On interpolation by rational functions with prescribed poles with applications to multivariate interpolation

G. MÜHLBACH
Institut für Angewandte Mathematik, University of Hannover, Welfengarten 1, D-3000 Hannover, FRG

Received 24 September 1989
Revised 27 December 1989

Abstract: This paper is concerned with interpolation in the sense of Hermite by certain rational functions of one or several complex variables. In the univariate setting the interpolants are generalized polynomials of a Cauchy-Vandermonde space, whereas in the multivariate setting the interpolants are elements of suitable subspaces of tensor products of Cauchy-Vandermonde spaces.

A Newton-type algorithm is given computing an interpolating univariate rational function with prescribed poles with no more than \(O(M^2)\) arithmetical operations where \(M\) is the number of nodes. It is proved that the generalized divided differences are analytic functions of the nodes if the function to be interpolated is analytic. The algorithm will be extended to the multivariate setting. For subsets of grids possessing the rectangular property and for certain subspaces of a tensor product of Cauchy-Vandermonde spaces an algorithm computing an interpolating rational function of two variables is given whose complexity is \(O(m^2n + n^2m)\), where \(m\) and \(n\) are the numbers of interpolation points in the \(x\)- and \(y\)-direction, respectively.

Keywords: Interpolation, rational functions, prescribed poles, multivariate rational interpolation.

1. Cauchy–Vandermonde systems

Let \((C_j)_{j=1,2,\ldots}\) be a given finite or countable infinite partition of the complex plane \(\mathbb{C}\). If we set \(C_0 := \{\infty\}\), then \(C := (C_j)_{j=0,1,\ldots}\) defines a partition of the extended complex plane \(\mathbb{C} \cup \{\infty\}\). Let \(B := (B_k)_{k=1,2,\ldots}\) be a given finite or infinite sequence of points of \(\mathbb{C} \cup \{\infty\}\). Its points will be used as "prescribed poles". \(C\) generates a partition of \(B\) into subsequences \((b^j_i)_{j=1,2,\ldots}\) of points \(b^j_i \in C_i\), \(i = 1, 2, \ldots\), \(j = 0, 1, \ldots\) \((b^j_i)_{j=1,2,\ldots}\) will be referred to as the \(i\)th cluster of \(B\) with respect to \(C\). Division of poles into clusters proves convenient for applications to civil engineering [2]. By the algorithm to be presented, a representation of a polynomial is easily derived such that an inclusion theorem [2,3] applies, giving rather sharp estimates of its zeros. Notice that repetition of points is allowed. With a pair \((B, C)\) we associate a system \(U = (u_k)_{k=1,2,\ldots}\) of rational functions defined by

\[
u_k(z) = \begin{cases} 
  z^{j-1}, & \text{if } B_k = b^j_0 \in C_0 = \{\infty\}, \\
  \frac{1}{\prod_{h=1}^n (z - b^j_h)}, & \text{if } B_k = b^j_i \in C_i \subset \mathbb{C}.
\end{cases}
\]

0377-0427/90/$03.50 © 1990 – Elsevier Science Publishers B.V. (North-Holland)
The system $U$ of functions (1) will be referred to as the Cauchy–Vandermonde system (in short: CV-system) generated by $(B, C)$. The span of a CV-system will be called a Cauchy–Vandermonde space (in short: CV-space). An initial section $B_M = (B_1, \ldots, B_M)$ of a pole sequence $B = (B_j)_{j=1,2,\ldots}$ will be called consistently ordered subordinate to $C$ if there is a permutation $C' = (C'_j)_{j=0,1,\ldots}$ of $C$ such that

$$B_M = \left( \beta_0^1, \ldots, \beta_0^{n_0}, \beta_1^1, \ldots, \beta_q^{n_q} \right),$$

with $\beta_i^j \in C'_i$ for all $i, j$. Here $n_i$ are positive integers adding up to $M$. Then also the system $U_M = (u_0^1, \ldots, u_0^{n_0}, u_1^1, \ldots, u_q^{n_q})$ generated by $(B_M, C')$ is called a consistently ordered CV-system of length $M$ subordinate to $C$. Consistently ordered systems will be used only to get simple sign factors in determinants. For convenience, sometimes we assume $C'_0 = C_0$ or $C'_q = C_0$ having the cluster of infinity poles in front or at the end of $B_M$. We are going to show that interpolation by elements of CV-spaces is close to interpolation by polynomials in a twofold sense.

