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THE INDEX OF *grad* $f(x, y)$

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Let $f(x, y)$ be a real polynomial of degree d with isolated critical points, and let i be the index of *grad* f around a large circle containing the critical points. An elementary argument shows that $|i| \leqslant d-1$. In this paper we show that $i \leqslant \max\{1, d-3\}$. We also show that if all the level sets of f are compact, then $i = 1$, and otherwise $|i| \leqslant d_{\mathbf{R}} - 1$ where $d_{\mathbf{R}}$ is the sum of the multiplicities of the real linear factors in the homogeneous term of highest degree in f. The technique of proof involves computing i from information at infinity. The index i is broken up into a sum of components $i_{p,c}$ corresponding to points p in the real line at infinity and limiting values $c \in \mathbf{R} \cup \{\infty\}$ of the polynomial. The numbers $i_{p,c}$ are computed in three ways: geometrically, from a resolution of $f(x, y)$, and from a Morsification of $f(x, y)$. The $i_{p,c}$ also provide a lower bound for the number of vanishing cycles of $f(x, y)$ at the point p and value c.

1. INTRODUCTION

Let $f(x, y)$ be a real polynomial with isolated critical points. Let i be the index of the gradient vector field of $f(x, y)$ around a large circle C centered at the origin and containing the critical points, oriented in the counterclockwise direction. If the critical points of f are nondegenerate, then the index i is the number of local extrema minus the number of saddles.

What bounds can be placed on the index i in terms of the degree d of the polynomial? It follows easily from Bezout's theorem that [4, Proposition 2.5]

$$|i| \leqslant d-1.$$

It is easy to find polynomials satisfying the lower bound of this inequality; for example if $f = l_1, \ldots, l_d$ where the l_i are equations of lines in general position, then $i = 1-d$, as can be seen by looking at how the gradient vector field turns on the circle C, or by counting critical points [4, Section 4].

The upper bound is more mysterious. In the first place, polynomials with $i > 1$ are hard to find. (The dubious reader should try to do so!) A simple example with two local extrema and no other critical points ($i = 2$) is $f(x, y) = y^5 + x^2y^3 - y$. A polynomial of degree five can have as many as sixteen critical points in the complex plane; a generic polynomial of degree five will have exactly this number. The above polynomial, however, has only four critical points in the plane (two real and two complex), so it is not generic. In fact this behavior is typical for polynomials with $i > 1$ [4, Theorem 6.2].

There are polynomials of degree d with i arbitrarily large (see Example 2.5), but they have $i \approx (1/3)d$. So evidently there is a large gap between the theoretical upper bound and examples. One of the goals of this paper is to give a modest improvement of this upper bound. We will show

THEOREM 7.11. *If* $f(x, y)$ *is a real polynomial of degree* d *with isolated critical points, and* i *is the index of grad* f *around a large circle containing the critical points,* *then*

$$i \leqslant \max\{1, d-3\}.$$

In particular this result implies that the minimum degree for a polynomial with $i > 1$ is five, as in the example above. In fact, the bound is often better. (See, for example, Proposition 7.4.)

Let the *real degree* $d_{\mathbf{R}}$ of f be the sum of the multiplicities of the real linear factors in the homogeneous term of highest degree in f. (Thus $d_{\mathbf{R}} \leqslant d$.) We will also show

THEOREM 7.8. *If all the level sets of the polynomial $f(x, y)$ are compact, then $i = 1$. Otherwise*

$$|i| \leqslant d_{\mathbf{R}} - 1.$$

It is easy to find polynomials realizing the lower bound (Corollary 5.5), but the upper bound still appears high.

The basic idea of the proofs is to compute the index i from "information at infinity". We write i as

$$i = 1 + \sum_{\substack{p \in \mathbf{L} \\ c \in \mathbf{R} \cup \{\infty\}}} i_{p,c}.$$

The terms $i_{p,c}$ are defined as follows: The number $\pm 1/2$ is assigned to a point q where the circle C is tangent to a level set of the polynomial according as whether the level set is locally inside or outside C at q. The circle is then made larger and larger. The point q where the level set is tangent to the circle approaches a limiting point p on the line at infinity in real projective space, and the value of the polynomial $f(q)$ approaches a limiting value c. The term $i_{p,c}$ is the sum of all the numbers $\pm 1/2$ associated to p and c in this manner. This material is in Section 3. We also show (Proposition 5.6) that the family of circles can be replaced by the level sets of any reasonable function, and the $i_{p,c}$ will remain the same.

The polynomial f extends to a function on projective space which is not well-defined at certain points on the line at infinity. Blowing up these points gives a well-defined function $\tilde{f}$. We use this technique in Section 4 to derive some simple properties of the level curves of f.

In Section 5, we use Morse theory to show that the $i_{p,c}$ can be computed from the critical points of $\tilde{f}$ and information about the exceptional sets. The process of blowing up and computing the index is easy to carry out in specific examples.

The polynomial f can also be deformed into what we call a "Morsification", a polynomial whose real critical points are nondegenerate and whose homogeneous term of highest degree has no repeated real linear factors (Section 6). There is a simple formula relating the index of the original polynomial, the index of the new polynomial, and the index of the newly created critical points. The deformation process is not too well understood, and this section contains some examples and a conjecture.

The computations of these sections are used in Section 7 to establish bounds on the $i_{p,c}$. These local bounds are sharp. The global bounds on i follow from the local bounds and some delicate arguments. However, the global bounds are not sharp and there still is a big gap between the global bounds and the examples.

In Section 8 we relate $i_{p,c}$ to the "jump" at c in the Milnor number of the family $f(x, y) = t$ at the point p on the line at infinity.

Throughout this paper the techniques are those of basic topology (Morse Theory) and basic algebraic geometry (Bezout's theorem, explicit computation of intersection multiplicities, etc.) Computer algebra programs were used to find critical points, contour plots and the $i_{p,c}$. Although many of the results and techniques are valid in higher dimensions, the exposition is in dimension two for reasons of clarity.

The author's interest in these questions started in 1989 when he worked with a group of undergraduates in the Mount Holyoke Summer REU [4]. Another group of students continued this work in 1992; one of their results was the construction of polynomials with an arbitrarily large number of local maxima and no other critical points [9]. (These polynomials have $i \approx d/4$.)

Shustin [10] has studied polynomials all of whose critical points lie in the complex plane. He finds polynomials of this type with almost all arbitrarily prescribed numbers of local maxima, minima and saddles. These polynomials have $i = 1 - d_{\mathbf{R}}$ and, in particular, $i \leqslant 1$. They are stable in the sense that nearby polynomials have the same number and type of critical points. The primary focus of this paper is polynomials f with $i > 1$; these polynomials are not stable. In fact, [4, Theorem 6.2] says that

$$i \leqslant \tfrac{1}{2}m + 1$$

where m is the sum over p in the line at infinity in real projective space of the intersection multiplicities at p of the completions of $f_x = 0$ and $f_y = 0$.

This paper is real counterpart of the study by many people of "critical points at infinity" for complex polynomials; see [5] for further references.

Some notation which will be used throughout the paper: We let

$$L = \{[x, y, z] \in \mathbf{P}^2 \colon z = 0\}$$

be the line at infinity in real projective space $\mathbf{P}^2$, and $\mathbf{L}_{\mathbf{C}}$ be the line at infinity in complex projective space $\mathbf{CP}^2$. We use d for the degree of the polynomial $f(x, y)$, and f_d for the homogeneous term of degree d in f.

2. A POLYNOMIAL ZOO

A number of polynomials with strange properties are used as examples throughout this paper. These are described in this section.

Example 2.1. The polynomial $y(xy - 1)$, which has no critical points in the plane, is the standard example of a polynomial with a "critical point at infinity" (at $[1, 0, 0]$). The "critical value" (jump in the Milnor number) is at 0. This polynomial perhaps first appeared in [2].

Example 2.2. The polynomial $x(y^2 - 1)$ has saddles at (0, 1) and (0, -1). The family of level curves at $[1, 0, 0]$ is equisingular; there is no "critical point" at $[1, 0, 0]$.

Example 2.3. The parabola $y^2 - x$ is the simplest example of a polynomial with a "critical point" at $[1, 0, 0]$ with "critical value" ∞ [5].

Example 2.4. The polynomial $y^5 + x^2y^3 - y$ has a local minimum at $(0, -1/\sqrt[4]{5})$, a local maximum at $(0, 1/\sqrt[4]{5})$ and no other critical points. This polynomial was found by an REU group of undergraduates at Mount Holyoke College in the summer of 1989 [4].

Example 2.5. The polynomial $(y(x^2+1)-1)(y(x^2+2)-1)\ldots(y(x^2+k)-1)$ has $k-1$ local extrema and no other real critical points; for $k=2$ there is a local minimum, for $k=3$ there is a local minimum and a local maximum, for $k=4$ there are two local minima and a local maximum, and so forth. This polynomial was also found by the REU group [4].

Example 2.6. The polynomial $(xy^2-y-1)^2+(y^2-1)^2$ from [11] has local minima at $(2,1)$ and $(0,-1)$, and no other critical points. Note the asymmetry of this polynomial compared with the previous ones.

Example 2.7. The polynomial $x^2(1+y)^3+y^2$ has its sole critical point at the origin. This critical point is a local maximum, but not an absolute maximum [3].

Example 2.8. The polynomial $y-(xy-1)^2$ has a saddle at $(-1/2,0)$ and no other critical points. At $[1,0,0]$ the level set $f=0$ has one branch, but the general level set has two branches [7].

Example 2.9. The polynomial $f(x,y)=(x-y^2)((x-y^2)(y^2+1)-1)$ has its zero locus along the parabola $x=y^2$ and the curve $x=y^2+1/(y^2+1)$ which is asymptotic to this parabola. Its only critical point is a minimum at $(1/2,0)$. The level curves intersect $\mathbf{L}$ only at $[1,0,0]$, and they are tangent to $\mathbf{L}$ at this point. (The "curve of tangencies" (see the next section) is also tangent to $\mathbf{L}$ at $[1,0,0]$.)

