Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 3537–3544
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Binary relations and reduced hypergroups☆

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Received 28 March 2007; received in revised form 2 July 2007; accepted 4 July 2007
Available online3 August 2007

Abstract

Different partial hypergroupoids are associated with binary relations defined on a set H. In this paper we find sufficient and necessary conditions for these hypergroupoids in order to be reduced hypergroups. Given two binary relations ρ and σ on H we investigate when the hypergroups associated with the relations $\rho \cap \sigma$, $\rho \cup \sigma$ and $\rho\sigma$ are reduced. We also determine when the cartesian product of two hypergroupoids associated with a binary relation is a reduced hypergroup.

MSC: 20N20; 04A05

Keywords: Binary relation; Reduced hypergroup

1. Introduction and preliminaries

The first step in the history of the development of Hyperstructures Theory was the 8th Congress of Scandinavian Mathematicians from 1934, when Marty [12] introduced the notion of hypergroup, analyzed its properties and applied them to non-commutative groups, algebraic functions, rational fractions. Nowadays the hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, ethnology, etc. (see [6]).

Till now, several connections between hyperstructures and binary relations are established and studied by many researchers: Rosenberg [13], Corsini [3,4], Corsini and Leoreanu [5], Chvalina [1], Konstantinidou and Serafimidis [11], Spartalis [14–16], De Salvo and Lo Faro [8] and so on. In this paper we deal with the hypergroupoids associated with binary relations introduced by Rosenberg [13] and studied then by Corsini and Leoreanu [3–5].

In the following we present some results obtained on this argument.

For a non-empty set H, we denote by $\mathscr{P}^*(H)$ the set of all non-empty subsets of H.

☆ Work partially supported by the CEEX Programme of the Romanian Ministry of Education and Research, Grant CEX 05D11-11/04.10.2005.
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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.07.011

Definition 1.1. A non-empty set H, endowed with a mapping, called *hyperoperation*, $\circ : H^2 \longrightarrow \mathscr{P}^*(H)$ is named *hypergroupoid*. A hypergroupoid which verifies the following conditions:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$,
(ii) $x \circ H = H = H \circ x$, for all $x \in H$

is called *hypergroup*.

If A and B are non-empty subsets of H, then $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$.

Rosenberg [13] has associated a partial hypergroupoid $\mathbb{H}_\rho = \langle H, \circ \rangle$ with a binary relation ρ defined on a set H, where, for any $x, y \in H$,

$$x \circ x = L_x = \{z \in H \mid (x, z) \in \rho\} \quad \text{and} \quad x \circ y = L_x \cup L_y.$$

Definition 1.2. An element $x \in H$ is called *outer element* of ρ if there exists $h \in H$ such that $(h, x) \notin \rho^2$.

We need some of Rosenberg results that we recall in the next theorem.

Theorem 1.3 (*Roserberg [13, Proposition 2]*). $\mathbb{H}_\rho$ *is a hypergroup if and only if*

(i) *ρ has full domain*;
(ii) *ρ has full range*;
(iii) $\rho \subset \rho^2$;
(iv) *if $(a, x) \in \rho^2$ then $(a, x) \in \rho$, whenever x is an outer element of ρ.*

Remark. If ρ is a quasiorder relation, then the hypergroupoid $\mathbb{H}_\rho$ associated with H is a hypergroup.

Theorem 1.4 (*Corsini and Leoreanu [5, Proposition 1.1, Corollary 1.2, Remark 1.3]*). *Let ρ be a relation defined on a set H and $a, x \in H$. Let "$\circ$" be the partial hyperoperation defined above.*

(i) *If $\rho \subset \rho^2$, then $(a, x) \in \rho^2$ if and only if $x \in a \circ a \circ a$.*
(ii) *If $\rho \subset \rho^2$, then x is an outer element for ρ if and only if there exists $a \in H$ such that $x \notin a \circ a \circ a$.*
(iii) *If $\rho \subset \rho^2$, then there are no outer elements for ρ if and only if for any $a \in H$, we have $a \circ a \circ a = H$.*

Theorem 1.5 (*Corsini [3, Theorem 1.3]*). *If $\mathbb{H}_\rho$ is a hypergroup, then the following statements hold*:

(i) *ρ^2 is a transitive relation.*
(ii) *If ρ is symmetric, then ρ^2 is an equivalence relation on H.*
(iii) *If ρ is symmetric and $| H/\rho^2 | > 1$, then ρ is an equivalence relation on H.*

Corollary 1.6. *Let ρ be a reflexive, symmetric and non-transitive relation on H. The following assertions are equivalent*:

(i) *$\mathbb{H}_\rho$ is a hypergroup.*
(ii) *For any $x \in H$ we have $x \circ x \circ x = H$.*
(iii) *There are no outer elements for ρ.*
(iv) $\rho^2 = H \times H$.

