\operatorname{reg}^0(\mathcal{F} \otimes \mathcal{G}) > \operatorname{reg}^0(\mathcal{F}) + \operatorname{reg}^0(\mathcal{G})$.

3. Ample filters of nearly locally free sheaves

In this section, we take care of some technicalities regarding ample filters. First, we must define them.

A filter $\mathcal{P}$ is a partially ordered set such that:

$$\text{for all } \alpha, \beta \in \mathcal{P},\ \text{there exists } \gamma \in \mathcal{P} \text{ such that } \alpha < \gamma \text{ and } \beta < \gamma.$$

Let X be a projective scheme. If a set of coherent sheaves is indexed by a filter, then we will call that set a *filter of sheaves*. An element of such a filter will be denoted $\mathcal{G}_\alpha$ for $\alpha \in \mathcal{P}$. The indexing filter $\mathcal{P}$ will usually not be named.

Definition 3.1. Let X be a projective scheme over a field. Let $\mathcal{P}$ be a filter. A filter of coherent sheaves $\{\mathcal{G}_\alpha\}$ on X, with $\alpha \in \mathcal{P}$, will be called a *t-ample filter* if for every locally free sheaf $\mathcal{E}$, there exists α_0 such that

$$H^q(X, \mathcal{E} \otimes \mathcal{G}_\alpha) = 0, \quad q > t,\ \alpha \geqslant \alpha_0.$$

If $\mathcal{P} \cong \mathbb{N}$ as filters, then a t-ample filter $\{\mathcal{G}_\alpha\}$ is called a *t-ample sequence*. An *ample filter* is a 0-ample filter.

Note that if X has characteristic $p > 0$, then the F-amplitude $\phi(\mathcal{G}) \leqslant t$ if and only if the sequence $\{\mathcal{G}, \mathcal{G}^{(p)}, \mathcal{G}^{(p^2)}, \ldots\}$ is a t-ample sequence. Also, if $\mathcal{E}$ is a semiample vector bundle, then $\mathcal{E}$ is t-ample if and only if $S^n(\mathcal{E})$ is a t-ample sequence [L, Example 6.2.19].

The definition above will be useful for studying the connection between ample filters and F-amplitude. However, one needs stronger vanishing theorems to imitate the category equivalence of Serre Correspondence. (See, for instance, [VdB, Definition 5.1].) Fortunately, when an ample filter consists of sheaves which are nearly locally free, we have a stronger vanishing statement. We now make precise what we mean by nearly locally free.

Definition 3.2. Let X be a projective scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf. Then $\mathcal{F}$ is *locally free in codimension n* (or *lfc-n*) if there exists a closed subscheme Y of dimension n such that $\mathcal{F}|_{X\setminus Y}$ is locally free. If $\mathcal{F}$ is locally free, we say it is lfc-(-1).

It is clear that an lfc-n sheaf is lfc-$(n+1)$. Also, note that if X is an integral scheme of dimension d, then any coherent sheaf is lfc-$(d-1)$ [H2, Exercise II.5.8].

Lemma 3.3. *Let X be a projective scheme, and let $\{\mathcal{G}_\alpha\}$ be a t-ample filter of lfc-$(t+2)$ sheaves. Then for any coherent sheaf $\mathcal{F}$, there exists α_0 such that*

$$H^q(X, \mathcal{F} \otimes \mathcal{G}_\alpha) = 0, \quad q > t,\ \alpha \geqslant \alpha_0.$$

Proof. Choose a very ample invertible sheaf $\mathcal{O}_X(1)$. Since $\{\mathcal{G}_\alpha\}$ is a t-ample filter, we can find α_0 such that $\operatorname{reg}^t(\mathcal{G}_\alpha) < -\operatorname{reg}^0(\mathcal{F}) - (\dim X - t - 1)(R-1)$. Then by Lemma 2.2 and Proposition 2.8, we have the desired vanishing. □

The following lemma allows us to study vanishing for lfc-$(t+1)$ sheaves. Note that if $\mathcal{F}$ is lfc-$(t+1)$, then for any coherent $\mathcal{G}$, $\mathcal{T}or_i(\mathcal{F}, \mathcal{G}), i > 0$ is supported on a subscheme of dimension $\leqslant t+1$.

Lemma 3.4. *Let X be a projective scheme, let $\mathcal{O}_X(1)$ be ample and generated by global sections, and let $0 \to \mathcal{K} \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$ be an exact sequence of coherent sheaves. Suppose $\dim \operatorname{Supp} \mathcal{K} \leqslant t+1$. Then*

$$\operatorname{reg}^t(\mathcal{F}) \leqslant \max\bigl(\operatorname{reg}^t(\mathcal{F}''), \operatorname{reg}^t(\mathcal{F}')\bigr).$$

Proof. This is immediate since $H^q(\mathcal{K}(m-q)) = 0, q > t+1$ for any m. □

We will now state three standard lemmas which will help to reduce questions about ample filters to the case where X is an integral scheme. These all follow from applying Lemmas 3.3 and 3.4, the methods of [H2, Exercise III.5.7], and the fact that even if $\mathcal{G}_\alpha$ is not locally free, the projection formula $f_*(\mathcal{F} \otimes f^*\mathcal{G}_\alpha) = f_*\mathcal{F} \otimes \mathcal{G}_\alpha$ still holds for f finite [A1, Lemma 5.6].