2. Interpolation by Cauchy–Vandermonde systems

Let $A = (A_j)_{j=1,2,\ldots}$ be a given fixed sequence of complex numbers which later will serve as interpolation points or nodes taking into account multiplicities. With such a node system $A$, naturally there is associated a system $L = (L_j)_{j=1,2,\ldots}$ of Hermite type linear functionals $u \mapsto L_i u := D^{\alpha_i} u(A_i)$ where $D = d/dz$ denotes complex differentiation and $\mu(A_i)$ := multiplicity of $A_i$ in the initial section $A_{i-1} := (A_1, \ldots, A_{i-1})$. A node system $A_M$ will be called consistently ordered if

$$A_M = \left( a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_p, \ldots, a_p \right),$$

$$m_1 \quad m_2 \quad \cdots \quad m_p$$

where $a_1, \ldots, a_p \in \mathbb{C}$ are pairwise distinct with multiplicities $m_1, \ldots, m_p \geq 1$ adding up to $M$. With $A_M$ we associate the number

$$\text{mult}(A_M) := \prod_{k=1}^{M} \mu(A_k) !,$$

which in some sense is a measure of “repetition of nodes” in $A_M$.

It is well known [5,12] that the problem of Hermite interpolation

$(H_M)$: given a CV-system $U$ generated by $(B, C)$; given a node system $A_M = (A_1, \ldots, A_M)$ and given a complex function $f$ which is sufficiently often differentiable at the multiple points of $A_M$, find a generalized polynomial

$$p_M f = \sum_{k=1}^{M} c_k u_k$$

that satisfies the interpolation conditions

$$D^{\mu(A_i)} (p_M f - f)(A_i) = 0, \quad i = 1, \ldots, M,$$
has a unique solution provided the nodes and the poles do not intersect. Indeed, problem \((H_M)\)
can be reduced to interpolation by ordinary polynomials. Let

\[ Q(z) = \prod_{k=1}^{M} (z - B_k) \quad \text{and} \quad Q_i(z) = \prod_{k \in \mathbb{C}_i \cup C_i} (z - B_k), \]

where \(\prod^*\) means that in the product each infinity factor is to be replaced by a factor one. Suppose that \(P\) is the polynomial of degree less than \(M\) that interpolates the product \(Qf\) at \(A_M\). Assuming that (2) is a consistent ordering of \(B_M = (B_1, \ldots, B_M)\) with \(C'_0 = \{\infty\}\), then \(Qf := P + R\) where \(R\) is the remainder term and where \(P\) can be rewritten in the form

\[ P(z) = \sum_{i=0}^{q} \sum_{j=1}^{n_i} c^i/(z - \beta^i_{j+1}) \cdots (z - \beta^i_{n_j}) Q_i(z) + \sum_{j=1}^{n_0} c^0_j z^{-1} Q_0(z). \]

Division by \(Q = Q_0\) yields the equation \(f = p_M f + R/Q\). Here

\[ p_M f = \frac{P}{Q} = \sum_{i=0}^{q} \sum_{j=1}^{n_i} c^i_j u^i_j \quad (5) \]
solves problem \((H_M)\) since from Leibniz’ rule we infer

\[ D^{\mu+\nu}(A_i)(P - Qf)(A_i) = 0, \quad i = 1, \ldots, M, \quad \Leftrightarrow \quad D^{\mu+\nu}(A_i)\left(\frac{P}{Q} - f\right)(A_i) = 0, \quad i = 1, \ldots, M. \]

Moreover, the coefficients \(c^i_j\) belonging to finite poles can be expressed as certain ordinary divided differences:

\[ c^i_j = \left[ \beta^i_1, \beta^i_2, \ldots, \beta^i_{n_j} \right] \left( \frac{P}{Q} \right) \quad \text{for} \quad i = 1, \ldots, q, \quad j = 1, \ldots, n_j. \quad (6) \]

In fact, by applying the ordinary divided difference \([\beta^i_1, \beta^i_2, \ldots, \beta^i_{n_j}](\cdot)\) to

\[ \left( \frac{P}{Q} \right) (z) = \sum_{k=1}^{n_i} c^i_k (z - \beta^i_{k+1}) \cdots (z - \beta^i_{n_j}) + S_i(z), \]

we get (6) for \(i = 1, \ldots, q\) and \(j = 1, \ldots, n_i\), since the polynomial \(S_i\) vanishes at all points of the \(i\)th cluster. In particular, by Newton’s remainder formula

\[ c^i_{n_i} = \frac{P(\beta^i_{n_i})}{Q_i(\beta^i_{n_i})} = \left[ A_1, \ldots, A_M, \beta^i_{n_i} \right] (Qf) \frac{\prod_{j=1}^{M} (\beta^i_{n_i} - A_j)}{Q_i(\beta^i_{n_i})}, \quad i = 1, \ldots, q. \quad (7) \]

In addition, by comparing the leading coefficients (that of \(z^{M-1}\)) in (5), we find

\[ c^0_{n_0} = \left[ A_1, \ldots, A_M \right] (Qf) - \sum_{i=1}^{q} c^i_1, \quad (8) \]
since \([A_1, \ldots, A_M] P = [A_1, \ldots, A_M] (Qf)\).