Proposition 1.7 (*Corsini [3, Theorem 2.5]*). *Let ρ and σ be two binary relations on H with full domain and full range such that $\rho^2 = \rho$, $\sigma^2 = \sigma$ and $\rho\sigma = \sigma\rho$. Then $\mathbb{H}_{\rho\sigma}$ is a hypergroup.*

It may happen that the hyperoperation "$\circ$" does not discriminate between a pair of elements of H, when two elements play interchangeable roles with respect to the hyperoperation. On a hypergroupoid $\langle H, \circ \rangle$, the following three equivalence relations, called *the operational equivalence, the inseparability* and *the essential indistinguishability*, respectively, may be defined (see [9,10,7]):

- $x \sim_o y \iff x \circ a = y \circ a$ and $a \circ x = a \circ y$, for any $a \in H$;
- $x \sim_i y \iff$ for $a, b \in H$, we have $x \in a \circ b \iff y \in a \circ b$;
- $x \sim_e y \iff x \sim_o y$ and $x \sim_i y$.

For any $x \in H$, let $\widehat{x_o}$, $\widehat{x_i}$ and $\widehat{x_e}$, respectively, denote the equivalence classes of x with respect to the relations $\sim_o$, $\sim_i$ and $\sim_e$.

We say that a hypergroup $\langle H, \circ \rangle$ is *reduced* if and only if, for any $x \in H$, $\widehat{x_e} = \{x\}$.

Proposition 1.8 (*Jantosciak [10, Proposition 3]*). *For any hypergroup $\langle H, \circ \rangle$, the quotient hypergroup $\langle H/\sim_e, \star \rangle$ is a reduced hypergroup, where the hyperoperation $\star$ on $H/\sim_e$ is defined by*

$$\widehat{x_e} \star \widehat{y_e} = \{\widehat{z_e} \mid z \in x \circ y\}.$$

The quotient hypergroup $\langle H/\sim_e, \star \rangle$ is called the *reduced form* of the hypergroup $\langle H, \circ \rangle$.

It is known that the study of hypergroups falls into two parts: the study of reduced hypergroups and the study of all hypergroups having the same reduced form.

Our goal is to determine necessary and sufficient conditions such that the hypergroup $\mathbb{H}_\rho$, associated with a binary relation ρ, is reduced. Moreover, given two binary relations ρ and σ defined on H, we investigate when the hypergroups $\mathbb{H}_{\rho\cap\sigma}$, $\mathbb{H}_{\rho\cup\sigma}$, $\mathbb{H}_{\rho\sigma}$ are reduced. In the last part of the paper we talk about the cartesian product of the reduced hypergroups.

2. Basic properties

Let ρ be a binary relation defined on a non-empty set H.

For any $x \in H$, we denote $L_x^\rho = \{z \in H \mid (x, z) \in \rho\}$ and $R_x^\rho = \{z \in H \mid (z, x) \in \rho\}$.

If it is clear what is the relation we talk about, then we use the notations L_x and R_x instead of L_x^ρ and R_x^ρ.

If ρ is a relation such that the associated hypergroupoid $\mathbb{H}_\rho$ is a hypergroup, then, for any $x \in H$, we have $L_x \neq \emptyset$ and $R_x \neq \emptyset$.

It is easy to see that

(1) ρ is reflexive if and only if, for any $x \in H$, $x \in L_x$;
(2) ρ is symmetric if and only if, for any $x \in H$, $L_x = R_x$;
(3) ρ is transitive if and only if, for any $x, y \in H$ with $L_x \cap R_y \neq \emptyset$ it results $y \in L_x$.

Let ρ and σ be two distinct binary relations defined on H. One verifies that:

(i)
$$L_x^{\rho\cap\sigma} = \{z \in H \mid (x, z) \in \rho \cap \sigma\} = L_x^\rho \cap L_x^\sigma,$$
$$R_x^{\rho\cap\sigma} = \{z \in H \mid (z, x) \in \rho \cap \sigma\} = R_x^\rho \cap R_x^\sigma.$$

(ii)
$$L_x^{\rho\cup\sigma} = \{z \in H \mid (x, z) \in \rho \cup \sigma\} = L_x^\rho \cup L_x^\sigma,$$
$$R_x^{\rho\cup\sigma} = \{z \in H \mid (z, x) \in \rho \cup \sigma\} = R_x^\rho \cup R_x^\sigma.$$

(iii)
$$L_x^{\rho\sigma} = \{z \in H \mid (x, z) \in \rho\sigma\} = \{z \in H \mid \exists t \in H : (x, t) \in \rho, (t, z) \in \sigma\} = \{z \in L_t^\sigma \mid t \in L_x^\rho\},$$
$$R_x^{\rho\sigma} = \{z \in H \mid (z, x) \in \rho\sigma\} = \{z \in H \mid \exists t \in H : (z, t) \in \rho, (t, x) \in \sigma\} = \{z \in R_t^\rho \mid t \in R_x^\sigma\}.$$

(iv) If, for any $x \in H$, $L_x^\rho = L_x^\sigma$, then $\rho = \sigma$.