Lemma 3.5. *Let $f : Y \to X$ be a finite morphism of projective schemes and let $\{\mathcal{G}_\alpha\}$ be a filter of lfc-$(t+1)$ sheaves on X. If $\{\mathcal{G}_\alpha\}$ is a t-ample filter on X, then $\{f^*\mathcal{G}_\alpha\}$ is a t-ample filter on Y. If f is surjective and $\{f^*\mathcal{G}_\alpha\}$ is a t-ample filter on Y, then $\{\mathcal{G}_\alpha\}$ is a t-ample filter on X.*

Lemma 3.6. *Let X be a projective scheme with reduction $i : X_{\mathrm{red}} \to X$, and let $\{\mathcal{G}_\alpha\}$ be a filter of lfc-$(t+1)$ coherent sheaves on X. Then $\{\mathcal{G}_\alpha\}$ is a t-ample filter if and only if $\{i^*\mathcal{G}_\alpha\}$ is a t-ample filter.*

Lemma 3.7. *Let X be a reduced scheme with irreducible components $f_j : X_j \to X$, $j = 1, \ldots, n$, and let $\{\mathcal{G}_\alpha\}$ be a filter of lfc-$(t+1)$ sheaves on X. Then $\{\mathcal{G}_\alpha\}$ is a t-ample filter if and only if $\{f_j^*\mathcal{G}_\alpha\}$ is a t-ample filter for each j.*

We will also need to change the base field so that we can work over a perfect field.

Lemma 3.8. *Let $k \subseteq k'$ be fields, let X be a projective scheme over k, and let $\{\mathcal{G}_\alpha\}$ be a filter of coherent sheaves on X. Then $\{\mathcal{G}_\alpha\}$ is a t-ample filter if and only if $\{\mathcal{G}_\alpha \otimes_k k'\}$ is a t-ample filter.*

Proof. Let $f : X \times_k k' \to X$ be the base change, and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Then $f^*O_X(1)$ is very ample on $X \times k'$ [EGA, II, 4.4.10]. Now since $k \to k'$ is a flat morphism,

$$H^q\bigl(X \times_k k', f^*\mathcal{G}_\alpha \otimes f^*O_X(b)\bigr) = H^q\bigl(X, \mathcal{G}_\alpha \otimes O_X(b)\bigr) \otimes_k k'$$

for any $b \in \mathbb{Z}$ [H2, Proposition III.9.3]. Then by the definition of t-ample filter and the faithful flatness of $k \to k'$, we have the lemma. $\square$

We now wish to extend some of the basic F-ampleness results of [A1] from the locally free case to the lfc-n case.

Since the definition of F-ampleness in characteristic zero depends on arithmetic thickenings, we must make sure that the property lfc-n behaves well in such cases.

Lemma 3.9. *Let X be a projective scheme over a field of characteristic* 0*, and let $\mathcal{F}$ be an lfc-n coherent sheaf. Then there exists an arithmetic thickening $f : \tilde{X} \to \operatorname{Spec} A$ such that for every fiber $\tilde{X}_x$, the sheaf $\tilde{\mathcal{F}}_x$ is lfc-n.*

Proof. Let Y be a closed subscheme of X such that $\mathcal{F}|_U$ is locally free where $U = X \setminus Y$. Then $\dim Y \leqslant n$ by hypothesis. We can find a thickening $\tilde{Y} \xrightarrow{i} \tilde{X} \xrightarrow{f} \operatorname{Spec} A$ such that i is a closed immersion and each fiber of $f \circ i$ has dimension $\leqslant n$. We can shrink the thickening so that $\mathcal{F}$ has a thickening $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}|_{\tilde{U}}$ is locally free, where $\tilde{U} = \tilde{X} \setminus \tilde{Y}$. Then for every $y \in \operatorname{Spec} A$, we have $\tilde{\mathcal{F}}|_{\tilde{U}_y}$ locally free and $\dim \tilde{Y}_y \leqslant n$. Thus $\tilde{\mathcal{F}}_y$ is lfc-n, as desired. $\square$

Because the study of F-ampleness relies on pulling back by the Frobenius morphism F, we also need to see how the lfc-n property behaves under pullback.

Lemma 3.10. *Let X be a projective scheme, and let $\mathcal{F}$ be an lfc-n coherent sheaf on X. If $f : X' \to X$ is a quasi-finite morphism* (*that is, a morphism with finite fibers*)*, then $f^*\mathcal{F}$ is lfc-n.*

Proof. Let $Y \subset X$ be a closed subscheme such that $\dim Y \leqslant n$ and $\mathcal{F}_{X \setminus Y}$ is locally free. Then $f^*(\mathcal{F}_{X \setminus Y}) = f^*\mathcal{F}_{X' \setminus f^{-1}(Y)}$ is locally free and $\dim f^{-1}(Y) \leqslant n$ since f is quasi-finite. $\square$

We now turn our attention to F-amplitude (see Definition 1.2). Note that the definition only requires vanishing when tensoring with a locally free sheaf. However, if $\mathcal{F}$ is locally free in codimension 2, we can get more, generalizing [A1, Lemma 2.2].

Lemma 3.11. *Let X be a projective scheme over a field of characteristic $p > 0$, and let $\mathcal{F}$ be an lfc-$(t + 2)$ coherent sheaf. Then for any coherent sheaf $\mathcal{G}$, there exists n_0 such that*

$$H^i\big(X, \mathcal{G} \otimes \mathcal{F}^{(p^n)}\big) = 0, \quad i > \phi(\mathcal{F}),\ n \geqslant n_0.$$

Proof. The proof uses the same methods as Lemma 3.3. $\square$

4. Ample filters and *F*-ampleness: smooth case

Our main goal in this section is to prove Theorem 1.3. For this section, X will be a projective variety (that is, an integral scheme), smooth over a perfect field k.

We will first need a lemma which is a slight generalization of [A2, Theorem 5.4]. The concept of "sufficiently ample line bundle" is defined in [A2, §1]. For our purposes, we need only know that given any ample line bundle $\mathcal{L}$, there exists n_0 such that $\mathcal{L}^n$ is sufficiently ample for $n \geqslant n_0$ [A2, Theorem 1.2].

Lemma 4.1. *Let X be a projective variety over a perfect field of characteristic $p > 0$. Let $\mathcal{O}_X(1)$ be a sufficiently ample line bundle, and let $\mathcal{E}, \mathcal{F}$ be coherent sheaves, with at least one of $\mathcal{E}, \mathcal{F}$ locally free. If $q > \lambda(\mathcal{E}(-\dim X))$ and if $p^N \geqslant \operatorname{reg}^0(\mathcal{F})$, then*

$$H^q\big(X, \mathcal{E}^{(p^N)} \otimes \mathcal{F}\big) = 0.$$

Proof. The proof proceeds exactly as in [A2, Theorem 5.4], where it is assumed both $\mathcal{E}, \mathcal{F}$ are locally free. We need only make sure that a certain exact sequence remains exact when tensoring with $\mathcal{E} \otimes \mathcal{F}$.