In this sense interpolation by CV-systems actually is close to polynomial interpolation. But this relation does not give a practical recursive procedure to compute the coefficients since they depend on all poles of \(B_M = (B_1, \ldots, B_M)\) and on all nodes of \(A_M = (A_1, \ldots, A_M)\). If later another pole and another node are added, the whole procedure has to be restarted with respect to
the new pole system and to the new node system. On the other side, problem \((H_M)\) is interpolation by generalized polynomials. This point of view in fact leads to simple computation methods as we are going to show in the next sections.

3. Cauchy–Vandermonde determinants

Alternatively, the unisolvency of problem \((H_M)\) can be proved by computing the determinant of the associated system of linear equations for the coefficients of \(p_{Mf}\). Knowing the generalized Vandermonde determinant

\[ V | U_M; A_M | = \begin{vmatrix} u_1 & \cdots & u_M \\ A_1 & \cdots & A_M \end{vmatrix} := \det(D^{\mu(\lambda)} u_j(A_i)) \]

as an explicit function of the node system \(A_M = (A_1, \ldots, A_M)\) and of the pole system \(B_M = (B_1, \ldots, B_M)\) generating the CV-system \(U_M = (u_1, \ldots, u_M)\), we are able to derive algorithms computing \(p_{Mf}\) recursively. They are similar to those for ordinary polynomial interpolation. This is the second aspect under which interpolation by CV-systems resembles classical polynomial interpolation. To simplify notations let us adopt the following conventions: throughout void sums are set equal to zero, void products and void determinants are set equal to one. If \((f_j)_{j \in J}\) is any finite system of extended complex numbers, by

\[ \prod_{j \in J} f_j^* := \prod_{j \in J} f_j^* \]

we denote the product where each factor equal to zero or equal to infinity is to be replaced by one:

\[ f_j^* = \begin{cases} 1, & \text{if } f_j = 0 \text{ or if } f_j = \infty, \\ f_j, & \text{else}. \end{cases} \]

In this context, with respect to a fixed partition \(C = (C_j)\) of \(C\) we set \(b_i^j - b_j^i := 0\) for all \(i, j, j_2, \ldots, j_k\), that means in a product \(\prod^*\) the elements of each cluster are identified.

**Theorem 1.** Let the CV-system \(U_M^0 = (u_1, \ldots, u_M)\) be generated by \((B, C)\) where \(B = B_M^0 = (B_1, \ldots, B_M)\) is consistently ordered with respect to \(C\). Let \(A_M = A_M^0 = (A_1, \ldots, A_M)\) be a consistently ordered node system such that the nodes and the poles do not intersect. Then

\[
V | U_M^0; A_M^0 | = \text{mult}(A_M^0) \frac{\prod_{k, j = 1}^M (A_k - A_j) \prod_{k, j = 1}^M (B_k - B_j)}{\prod_{k, j = 1}^M (A_k - B_j) \prod_{k, j = 1}^M (B_k - A_j)},
\]

with \(\text{mult}(A_M^0)\) defined by (4).
Corollary. Any CV-system \( U \) generated by a pole system \( B \) subordinate to a partition \( C \) of \( \mathbb{C} \) is an extended complete Chebyshev-system (ECT-system) on any subset of the complex plane not intersecting the pole set.

Theorem 1 generalizes a result of [5]. For a proof and for alternative representations of the Cauchy–Vandermonde determinants we refer to [10]. Since (9) contains as particular cases the classical Vandermonde determinants and also Cauchy’s determinant, it will be called a Cauchy–Vandermonde determinant.