Proposition 2.1. *Let $\mathbb{H}_\rho$ be the hypergroup associated with the binary relation ρ defined on H. For any $x, y \in H$, the following implications hold*:

(1) $x \sim_o y \Longleftrightarrow L_x = L_y$,
(2) $x \sim_i y \Longleftrightarrow R_x = R_y$.

Proof. (1) By the definition of the relation "$\sim_o$", we have that $x \sim_o y$ is equivalent with $x \circ a = y \circ a$, for any $a \in H$, which means $L_x \cup L_a = L_y \cup L_a$. If $L_x = L_y$, it is clear that $x \sim_o y$.

Now we suppose $x \sim_o y$, hence, for any $a \in H$, $L_x \cup L_a = L_y \cup L_a$.

- For $a = x$ it results $L_x = L_x \cup L_y$, so $L_y \subseteq L_x$.
- For $a = y$ it results $L_x \cup L_y = L_y$, so $L_x \subseteq L_y$.

We conclude that $L_x = L_y$.

(2) Take $x, y \in H$, $x \sim_i y$. This means that $x \in a \circ b \Longleftrightarrow y \in a \circ b$, for $a, b \in H$, that is $x \in L_a \cup L_b \Longleftrightarrow y \in L_a \cup L_b$. But, for any $x \in H$, $R_x \neq \emptyset$, therefore there exists $a \in H$ such that $a \in R_x$, that is $x \in L_a$; it follows $x \in L_a = a \circ a$ and since $x \sim_i y$, we obtain $y \in L_a$, that is $a \in R_y$. Similarly we obtain $R_y \subseteq R_x$ and then $R_x = R_y$. □

Now, if $R_x = R_y$ we have $x \in L_z \Longleftrightarrow y \in L_z$, for $z \in H$, therefore $x \in z \circ t \Longleftrightarrow y \in z \circ t$, for $z, t \in H$, which means $x \sim_i y$.

In the following, we investigate when two different elements $x, y \in H$ are in the relation $x \sim_e y$ in the hypergroups $\mathbb{H}_{\rho \cap \sigma}$ and $\mathbb{H}_{\rho\sigma}$.

Proposition 2.2. *Let ρ and σ be two quasiorder relations on a non-empty set H. For any $x, y \in H$, $x \sim_e y$ in $\mathbb{H}_{\rho \cap \sigma}$ if and only if $x \sim_e y$ in $\mathbb{H}_\rho$ and $x \sim_e y$ in $\mathbb{H}_\sigma$.*

Proof. Since ρ and σ are two quasiorder relations, the hypergroupoids associated with ρ, σ and $\rho \cap \sigma$ are hypergroups.

First, we suppose $x \sim_e y$ in $\mathbb{H}_\rho$ and $x \sim_e y$ in $\mathbb{H}_\sigma$; by the previous proposition we have $L_x^\rho = L_y^\rho$, $R_x^\rho = R_y^\rho$, $L_x^\sigma = L_y^\sigma$ and $R_x^\sigma = R_y^\sigma$, so $L_x^{\rho\cap\sigma} = L_y^{\rho\cap\sigma}$ and $R_x^{\rho\cap\sigma} = R_y^{\rho\cap\sigma}$, that is $x \sim_e y$ in $\mathbb{H}_{\rho\cap\sigma}$.

Conversely, suppose $x \sim_e y$ in $\mathbb{H}_{\rho\cap\sigma}$, that is $x \sim_o y$ and $x \sim_i y$ in $\mathbb{H}_{\rho\cap\sigma}$. It is enough to show the implications:

(1) $L_x^\rho \cap L_x^\sigma = L_y^\rho \cap L_y^\sigma \Longrightarrow L_x^\rho = L_y^\rho$ and $L_x^\sigma = L_y^\sigma$;
(2) $R_x^\rho \cap R_x^\sigma = R_y^\rho \cap R_y^\sigma \Longrightarrow R_x^\rho = R_y^\rho$ and $R_x^\sigma = R_y^\sigma$.

We will prove the first one, the second one has a similar proof.

Since ρ and σ are reflexive relations, we write $x \in L_x^\rho \cap L_x^\sigma$, so $x \in L_y^\rho \cap L_y^\sigma$, that is $(y, x) \in \rho \cap \sigma$ and similarly, $(x, y) \in \rho \cap \sigma$.