More specifically, Arapura sets $f' = F^n$, the nth power of the absolute Frobenius. Then $f' : X \to X$ is not k-linear, but since k is perfect, we can let $X' = X \times_{\operatorname{Spec} k} \operatorname{Spec} k$ over the p^nth power map of k. There are then a k-linear map f and natural map g:

$$X \xrightarrow{f} X' \xrightarrow{g} X,$$

where $f' = g \circ f$. Set $\mathcal{E}' = g^*\mathcal{E}$.

We then let $\gamma : X \to X' \times X$ be the morphism for which $p_1 \circ \gamma = f$ and $p_2 \circ \gamma = id$. Arapura constructs a locally free resolution of $\gamma_* \mathcal{O}_X$, which he tensors with $\mathcal{E}' \boxtimes \mathcal{F} = p_1^*\mathcal{E}' \otimes p_2^*\mathcal{F}$. If we can verify that $\mathcal{T}or_i(\mathcal{E}' \boxtimes \mathcal{F}, \gamma_*\mathcal{O}_X) = 0$, $i > 0$, then we are done. Since one of $\mathcal{E}', \mathcal{F}$ is locally free, it suffices to verify that $\mathcal{T}or_i(p_1^*\mathcal{E}', \gamma_*\mathcal{O}_X) = 0$ and $\mathcal{T}or_i(p_2^*\mathcal{F}, \gamma_*\mathcal{O}_X) = 0$.

Let $0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{E} \to 0$ be exact with $\mathcal{G}$ locally free. Set $\mathcal{K}' = g^*\mathcal{K}, \mathcal{G}' = g^*\mathcal{G}$. Then since p_1 is flat, there is a short exact sequence

$$0 \to p_1^*\mathcal{K}' \to p_1^*\mathcal{G}' \to p_1^*\mathcal{E}' \to 0$$

and hence an exact sequence

$$0 \to \mathcal{T}or_1\big(p_1^*\mathcal{E}', \gamma_*\mathcal{O}_X\big) \to p_1^*\mathcal{K}' \otimes \gamma_*\mathcal{O}_X \to p_1^*\mathcal{G}' \otimes \gamma_*\mathcal{O}_X \to p_1^*\mathcal{E}' \otimes \gamma_*\mathcal{O}_X \to 0.$$

However, since X, X' are smooth (and hence regular) [AK, Prop. VII.6.3] and f is finite, we have that f is flat [AK, Corollary V.3.6]. Since γ is also finite, there is an exact sequence

$$0 \to \gamma_*\big(f^*\mathcal{K}'\big) \to \gamma_*\big(f^*\mathcal{G}'\big) \to \gamma_*\big(f^*\mathcal{E}'\big) \to 0.$$

Since γ is a finite morphism, the projection formula holds for any coherent sheaf [A1, Lemma 5.6]. That is, $p_1^*\mathcal{E}' \otimes \gamma_*\mathcal{O}_X \cong \gamma_*(\gamma^* p_1^*\mathcal{E}') \cong \gamma_*(f^*\mathcal{E}')$. So we have

$$0 \to \mathcal{T}or_1\big(p_1^*\mathcal{E}', \gamma_*\mathcal{O}_X\big) \to \gamma_*\big(f^*\mathcal{K}'\big) \to \gamma_*\big(f^*\mathcal{G}'\big) \to \gamma_*\big(f^*\mathcal{E}'\big) \to 0.$$

Since all these exact sequences were derived by application of functors, we have $\mathcal{T}or_1(p_1^*\mathcal{E}', \gamma_*\mathcal{O}_X) = 0$.

We also have $\mathcal{T}or_i(p_1^*\mathcal{K}', \gamma_*\mathcal{O}_X) \cong \mathcal{T}or_{i+1}(p_1^*\mathcal{E}', \gamma_*\mathcal{O}_X)$ for $i > 0$. Allowing $\mathcal{K}'$ to play the role of $\mathcal{E}'$ and using induction, we have $\mathcal{T}or_i(p_1^*\mathcal{E}', \gamma_*\mathcal{O}_X) = 0$ for $i > 0$.

The case of $\mathcal{T}or_i(p_2^*\mathcal{F}, \gamma_*\mathcal{O}_X) = 0$ is similar. □

We immediately have a generalization of [A2, Corollary 5.5].

Corollary 4.2. *Let X be a smooth projective variety defined over a perfect field k of arbitrary characteristic. Let $\mathcal{O}_X(1)$ be sufficiently ample and let $\mathcal{E}$ be a coherent sheaf on X. Then the Frobenius amplitude $\phi(\mathcal{E})$ satisfies*

$$\phi(\mathcal{E}) \leqslant \lambda\big(\mathcal{E}(-\dim X)\big).$$

Proof. Using an arithmetic thickening, we may assume k has characteristic $p > 0$. Then letting $\mathcal{F}$ be an arbitrary locally free sheaf, the result follows from Lemma 4.1. □

We can now prove one direction of Theorem 1.3.

Theorem 4.3. *Let X be a projective variety, smooth over a perfect field k. Let $\{\mathcal{G}_\alpha\}$ be a t-ample filter. Then for any locally free coherent sheaf $\mathcal{E}$, there exists α_0 such that $\phi(\mathcal{G}_\alpha \otimes \mathcal{E}) \leqslant t$ for $\alpha \geqslant \alpha_0$.*

Proof. Since $\{\mathcal{G}_\alpha\}$ is a t-ample filter, there exists an index α_0 such that $\lambda(\mathcal{G}_\alpha \otimes \mathcal{E}(-\dim X)) \leqslant t$ for $\alpha \geqslant \alpha_0$. We then apply Corollary 4.2. □

We now turn to proving the other half of Theorem 1.3. The main tool is the following exact sequence. When k is algebraically closed, the surjectivity of the trace map below is mentioned, but not proven, in [F2, §1]. It is presumably well known, but we provide a proof for lack of reference.