3. A FORMULA FOR *i* FROM THE GEOMETRY OF *grad f*

Let $f(x,y)$ be a real polynomial with isolated critical points. (Note that f is thus not constant.) Let i be the index of the gradient vector field of $f(x,y)$ around a large circle C centered at the origin and containing the critical points, oriented in the counterclockwise direction. (Recall that the index is the topological degree of the map $C\to S^1$ defined by $t\mapsto \mathit{grad}f(\alpha(t))/|\mathit{grad}f(\alpha(t))|$, where $t\mapsto\alpha(t)$ is a parameterization of C.) This section contains the fundamental geometric decomposition of the index i (Proposition 3.3).

To each point $q\in C$ where a smooth level curve of f is tangent to C at q we assign the number $\pm 1/2$ or 0 as follows: If the level curve of f is outside the circle C near q, this number is $-1/2$. If it is inside C near q, the number is $+1/2$. (These conditions are topological; the tangency may be algebraically degenerate.) If one side is outside and the other inside, or if the level set is contained in C near q (in which case C is a connected component of the level set), the number is 0. (See Fig. 1; the circle C is dotted, and the level curves of f are solid lines.)

The points q where the level sets of f are tangent to C are the zeros of the (real) *curve of tangencies*

$$f_x y - f_y x = 0.$$

This curve may have reducible components.

Choose the circle C large enough so that it contains the compact components and the isolated singular points of the curve of tangencies and their points of common tangency. In the exterior of C the curve of tangencies is a union of connected components. Each component γ is a smooth arc which goes to infinity; we call this an *end* of the curve of tangencies. Choose the circle C large enough so that the numbers $\pm 1/2, 0$ assigned above are constant along each end γ. (This is possible since the intersection multiplicity of C and

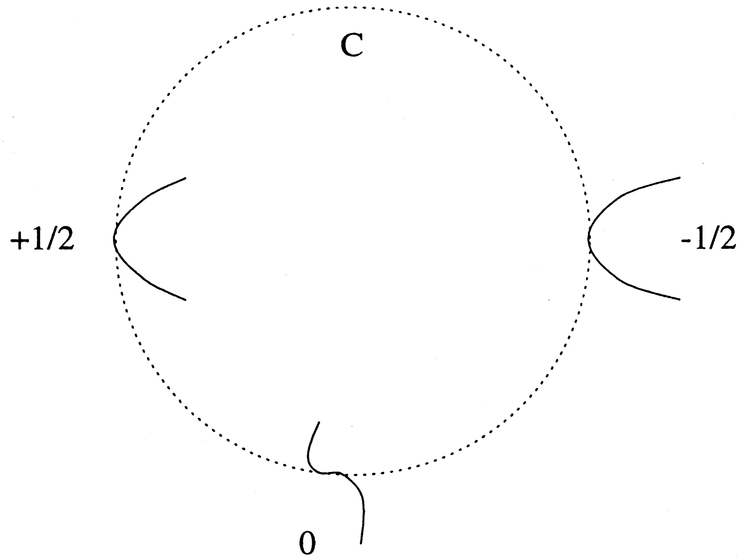

Fig. 1. Assigning $\pm 1/2$ or 0 to a point of tangency.

the level sets of f is constant along each end γ for C large.) Let $i(\gamma)$ be the number $\pm 1/2$ or 0 assigned to γ in this fashion.

Let $p(\gamma) \in \mathbf{L}$ be the endpoint of the closure of γ, and let $c(\gamma) \in \mathbf{R} \cup \{\infty\}$ be the limiting value of $f(q)$ as q goes to infinity along γ.

LEMMA 3.1. *For each end γ of the curve of tangencies, the number $c(\gamma)$ exists. In fact, the function f restricted to γ is strictly increasing or decreasing.*

Proof. Let $\Gamma(f) \subset \mathbf{P}^2 \times \mathbf{P}$ be the closure of the graph of f. The end γ lifts uniquely to $\Gamma(f)$, intersecting the fiber over p at a point (p, c). The number c is $c(\gamma)$. The function f is strictly increasing or decreasing since γ is perpendicular to the level sets of f. $\square$

For $p \in \mathbf{L}$ and $c \in \mathbf{R} \cup \{\infty\}$, we let

$$i_{p,c} = \sum i(\gamma)$$

where the sum is over all ends γ with $p(\gamma) = p$ and $c(\gamma) = c$. We also let

$$i_p = \sum_{c \in \mathbf{R} \cup \{\infty\}} i_{p,c}, \quad i_{\mathbf{L},\infty} = \sum_{p \in \mathbf{L}} i_{p,\infty}.$$

LEMMA 3.2. *The numbers $i_{p,c}$ are integers (not just half-integers).*

Proof. The curve of tangencies can be lifted to $\Gamma(f)$, the graph of f. A real branch of this curve at $(p, c) \in \Gamma(f)$ is a pair of ends $\gamma \neq \gamma'$. If $i(\gamma) = 0$, then the intersection of the level sets of f with the family of circles is degenerate along γ, and hence $i(\gamma') = 0$ as well. $\square$

PROPOSITION 3.3. *If $f(x, y)$ is a real polynomial with isolated critical points, then*

$$i = 1 + \sum_{\substack{p \in \mathbf{L} \\ c \in \mathbf{R}}} i_{p,c} + i_{\mathbf{L},\infty}.$$

Proof. If no connected component of a level set of f is contained in the large circle C, then we have that

$$i = 1 + \sum i(\gamma) \tag{1}$$

where the sum is over all ends γ of the curve of tangencies: This is clearly true if all the points of tangency are regular values for the map $C \to S^1$ defined by $t \mapsto \mathit{grad} f(\alpha(t))/|\mathit{grad} f(\alpha(t))|$, where $t \mapsto \alpha(t)$ is a parameterization of C. If a point of tangency is not a regular value for this map (e.g. for $f(x, y) = y^3 + x$, or $y^4 + x$), then a small (topological) deformation shows that it still holds. The expression of the proposition is just a decomposition of (1).

Now suppose that a connected component of a level set $f = c$ is contained in C. We may assume without loss of generality that $c \gg 0$. Since one component of $f = c$ is compact, all components are compact by Proposition 4.4. Thus by Proposition 5.2, $i = 1$ (which is obvious here), $i_{\mathbf{L},\infty} = 0$ and $i_{p,c} = 0$ for all $p \in \mathbf{L}$ and $c \in \mathbf{R}$. □

A corollary of the Proposition is that

$$i = 1 + \sum_{p \in \mathbf{L}} i_p. \tag{2}$$

The process of decomposing the index for the polynomial $f(x, y) = y(xy - 1)$ of Example 2.1 is pictured in Fig. 2. (The circle is dotted, the solid lines are the level sets of f, the dashed lines are the ends γ of the curve of tangencies, and the numbers are $i(\gamma)$.)

A geometrically obvious example of the decomposition of the index is for a polynomial f for which the real linear factors of f_d are irreducible. In this case $i = 1 - d_{\mathbf{R}}$, $i_{p,c} = 0$ for $p \in \mathbf{L}$ and $c \in \mathbf{R}$, and $i_{\mathbf{L},\infty} = -d_{\mathbf{R}}$. (This will be proved formally in Corollary 5.5.)

The invariants of Proposition 3.3 for selected polynomials are given in Table 1; all the nonzero $i_{p,c}$ for $c \in \mathbf{R}$ are listed.

Note that the sum of the $i(\gamma)$'s making up $i_{p,c}$ is over ends γ where $\mathit{grad} f$ points both out of and into the circle C; the process of decomposing the index described below does not work if the sum is just over those points where the gradient points out, as can be seen in the example $f(x, y) = y(x^2 y - 1)$.

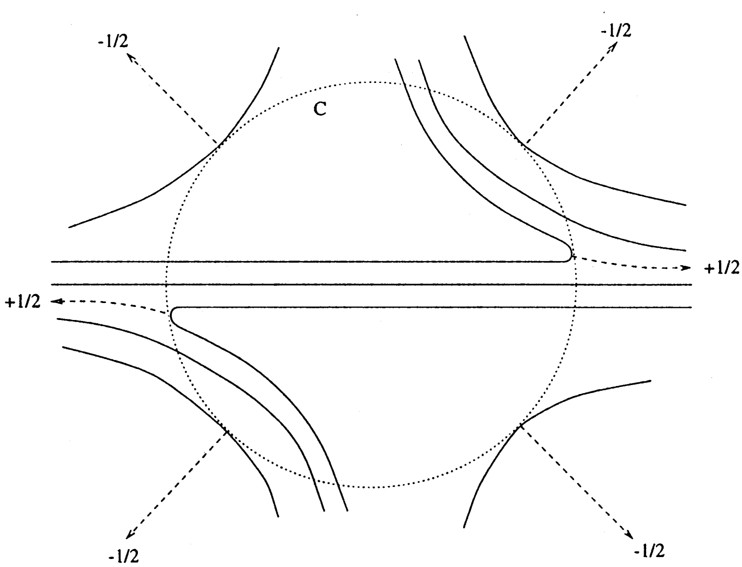

Fig. 2. The index computation for the polynomial $y(xy - 1)$.

Table 1. Index invariants of selected polynomials

$f(x, y)$	$i_{\mathbf{L},\infty}$	$p \in \mathbf{L}$	c	$i_{p,c}$	i
Example 2.1: $y(xy-1)$	-2	$[1, 0, 0]$	0	1	0
Example 2.2: $x(y^2-1)$	-3				-2
Example 2.3: y^2-x	-1				0
Example 2.4: $y^5+x^2y^3-y$	-1	$[1, 0, 0]$	0	2	2
Example 2.9	-1	$[1, 0, 0]$	0	1	1
Example 2.8: $y-(xy-1)^2$	-2	$[1, 0, 0]$	0	0	-1
Example 2.6	-1	$[1, 0, 0]$	1	1	2
			2	1	
$y(x^2y-1)$	-2	$[1, 0, 0]$	0	1	0

It is useful to have both the expression of Proposition 3.3 where the limiting value $c=\infty$ is separated out and put into $i_{\mathbf{L},\infty}$ (see, for example, Proposition 5.3), as well as the expression of Eq. (2), where these values are grouped by p into i_p (see, e.g. Lemma 7.1).