Let us consider $z \in L_x^\rho$, that is $(x, z) \in \rho$ and since $(y, x) \in \rho$, by the transitivity of ρ, it results $(y, z) \in \rho$, $z \in L_y^\rho$. We have $L_x^\rho \subseteq L_y^\rho$ and similarly $L_y^\rho \subseteq L_x^\rho$. We obtain $L_x^\rho = L_y^\rho$ and, in the same way, $L_x^\sigma = L_y^\sigma$. □

Proposition 2.3. *Let ρ and σ be two binary relations on H with full domain and full range such that $\rho^2 = \rho$, $\sigma^2 = \sigma$ and $\rho\sigma = \sigma\rho$. If, for $x, y \in H$, $x \sim_o y$ in $\mathbb{H}_\rho$ and $x \sim_i y$ in $\mathbb{H}_\sigma$, then $x \sim_e y$ in $\mathbb{H}_{\rho\sigma}$.*

Moreover, $x \sim_e y$ in $\mathbb{H}_\rho$ and $x \sim_e y$ in $\mathbb{H}_\sigma$ lead to $x \sim_e y$ in $\mathbb{H}_{\rho\sigma}$.

Proof. In this hypothesis, the hypergroupoids $\mathbb{H}_\rho$, $\mathbb{H}_\sigma$ and $\mathbb{H}_{\rho\sigma}$ are hypergroups.

Let us consider $x, y \in H$ such that $x \sim_o y$ in $\mathbb{H}_\rho$ and $x \sim_i y$ in $\mathbb{H}_\sigma$, so we have $L_x^\rho = L_y^\rho$ and $R_x^\sigma = R_y^\sigma$. It is enough to prove the implications

(1) $L_x^\rho = L_y^\rho \Longrightarrow L_x^{\rho\sigma} = L_y^{\rho\sigma}$;
(2) $R_x^\sigma = R_y^\sigma \Longrightarrow R_x^{\rho\sigma} = R_y^{\rho\sigma}$.

Let $z \in L_x^{\rho\sigma}$; there exists $t \in L_x^{\rho}$ such that $z \in L_t^{\sigma}$, so there exists $t \in L_y^{\rho}$ such that $z \in L_t^{\sigma}$; therefore $z \in L_y^{\rho\sigma}$. Similarly $L_y^{\rho\sigma} \subseteq L_x^{\rho\sigma}$.

In the same way we can show the second implication.

Thus, if $x \sim_o y$ in $\mathbb{H}_\sigma$ and $x \sim_i y$ in $\mathbb{H}_\rho$, it results $x \sim_e y$ in $\mathbb{H}_{\sigma\rho}$ and since $\rho\sigma = \sigma\rho$ we obtain the last assertion of the proposition. □

3. Reduced hypergroups associated with binary relations

In this section, first, we determine a necessary and sufficient condition for the hypergroup $\mathbb{H}_\rho$ in order to be reduced; then we analyze this condition for different types of relations. Secondly, we prove that the hypergroupoid $\mathscr{H}_\rho$ associated with a binary relation defined by Corsini [4] is not a reduced hypergroup.

Theorem 3.1. *The hypergroup $\mathbb{H}_\rho$ is reduced if and only if, for any $x, y \in H$, x different from y, either $L_x \neq L_y$ or $R_x \neq R_y$.*

Proof. By the definition, the hypergroup $\mathbb{H}_\rho$ is reduced if and only if, for any $x \neq y$, it is true $x \not\sim_o y$ or $x \not\sim_i y$ and by the Proposition 2.1 this is equivalent with $L_x \neq L_y$ or $R_x \neq R_y$. □

For some particular relations, the condition expressed in the previous theorem is simpler, as we see in the following results.

Proposition 3.2. *If ρ is an equivalence on H, then the hypergroupoid $\mathbb{H}_\rho$ is a reduced hypergroup if and only if $\rho = \Delta_H = \{(x, x) \mid x \in H\}$.*

Proof. If ρ is an equivalence on H, then $\langle \mathbb{H}_\rho, \circ \rangle$ is a hypergroup.

Since ρ is symmetric, we have, for any $x \in H$, $L_x = R_x$ and then, $\mathbb{H}_\rho$ is reduced if and only if, for any $x \neq y$, $L_x \neq L_y$. We show that this condition is equivalent with the following one: for any $x \in H$, $L_x = \{x\}$ and then, it is clear $\rho = \Delta_H$.

If, for any $x \in H$, $L_x = \{x\}$, it results for all $x \neq y$ that $L_x \neq L_y$.