4. Newton’s interpolation formula

Let \( M \geq 2 \). Suppose that \( A_M = (A_1, \ldots, A_M) \) is a fixed system of \( M \) nodes which again will be thought of as an initial section of a given node sequence \( A = (A_j)_{j=1,2,\ldots} \). Let \( U_M \) be a CV-system generated by a pole system \( B_M \) which, when consistently ordered, has form (2). It is clear that by dropping in any nonempty cluster \((B_i^1, \ldots, B_i^q)\), \( i = 0, \ldots, q \), the “last pole” \( B_i^q \), the associated shorter CV-system \( U_{M-1} \) is a subsystem of \( U_M \) differing from that system by the function \( u_i^q \). This process may be repeated. There is a variety of chains \( U_1 \subset U_2 \subset \cdots \subset U_m \) where \( U_j \) are CV-systems of length \( j \). From the other direction, starting from a given pole sequence \( B = (B_k)_{k=1,2,\ldots} \), subordinate to a given partition \( C = (C_j)_{j=1,2,\ldots} \) of \( \mathbb{C} \) as in Section 1, by constructing the system \( U = (u_j)_{j=1,2,\ldots} \) of functions \( u_j \) defined by (1), automatically the subsystems \( U_k := (u_1, \ldots, u_k) \) form such chains \( U_1 \subset U_2 \subset \cdots \) of CV-systems \( U_k \), which are generated correspondingly by the subsystems \( B_k = (B_1, \ldots, B_k) \) of the given pole sequence \( B \).

It is known [1,6,7,9] that in this situation the solution \( p_M f \) of the Hermite interpolation problem \( (H_M) \) can be computed recursively by a generalization of Newton’s classical interpolation formula

\[
p_M f = p_{M-1} f + [A_1, \ldots, A_M] f \cdot r_{M-1} u_M = \sum_{k=1}^{M} [A_1, \ldots, A_k] f \cdot r_{k-1} u_k. \tag{10}
\]

Here \( p_k f \in \text{span} \, U_k \) interpolates \( f \) at \( A_k = (A_1, \ldots, A_k) \) and

\[
r_{k-1} h(z) := h(z) - p_{k-1} h(z) =: r[A_1, \ldots, A_k] h = \begin{vmatrix}
  u_1 & \cdots & u_{k-1} & h \\
  A_1 & \cdots & A_{k-1} & z \\
  \vdots & \ddots & \vdots & \vdots \\
  A_1 & \cdots & A_{k-1} & u_{k-1}
\end{vmatrix},
\]

if \( z \not\in \{A_1, \ldots, A_{k-1}\} \),

is an interpolation remainder term satisfying

\[
\text{D}^\mu_{\alpha_\rho}(A_k) r_{k-1} h(A_k) = \begin{vmatrix}
  u_1 & \cdots & u_{k-1} & h \\
  A_1 & \cdots & A_{k-1} & A_k \\
  \vdots & \ddots & \vdots & \vdots \\
  u_1 & \cdots & u_{k-1} & A_k
\end{vmatrix}
\]

for all \( A_k \in \mathbb{C} \), \( \forall \alpha \in \mathbb{R}^+ \).
and

\[
[A_1, \ldots, A_k] f := \begin{bmatrix}
  u_1 & \cdots & u_k \\
  A_1 & \cdots & A_k
\end{bmatrix}
\]

\[
= \begin{vmatrix}
  u_1 & \cdots & u_{k-1} & f \\
  A_1 & \cdots & A_{k-1} & A_k \\
  u_1 & \cdots & u_{k-1} & u_k \\
  A_1 & \cdots & A_{k-1} & A_k
\end{vmatrix}
\]

(13)

are generalized divided differences with respect to the systems \(U_k\) and \(A_k\). Notice that the one-line notation of divided differences always is with respect to the “natural ordering” of the functions beginning with \(u_1, u_2, \ldots\) but with respect to different node systems as indicated. For \(k = M\) the divided difference (13) is the “leading coefficient” (before \(u_M\)) of the interpolating polynomial (10). Hence it is invariant under permutations of the nodes \(A_1, \ldots, A_M\) and of the basic functions \(u_1, \ldots, u_{M-1}\). Thus for interpolation by CV-systems in considering divided differences we may assume that the underlying CV- and the node system are consistently ordered. In contrast, the basic functions \(r_{k-1}u_k\) in the Newton formula (10) depend essentially on the ordering of the \(u_i\) or the poles, respectively. In general, for ECT-systems the divided differences can be computed recursively [6,7]:

\[
[A_1, \ldots, A_M] f = \begin{cases}
  \frac{[A_2, \ldots, A_M] f - [A_1, \ldots, A_{M-1}] f}{[A_2, \ldots, A_M] u_M - [A_1, \ldots, A_{M-1}] u_M}, & \text{if } A_1 \neq A_M, \\
  \begin{vmatrix}
    u_1 & \cdots & u_{M-1} & f \\
    A_1 & \cdots & A_{M-1} & A_M \\
    u_1 & \cdots & u_{M-1} & u_M \\
    A_1 & \cdots & A_{M-1} & A_M
  \end{vmatrix}, & \text{if } A_1 = \cdots = A_M = a.
\end{cases}
\]