Lemma 4.4. *Let X be a projective variety, smooth over a perfect field of characteristic $p > 0$. Let $F : X \to X$ be the absolute Frobenius morphism. Then there is an exact sequence of coherent, locally free sheaves*

$$0 \to \mathcal{K} \to F_*\omega \to \omega \to 0$$

where ω is the canonical sheaf of X.

Proof. Since X is smooth (and hence regular) [AK, Proposition VII.6.3] and F is finite, we have that F is finite and flat [AK, Corollary V.3.6]. Thus $F_*\mathcal{O}_X$ is a locally free sheaf. Consider $\phi : \mathcal{O}_X \to F_*\mathcal{O}_X$ with the morphism ϕ locally given by $a \mapsto a^p$. The morphism ϕ remains injective locally at $x \in X$ upon tensoring with any residue field $k(x)$, so the cokernel of ϕ is also locally free [H2, Exercise II.5.8]. Let $\omega = \bigwedge^{\dim X} \Omega^1_{X/k}$ [AK, Theorem VII.5.1]. Dualizing via [H2, Exercises III.6.10, 7.2], we have the desired short exact sequence of locally free sheaves. □

We can now exploit the method of [F1, Lemma 5.8], [F2, §1] to prove a version of Fujita's Vanishing Theorem [F2, Theorem 1] which is a generalization of Serre Vanishing.

Theorem 4.5. *Let X be a projective variety, smooth over a perfect field k with ample invertible sheaf $\mathcal{L}$. Let $\mathcal{E}$ be a locally free coherent sheaf. Then there exists m_0 such that*

$$H^q\big(X, \mathcal{E} \otimes \mathcal{L}^m \otimes \mathcal{G}\big) = 0, \quad q > t,\ m \geqslant m_0,$$

for any coherent $\mathcal{G}$ with $\phi(\mathcal{G}) \leqslant t$. (Note that m_0 does not depend on $\mathcal{G}$.)

Proof. First suppose that the characteristic of k is $p > 0$. Let $0 \to \mathcal{K} \to F_*\omega \to \omega \to 0$ be the exact sequence of Lemma 4.4. Let $q > t$ be given. Suppose by descending induction on q that

$$H^{q+1}\big(X, \mathcal{K} \otimes \mathcal{L}^m \otimes \mathcal{G}\big) = 0, \quad m \geqslant m_1,\ \phi(\mathcal{G}) \leqslant t.$$

(This is certainly true for $q \geqslant \dim X$ because $\dim X$ is the cohomological dimension of X.)

Let $h^q(\mathcal{F}) = \dim_k H^q(X, \mathcal{F})$. Then we have

$$h^q\big(F_*\omega \otimes \mathcal{L}^m \otimes \mathcal{G}\big) = h^q\big(F_*\big(\omega \otimes \mathcal{L}^{pm} \otimes \mathcal{G}^{(p)}\big)\big) \geqslant h^q\big(\omega \otimes \mathcal{L}^m \otimes \mathcal{G}\big)$$

for $m \geqslant m_1$ and *any* $\mathcal{G}$ with $\phi(\mathcal{G}) \leqslant t$. But if $\phi(\mathcal{G}) \leqslant t$, then $\phi(\mathcal{L}^m \otimes \mathcal{G}^{(p^n)}) \leqslant t$ for any $n \geqslant 0$, $m \geqslant 0$ [A1, Theorem 4.5]. So then $h^q(\omega \otimes \mathcal{L}^{p^e m} \otimes \mathcal{G}^{(p^e)}) \geqslant h^q(\omega \otimes \mathcal{L}^m \otimes \mathcal{G})$ for all $e \geqslant 0$. Since $\phi(\mathcal{L}^m \otimes \mathcal{G}^{(p^n)}) \leqslant t$, these dimensions equal 0 for e sufficiently large. But then $h^q(\omega \otimes \mathcal{L}^m \otimes \mathcal{G}) = 0$, $q > t$, $m \geqslant m_1$.

Now suppose the characteristic of k is 0. Then a version of the Kodaira Vanishing Theorem holds. That is, if $\phi(\mathcal{G}) \leqslant t$, then $H^q(X, \omega \otimes \mathcal{G}) = 0$, $q > t$ [A1, Corollary 8.6].

Now let the characteristic of k be arbitrary. Let $\mathcal{E}$ be a locally free coherent sheaf. Let $\mathcal{O}_X(1)$ be very ample. By taking m_2 sufficiently large, we can make $\operatorname{reg}^0(\mathcal{E} \otimes \omega^{-1} \otimes \mathcal{L}^{m_2}) < 0$. And by the above arguments, by taking m_3 sufficiently large, $\operatorname{reg}^t(\omega \otimes \mathcal{L}^m \otimes \mathcal{G}) < -(\dim X - t - 1)(R - 1)$ where $R = \max\{1, \operatorname{reg}^0(\mathcal{O}_X)\}$ and $m \geqslant m_3$. Then by Proposition 2.8, we have $\operatorname{reg}^t(\mathcal{E} \otimes \mathcal{L}^m \otimes \mathcal{G}) < 0$ for $m \geqslant m_2 + m_3$. This immediately gives the theorem. □

We can now prove the other half of Theorem 1.3.