Conversely, let $y \neq x$, $y \in L_x$; we obtain $\{x, y\} \subseteq L_y$. For any $z \in L_y \setminus \{x, y\}$ we have $(y, z) \in \rho$, $(x, y) \in \rho$ and by transitivity it follows $(x, z) \in \rho$, so $z \in L_x$. Similarly, it results $L_x \subseteq L_y$, thus $L_x = L_y$ which is in contradiction with the hypothesis. □

Proposition 3.3. *If ρ is a non-symmetric quasiorder on H, then the hypergroup $\langle \mathbb{H}_\rho, \circ \rangle$ is reduced if and only if, for any $x \neq y$, $L_x \neq L_y$.*

Proof. If ρ is a quasiorder on H then, for any $x \neq y \in H$, we have the implication $x \sim_o y \Longrightarrow x \sim_i y$.

Indeed, if we suppose $L_x = L_y$ and $R_x \neq R_y$, there exists $z \in R_x$, $z \notin R_y$; then $(z, x) \in \rho$ and $(z, y) \notin \rho$. But ρ is reflexive and then $y \in L_y = L_x$; thus $(x, y) \in \rho$ and by transitivity we obtain $(z, y) \in \rho$, which is false.

So, for any $x \neq y$, the condition "$L_x \neq L_y$ or $R_x \neq R_y$" is equivalent with "$L_x \neq L_y$". □

Proposition 3.4. *If ρ is a reflexive symmetric non-transitive relation on H, such that $\rho^2 = H \times H$, then the hypergroup $\langle \mathbb{H}_\rho, \circ \rangle$ is reduced if and only if $L_x \neq L_y$, for all $x, y \in H$, x different from y.*

Proof. As in the previous proposition it is enough to prove that, for any $x \neq y$, $x \sim_o y \Longrightarrow x \sim_i y$.

If we suppose there exists $a \in H$ such that $x \in L_a$ and $y \notin L_a$, then, by the symmetry, we have $a \in L_x = L_y$ and thus $a \in L_y$, so $y \in L_a$, contradiction.

Given a binary relation ρ on H, Corsini [4] has defined another hyperoperation: for any $x, y \in H$,

$$x \otimes_\rho y = L_x \cap R_y,$$

and he has proved that $\mathscr{H}_\rho = \langle H, \otimes_\rho \rangle$ is a hypergroupoid if and only if $\rho^2 = H \times H$.

In case that the Corsini hyperoperation $\otimes_\rho$ is left or right reproductive, then $\mathscr{H}_\rho$ is the total hypergroup (see [16, Remark 2.4]). So, the unique hypergroup obtained in this manner is the total hypergroup, which clearly is not reduced. □

4. The hypergroups $\mathbb{H}_{\rho\cap\sigma}$, $\mathbb{H}_{\rho\cup\sigma}$, $\mathbb{H}_{\rho\sigma}$ as reduced hypergroups

Let ρ and σ be two binary relations defined on a non-empty set H. The hypergroups $\mathbb{H}_{\rho\cap\sigma}$, $\mathbb{H}_{\rho\sigma}$ and $\mathbb{H}_{\rho\cup\sigma}$ are reduced independently if $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are or are not reduced hypergroups, as we will see in the following results.

Proposition 4.1. *Let ρ and σ be two quasiorder relations on H. If the hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced, then the hypergroup $\mathbb{H}_{\rho\cap\sigma}$ is reduced too.*

Proof. If we suppose that the hypergroup $\mathbb{H}_{\rho\cap\sigma}$ is not reduced, then it results there exist $x \neq y$ in H such that $x \sim_e y$ in $\mathbb{H}_{\rho\cap\sigma}$ and therefore $x \sim_e y$ in $\mathbb{H}_\rho$, $x \sim_e y$ in $\mathbb{H}_\sigma$, which is impossible because the hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced. □

Remark. If the hypergroup $\mathbb{H}_{\rho\cap\sigma}$ is reduced, then the hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ can be reduced or not, as one sees from the following examples.

Example 4.2. Let $H = \{1, 2, 3, 4\}$.

(i) If $\rho \cap \sigma = \Delta_H = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and ρ, σ are equivalences on H different from the diagonal relation Δ_H, then the hypergroup $\mathbb{H}_{\rho\cap\sigma}$ is reduced, but neither $\mathbb{H}_\rho$ nor $\mathbb{H}_\sigma$ is a reduced hypergroup (see Proposition 3.2).
(ii) Set $\rho = \Delta_H \cup \{(1, 2)\}$ and $\sigma = \Delta_H \cup \{(1, 3)\}$. Then $\rho \cap \sigma = \Delta_H$, so $\mathbb{H}_{\rho\cap\sigma}$ is a reduced hypergroup and also $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$.
(iii) Set $\rho = \Delta_H \cup \{(1, 2), (2, 1), (1, 3), (2, 3)\}$, $\sigma = \Delta_H \cup \{(1, 2), (3, 4)\}$, so $\rho \cap \sigma = \Delta_H \cup \{(1, 2)\}$. It results the hypergroups $\mathbb{H}_{\rho\cap\sigma}$ and $\mathbb{H}_\sigma$ are reduced, but the hypergroup $\mathbb{H}_\rho$ is not ($L_1^\rho = L_2^\rho$, $R_1^\rho = R_2^\rho$).