(14)

If there are two distinct nodes in \(A\), with no loss of generality, we may assume \(A_1 \neq A_M\). Brezinski [1] has observed that in this case the numerator of the right-hand side of (14) can be expressed by

\[
[A_2, \ldots, A_M] f - [A_1, \ldots, A_{M-1}] f
= \begin{vmatrix}
  u_1 & \cdots & u_{M-2} & f \\
  A_2 & \cdots & A_{M-1} & A_M \\
  u_1 & \cdots & u_{M-2} & u_M \\
  A_2 & \cdots & A_{M-1} & A_M
\end{vmatrix}
\]

(15)

This is an immediate consequence of Sylvester’s identity on determinants. By inserting \(f = u_M\) in (15) and by dividing (15) by the expression thus obtained the recurrence relation (14) is established (cf. [1]).

From (12) and (15), if \(A_1 \neq A_M\), an alternative representation of the left-hand side of (15) can be derived:

\[
[A_2, \ldots, A_M] f - [A_1, \ldots, A_{M-1}] f = \frac{D^\mu_{A_M}(A_M) f(A_M)}{D^\mu_{A_M}(A_M) f[A_2, \ldots, A_{M-1}] u_{M-1}(A_M)},
\]

(16)

since by assumption the multiplicity of \(A_M\) in \((A_2, \ldots, A_{M-1})\) equals \(\mu_{A_M}(A_M)\).
With the aid of (14) in general the complete table of the divided differences for any ECT-system can be computed with $O(M^3)$ arithmetical operations. In case of a CV-system this procedure simplifies considerably where the complexity is reduced to $O(M^2)$. This is a consequence of the following theorem.

**Theorem 2.** Suppose $M \geq 2$. Let $U$ be a CV-system generated by $(B, C)$. Then for any system $A_M = (A_1, \ldots, A_M)$ of $M$ nodes not intersecting the pole set, if $A_1 \neq A_M$, with respect to $U_{M-1} = (u_1, \ldots, u_{M-1})$, we have

$$\begin{bmatrix} A_2, \ldots, A_M \end{bmatrix} u_M - \begin{bmatrix} A_1, \ldots, A_{M-1} \end{bmatrix} u_M = \frac{A_M - A_1}{(A_M - B_M)(B_M - A_1)} F_M,$$

with

$$F_M := F_M(A_2, \ldots, A_{M-1}; B_1, \ldots, B_M) = \frac{\gamma_{M,M}}{\gamma_{M,M-1}},$$

$$\gamma_{M,k} := \frac{\prod_{j=1}^{k-1} (B_k - B_j)}{M-1}, \quad k = M-1, M,$$

and

$$D^\mu u(A_M) r_{M-1} u_M(A_M) = \mu_{A_M}(A_M)! \frac{\gamma_{M,M}}{\prod_{j=1}^{M-1} (A_M - A_j)} \frac{\prod_{j=1}^{M} (A_M - A_j)}{\prod_{j=1}^{M-1} (A_M - B_j)}$$

for any $A_M \in \mathbb{C}$. In particular, if $z = A_M \notin \{A_1, \ldots, A_{M-1}\}$, then

$$r_{M-1} u_M(z) = \frac{\gamma_{M,M}}{\prod_{j=1}^{M-1} (z - A_j)} \frac{\prod_{j=1}^{M} (z - A_j)}{\prod_{j=1}^{M-1} (z - B_j)}.$$

**Proof.** Let us prove (20) first. By (12) we know

$$D^\mu u(A_M) r_{M-1} u_M(A_M) = \begin{vmatrix} u_1 & \cdots & u_{M-1} & u_M \\ A_1 & \cdots & A_{M-1} & A_M \end{vmatrix}.$$
To prove (17) we start from (16) applied to \( f = u_M \). By using (20) twice we get

\[
[A_2, \ldots, A_M]u_M - [A_1, \ldots, A_{M-1}]u_M = \prod_{j=1}^{M-1} (A_M - A_j) \gamma_{M,M} \prod_{j=1}^{M-2} (A_M - A_{j+1}) (B_M - A_1)^* \]

\[
\times \prod_{j=1}^{M-1} (A_M - B_j) \tilde{\gamma}_{M-1,M-1} \frac{1}{\prod_{j=2}^{M-2} (B_M - A_{j+1})}
\]

with

\[
\tilde{\gamma}_{M-1,M-1} := \frac{\prod_{j=1}^{M-2} (B_{M-1} - B_j)}{\prod_{j=2}^{M-2} (B_M - A_{j+1})}
\]