The decomposition of i into the $i_{p,c}$ reflects the geometry of f near infinity. It is apparently not related to the finite critical points of the polynomial and their critical values.

4. RESOLUTIONS AND THE GEOMETRY OF LEVEL SETS

In this section we describe the resolution of the points of indeterminacy of a polynomial on the line at infinity, and use this concept to establish some simple properties of its affine level curves.

A polynomial

$$f:\mathbf{R}^2 \to \mathbf{R}$$

extends to a map of real projective spaces

$$\hat{f}:\mathbf{P}^2 \to \mathbf{P}$$

which is undefined at a finite number of points on the line at infinity $\mathbf{L}$. By blowing up these points one gets a manifold M and a map

$$\pi: M \to \mathbf{P}^2$$

such that the map

$$\tilde{f}: M \to \mathbf{P}$$

lifting $\hat{f}$ is everywhere defined. We call the map $\tilde{f}$ a *resolution of f*. (We avoid the use of minimal resolutions, though this concept could be used to provide alternate proofs of some of the results below.)

Any resolution $\tilde{f}: M \to \mathbf{P}$ factors through

$$\bar{f}:\Gamma(f) \to \mathbf{P}$$

where $\Gamma(f)$ is the graph of f as defined above; note that the function $\bar{f}$ is everywhere defined.

For example, a resolution of $y(xy-1)$ is given in Fig. 3. The proper transforms of level curves of f are ordinary lines; the exceptional sets are thick lines. The number c^m next to a divisor means that at each smooth point of the divisor there are local coordinates (u, v) in a neighborhood of the point such that the divisor is $u=0$ and $\tilde{f}(u, v) = (u-c)^m$.

Let f be a polynomial, and let $\tilde{f}$ be a resolution of f. By $A \gg 0$, we mean as usual that A is large, but more precisely in this context we mean that A is greater than the absolute

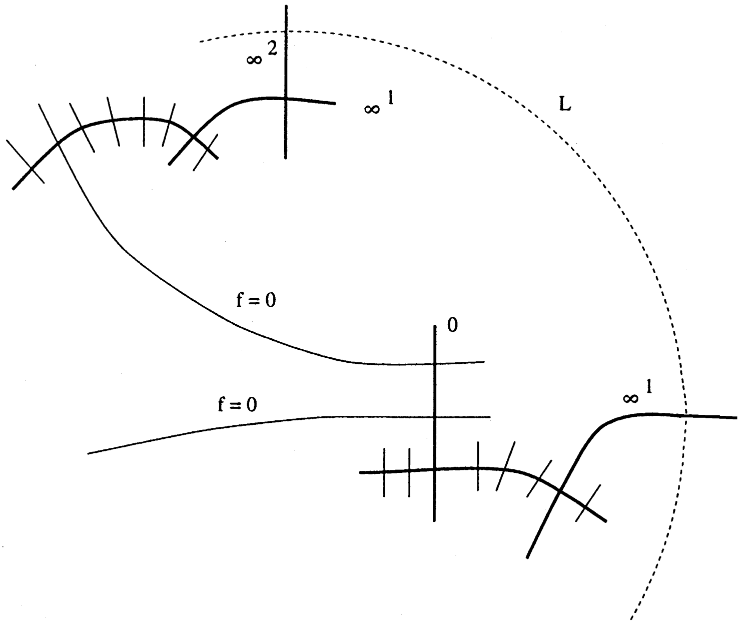

Fig. 3. A resolution of $y(xy-1)$.

value of all the critical values of f, and that if $|t| \geqslant A$, then the level sets $\tilde{f} = t$ are smooth and transversally intersect the exceptional sets of $\tilde{f}$. In particular, this means that the topological type of the level sets $f(x, y) = A$ and $f(x, y) = -A$ are independent of A.

Fix a polynomial f. Let

$$\mathbf{L}' = \{[a, b, 0] \in \mathbf{L} : f_d(a, b) = 0\}$$

and let

$$\mathbf{L}'' = \{p \in \mathbf{L}\text{: There is a } t \in \mathbf{R} \text{ with } p \text{ in the closure of } f(x, y) = t\}.$$

Both $\mathbf{L}'$ and $\mathbf{L}''$ are finite sets of points.

Lemma 4.1. *If* $p \in \mathbf{L}''$, *then*

1. *The point p is in the closure of level sets $|f| = A$ for $A \gg 0$.*
2. *If $\tilde{f}$ is a resolution of f, there is an exceptional set over p on which $\tilde{f}$ is not constant.*
3. $p \in \mathbf{L}'$.

Proof. Choose a resolution $\tilde{f}$ of f. There is an exceptional set E with $\pi(E) = p$ such that $\tilde{f} = t$ intersects E. The function $\tilde{f}$ restricted to E is a real rational function.

If $\tilde{f}$ restricted to E is not constant, then there is a $q \in E$ such that $\tilde{f}$ restricted to $E - \{q\}$ is a polynomial. This implies (1) and (2) in this case.

Now suppose that $\tilde{f}$ restricted to E is constant. The (real) exceptional sets over p form a connected tree. Since the value of $\tilde{f}$ is t at one point on the tree, and is infinity at the points where the tree intersects the proper transform of $\mathbf{L}$, there is a component in the tree where $\tilde{f}$ takes a continuum of large values. This implies (1) and (2) in this case.

Part (3) follows since the level curves in $\mathbf{C}^2$ of the complexified polynomial f intersect $\mathbf{L}_\mathbf{C}$ complex at exactly the zeros of the complexified f_d. □

Note that the converse of (3) above is not true: For the polynomial $f(x, y) = y^4 + x^2$, for example, $f_4(1, 0) = 0$, but $[1, 0, 0]$ is not contained in the closure of any real level curve.

Given a real polynomial $f(x, y)$, we let l'' be the number of points in $\mathbf{L}''$. If $\tilde{f}$ is a resolution of f, we let $\xi_{\mathbf{L},\mathbf{nc}}(\tilde{f})$ be the number of (real) exceptional sets E of $\tilde{f}$ such that $\tilde{f}|E$ is nonconstant. (In the pictures, these exceptional sets are cross hatched by level curves of the polynomial.)

COROLLARY 4.2. *If $\tilde{f}$ is a resolution of f, then $l'' \leqslant \xi_{\mathbf{L},\mathbf{nc}}(\tilde{f})$.*

The inequality may be strict: The polynomial $x(y + 1)(y + 2) \ldots (y + k)$ has $l'' = 2$ and $\xi_{\mathbf{L},\mathbf{nc}}(\tilde{f}) = k + 1$.

PROPOSITION 4.3. *Let $f(x, y)$ be a real polynomial, let $\tilde{f}$ be a resolution of f, and let $A \gg 0$. The following are equivalent:*

1. *The set $|f| = A$ is compact.*
2. *The set $f = t$ is compact, for all t.*
3. $\xi_{\mathbf{L},\mathbf{nc}}(\tilde{f}) = 0$.

Furthermore, if any of the above is true, then

1. *One of $f = A$ and $f = -A$ is homeomorphic to a circle, and the other is empty.*
2. *f_d has no real linear factors ($d_{\mathbf{R}} = 0$).*

Proof. (1) implies (2) by Lemma 4.1. (2) is equivalent to (3) since $f = t$ is compact for all t if and only if $\tilde{f}|E = \infty$ for all exceptional sets E. The additional conclusions are obvious. □

PROPOSITION 4.4. *Let $f(x, y)$ be a real polynomial, let $\tilde{f}$ be a resolution of f, and let $A \gg 0$. If $|f| = A$ is not compact, then*

1. *All the connected components of $|f| = A$ are noncompact.*
2. *The number of connected components of $|f| = A$ is $2\xi_{\mathbf{L},\mathbf{nc}}(\tilde{f})$.*

Proof. The geometry of the resolution implies (1). If E is an exceptional set on which $\tilde{f}$ is not constant, then $\tilde{f}$ restricted to E either takes the value $+A$ exactly twice, the value $-A$ exactly twice or the values $+A$ and $-A$ each once. Since the level sets $f = \pm A$ are transverse to E, this proves (2). □

5. A FORMULA FOR *i* IN TERMS OF A RESOLUTION

This section gives a formula (Proposition 5.3) for computing the index i and the terms $i_{p,c}$ in the decomposition of i in terms of a resolution of the polynomial. In Section 7 this proposition will play a role in finding bounds on i.

LEMMA 5.1. *Let $c \in \mathbf{R}$. If $p \in \mathbf{L}$ is not in the closure of the level set $f = c$, then $i_{p,c} = 0$. (Furthermore, $i^{\mathrm{abs}}_{p,c} = 0$, in the notation of Section 7.)*

Proof. Suppose $i_{p,c} \neq 0$. There is an end γ of the curve of tangencies passing through p, and f has limiting value c along γ. Let $\tilde{f}$: $M \to \mathbf{P}^2$ be a resolution of f. The curve γ lifts to

M and passes through some $q \in M$ with $\pi(q) = p$. Also $\tilde{f}(q) = c$. Hence the closure of $f = c$ intersects some exceptional set over p, so p is in the closure of $f = c$. This is a contradiction. □

The precise meaning of the notation $A \gg 0$ can be found in the previous section.

PROPOSITION 5.2. *If $f(x, y)$ is a real polynomial with isolated critical points, and if $|f| = A$ is compact for $A \gg 0$, then*

1. $i = 1$.
2. $i_{\mathbf{L},\infty} = 0$.
3. $i_{p,c} = 0$ *for all* $p \in \mathbf{L}$ *and* $c \in \mathbf{R}$.