Proposition 4.3. *Let ρ and σ be two binary relations on H with full domain and full range such that $\rho^2 = \rho$, $\sigma^2 = \sigma$ and $\rho\sigma = \sigma\rho$. If the hypergroup $\mathbb{H}_{\rho\sigma}$ is reduced, then both hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced.*

Proof. From the Proposition 2.3 we have the implications:

(1) $L_x^{\rho\sigma} \neq L_y^{\rho\sigma} \Longrightarrow L_x^\rho \neq L_y^\rho$ and $L_x^\sigma \neq L_y^\sigma$;
(2) $R_x^{\rho\sigma} \neq R_y^{\rho\sigma} \Longrightarrow R_x^\rho \neq R_y^\rho$ and $R_x^\sigma \neq R_y^\sigma$.

If $\mathbb{H}_{\rho\sigma}$ is a reduced hypergroup then, for any $x \neq y$, we have $x \not\sim_e y$, so, for any $x \neq y$, $L_x^{\rho\sigma} \neq L_y^{\rho\sigma}$ or $R_x^{\rho\sigma} \neq R_y^{\rho\sigma}$. It follows ($L_x^\rho \neq L_y^\rho$ and $L_x^\sigma \neq L_y^\sigma$) or ($R_x^\rho \neq R_y^\rho$ and $R_x^\sigma \neq R_y^\sigma$) and therefore the hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced. □

Remark. In the same hypothesis as in the Proposition 4.3, if $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced hypergroups, then the hypergroup $\mathbb{H}_{\rho\sigma}$ is reduced or not, as the following examples show.

Example 4.4. We consider the following two situations.

(1) Set $H = \{1, 2, 3, 4\}$, $\rho = \Delta_H \cup \{(1, 2)\} = \rho^2$ and $\sigma = \Delta_H \cup \{(1, 3)\} = \sigma^2$. Clearly, $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced hypergroups (see the Proposition 3.3); since $\rho\sigma = \Delta_H \cup \{(1, 2), (1, 3)\} = \sigma\rho$, it results that the hypergroup $\mathbb{H}_{\rho\sigma}$ is reduced.
(2) Set $H = \{1, 2, 3\}$, $\rho = \Delta_H \cup \{(2, 1), (2, 3)\} = \rho^2$ and $\sigma = \Delta_H \cup \{(1, 3), (1, 2)\} = \sigma^2$. Again it results that $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced hypergroups; we obtain $\rho\sigma = \Delta_H \cup \{(1, 2), (1, 3), (2, 1), (2, 3)\} = \sigma\rho$, and then $L_1^{\rho\sigma} = H = L_2^{\rho\sigma}$, $R_1^{\rho\sigma} = \{1, 2\} = R_2^{\rho\sigma}$, therefore the hypergroup $\mathbb{H}_{\rho\sigma}$ is not reduced.

Remark. Let ρ and σ be two binary relations defined on H such that the hypergroupoids $\mathbb{H}_\rho$, $\mathbb{H}_\sigma$ and $\mathbb{H}_{\rho\cup\sigma}$ are hypergroups. If $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced hypergroups, then the hypergroup $\mathbb{H}_{\rho\cup\sigma}$ can be reduced or not and conversely, if $\mathbb{H}_{\rho\cup\sigma}$ is a reduced hypergroup, it does not result that the hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced, too, as it follows from the following examples.

Example 4.5. We present the following situations.

(1) Set $H=\{1,2,3\}$, $\rho=\Delta_H\cup\{(1,2)\}=\rho^2$ and $\sigma=\Delta_H\cup\{(2,1)\}=\sigma^2$; we find $\rho\cup\sigma=\Delta_H\cup\{(1,2),(2,1)\}=(\rho\cup\sigma)^2$. It is clear that $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ are reduced hypergroups, but the hypergroup $\mathbb{H}_{\rho\cup\sigma}$ is not reduced, since $L_1^{\rho\cup\sigma}=\{1,2\}=L_2^{\rho\cup\sigma}$, $R_1^{\rho\cup\sigma}=\{1,2\}=R_2^{\rho\cup\sigma}$ (see the Proposition 3.3).

(2) Set $H=\{1,2,3\}$, $\rho=\Delta_H\cup\{(1,2)\}=\rho^2$ and $\sigma=\Delta_H\cup\{(1,3)\}=\sigma^2$; we obtain $\rho\cup\sigma=\Delta_H\cup\{(1,2),(1,3)\}=(\rho\cup\sigma)^2$. It follows that all the three hypergroups $\mathbb{H}_\rho$, $\mathbb{H}_\sigma$ and $\mathbb{H}_{\rho\cup\sigma}$ are reduced.