Since \( \gamma_{M,M-1} = \tilde{\gamma}_{M-1,M-1}/(B_{M-1} - A_2)^* \) after cancelling all common factors, it is easily seen that the last expression equals the right-hand side of (17). This ends the proof of Theorem 2. \( \square \)

Let us mention that in case of only simple nodes (17) has been derived by a different approach in [11]. Notice that starting from generalized Taylor’s coefficients

\[
[a, \ldots, a] f = \begin{bmatrix} u_1 & \cdots & u_j \\ a & \cdots & a \end{bmatrix} f, \quad j = 1, \ldots, k.
\]

by using (14) and (17), the complete table of generalized divided differences of a function \( f \) with respect to a CV-system \( U_M \) and to any system \( A_M \) of nodes not intersecting the pole set corresponding to \( U_M \), can be computed recursively with an amount of arithmetical operations bounded by \( O(M^2) \). Let us assume that we are computing the divided difference table according to the Neville graph. Then two consecutive factors (18)

\[
F_M = F_M(A_2, \ldots, A_{M-1}; B_1, \ldots, B_M) \quad \text{and} \quad F_M' = F_M(A_3, \ldots, A_M; B_1, \ldots, B_M)
\]

of the denominators (17) in the \( M \)th column are related by

\[
F_M' = F_M \frac{(B_M - A_2)^* (B_{M-1} - A_M)^*}{(B_{M-1} - A_2)^* (B_M - A_M)^*},
\]

as an easy calculation shows. Consequently, two consecutive denominators in the \( M \)th column of the divided difference table are related by

\[
\frac{[A_3, \ldots, A_{M+1}]u_M - [A_2, \ldots, A_M]u_M}{[A_2, \ldots, A_M]u_M - [A_1, \ldots, A_{M-1}]u_M} = \frac{A_{M+1} - A_2}{A_{M-1} - A_1} \frac{(B_{M-1} - A_M)^*}{(B_{M-1} - A_2)^*} \frac{(B_M - A_1)^*}{(B_M - A_{M+1})^*},
\]

provided \( A_{M+1} \neq A_2 \) and \( A_M \neq A_1 \).
On the other side each starting value \( (22) \) also can be computed from the given data \( a, B_i, D^{-1}f(a), i = 1, \ldots, k \), with no more than \( O(k^2) \) arithmetical operations as we are going to show now.

Let \( U_k = (u_1, \ldots, u_k) \) be the CV-system generated by \( B_k = (B_1, \ldots, B_k) \). We may assume that both systems are consistently ordered. We consider also the related system \( \tilde{U}_k = (\tilde{u}_1, \ldots, \tilde{u}_k) \) defined by

\[
\tilde{u}_j(z) = \frac{(z-a)^j}{\prod_{h=1}^{\star} (z-B_h)}, \quad j = 1, \ldots, k.
\]

By partial fraction decomposition we see that \( \tilde{U}_k = C U_k^T \) where \( C = (c_{i,j}) \) is a lower triangular matrix with diagonal entries

\[
c_{j,j} = \frac{(B_j - a)^{j-1}}{\prod_{h=1}^{\star} (B_j - B_h)}, \quad j = 1, \ldots, k.
\]

A simple method to calculate the starting values \( (22) \) now proceeds as follows. Compute first the coefficients

\[
\tilde{\varepsilon}_j = \begin{bmatrix} \tilde{u}_1 & \cdots & \tilde{u}_j \end{bmatrix} \quad \text{of} \quad p_k f = \sum_{j=1}^{k} \tilde{\varepsilon}_j \tilde{u}_j
\]

from

\[
D^{-1} p_k f(a) = D^{i-1} f(a), \quad i = 1, \ldots, k. \quad (23)
\]

Since \( (23) \) is a lower triangular linear system for the coefficients \( \tilde{\varepsilon}_j \) they can be computed by \( O(k^2) \) arithmetical operations provided the matrix \( (D^{-1} \tilde{u}_j(a)) \) of \( (23) \) is known. In [5, Theorem 10] an algorithm is given which can be adapted to compute this matrix from the data \( B_1, \ldots, B_k, a \) with no more than \( O(k^2) \) arithmetical operations. Now

\[
\begin{bmatrix} u_1 & \cdots & u_j \end{bmatrix} = c_{j,j} \begin{bmatrix} \tilde{u}_1 & \cdots & \tilde{u}_j \end{bmatrix}, \quad j = 1, \ldots, k,
\]