Proof. By Proposition 4.3, either $f(x, y) = A$ or $f(x, y) = -A$ is homeomorphic to a circle; let us assume the former. Clearly $i = 1$ and $i_{\mathbf{L},\infty} = 0$. Also $f(x, y) = c$ is compact for all c, so Lemma 5.1 implies that $i_{p,c} = 0$. □

Let $\tilde{f}$ be a resolution of f. For $p \in \mathbf{L}$ and $c \in \mathbf{R} \cup \{\infty\}$, let

- $i_{p,c}(\tilde{f})$ be the sum of the indices of $\tilde{f}$ at critical points $q \in M$ of $\tilde{f}$ such that $\tilde{f}(q) = c$ and $\pi(q) = p$.
- $\xi_{p,c}(\tilde{f})$ be the number of (real) exceptional sets E of $\tilde{f}$ with $\pi(E) = p$ and $\tilde{f}|E = c$.

Recall that $\xi_{\mathbf{L},nc}(\tilde{f})$ is the number of (real) exceptional sets on which $\tilde{f}$ is nonconstant.

PROPOSITION 5.3. *If $f(x, y)$ is a real polynomial with isolated critical points and $\tilde{f}$ is a resolution of f, then*

1. $i = 1 - \sum_{\substack{p\in\mathbf{L}\\ c\in\mathbf{R}}} (i_{p,c}(\tilde{f}) + \xi_{p,c}(\tilde{f})) - \xi_{\mathbf{L},nc}(\tilde{f})$.
2. $i_{p,c} = -i_{p,c}(\tilde{f}) - \xi_{p,c}(\tilde{f})$, *for* $p \in \mathbf{L}$ *and* $c \in \mathbf{R}$.
3. $i_{\mathbf{L},\infty} = -\xi_{\mathbf{L},nc}(\tilde{f})$.

By Proposition 3.3, any two parts of this proposition imply the third, but proving each part separately is more instructive. In fact, Part (1) follows from a straightforward application of Morse theory, Part (2) follows from Morse theory on a manifold with boundary, and Part (3) follows from the geometry of the large level curves.

Proof. Proof of (1): Let

- $i_{\mathbf{L}}(\tilde{f}) = \sum_{\substack{p\in\mathbf{L}\\ c\in\mathbf{R}}} i_{p,c}(\tilde{f})$;
- $\xi_{\mathbf{L},\infty}(\tilde{f}) = \sum_{p\in\mathbf{L}} \xi_{p,\infty}(\tilde{f})$;
- $\xi_{\mathbf{L}}(\tilde{f}) = \sum_{\substack{p\in\mathbf{L}\\ c\in\mathbf{R}}} \xi_{p,c}(\tilde{f})$.

We will do Morse theory on the function $\tilde{f}: M \to \mathbf{R} \cup \{\infty\}$. Suppose $c_1 < c_2 < \cdots < c_r$ in $\mathbf{R}$ are the critical values of $\tilde{f}$ restricted to the inverse image of $\mathbf{R}$. Choose $\varepsilon > 0$ so that $c_i + \varepsilon < c_{i+1} - \varepsilon$ for $1 \leqslant i < r$. Choose $A > 0$ so that $-A < c_1$ and $c_r < A$. Since a level set of $\tilde{f}$ corresponding to a regular value is a union of circles,

$$\chi(M) = \chi(\{\tilde{f} \leqslant -A\} \cup \{\tilde{f} \geqslant A\}) + \sum_i \chi(\{c_i - \varepsilon \leqslant \tilde{f} \leqslant c_i + \varepsilon\}) \tag{3}$$

where χ denotes Euler characteristic. The set $\{\tilde{f} \leqslant -A\} \cup \{\tilde{f} \geqslant A\}$ is homotopy equivalent to the set $\tilde{f}^{-1}(\infty)$. This is a connected set, and is homotopy equivalent to a join of circles. These circles are the exceptional sets where $\tilde{f} = \infty$ together with the proper transform of $\mathbf{L}$. Thus

$$\chi(\{\tilde{f} \leqslant -A\} \cup \{\tilde{f} \geqslant A\}) = -\xi_{\mathbf{L},\infty}(\tilde{f}).$$

Next, M is a connected sum of copies of $\mathbf{P}^2$, so

$$\chi(M) = 1 - (\xi_{\mathbf{L}}(\tilde{f}) + \xi_{\mathbf{L},\infty}(\tilde{f}) + \xi_{\mathbf{L},\mathrm{nc}}(\tilde{f})).$$

At a critical value c_i, $\chi(\{c_i - \varepsilon \leqslant \tilde{f} \leqslant c_i + \varepsilon\})$ is the sum of the indices of the corresponding critical points, by Morse theory. The sum of all these indices can be split into the parts coming from critical points in the finite plane and the line at infinity. Using this fact and the two equations above changes equation (3) to

$$1 - \xi_{\mathbf{L}}(\tilde{f}) - \xi_{\mathbf{L},\infty}(\tilde{f}) - \xi_{\mathbf{L},\mathrm{nc}}(\tilde{f}) = -\xi_{\mathbf{L},\infty}(\tilde{f}) + i + i_{\mathbf{L}}(\tilde{f})$$

which proves (1).

Proof of (2): (See Figs 4 and 5). Choose $\varepsilon > 0$ so that c is the only critical value in $(c - \varepsilon, c + \varepsilon)$. Let C' be the (closed) exterior of the circle C in the plane. Let N' be the connected component of $\{(x, y) \in \mathbf{R}^2 : c - \varepsilon \leqslant f(x, y) \leqslant c + \varepsilon\} \cap C'$ containing p in its closure. Choose the circle C large enough so that each boundary component of N' consists of an arc of $f = c \pm \varepsilon$ followed by an arc of C followed by an arc of $f = c \pm \varepsilon$.

Let

$$N = \overline{\pi^{-1}(N')} \subset M.$$

We assume that N is connected; if it is disconnected the proof is similar.

We need a variant of the Poincaré–Hopf theorem for vector fields on a manifold, or more properly, a variant of Morse theory on manifolds with boundary (see, for instance, [8,

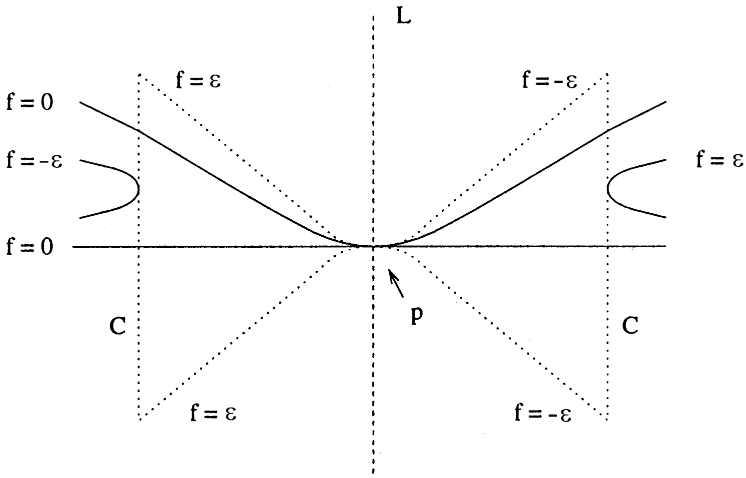

Fig. 4. The region N' (bounded by dotted lines) for the polynomial $y(xy - 1)$ at $p = [1, 0, 0]$ and $c = 0$.

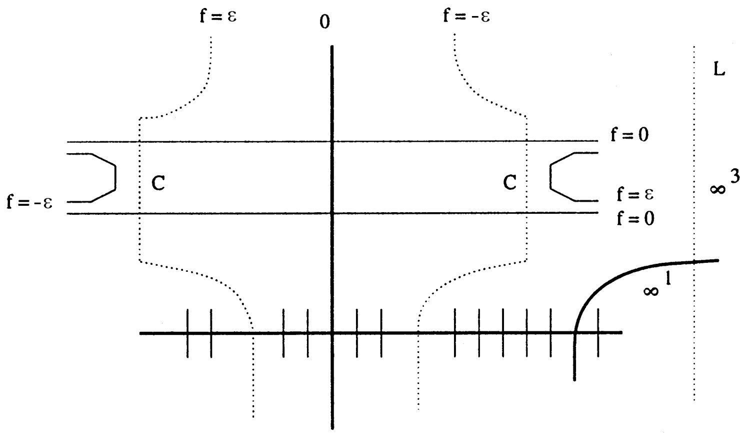

Fig. 5. The region N (bounded by dotted lines).

p. 35]). For an oriented manifold X with boundary, the Euler characteristic $\chi(X)$ is given by

$$\chi(X) = \sum \{\text{indices of internal critical points}\} + \{\text{index on boundary}\}$$

where the index of the vector field on the boundary is measured with respect to the outward pointing normal vector. This result is true for a gradient vector field on a nonorientable two-manifold X without boundary, provided that the index is defined to be $+1$ at a local extremum and -1 at a saddle, or more generally defined at an arbitrary critical point using the result of Arnold [1] that the index of a polynomial $f(x, y)$ at a point p is $1 - r$, where r is the number of real branches at p of the curve $f(x, y) = f(p)$. If X has a boundary with an orientable collar neighborhood, then the result is still true, provided that the index on the boundary is measured according as Fig. 1. Finally, the form we will use for N is

$$\chi(N) = 1 + \sum \{\text{indices of internal critical points}\} + \{\text{index on boundary}\}.$$

The term $+1$ comes from the fact that N has four corners (see Fig. 5).

Choose a Riemannian metric on N so that it agrees, on the boundary components of N consisting of arcs of C, with the standard metric on the plane. We apply the above result to the gradient vector field of $\tilde{f}$. In the interior of N there are the exceptional sets with $\tilde{f} = c$ and those critical points of $\tilde{f}$ which have critical value c. Since N retracts to the exceptional sets contained in it,

$$\chi(N) = 1 - \xi_{p,c}(\tilde{f}).$$

The index of the internal critical points of $\tilde{f}$ is $i_{p,c}(\tilde{f})$. Finally, the index of the gradient vector field on the boundary of N is $i_{p,c}$. Combining these facts proves (2).