Theorem 4.6. *Let X be a projective variety, smooth over a perfect field k. Let $\{\mathcal{G}_\alpha\}$ be a filter of coherent sheaves. Suppose that for any locally free sheaf $\mathcal{E}$, there exists α_0 such that the Frobenius amplitude $\phi(\mathcal{G}_\alpha \otimes \mathcal{E}) \leqslant t$ for $\alpha \geqslant \alpha_0$. Then $\{\mathcal{G}_\alpha\}$ is a t-ample sequence.*

Proof. Let $\mathcal{L}$ be an ample invertible sheaf and let $\mathcal{F}$ be a coherent locally free sheaf. Then by Theorem 4.5, there exists m such that $H^q(X, \mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{G}) = 0$ for all coherent $\mathcal{G}$ with $\phi(\mathcal{G}) \leqslant t$. For sufficiently large α, we have $\phi(\mathcal{L}^{-m} \otimes \mathcal{G}_\alpha) \leqslant t$, and we are done. □

5. Ample filters and *F*-ampleness: general case

In this section, we will obtain some generalizations of Theorem 1.3 to the case of X a projective scheme over a general field. Unfortunately, we are unable to obtain Theorem 1.3 in full generality. If X is not smooth, then the Frobenius morphism is not flat [Ku, Theorem 2.1]. We also lose the invertibility of ω_X [AK, Theorem VII.5.3]. These facts lead to weaker results.

The following is a partial generalization of Theorem 4.3. Note that the proof only works for 0-ample filters.

Proposition 5.1. *Let X be a projective scheme over a field k, and let $\{\mathcal{G}_\alpha\}$ be an ample filter of lfc-2 sheaves. Then for any invertible sheaf $\mathcal{H}$, there exists α_0 such that $\mathcal{H} \otimes \mathcal{G}_\alpha$ is F-ample for all $\alpha \geqslant \alpha_0$.*

Proof. Since $\{\mathcal{H} \otimes \mathcal{G}_\alpha\}$ is an ample filter, we may assume that $\mathcal{H} = \mathcal{O}_X$. Let $d = \dim X$, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf, and let $R = \max\{1, \operatorname{reg}(\mathcal{O}_X)\}$. Since $\{\mathcal{G}_\alpha\}$ is an ample filter, there exists α_0 such that $\operatorname{reg}(\mathcal{G}_\alpha) < -(d-1)R$ for $\alpha \geqslant \alpha_0$. Then by Lemma 2.7, there is a locally free resolution $\mathcal{E}_\bullet \to \mathcal{G}_\alpha \to 0$ where $\mathcal{E}_i$ is a direct sum of very ample invertible sheaves, $i = 0, \ldots, d-1$.

If $\operatorname{char} k = p > 0$, then pulling back $\mathcal{E}_\bullet \to \mathcal{G}_\alpha \to 0$ via F^{*n} for large n, and using Lemma 2.4, we see that $\lim_{n\to\infty} \operatorname{reg}(F^{*n}\mathcal{G}_\alpha) = -\infty$. Thus Proposition 2.8 shows that for any coherent locally free sheaf $\mathcal{E}$, the higher cohomology of $\mathcal{E} \otimes F^{*n}\mathcal{G}_\alpha$ vanishes for large n, so $\mathcal{G}_\alpha$ is F-ample. If $\operatorname{char} k = 0$, then one can choose an arithmetic thickening so that $\tilde{\mathcal{E}}_\bullet \to \tilde{\mathcal{G}}_\alpha \to 0$ is still a resolution by direct sums of very ample invertible sheaves, at least for $i = 0, \ldots, d-1$. Then the above argument on each closed fiber of the thickening shows that $\mathcal{G}_\alpha$ is F-ample for $\alpha \geqslant \alpha_0$. □

For the remainder of the section, we will prove a partial generalization of Theorem 4.6. The main idea of the proof is to reduce to the smooth case via alteration of singularities. Thus, we must study the higher direct images of a projective morphism $f : X \to Y$.

Lemma 5.2. *Let $f : X \to Y$ be a projective morphism of projective schemes over a field. Let $\mathcal{L}$ be an f-ample invertible sheaf on X and let $\mathcal{F}$ be a coherent sheaf on X. Then there exists m_0 such that*

$$f_*(\mathcal{F} \otimes \mathcal{L}^m) \otimes \mathcal{G} \cong f_*(\mathcal{F} \otimes \mathcal{L}^m \otimes f^*\mathcal{G}),$$
$$R^i f_*(\mathcal{F} \otimes \mathcal{L}^m) \otimes \mathcal{G} = R^i f_*(\mathcal{F} \otimes \mathcal{L}^m \otimes f^*\mathcal{G}) = 0, \tag{5.3}$$

for all $i > 0$, $m \geqslant m_0$, and coherent $\mathcal{G}$ on Y. (Note that m_0 is independent of $\mathcal{G}$.)

Thus also

$$H^q(Y, f_*(\mathcal{F} \otimes \mathcal{L}^m) \otimes \mathcal{G}) \cong H^q(X, \mathcal{F} \otimes \mathcal{L}^m \otimes f^*\mathcal{G})$$

for $q \geqslant 0$, $m \geqslant m_0$, and all coherent $\mathcal{G}$ on Y.

Proof. Our proof is similar to the proof of [A1, Lemma 5.7]. We proceed by descending induction on i. By Grothendieck's Vanishing Theorem [H2, Theorem III.2.7], the higher direct images are identically 0 for large i. So assume we have (5.3) for $i+1$.

Let $\mathcal{G}$ be a coherent sheaf on Y and let $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{G} \to 0$ be a resolution of $\mathcal{G}$ by locally free sheaves. Let $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{L}^m$. Then for some m_0, we have a commutative diagram

$$\begin{array}{ccccc} R^i f_*(\mathcal{F}(m)) \otimes \mathcal{E}_1 & \longrightarrow & R^i f_*(\mathcal{F}(m)) \otimes \mathcal{E}_0 & \longrightarrow & R^i f_*(\mathcal{F}(m)) \otimes \mathcal{G} \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ R^i f_*(\mathcal{F}(m) \otimes f^*\mathcal{E}_1) & \longrightarrow & R^i f_*(\mathcal{F}(m) \otimes f^*\mathcal{E}_0) & \longrightarrow & R^i f_*(\mathcal{F}(m) \otimes f^*\mathcal{G}) \to 0 \end{array}$$

for $m \geqslant m_0$. The first two vertical arrows are isomorphisms by the usual projection formula [H2, Exercise III.8.3] and thus the third one is as well.