since

\[
\begin{bmatrix} \tilde{u}_1 & \cdots & \tilde{u}_j \end{bmatrix} = \frac{V \begin{bmatrix} \tilde{u}_1 & \cdots & \tilde{u}_{j-1} & f \\ a & \cdots & a & a \end{bmatrix}}{V \begin{bmatrix} \tilde{u}_1 & \cdots & \tilde{u}_{j-1} & \tilde{u}_j \\ a & \cdots & a & a \end{bmatrix}}
\]

\[
= \frac{V \begin{bmatrix} c_{1,1} u_1 & \cdots & c_{j-1,j-1} u_{j-1} & f \\ a & \cdots & a & a \end{bmatrix}}{V \begin{bmatrix} c_{1,1} u_1 & \cdots & c_{j-1,j-1} u_{j-1} & c_{j,j} u_j \\ a & \cdots & a & a \end{bmatrix}}
\]

\[
= \frac{1}{c_{j,j}} \begin{bmatrix} u_1 & \cdots & u_j \end{bmatrix} f.
\]
5. Divided differences are analytic functions of the nodes

It is well known that ordinary divided differences (with respect to the monomials $1, x, x^2, \ldots$) are continuous functions of the nodes if the function to be interpolated is sufficiently smooth. We are going to show that generalized divided differences with respect to CV-systems are not only continuous but analytic functions of the nodes provided the function to be interpolated is analytic.

**Theorem 3.** Suppose that $U = (u_j)_{j=1}^{\infty}$ is a CV-system generated by $(B, C)$. For any positive integer $M$ by $B_M = (B_1, \ldots, B_M)$ and $U_M = (u_1, \ldots, u_M)$ we denote the corresponding initial sections of $B$ and $U$. Let $f$ be a complex function which is analytic in a region $G \subset \mathbb{C} \setminus \{B_1, \ldots, B_{M+1}\}$. Then the divided difference of $f$ with respect to $U_M$,

$$G^M \ni (A_1, \ldots, A_M) \mapsto \begin{bmatrix} u_1 & \cdots & u_M \\ A_1 & \cdots & A_M \end{bmatrix} f$$

is an analytic function of its nodes. Moreover, in $G^M$,

$$\frac{\partial}{\partial A_M} \begin{bmatrix} u_1 & \cdots & u_M \\ A_1 & \cdots & A_M \end{bmatrix} f = \begin{bmatrix} u_1 & \cdots & u_{M-1} & u_M & u_{M+1} \\ A_1 & \cdots & A_{M-1} & A_M & A_M \end{bmatrix} f$$

$$\times \frac{\gamma_{M+1,M+1}}{\gamma_{M+1,M} (A_M - B_{M+1})^* (B_{M+1} - A_M)^*},$$

where

$$\gamma_{M+1,k} := \frac{\prod_{j=2}^{M+1} (B_k - B_j)}{\prod_{j=2}^{M+1} (B_k - B_{j-1})}, \quad k = M, M + 1.$$ 

**Proof.** For any positive integer $M$ let

$$D^M := \{(A_1, \ldots, A_M) \in \mathbb{C}^M: A_1, \ldots, A_M \in G \text{ are pairwise distinct}\}.$$ 

Clearly, $D^M$ is an open subset of $\mathbb{C}^M$. In a first step we show by induction on $M$ that (24) is an analytic function of $(A_1, \ldots, A_M) \in D^M$. For $M = 1$,

$$A_1 \mapsto \begin{bmatrix} u_1 \\ A_1 \end{bmatrix} f = \begin{cases} \frac{f(A_1)}{u_1(A_1)} = (A_1 - B_1) f(A_1), & \text{if } B_1 \in \mathbb{C}, \\
& f(A_1), \quad \text{if } B_1 = \infty, \end{cases}$$

indeed is analytic on $G$, by assumption on $f$ and on $G$.

Let the assertion be proved for all indices less than or equal to $M$. If $A = (A_1, \ldots, A_{M+1}) \in D^{M+1}$, then by (17) and the induction hypothesis,

$$\begin{bmatrix} u_1 & \cdots & u_{M+1} \\ A_1 & \cdots & A_{M+1} \end{bmatrix} f = \frac{[A_2, \ldots, A_{M+1}] f - [A_1, \ldots, A_M] f}{(A_{M+1} - A_1) \gamma_{M+1,M+1}} \times \frac{\gamma_{M+1,M+1}}{(A_{M+1} - B_{M+1})^* (B_{M+1} - A_1)^* \gamma_{M+1,M}}$$

is an analytic function of $(A_1, \ldots, A_{M+1}) \in D^{M+1}$. 