Proof of (3): (See Fig. 6.) If $|f| = A$ is compact for $A \gg 0$, then Proposition 4.3 and Proposition 5.2 prove the result. Hence we can suppose that $|f| = A$ is not compact. If γ is an end of the curve of tangencies and $c(\gamma) = \infty$, then γ intersects $|f| = A$ for $A \gg 0$.

Let I be a connected component of $|f| = A$ in $\mathbf{R}^2$. Each γ which meets I has $c(\gamma) = \infty$. Since I begins and ends outside C by Proposition 4.4 part (1), clearly the sum of the $i(\gamma)$ over all γ meeting I is $-1/2$.

By Proposition 4.4 part (2), there is a two-to-one correspondence between connected components of $|f| = A$ and exceptional sets where $\tilde{f}$ is nonconstant. This proves (3). □

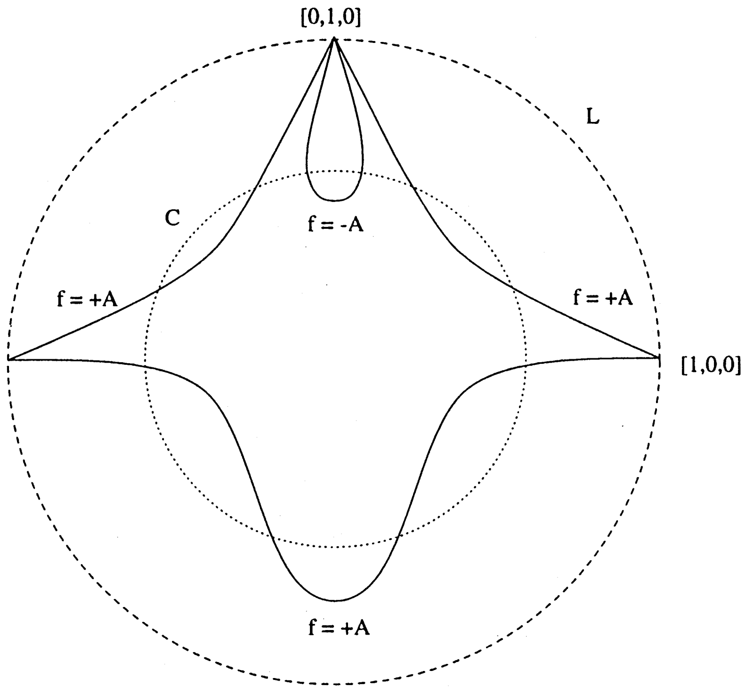

Fig. 6. The level curves of $f(x, y) = y(x^2y - 1)$ in $\mathbf{P}^2$.

Example 5.4. Consider the resolution $\tilde{f}$ of the polynomial $y(xy - 1)$ shown in Fig. 3. Over $p = [1, 0, 0]$, $\tilde{f}$ has two saddle points with critical value 0, so $i_{[1,0,0],0}(\tilde{f}) = -2$. There is one exceptional set E over $[1, 0, 0]$ with $\tilde{f}|E = 0$, so $\xi_{[1,0,0],0}(\tilde{f}) = 1$, and one exceptional set where $\tilde{f}$ is nonconstant, so $\xi_{[1,0,0],\mathrm{nc}}(\tilde{f}) = 1$. Over $p = [0, 1, 0]$, there is just one exceptional set where $\tilde{f}$ is nonconstant, so $\xi_{[0,1,0],\mathrm{nc}}(\tilde{f}) = 1$.

Recall that f_d is the homogeneous term of highest degree of the polynomial f, and that $d_{\mathbf{R}}$ is the real degree of f as defined in the Introduction.

Corollary 5.5. *If f_d has no repeated real linear factors, then $i = 1 - d_{\mathbf{R}}$. Also, $i_{p,c} = 0$ for $p \in \mathbf{L}$ and $c \in \mathbf{R}$, and $i_{\mathbf{L},\infty} = -d_{\mathbf{R}}$.*

Proof. This "geometrically obvious" result follows from Proposition 5.3, since over each point where the level curves of f intersect $\mathbf{L}$ the resolution is as in Fig. 7. □

Let $h(x, y)$ be a real polynomial whose homogeneous term of highest degree is a product of irreducible real quadratic factors (i.e. $d_{\mathbf{R}} = 0$). Instead of using a family of concentric circles (the level curves of the polynomial $x^2 + y^2$) to define the $i_{p,c}$, we could use the level curves of the polynomial $h(x, y)$. We let $i^h_{p,c}$ be the decomposition of the index defined this way.

Proposition 5.6. *Let $f(x, y)$ be a real polynomial with isolated critical points. If $h(x, y)$ is a real polynomial with $d_{\mathbf{R}} = 0$, then $i^h_{p,c} = i_{p,c}$ for all $p \in \mathbf{L}$ and $c \in \mathbf{R}$.*

Proof. Let e be the degree of h. We have that $i_{p,c} = i^{h'}_{p,c}$ for $h' = (x^2 + y^2)^{e/2}$.

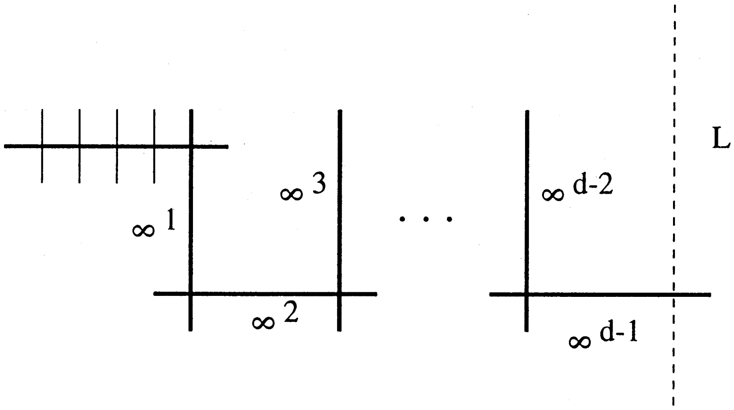

Fig. 7. Resolution of a point where $d_p = 1$.

Choose a family h^s of polynomials with $h^0 = h$ and $h^1 = h'$, and so that h^s is a polynomial in x, y and s with degree e in x and y and the homogeneous term of highest degree $(h^s)_e$ is a product of irreducible real quadratic factors. The curve of tangencies for h^s is $f_x h^s_y - f_y h^s_x = 0$. Choose a resolution $\tilde{f}: M \to \mathbf{R}$ of f, with $\pi: M \to \mathbf{P}^2$. The curve of tangencies for h^s lifts to M.

Let q^s be an intersection point of the proper transform of the lifted curve of tangencies with the exceptional set $\pi^{-1}(\mathbf{L})$. If $\pi(q^s) = p$ and $\tilde{f}(q^s) = c$, then q^s contributes to $i^{h^s}_{p,c}$. The point q^s varies continuously with s. Clearly $\pi(q^s)$ is independent of s. We will show that $\tilde{f}(q^s)$ is also independent of s.

If q^s does not move as a function of s, then this is true. Suppose q^s moves. Fix an s and call it s_0. Suppose that q^{s_0} is contained in an exceptional set E. We will show that $\tilde{f}|E$ is constant. The function $\tilde{f}|E$ is rational. By moving s_0 a little, we may assume that q^{s_0} is not an intersection point of $\pi^{-1}(\mathbf{L})$ and that q^{s_0} is not a critical point of $\tilde{f}|E$. For each s, the level sets of $\tilde{h}^s$ form a system of regular neighborhoods of $\pi^{-1}(\mathbf{L})$. The level curves of $\tilde{h}^s$ and $\tilde{f}$ are tangent along the curve of tangencies for h^s. Thus the level curves of $\tilde{f}$ become tangent to E in a neighborhood of s_0. Hence $\tilde{f}$ is constant in a neighborhood of s_0, and hence on E. □

The proposition is obviously not true for $i_{p,\infty}$ with $p \in \mathbf{L}'$. Lastly, is there a polynomial f with a resolution $\tilde{f}$ and a point q in the exceptional set such that $\tilde{f}$ has a local extremum at q?

6. A FORMULA FOR i IN TERMS OF A MORSIFICATION

This section contains a formula (Proposition 6.2) for computing the index i of a polynomial and the $i_{p,c}$ from Section 3 in terms of a Morsification. The proof is straightforward, and the results will not be used later. Some examples and a conjecture are included.

Definition 6.1. A *deformation* of a real polynomial $f(x, y)$ of degree d with isolated critical points is a real polynomial $h(x, y, s)$ of degree d in x and y with $h(x, y, 0) = f(x, y)$. We let $f^s(x, y) = h(x, y, s)$. A deformation is a *Morsification* if for small $s \neq 0$, $(f^s)_d$ (the homogeneous term of degree d of f^s) has no repeated real linear factors and the critical points of f^s in $\mathbf{R}^2$ are nondegenerate.

It is easy to show that a polynomial of degree d has a Morsification, and that the set of Morsifications is a dense open subset of the set of polynomials of degree $\leqslant d$.

If f^s is a Morsification of f and $p \in \mathbf{L}$ is in the closure of some level set of f and $c \in \mathbf{R} \cup \{\infty\}$, we let

- $\tilde{d}_p(f^s)$ be the number of real linear factors in $(f^s)_d$ which are deformations of the factor corresponding to p in f_d. (The number $\tilde{d}_p(f^s)$ is also the number of points on $\mathbf{L}$ through which the level sets of the Morsification pass and which go to p as $s \to 0$.)
- $i^\infty_{p,c}(f^s)$ be the index of the critical points of f^s which go to p and whose critical value goes to c as $s \to 0$.
- $i^\infty_p(f^s) = \sum_{c \in \mathbf{R} \cup \{\infty\}} i^\infty_{p,c}(f^s)$.
- $i^\infty(f^s) = \sum_{p \in \mathbf{L}} i^\infty_p(f^s)$.