(3) Set again $H=\{1,2,3\}$ and the relations $\rho=\Delta_H\cup\{(1,2),(2,1)\}=\rho^2$, $\sigma=\Delta_H\cup\{(1,3)\}=\sigma^2$, therefore $\rho\cup\sigma=\Delta_H\cup\{(1,2),(2,1),(1,3)\}$ which is different from $(\rho\cup\sigma)^2=\Delta_H\cup\{(1,2),(2,1),(1,3),(2,3)\}$. In this case the hypergroup $\mathbb{H}_\rho$ is not reduced, the hypergroup $\mathbb{H}_\sigma$ is reduced and the hypergroup $\mathbb{H}_{\rho\cup\sigma}$ is reduced, too. The hypergroupoid $\mathbb{H}_{\rho\cup\sigma}$ is a hypergroup because $\rho\cup\sigma\subset(\rho\cup\sigma)^2$ and for the outer elements 1 and 2 of $\rho\cup\sigma$, condition (iv) of the Theorem 1.3 holds.

5. The cartesian product of the reduced hypergroups

Let $\langle H_1,\circ_1\rangle$, $\langle H_2,\circ_2\rangle$ be two hypergroups. On the cartesian product $H_1\times H_2$ we define the hyperoperation

$$(x_1,x_2)\otimes(y_1,y_2)=(x_1\circ_1 y_1, x_2\circ_2 y_2)$$

and we obtain the hypergroup $\langle H_1\times H_2,\otimes\rangle$ [2].

Proposition 5.1. *In the hypergroup $\langle H_1\times H_2,\otimes\rangle$, the following implications hold*:

(i) $(x_1,x_2)\sim_o(y_1,y_2)\Longleftrightarrow x_1\sim_o y_1$ *in* H_1 *and* $x_2\sim_o y_2$ *in* H_2;
(ii) $(x_1,x_2)\sim_i(y_1,y_2)\Longleftrightarrow x_1\sim_i y_1$ *in* H_1 *and* $x_2\sim_i y_2$ *in* H_2.

Proof. (i) By the definition of the relation $\sim_o$ we have $(x_1,x_2)\sim_o(y_1,y_2)$ if and only if, for any $(a_1,a_2)\in H_1\times H_2$, it is true: $(x_1,x_2)\otimes(a_1,a_2)=(y_1,y_2)\otimes(a_1,a_2)$ and $(a_1,a_2)\otimes(x_1,x_2)=(a_1,a_2)\otimes(y_1,y_2)$, which is equivalent with $x_1\circ_1 a_1=y_1\circ_1 a_1$, $x_2\circ_2 a_2=y_2\circ_2 a_2$ and $a_1\circ_1 x_1=a_1\circ_1 y_1$, $a_2\circ_2 x_2=a_2\circ_2 y_2$, that is, $x_1\sim_o y_1$ and $x_2\sim_o y_2$.

(ii) By the definition of the relation $\sim_i$ we get $(x_1,x_2)\sim_i(y_1,y_2)$ if and only if, for $(a_1,a_2),(b_1,b_2)\in H_1\times H_2$, we have $(x_1,x_2)\in(a_1,a_2)\otimes(b_1,b_2)$ equivalently $(y_1,y_2)\in(a_1,a_2)\otimes(b_1,b_2)$, therefore $x_1\in a_1\circ_1 b_1$ and $x_2\in a_2\circ_2 b_2$ if and only if $y_1\in a_1\circ_1 b_1$ and $y_2\in a_2\circ_2 b_2$, that is, $x_1\sim_i y_1$ and $x_2\sim_i y_2$. $\square$

Theorem 5.2. *The hypergroup $\langle H_1\times H_2,\otimes\rangle$ is reduced if and only if the hypergroups $\langle H_1,\circ_1\rangle$ and $\langle H_2,\circ_2\rangle$ are reduced.*

Proof. First, we suppose that $\langle H_1\times H_2,\otimes\rangle$ is a reduced hypergroup and that H_1 is not reduced. Then there exists $x_1\neq y_1$ in H_1 such that $x_1\sim_e y_1$, that is, $x_1\sim_o y_1$ and $x_1\sim_i y_1$. It follows that, for any $x_2\in H_2$, we have $(x_1,x_2)\sim_o(y_1,x_2)$ and $(x_1,x_2)\sim_i(y_1,x_2)$ (by the previous proposition), that is $(x_1,x_2)\sim_e(y_1,y_2)$ with $(x_1,x_2)\neq(y_1,x_2)$; this means that $\langle H_1\times H_2,\otimes\rangle$ is not reduced, which is in contradiction with the hypothesis.