Now if $i > 0$, then we may increase m_0 so that $R^i f_*(\mathcal{F} \otimes \mathcal{L}^m) = 0$ for $m \geqslant m_0$ [L, Theorem 1.7.6]. This completes the proof of (5.3). The last statement of the lemma follows immediately from [H2, Exercise III.8.1]. □

Since the ampleness of a line bundle is not preserved under pull-backs, F-ampleness cannot be preserved either. However, if $\mathcal{L}$ is ample, then $f^*\mathcal{L}$ is nef, and hence $\mathcal{H} \otimes f^*\mathcal{L}$ is ample for any ample line bundle $\mathcal{H}$. This does generalize.

Lemma 5.4. *Let X, Y be projective schemes over a perfect field k and let $f : X \to Y$ be a projective morphism. Let $\mathcal{F}$ be an lfc-$(t+1)$ coherent sheaf on Y with $\phi(\mathcal{F}) \leqslant t$. Then for any ample line bundle $\mathcal{H}$ on X, we have $\phi(\mathcal{H} \otimes f^*\mathcal{F}) \leqslant t$.*

Proof. Suppose $\operatorname{char} k = p > 0$. Since k is perfect, the absolute Frobenius morphism is finite. Thus $\phi(\mathcal{H} \otimes f^*\mathcal{F}) = \phi(\mathcal{H}^{p^n} \otimes f^*\mathcal{F}^{(p^n)})$ by Lemma 3.5. Thus we may assume that $\mathcal{H}$ is very ample. Define $\mathcal{O}(1) = \mathcal{H}$.

Let $\mathcal{E}$ be a locally free coherent sheaf on X. Any ample line bundle is f-ample [L, Proposition 1.7.10]. So by Lemma 5.2, there exists m_0 such that

$$H^q\big(Y, f_*\big(\mathcal{E}(m-q)\big) \otimes \mathcal{G}\big) \cong H^q\big(X, \mathcal{E}(m-q) \otimes f^*\mathcal{G}\big)$$

for all $q \geqslant 0$, $m \geqslant m_0$ and coherent $\mathcal{G}$ on Y. Since $\phi(\mathcal{F}) \leqslant t$ and $\mathcal{F}$ is lfc-$(t+2)$, there exists n_0 such that

$$H^q\big(Y, f_*\big(\mathcal{E}(m_0-q)\big) \otimes \mathcal{F}^{(p^n)}\big) = 0$$

for $q > t$, $n \geqslant n_0$ by Lemma 3.11. Thus $\mathcal{E} \otimes f^*\mathcal{F}^{(p^n)}$ is (m,t)-regular for $m \geqslant m_0$, $n \geqslant n_0$. For $n_1 \geqslant n_0$ sufficiently large, we have $p^{n_1} \geqslant m_0 + \dim X$. Thus for $n \geqslant n_1$, $q > t$, we have

$$H^q\big(X, \mathcal{E}\big(p^n\big) \otimes f^*\mathcal{F}^{(p^n)}\big) = 0.$$

Thus $\phi(\mathcal{H} \otimes f^*\mathcal{F}) \leqslant t$.

If $\operatorname{char} k = 0$, the result follows immediately. □

We may now prove another generalization of Serre–Fujita Vanishing by reducing to Theorem 4.5.

Theorem 5.5. *Let X be a projective scheme over a field k with ample invertible sheaf $\mathcal{L}$. Let $\mathcal{E}$ be a coherent sheaf. Then there exists m_0 such that*

$$H^q\big(X, \mathcal{E} \otimes \mathcal{L}^m \otimes \mathcal{G}\big) = 0, \quad q > t,\ m \geqslant m_0,$$

for any coherent lfc-$(t+1)$ $\mathcal{G}$ with $\phi(\mathcal{G}) \leqslant t$. (Note that m_0 does not depend on $\mathcal{G}$.)

Proof. Let $\Sigma = \{\mathcal{L}^m \otimes \mathcal{G}\colon\ \phi(\mathcal{G}) \leqslant t,\ \ \mathcal{G} \text{ lfc-}(t+1)\}$. Partially order Σ by $\mathcal{L}^{m_1} \otimes \mathcal{G}_1 < \mathcal{L}^{m_2} \otimes \mathcal{G}_2$ if $m_1 < m_2$. If we can show that Σ is an ample filter, then we are done by Lemma 3.3. By Lemmas 3.6, 3.7, and 3.8, we may assume that X is an integral scheme over a perfect field.

Let $f : \tilde{X} \to X$ be an alteration of singularities, so that $\tilde{X}$ is smooth over k and f is surjective [D]. Let $\mathcal{H}$ be an ample line bundle on $\tilde{X}$. Then by Theorem 4.5 and Lemma 5.4, there exists m_0 such that

$$H^q\big(\tilde{X}, \mathcal{H}^m \otimes f^*\mathcal{G}\big) = 0$$

for $q > t$, $m \geqslant m_0$, $\phi(\mathcal{G}) \leqslant t$, with $\mathcal{G}$ lfc-$(t+1)$. Then by Lemma 5.2, for some $m_1 \geqslant m_0$, we have

$$H^q\big(X, f_*\big(\mathcal{H}^m\big) \otimes \mathcal{G}\big) = 0$$

for $q > t$, $m \geqslant m_1$, $\phi(\mathcal{G}) \leqslant t$, with $\mathcal{G}$ lfc-$(t+1)$.

Fix m. Since f is surjective, we have $\operatorname{Supp} f_*(\mathcal{H}^m) = X$. Thus the sheaf $\mathcal{H}om(f_*(\mathcal{H}^m), \mathcal{O}_X)$ contains a non-zero (and hence surjective) local homomorphism at the generic point. For N sufficiently large,

$$\mathcal{H}om\big(f_*\big(\mathcal{H}^m\big), \mathcal{O}_X\big) \otimes \mathcal{L}^N \cong \mathcal{H}om\big(f_*\big(\mathcal{H}^m\big), \mathcal{L}^N\big)$$

is generated by global sections. Therefore there exists a global homomorphism $\psi : f_*(\mathcal{H}^m) \to \mathcal{L}^N$ which is surjective at the generic point. Thus $\operatorname{Coker}\psi$ is a torsion sheaf.