Note that these invariants depend not just on the expression for f^s but also on the sign of s.

Proposition 6.2. *Let f^s be a Morsification of f.*

1. $i = 1 - d_{\mathbf{R}}(f^s) - i^\infty(f^s)$.
2. $i_p = 1 - \tilde{d}_p(f^s) - i^\infty_p(f^s)$ for $p \in \mathbf{L}''$ (i.e. *for p in the closure of some level set of f*).

Proof. (1): We have $i(f^s) = i^\infty(f^s) + i$ by summing the indices of the critical points of f^s, and $i(f^s) = 1 - d_{\mathbf{R}}(f^s)$ by Corollary 5.5.

(2): We may assume that $p = [1, 0, 0]$. Choose a four-sided region on the $x > 0$ side of the plane containing the critical points of f^s on that side which go to p as $s \to 0$, and such that the left side of the region is a segment of the circle C containing the points of tangency q which approach p, the top and bottom are level sets of f^s, and whose right side is a segment of a larger circle C'. Orient the boundary of this region counterclockwise. Choose a similar region on the left side of the plane. The index of $grad\, f^s$ about these two regions is clearly $1 - i_p - \tilde{d}_p(f^s)$. It is also the sum of the indices of the critical points in the interior of the regions, which is $i^\infty_p(f^s)$. □

There is no obvious formula for $i_{p,c}$; see Conjecture 6.6 at the end of this section.

Example 6.3. Let $f(x, y) = y^2 - x$ and $p = [1, 0, 0]$. Here $i_{p,c} = 0$ for $c \in \mathbf{R} \cup \{\infty\}$. The Morsification $f^s(x, y) = y^2 + sx^2 - x$ has a critical point at $(1/2s, 0)$ with critical value $-1/4s$. If $s > 0$ then $\tilde{d}_p(f^s) = 0$ and the critical point is a mininum, so $i^\infty_{p,0}(f^s) = 1$. If $s < 0$ then $\tilde{d}_p(f^s) = 2$ and the critical point is a saddle, so $i^\infty_{p,0}(f^s) = -1$.

Example 6.4. Let $f(x, y) = y(xy - 1)$ and $p = [1, 0, 0]$. Here $i_{p,0} = 1$ and $i_{p,c} = 0$ for $c \neq 1$. Define a Morsification by $f^s(x, y) = (y - sx)(xy - 1)$. (This deformation simply tilts the line in the zero locus of f.) We have $\tilde{d}_p(f^s) = 2$. For $s > 0$, f^s has two real nondegenerate critical points:

Critical point	Type	Critical value
$(+1/\sqrt{s},\ +\sqrt{s})$	Saddle	0
$(-1/\sqrt{s},\ -\sqrt{s})$	Saddle	0

Thus $i^\infty_{p,0}(f^s) = -2$.

Example 6.5. Let $f(x, y) = x(y^2 - 1) = x(y+1)(y-1)$ and $p = [1, 0, 0]$. Here $i_{p,\infty} = -1$ and $i_{p,c} = 0$ for $c \in \mathbf{R}$. We give two Morsifications. The first is $f^s(x, y) = x(y - sx + 1))\ (y + sx - 1)$. We have $\tilde{d}_p(f^s) = 2$. The function f^s has critical points:

Critical point	Type	Critical value
$(0, \pm 1)$	Saddle	0
$(1/s, 0)$	Saddle	0
$(1/(3s), 0)$	Minimum $(s > 0)$ Maximum $(s < 0)$	$-4/(27\, s)$

Thus $i^{\infty}_{p,0} = -1$ and $i^{\infty}_{p,\infty} = 1$. The second Morsification is $f^s(x, y) = x(y^2 - 1) + sx^3$. For $s > 0$, $\tilde{d}_p(f^s) = 0$ and f^s has critical points

Critical point	Type	Critical value
$(0, \pm 1)$	Saddle	0
$(1/\sqrt{3s}, 0)$	Minimum	$-2/3\sqrt{3s}$
$(-1/\sqrt{3s}, 0)$	Maximum	$2/3\sqrt{3s}$

Thus $i^{\infty}_{p,\infty} = 2$. (If $s < 0$, the only real critical points of f^s are at $(0, \pm 1)$, and $\tilde{d}_p(f^s) = 2$.)

There is no obvious formula for $i_{p,c}$ in terms of a deformation, as can be seen in the case of $x(y^2 - 1)$ above. However, the following conjecture seem reasonable:

CONJECTURE 6.6. *If f^s is a deformation of f and $p \in \mathbf{L}''$, then $i_{p,c} \leqslant -i^{\infty}_{p,c}(f^s)$ for $c \in \mathbf{R}$, and $i_{p,\infty} \leqslant 1 - i^{\infty}_{p,\infty}(f^s)$.*

For a deformation f^s of f it is easy to find bounds on the number of local maxima, minima and saddles near a point $p \in \mathbf{L}$. It would be interesting to see what possible combinations of these can occur, similar to the investigation in [4] or [10].

7. BOUNDS ON i

This section contains the main results of this paper, the bounds on the index i of the gradient vector field of a real polynomial. The main tool is a bound on i_p (Lemma 7.1). This, together with the interpretation of $i_{\mathbf{L},\infty}$ in terms of a resolution (Proposition 5.3) and some lemmas using techniques from Section 4 give the first main result (Theorem 7.8). We next give a refinement (Lemma 7.9) of Lemma 7.1. This and a number of technical details gives the second main result (Theorem 7.11). As remarked in the Introduction, there is still a large gap between the upper bounds and known examples.

Let $f(x, y)$ be a real polynomial of degree d with isolated critical points. For $p \in \mathbf{L}$ and $c \in \mathbf{R} \cup \{\infty\}$, recall that

$$i_{p,c} = \sum i(\gamma)$$

where the sum is over ends γ of the curve of tangencies with $p(\gamma) = p$ and $c(\gamma) = c$. We let

$$i^{\mathrm{abs}}_{p,c} = \sum |i(\gamma)| \quad \text{and} \quad i^{\mathrm{abs}}_{p} = \sum_{c \in \mathbf{R} \cup \{\infty\}} i^{\mathrm{abs}}_{p,c}.$$

These invariants can be computed from a resolution of f, and in particular are integers, although this is not evident from the proof of Proposition 5.3. (The invariants $i^N_{p,c}$ and so forth later in this section can also be computed from a resolution.)

Recall that $\mathbf{L}'$ and $\mathbf{L}''$ were introduced in Section 4, that $\mathbf{L}'' \subset \mathbf{L}'$ and that l'' is the number of elements in $\mathbf{L}''$. If $p = [a, b, 0] \in \mathbf{L}$, we let d_p be the multiplicity of the factor $(bx - ay)$ in f_d. The next result is the local analogue of the estimate $|i| \leqslant d - 1$ from the Introduction. This estimate, like the global one, is proved by relating the index to an algebraic intersection number.

Lemma 7.1. *If* $p \in \mathbf{L}'$ (*i.e. if* $d_p > 0$), *then*

$$i_p^{\mathrm{abs}} \leqslant d_p - 1.$$

This follows from the next two results. We let Γ be the projective completion of the curve of tangencies, and $\Gamma_{\mathbf{C}}$ be its complexification. We use $(A \cdot B)_p$ to denote the intersection number of the curves A and B at p.

Lemma 7.2. $i_p^{\mathrm{abs}} \leqslant (\Gamma_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p$.

Proof. The number i_p^{abs} is at most one half the number of ends γ of the curve of tangencies with $p(\gamma) = p$. This number is the number of real branches of the completion of the curve of tangencies at p, which is at most the number of branches of $\Gamma_{\mathbf{C}}$ at p. This number is at most $(\Gamma_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p$, since no component of the curve of tangencies is contained in $\mathbf{L}_{\mathbf{C}}$. □

Lemma 7.3. $(\Gamma_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p = d_p - 1$.

Proof. Without loss of generality, we may assume that $p = [1, 0, 0]$. We have that $f = y^{d_p} h(x, y) + \{\text{terms of lower order}\}$ where $d_p \geqslant 1$, $h(x, y)$ is homogeneous of degree $d - d_p$, and y does not divide $h(x, y)$. Changing coordinates to p and computing as in [6, III.3] shows that $(\Gamma_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p = d_p - 1$. □

Lemma 7.1 is sharp: The polynomial of Example 2.5 at $p = [1, 0, 0]$ has $d_p = k$ and $i_p^{\mathrm{abs}} = i_{p,0} = k - 1$. Another example is provided by the polynomial $x(y + 1)(y + 2) \cdots (y + k)$ at $p = [1, 0, 0]$, which has $d_p = k$ and $i_p^{\mathrm{abs}} = -i_{p,0} = k - 1$.