Conversely, we suppose that $\langle H_1,\circ_1\rangle$ and $\langle H_2,\circ_2\rangle$ are reduced hypergroups, but $\langle H_1\times H_2,\otimes\rangle$ is not. Then there exist $(x_1,x_2)\neq(y_1,y_2)\in H_1\times H_2$ such that $(x_1,x_2)\sim_e(y_1,y_2)$. By the previous proposition we find $x_1\sim_e y_1$ and $x_2\sim_e y_2$. Since $\langle H_1,\circ_1\rangle$ and $\langle H_2,\circ_2\rangle$ are reduced, it follows that $x_1=y_1$, $x_2=y_2$, thus $(x_1,x_2)=(y_1,y_2)$ which is false. $\square$

Proposition 5.3. *Let ρ_1, ρ_2 be two binary relations defined on the non-empty sets H_1, H_2 such that the associated hypergroupoids $(\mathbb{H}_1)_{\rho_1}$ and $(\mathbb{H}_2)_{\rho_2}$ are hypergroups.*

(i) *If $(\mathbb{H}_1)_{\rho_1}$ and $(\mathbb{H}_2)_{\rho_2}$ are reduced hypergroups and, for $j \in \{1, 2\}$, the implication $\rho_j^2 \neq H_j^2 \Longrightarrow \rho_{3-j} = \rho_{3-j}^2$ (that is $(H_1 \times H_2)_{\rho_1 \times \rho_2}$ is a hypergroup) ([4]) holds, then $(H_1 \times H_2)_{\rho_1 \times \rho_2}$ is a reduced hypergroup.*
(ii) *If $(H_1 \times H_2)_{\rho_1 \times \rho_2}$ is a reduced hypergroup, then at least one of the hypergroups $(\mathbb{H}_1)_{\rho_1}$ and $(\mathbb{H}_2)_{\rho_2}$ is reduced.*

Proof. (i) If we suppose that $(H_1 \times H_2)_{\rho_1 \times \rho_2}$ is not reduced, then there exist $(x_1, x_2) \neq (y_1, y_2) \in H_1 \times H_2$ such that $L_{(x_1,x_2)} = L_{(y_1,y_2)}$ and $R_{(x_1,x_2)} = R_{(y_1,y_2)}$, that is $L_{x_1} = L_{y_1}$, $L_{x_2} = L_{y_2}$, $R_{x_1} = R_{y_1}$, $R_{x_2} = R_{y_2}$. This implies that $x_1 \sim_e y_1$ in $(\mathbb{H}_1)_{\rho_1}$ and $x_2 \sim_e y_2$ in $(\mathbb{H}_2)_{\rho_2}$, but since $(\mathbb{H}_1)_{\rho_1}$ and $(\mathbb{H}_2)_{\rho_2}$ are reduced, it follows $x_1 = y_1$ and $x_2 = y_2$, therefore $(x_1, x_2) = (y_1, y_2)$, which is false.

(ii) Now, if $(H_1 \times H_2)_{\rho_1 \times \rho_2}$ is a reduced hypergroup and if we suppose that both hypergroups $(\mathbb{H}_1)_{\rho_1}$ and $(\mathbb{H}_2)_{\rho_2}$ are not reduced, it follows there exist $x_1 \neq y_1 \in H_1$ and $x_2 \neq y_2 \in H_2$ such that $x_1 \sim_e y_1$ in $(\mathbb{H}_1)_{\rho_1}$ and $x_2 \sim_e y_2$ in $(\mathbb{H}_2)_{\rho_2}$; we obtain $L_{x_1} = L_{y_1}$, $R_{x_1} = R_{y_1}$ and $L_{x_2} = L_{y_2}$, $R_{x_2} = R_{y_2}$, which lead to the relations $L_{(x_1,x_2)} = L_{(y_1,y_2)}$ and $R_{(x_1,x_2)} = R_{(y_1,y_2)}$. This is in contradiction with the hypothesis that $(H_1 \times H_2)_{\rho_1 \times \rho_2}$ is reduced. □

6. Conclusions

The hypergroup associated with a binary relation ρ in the sense of Rosenberg is reduced if and only if, for any $x, y \in H$, either $L_x \neq L_y$ or $R_x \neq R_y$. The unique equivalence relation ρ defined on H such that the hypergroup $\mathbb{H}_\rho$ is reduced is the diagonal relation Δ_H. Given two binary relations ρ and σ on H, the property of being reduced of the associated hypergroups $\mathbb{H}_\rho$ and $\mathbb{H}_\sigma$ may or may not influence the same property of the hypergroups $\mathbb{H}_{\rho\sigma}$, $\mathbb{H}_{\rho\cap\sigma}$, $\mathbb{H}_{\rho\cup\sigma}$ and conversely. Finally, we proved that the cartesian product of reduced hypergroups is a reduced hypergroup.

Acknowledgments

The authors are highly grateful to the referees for their constructive suggestions for improving the paper.

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