Now descending induction on q and noetherian induction applied to $\operatorname{Coker}\psi$ give $H^q(\mathcal{L}^n \otimes \mathcal{G}) = 0$ for $q > t$, $n \geqslant n_0$, $\phi(\mathcal{G}) \leqslant t$, with $\mathcal{G}$ lfc-$(t+1)$. If $\mathcal{E}$ is any coherent sheaf on X, we may find a resolution of $\mathcal{E}$ by direct sums of $\mathcal{L}^{-j}$. The theorem then follows by Lemma 2.4. □

We now immediately find the following partial generalization of Theorem 4.6, using the same method of proof.

Corollary 5.6. *Let X be a projective scheme over a field k. Let $\{\mathcal{G}_\alpha\}$ be a filter of lfc-$(t+1)$ coherent sheaves. Suppose that for any invertible sheaf $\mathcal{H}$, there exists α_0 such that the Frobenius amplitude $\phi(\mathcal{G}_\alpha \otimes \mathcal{H}) \leqslant t$ for $\alpha \geqslant \alpha_0$. Then $\{\mathcal{G}_\alpha\}$ is a t-ample filter.*

6. *F*-amplitude of an ample vector bundle

Let $\mathcal{E}$ be an ample vector bundle. If $\operatorname{char} k = 0$, then $\phi(\mathcal{E}) < \operatorname{rank}(\mathcal{E})$ [A1, Corollary 6.6]. We will now derive another bound on $\phi(\mathcal{E})$, which is independent of the characteristic of k. First, we need a lemma.

Lemma 6.1. *Let X be a projective scheme, let H be a very ample Cartier divisor, and let $\mathcal{E}$ be a vector bundle. Then*

$$\phi(\mathcal{E}_H) \leqslant \phi(\mathcal{E}) \leqslant \phi(\mathcal{E}_H) + 1.$$

Proof. The first inequality follows from Lemma 3.5.

If $\operatorname{char} k = p > 0$, for any $b \in \mathbb{Z}$ there exists n_0 such that $\mathcal{O}_X(b) \otimes \mathcal{E}_H^{(p^n)}$ is $(0, \phi(\mathcal{E}_H))$-regular for $n \geqslant n_0$. Thus there are exact sequences

$$0 = H^q\big(O_H(b+m-q) \otimes \mathcal{E}_H^{(p^n)}\big) \to H^{q+1}\big(O_X(b+m-q-1) \otimes \mathcal{E}^{(p^n)}\big)$$
$$\to H^{q+1}\big(O_X(b+m-q) \otimes \mathcal{E}^{(p^n)}\big) \to H^{q+1}\big(O_H(b+m-q) \otimes \mathcal{E}_H^{(p^n)}\big) = 0$$

for $q > \phi(\mathcal{E}_H)$, $m \geqslant 0$, $n \geqslant n_0$. By Serre Vanishing, $H^{q+1}(O_X(b+m-q) \otimes \mathcal{E}^{(p^n)}) = 0$ for $m \gg 0$. So by descending induction on m and [A1, Corollary 2.3], we have $\phi(\mathcal{E}) \leqslant \phi(\mathcal{E}_H) + 1$. The case of $\operatorname{char} k = 0$ is then immediate. □

It is now an easy matter to obtain our bound on $\phi(\mathcal{E})$ for ample $\mathcal{E}$. This generalizes [A1, Proposition 5.4].

Theorem 6.2. *Let X be a projective scheme over a field k with $\dim X > 0$, and let $\mathcal{E}$ be a vector bundle which is ample on some complete intersection curve. Then $\phi(\mathcal{E}) < \dim X$. In particular, this is true for ample $\mathcal{E}$.*

Proof. We may assume that k is algebraically closed (Lemma 3.8) and that X is reduced (Lemma 3.6), irreducible (Lemma 3.7), and normal (Lemma 3.5). If $\dim X = 1$, then the claim is [A1, Proposition 5.4]. If $\dim X > 1$, then induction on $\dim X$ and Lemma 6.1 yields the result. □

We can expect no better result. On $\mathbb{P}^n$, the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ is ample (and even p-ample, see Definition 7.1), yet $\phi(\mathcal{T}_{\mathbb{P}^n}) = n-1$ by [A1, Theorem 5.5], because $H^{n-1}(\mathcal{T}(-n-1)) \neq 0$ [A1, Example 5.9].

7. Other ampleness conditions

In this section we compare various alternative definitions of "ample filter" for locally free sheaves and indicate how they relate to the Frobenius morphism. We remind the reader of the definition of p-ample and ample locally free sheaf.

Definition 7.1. Let X be a projective scheme over a field k, and let $\mathcal{E}$ be a locally free coherent sheaf. If $\operatorname{char} k = p > 0$, then $\mathcal{E}$ is *p-ample* if for any coherent sheaf $\mathcal{F}$, there exists n_0 such that $\mathcal{F} \otimes \mathcal{E}^{(p^n)}$ is generated by global sections for $n \geqslant n_0$. If $\operatorname{char} k = 0$, we say $\mathcal{E}$ is p-ample if $\mathcal{E}$ is p-ample on every closed fiber of some arithmetic thickening.

If the sequence $\{S^n(\mathcal{E})\colon n \in \mathbb{N}\}$ is an ample sequence, then $\mathcal{E}$ is an *ample* locally free sheaf.

We now state some progressively weaker "ampleness" properties on filters of locally free sheaves.