Recall that the real degree of the polynomial is

$$d_{\mathbf{R}} = \sum_{p \in \mathbf{L}} d_p.$$

We let

$$\tilde{d}_{\mathbf{R}} = \sum_{p \in \mathbf{L}''} d_p.$$

This is the sum of the d_p's over those p in the line at infinity which are in the closure of some level set of the polynomial f. Note that

$$l'' \leqslant \tilde{d}_{\mathbf{R}} \leqslant d_{\mathbf{R}} \leqslant d. \tag{4}$$

Proposition 7.4. *If* $f(x, y)$ *is a real polynomial with isolated critical points, then*

$$i \leqslant 1 + \tilde{d}_{\mathbf{R}} - 2l''.$$

Proof. Let $\tilde{f}$ be a resolution of f. We have

$$i = 1 + \sum_{\substack{p \in \mathbf{L} \\ c \in \mathbf{R}}} i_{p,c} + i_{\mathbf{L},\infty} \leqslant 1 + \sum_{\substack{p \in \mathbf{L}'' \\ c \in \mathbf{R}}} i_{p,c}^{\text{abs}} - \xi_{\mathbf{L},nc}(\tilde{f})$$

$$\leqslant 1 + \sum_{p \in \mathbf{L}''} (d_p - 1) - \xi_{\mathbf{L},nc}(\tilde{f}) \leqslant 1 + \tilde{d}_{\mathbf{R}} - l'' - \xi_{\mathbf{L},nc}(\tilde{f}).$$

The first equality follows from Proposition 3.3. The first inequality follows since $i_{\mathbf{L},\infty} = -\xi_{\mathbf{L},\mathrm{nc}}(\tilde{f})$ by Part (3) of Proposition 5.3, $i_{p,c} \leqslant i_{p,c}^{\text{abs}}$ by definition, and $i_{p,c}^{\text{abs}} = 0$ for $p \in \mathbf{L} - \mathbf{L}''$ by Lemma 5.1. The second inequality follows from Lemma 7.1. The result follows from Corollary 4.2. □

To get a lower bound on the index, we need to compactify the plane $\mathbf{R}^2$ by the circle

$$\mathbf{S} = \{(a, b, 0) \in (\mathbf{R}^3 - 0)/\mathbf{R}^+\}.$$

The projection map

$$\mathbf{S} \to \mathbf{L}$$

which takes $(a, b, 0)$ to $[a, b, 0]$ will be denoted by $q \mapsto |q|$. If γ is an end of the curve of tangencies, we let $q(\gamma) \in \mathbf{S}$ be the endpoint of the closure of γ in $\mathbf{S}$.

For example, for $y - (xy - 1)^2$ there are ends γ and γ' with $q(\gamma) = (1, 0, 0)$, $q(\gamma') = (-1, 0, 0)$, $c(\gamma) = c(\gamma') = 0$ and $i(\gamma) = +1/2$, $i(\gamma') = -1/2$. The two cancel out to give $i_{[1,0,0],0} = 0$.

Let

$$\mathbf{S}'' = \{q \in \mathbf{S}\text{: There is a } t \in \mathbf{R} \text{ such that } q \text{ is in the closure of } f(x, y) = t\}.$$

This is a finite set of points; we let s'' denote the number of points in this set. Since the fibers of the projection map

$$\mathbf{S}'' \to \mathbf{L}''$$

consist of one or two points,

$$l'' \leqslant s'' \leqslant 2l''.$$

Thus the string of inequalities (4) becomes

$$0 \leqslant \tfrac{1}{2} l'' \leqslant \tfrac{1}{2} s'' \leqslant l'' \leqslant \tilde{d}_{\mathbf{R}} \leqslant d_{\mathbf{R}} \leqslant d. \tag{5}$$

The next two lemmas are preparation for proving Proposition 7.7.

Lemma 7.5.

$$\left| \sum_{q(\gamma) \in \mathbf{S}''} i(\gamma) \right| \leqslant \tilde{d}_{\mathbf{R}} - l''.$$

Proof.

$$\left| \sum_{q(\gamma) \in \mathbf{S}''} i(\gamma) \right| \leqslant \sum_{q(\gamma) \in \mathbf{S}''} |i(\gamma)| \leqslant \sum_{p \in \mathbf{L}''} \sum_{|q(\gamma)| = p} |i(\gamma)| \leqslant \sum_{p \in \mathbf{L}''} (d_p - 1) = \tilde{d}_{\mathbf{R}} - l''.$$ □

LEMMA 7.6.

$$\left| \sum_{q(\gamma) \in \mathbf{S} - \mathbf{S}''} i(\gamma) \right| \leqslant \tfrac{1}{2} s''.$$

Proof. If $\mathbf{S}''$ is empty ($s'' = 0$), then all the level sets of f are compact and the left-hand side sum is zero by Proposition 5.2.

Now suppose that $\mathbf{S}''$ is not empty. Fix a connected component V of $\mathbf{S} - \mathbf{S}''$. Let $A \geqslant 0$ and let $I(V)$ be the connected component of $|f| = A$ which goes to V as A goes to infinity. By Propositions 4.3 and 4.4, $I(V)$ is not compact and hence has its endpoints on $\mathbf{S}$. We have (see the proof of Part (3) of Proposition 5.3) that

$$\sum i(\gamma) = -\tfrac{1}{2}$$

where the sum is over all ends γ of the curve of tangencies which intersect $I(V)$. If an end γ has $q(\gamma) \in V$, then γ intersects $I(V)$, and no other connected component of $|f| = A$. (Since the closure of no level curves of f pass through the endpoint of γ, the function f approaches infinity monotonely on γ. (Lemma 3.1)) However, some of ends γ of the curve of tangencies may intersect $I(V)$ but have their endpoints on the endpoints of V. (I know of no examples of polynomials with this property, though.) The sum of the $i(\gamma)$ which intersect $I(V)$ and have endpoint a chosen endpoint of V is 0 or $+1/2$. The sum over both endpoints of V is thus 0, $+1/2$ or 1. Hence

$$\sum_{q(\gamma) \in V} i(\gamma) = -\tfrac{1}{2},\ 0 \text{ or } +\tfrac{1}{2}.$$

Since $\mathbf{S} - \mathbf{S}''$ has s'' connected components, this proves the lemma. □

PROPOSITION 7.7. *If $f(x, y)$ is a real polynomial with isolated critical points, then*

$$i \geqslant 1 - d_{\mathbf{R}}.$$

Proof. We have

$$i = 1 + \sum_{\gamma} i(\gamma) = 1 + \sum_{q(\gamma) \in \mathbf{S}''} i(\gamma) + \sum_{q(\gamma) \in \mathbf{S} - \mathbf{S}''} i(\gamma) \geqslant 1 + l'' - \tilde{d}_{\mathbf{R}} - \tfrac{1}{2} s'' \geqslant 1 - \tilde{d}_{\mathbf{R}} \geqslant 1 - d_{\mathbf{R}}.$$

The first inequality follows from the two lemmas above, and the last two by the string of inequalities (5). □

The following is our first main result.

THEOREM 7.8. *Let $f(x, y)$ be a real polynomial of real degree $d_{\mathbf{R}}$ with isolated critical points, and let i be the index of grad f around a large circle containing the critical points. If all the level sets of f are compact, then $i = 1$. Otherwise*

$$|i| \leqslant d_{\mathbf{R}} - 1.$$

Proof. If the level sets are compact then the result is obvious (see Proposition 5.2). If some level sets are not compact, then $l'' > 0$. The upper bound follows from Proposition 7.4, and the lower bound from Proposition 7.7. □

As remarked in the Introduction and Corollary 5.5, it is easy to find "generic" polynomials which realize the lower bound of this theorem. The upper bound appears too high;

the estimate of Proposition 7.4 is somewhat better. Finally, the result seems somewhat obvious and one could hope for a better proof.

We now further decompose $i_{p,c}$ and its refinements defined above. For $p \in \mathbf{L}$ and $c \in \mathbf{R} \cup \{\infty\}$, recall that $i_{p,c} = \sum i(\gamma)$, summed over all ends γ of the curve of tangencies with $p(\gamma) = p$ and $c(\gamma) = c$. We let $i^{\mathrm{T}}_{p,c}$ (respectively, $i^{\mathrm{N}}_{p,c}$) be the sum of the $i(\gamma)$'s such that the corresponding curve γ is tangent (respectively, not tangent) to $\mathbf{L}$ at p. Thus

$$i_{p,c} = i^{\mathrm{N}}_{p,c} + i^{\mathrm{T}}_{p,c}.$$

We similarly decompose $i^{\mathrm{abs}}_{p,c}$. As before, these numbers are all integers. For example, the polynomial $y(xy-1)$ of Example 2.1 has $i_{[1,0,0],0} = i^{\mathrm{N}}_{[1,0,0],0} = 1$, and the polynomial in Example 2.9 has $i_{[1,0,0],0} = i^{\mathrm{T}}_{[1,0,0],0} = 1$. We also let

$$i^{\mathrm{N,abs}}_{p} = \sum_{c \in \mathbf{R} \cup \{\infty\}} i^{\mathrm{N,abs}}_{p,c}$$

and define $i^{\mathrm{T,abs}}_{p}$ similarly. The following lemma is a refinement of Lemma 7.1.

Lemma 7.9. *If $p \in \mathbf{L}'$, then*

$$i^{\mathrm{N,abs}}_{p} + 2 i^{\mathrm{T,abs}}_{p} \leqslant d_p - 1.$$

Proof. We let Γ^{T} (respectively, Γ^{N}) be the product of the branches of the curve of tangencies Γ tangent (respectively, not tangent) to $\mathbf{L}$ at p, so that $\Gamma = \Gamma^{\mathrm{T}}\Gamma^{\mathrm{N}}$ near p. As in Lemma 7.2, we have that

$$i^{\mathrm{N,abs}}_{p} \leqslant (\Gamma^{\mathrm{N}}_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p. \tag{6}$$

Similarly

$$i^{\mathrm{T,abs}}_{p} \leqslant \tfrac{1}{2} (\Gamma^{\mathrm{T}}_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p \tag{7}$$

since these branches are tangent to $\mathbf{L}$ at p. Thus

$$i^{\mathrm{N,abs}}_{p} + 2 i^{\mathrm{T,abs}}_{p} \leqslant (\Gamma^{\mathrm{N}}_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p + (\Gamma^{\mathrm{T}}_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p = (\Gamma_{\mathbf{C}} \cdot \mathbf{L}_{\mathbf{C}})_p = d_p - 1.$$

as before. □

Some stronger local estimates are probably true. In fact, let $f = f^{\mathrm{N}} f^{\mathrm{T}}$ at p, where f^{T} (respectively, f^{N}) are the branches of $f = t$ tangent (respectively, not tangent) to $\mathbf{L}$ at p (which is independent of $t \in \mathbf{R}$), and let d^{T}_p (respectively, d^{N}_p) be the intersection number of $f^{\mathrm{T}} = t$ (respectively, $f^{\mathrm{N}} = t$) with $\mathbf{L}$ at p. Thus

$$d_p = d^{\mathrm{N}}_p + d^{\mathrm{T}}_p.$$

It seems reasonable to expect that $i^{\mathrm{N,abs}}_{p} \leqslant d^{\mathrm{N}}_p - 1$ and $i^{\mathrm{T,abs}}_{p} \leqslant (1/2) d^{\mathrm{T}}_p - 1$ for $p \in \mathbf{L}$, and that these estimates are sharp.

We need one more technical lemma:

Lemma 7.10. *Fix $p \in \mathbf{L}$ and let $\tilde{f}$ be a resolution of f. If there are $c, c' \in \mathbf{R}$ such that $i^{\mathrm{T,abs}}_{p,c} > 0$ and $i^{\mathrm{N,abs}}_{p,c'} > 0$, then $\xi_{p,\mathrm{nc}}(\tilde{f}) \geqslant 2$ (i.e. there are at least two exceptional sets over p on which $\tilde{f}$ is not constant).*

Proof. There are ends γ and γ' of the curve of tangencies with $p(\gamma) = p(\gamma') = p$, $c(\gamma) = c$ and $c(\gamma') = c'$, and with γ (respectively, γ') tangent (respectively, not tangent) to $\mathbf{L}$ at p. By the proof of Lemma 5.1, the closure of the level curve $f = c$ (respectively, $f = c'$) intersects an exceptional set E (respectively, E') over p. By the proof of Lemma 4.1, there is at least one exceptional set over p where $\tilde{f}$ is not constant. In fact, there are at least two such exceptional sets: Since γ and γ' have distinct tangents at p, the limit of f must be infinite on all but a finite number of tangent directions between these by [4, Proposition 1.3], so E and E' are distinct and the chain of exceptional sets connecting E and E' must have at least one member E_0 with $f|E_0 = \infty$. Thus there must be an exceptional set in the chain connecting E and E_0 where $\tilde{f}$ is nonconstant, and similarly between E' and E_0. $\square$

THEOREM 7.11. *If $f(x, y)$ is a real polynomial of degree d with isolated critical points, and i is the index of $grad\, f$ around a large circle containing the critical points, then*

$$i \leqslant \max\{1, d - 3\}.$$

The difficult part of this proof is the case when $l'' = 1$, i.e. when the closures of the level curves of the polynomial intersect the line at infinity at just one point.

Proof. If $l'' = 0$ then $i = 1$ by Lemma 5.2. If $l'' \geqslant 2$ then $i \leqslant d_{\mathbf{R}} - 3$ by Proposition 7.4. Thus we must treat the case $l'' = 1$. We may suppose without loss of generality that $p = [1, 0, 0]$ is the only point where the real level curves of f intersect $\mathbf{L}$. The point p will remain fixed for the rest of the proof.

Let $\tilde{f}$ be a resolution of f. From Proposition 3.3, Lemma 5.1 and part (3) of Proposition 5.3 we have that

$$i = 1 + \sum_{c \in \mathbf{R}} i_{p,c} - \xi_{p,\mathrm{nc}}(\tilde{f}). \tag{8}$$

Since $\xi_{\mathbf{L},\mathrm{nc}}(\tilde{f}) \geqslant l'' = 1$ by Corollary 4.2, we also have the weaker form of this equation:

$$i \leqslant \sum_{c \in \mathbf{R}} i_{p,c}.$$

Suppose $d_p < d$. We have that $d_p \leqslant d - 2$ since the roots of f_d other than p are complex and hence occur in conjugate pairs. Thus

$$i \leqslant \sum_{c \in \mathbf{R}} i_{p,c} \leqslant \sum_{c \in \mathbf{R}} i_{p,c}^{\mathrm{abs}} \leqslant i_p^{\mathrm{abs}} \leqslant d_p - 1 \leqslant d - 3$$

where the fourth inequality follows from Lemma 7.1.

Thus we may assume that $d_p = d$, so that

$$f(x, y) = \pm y^d + h(x, y)$$

where h has degree $e < d$. If h is a function of x alone, then from far away $f(x, y)$ looks like $\pm y^d \pm x^e$, which has $i = 0$ or ± 1. Thus we may assume that h is a nonconstant function of both x and y.

The rest of the proof is divided into three cases:

Case 1: Suppose $\sum_{c \in \mathbf{R}} i_{p,c}^{\mathrm{N}} = 0$. Then

$$i \leqslant \sum_{c \in \mathbf{R}} i_{p,c} = \sum_{c \in \mathbf{R}} i_{p,c}^{\mathrm{T}} \leqslant \sum_{c \in \mathbf{R}} i_{p,c}^{\mathrm{T,abs}} \leqslant i_p^{\mathrm{T,abs}} \leqslant \tfrac{1}{2}(d_p - 1) \leqslant \max\{1, d - 3\}$$

where the fourth inequality follows from Lemma 7.9.

Case 2: Suppose that $\sum_{c\in\mathbf{R}} i^{\mathrm{T}}_{p,c} = 0$. We have that

$$i \leqslant \sum_{c\in\mathbf{R}} i_{p,c} = \sum_{c\in\mathbf{R}} i^{\mathrm{N}}_{p,c} \leqslant \sum_{c\in\mathbf{R}} i^{\mathrm{N,abs}}_{p,c} \leqslant i^{\mathrm{N,abs}}_{p} \leqslant (\Gamma^{\mathrm{N}}_{\mathbf{C}}\cdot\mathbf{L}_{\mathbf{C}})_p$$

where the last inequality is equation (6).

Since $f(x, y) = \pm y^d + h(x, y)$ where h is a nonconstant function of both x and y of degree less than d, a computation shows that z divides the term of lowest degree in the curve of tangencies localized at p. Hence Γ^{T} is nonempty, so $(\Gamma^{\mathrm{T}}_{\mathbf{C}}\cdot\mathbf{L}_{\mathbf{C}})_p \geqslant 2$.

Thus

$$(\Gamma^{\mathrm{N}}_{\mathbf{C}}\cdot\mathbf{L}_{\mathbf{C}})_p = (\Gamma_{\mathbf{C}}\cdot\mathbf{L}_{\mathbf{C}})_p - (\Gamma^{\mathrm{T}}_{\mathbf{C}}\cdot\mathbf{L}_{\mathbf{C}})_p \leqslant (d_p - 1) - 2 = d_p - 3 = d - 3.$$

Case 3: Suppose that $i^{\mathrm{N,abs}}_{p,c} > 0$ and $i^{\mathrm{T,abs}}_{p,c'} > 0$ for some $c, c' \in \mathbf{R}$. We have by equation (8) and Lemma 7.10 that

$$\begin{aligned} i &\leqslant \sum_{c\in\mathbf{R}} i_{p,c} - 1 \leqslant \sum_{c\in\mathbf{R}} i^{\mathrm{abs}}_{p,c} - 1 \\ &= \sum_{c\in\mathbf{R}} i^{\mathrm{N,abs}}_{p,c} + 2\left(\sum_{c\in\mathbf{R}} i^{\mathrm{T,abs}}_{p,c}\right) - \sum_{c\in\mathbf{R}} i^{\mathrm{T,abs}}_{p,c} - 1 \\ &\leqslant (d-1) - 1 - 1 = d - 3 \end{aligned}$$

where the last inequality follows from Lemma 7.9. □

8. VANISHING CYCLES

Suppose $p\in\mathbf{L}$ and $c\in\mathbf{R}\cup\{\infty\}$. In this section we relate the term $i_{p,c}$ in the decomposition of the index i of a real polynomial $f(x, y)$ to the number of vanishing cycles $v_{p,c}$ of the corresponding complex polynomial at (p, c). This number is defined to be the "jump" in the Milnor number of the family $f(x, y) = t$ of complex polynomials at p when $t = c$. (For a detailed discussion of this notion, in particular the case $c = \infty$, the reader is referred to [5].) Recall that $i^{\mathrm{abs}}_{p,c}$ was defined in the last section, and that $i_{p,c} \leqslant i^{\mathrm{abs}}_{p,c}$.

PROPOSITION 8.1. *Let $f(x, y)$ be a real polynomial with isolated critical points. If $p\in\mathbf{L}'$ (i.e. p is a zero of f_d) and $c\in\mathbf{R}\cup\{\infty\}$, then*

$$i^{\mathrm{abs}}_{p,c} \leqslant v_{p,c}.$$

Proof. Suppose $c\in\mathbf{R}$. (The proof for $c = \infty$ is similar.) As in the proof of Lemma 7.2, the number $i^{\mathrm{abs}}_{p,c}$ is at most one half the number of ends γ of the curve of tangencies with $p(\gamma) = p$ and $c(\gamma) = c$. Since f is either strictly increasing or decreasing on each end γ (Lemma 3.1), we may assume without loss of generality (replace f by $-f$), that the number of ends γ with $p(\gamma) = p$, $c(\gamma) = c$ and $f|\gamma < c$ is at most the number of ends γ with $p(\gamma) = p$, $c(\gamma) = c$ and $f|\gamma > c$. Hence $i^{\mathrm{abs}}_{p,c}$ is at most the number v of ends γ with $p(\gamma) = p$, $c(\gamma) = c$ and $f|\gamma > c$.

Since f is strictly decreasing to c along γ, v is the number of intersection points in $\mathbf{R}^2$ of the curves $xf_y - yf_x = 0$ (the curve of tangencies) and $f = c + \varepsilon$ which approach p as $\varepsilon \downarrow 0$. If we assume without loss of generality that $p = [1, 0, 0]$, we may replace the curve $xf_y - yf_x = 0$ by the curve $f_y = 0$. Thus v is at most the number of intersection points of the complex curves $f_y = 0$ and $f = c + \varepsilon$ which approach p as $\varepsilon \downarrow 0$. This number is well known to be $v_{p,c}$ (see, e.g. [5, 2.13]). □

The inequality of the proposition can be strict; for example the polynomial $y(x^a y - 1)$ at $p = [1, 0, 0]$ and $c = 0$ has $i_{p,c} = 1$ and $v_{p,c} = a$.

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