Theorem 7.2. *Let X be a projective scheme, and let $\{\mathcal{E}_\alpha\}$ be a filter of locally free coherent sheaves. Consider the following properties.*

(1) *For any coherent sheaf $\mathcal{F}$, there exists α_0 such that $H^q(\mathcal{F} \otimes \mathcal{E}_\alpha) = 0$ for $q > 0$, $\alpha \geqslant \alpha_0$.*
(2) *For any invertible sheaf $\mathcal{H}$, there exists α_0 such that $\mathcal{H} \otimes \mathcal{E}_\alpha$ is an F-ample locally free sheaf for $\alpha \geqslant \alpha_0$.*
(3) *For any coherent sheaf $\mathcal{F}$, there exists α_0 such that $\mathcal{F} \otimes \mathcal{E}_\alpha$ is generated by global sections for $\alpha \geqslant \alpha_0$.*
(4) *For any invertible sheaf $\mathcal{H}$, there exists α_0 such that $\mathcal{H} \otimes \mathcal{E}_\alpha$ is a p-ample locally free sheaf for $\alpha \geqslant \alpha_0$.*
(5) *For any invertible sheaf $\mathcal{H}$, there exists α_0 such that $\mathcal{H} \otimes \mathcal{E}_\alpha$ is an ample locally free sheaf for $\alpha \geqslant \alpha_0$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof. (1) $\Leftrightarrow$ (2) is Proposition 5.1 and Corollary 5.6. For (1) implies (3), we may choose α_0 such that $\mathcal{F} \otimes \mathcal{E}_\alpha$ is 0-regular, and hence generated by global sections, for $\alpha \geqslant \alpha_0$.

For (3) implies (4), let $\mathcal{L}$ be an ample invertible sheaf, and let $\mathcal{H}$ be an arbitrary invertible sheaf. Choose α_0 such that $\mathcal{L}^{-1} \otimes \mathcal{H} \otimes \mathcal{E}_\alpha$ is generated by global sections for $\alpha \geqslant \alpha_0$. Then $\mathcal{H} \otimes \mathcal{E}_\alpha$ is the quotient of a finite direct sum of copies of $\mathcal{L}$. It is easy to see in characteristic p that $\bigoplus \mathcal{L}$ is p-ample and that quotients of p-ample sheaves are p-ample [H1, Proposition 6.4]. If $\operatorname{char} k = 0$, the surjection $\bigoplus \mathcal{L} \twoheadrightarrow \mathcal{H} \otimes \mathcal{E}_\alpha$ can be extended to some arithmetic thickening. Hence the closed fibers of $\tilde{\mathcal{H}} \otimes \tilde{\mathcal{E}}_\alpha$ are p-ample, so $\mathcal{H} \otimes \mathcal{E}_\alpha$ is p-ample for $\alpha \geqslant \alpha_0$.

For (4) implies (5), it is known in characteristic $p > 0$ that p-ample bundles are ample [H1, Proposition 6.3]. In characteristic 0, each of the closed fibers of a thickening $\tilde{\mathcal{H}} \otimes \tilde{\mathcal{E}}_\alpha$ is ample, hence $\mathcal{H} \otimes \mathcal{E}$ is ample by [EGA, III_1, 4.7.1]. $\square$

In certain special cases, all these statements are equivalent.

Proposition 7.3. *Let X be a projective scheme, and let $\{\mathcal{E}_\alpha\}$ be a filter of locally free coherent sheaves. Suppose one of the following holds.*

(1) $\dim X \leqslant 1$.
(2) *Each $\mathcal{E}_\alpha$ is an invertible sheaf.*
(3) *The filter is $\{S^n(\mathcal{E})\colon\ n \in \mathbb{N}\}$ for some fixed locally free coherent sheaf $\mathcal{E}$.*

Then all the statements of Theorem 7.2 *are equivalent.*

Proof. If $\dim X \leqslant 1$ or $\mathcal{E}_\alpha$ is invertible, then ample implies F-ample by Theorem 6.2 or [Ke3, Theorem 1.3]. Therefore, 7.2(5) implies 7.2(2). If $\{S^n(\mathcal{E})\colon\ n \in \mathbb{N}\}$ satisfies 7.2(5), then $S^m(\mathcal{E})$ is ample for some m. So then $\mathcal{E}$ is ample [H1, Proposition 2.4]. But the vanishing in 7.2(1) can be taken as a definition of an ample locally free sheaf, at least when X is proper [H1, Proposition 3.3]. $\square$

We note that the equivalence of statements still holds when the $\mathcal{E}_\alpha$ are invertible sheaves and X is proper over a commutative noetherian ring [Ke3, Theorem 1.3].

In general, the implications are not reversible, except possibly (3) $\Rightarrow$ (4).

Remark 7.4. Let $X = \mathbb{P}^2$ over an algebraically closed field of characteristic $p > 0$. Then there exists a locally free coherent sheaf $\mathcal{E}$ such that $\{\mathcal{E}^{(p^n)}\colon\ n \in \mathbb{N}\}$ satisfies 7.2(3), but not 7.2(2). There also exists a locally free coherent sheaf $\mathcal{F}$ such that $\{\mathcal{F}^{(p^n)}\colon\ n \in \mathbb{N}\}$ satisfies 7.2(5), but not 7.2(4).

Proof. Let $\mathcal{E}$ be p-ample, but not F-ample. Then the first claim is satisfied by definition. Let $\mathcal{F}$ be ample, but not p-ample. If the sequence $\{\mathcal{F}^{(p^n)}\}$ satisfies 7.2(4), then $\mathcal{F}^{(p^n)}$ is p-ample for some m. But then $\mathcal{F}$ is p-ample [H1, Proposition 6.4], a contradiction. However, $\{\mathcal{F}^{(p^n)}\}$ satisfies 7.2(5) [B, Proposition 3.1]. Such $\mathcal{E}$, $\mathcal{F}$ do exist on $\mathbb{P}^2$ [G]. An important specific example is the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ for $n \geqslant 2$. This bundle is p-ample, but not F-ample [A1, Example 5.9]. $\square$

Does 7.2(4) imply 7.2(3) in general? The answer is affirmative when X is smooth [Ke4].

Acknowledgments

We would like to thank D. Arapura for discussions on F-ampleness. We thank M. Chardin, R. Lazarsfeld, and J. Sidman for discussions on regularity. Also, H. Brenner kindly pointed out that the hypotheses of Theorem 6.2 could be weakened with no change to the proof.

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