In a second step we show that for every positive integer \( M \), (24) is a continuous function of \( (A_1, \ldots, A_M) \in G^M \). This is a consequence of the connections of the generalized divided differences with respect to CV-systems to ordinary divided differences which are given in (6)–(8). Since the \( Qf \) at the nodes \( A_1, \ldots, A_M \) interpolating ordinary polynomial \( P \) depends continuously on \( (A_1, \ldots, A_M) \), from (6) and (8) we see that in fact (24) is a continuous function of \( (A_1, \ldots, A_M) \in G^M \).

Next we show (25). Let \( (A_1, \ldots, A_M) \in G^M \) be arbitrary but fixed. Then, by (17),

\[
\lim_{H \to 0} \frac{\prod_{i=0}^{M-1} \gamma_{i+1,M+1}^{-1}}{\gamma_{M+1,M+1}^{-1}}
\]

\[
\gamma_{i+1,M+1}^{-1}
\]

where \( \gamma_{i+1,k} \) and \( \gamma_{i+1,k} \) are defined by (19) with respect to \( B_{M+1} \) and to \( \tilde{A}_{M+1} = (A_M, A_1, \ldots, A_{M-1}, A_M + H) \) and \( \tilde{A}_{M+1} = (A_M, A_1, \ldots, A_{M-1}, A_M) \), respectively. For the last equality we have used the continuity of the generalized divided difference of order \( M + 1 \) as a function of its nodes in \( G^{M+1} \). Since divided differences are symmetric functions of the nodes, we have proved that the function (24) in the open set \( G^M \) is partially holomorphic, the partial complex derivatives being continuous, hence by Hartogg’s theorem (cf. [4, p.135]) it is an analytic function in \( G^M \).

6. Extension to multivariate interpolation

Interpolation by generalized polynomials in several real or complex variables differs much from the univariate case. In general, only for particular node systems unisolvency of the corresponding interpolation problem can be expected. We are mainly interested in node systems giving rise to simple procedures calculating the interpolant recursively. More precisely, we want to extend Newton’s procedure to the multivariate setting. The simplest systems suitable for this purpose are rectangular grids and certain of their subsystems. Suppose that \( X = (x_1, \ldots, x_m) \) and \( Y = (y_1, \ldots, y_n) \) are systems of nodes belonging to regions \( G \) and \( H \) of the complex \( z \)- and \( w \)-plane, correspondingly, the points being not necessarily distinct. Then the ordered Cartesian product of \( X \) and \( Y \) taking into account multiplicities,

\[
X \otimes Y := \{(x_\mu, y_\nu) : (\mu, \nu) = (1, 1), \ldots, (m, n) \text{ in lexicographic order}\},
\]
will be referred to as the grid generated by $X$ and $Y$. Suppose we are given two CV-systems $U_m = (u_1, \ldots, u_m)$ and $V_n = (v_1, \ldots, v_n)$, generated correspondingly by $(B_m, C)$ and $(D_n, E)$. Here $B_m = (B_1, \ldots, B_m)$ is a pole system subordinate to a partition $C$ of the $z$-plane and $D_n = (D_1, \ldots, D_n)$ is a pole system subordinate to a partition $E = (E_j)_{j=1,2,\ldots}$ of the $w$-plane. Then, by $U_m \otimes V_n$ we denote the tensor product of these systems:

$$U_m \otimes V_n = \{(u_\mu, v_\nu) : (\mu, \nu) = (1,1), \ldots, (m,n)\} \text{ in lexicographic order,}$$

where

$$(u_\mu \otimes v_\nu)(z, w) = u_\mu(z) v_\nu(w).$$

The elements of span $U_m$, span $V_n$ and of span $U_m \otimes V_n$ will be referred to as $u$-, $v$- and $w$-polynomials, respectively. As usual, $D_1 = \partial / \partial z$ and $D_2 = \partial / \partial w$ denote partial complex derivatives with respect to the first and to the second variable. Consider the following problem of Hermite interpolation:

$$(H_{X \otimes Y}): \text{ given a grid } X \otimes Y = (x_1, \ldots, x_m) \otimes (y_1, \ldots, y_n), \text{ given a complex function } f \text{ of two complex variables which is sufficiently often partially differentiable at the multiple nodes of } X \otimes Y, \text{ find a } w\text{-polynomial }$$

$$pf = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} c_{\mu, \nu} \cdot u_\mu \otimes v_\nu$$

such that for all $\mu = 1, \ldots, m$, $\nu = 1, \ldots, n$,

$$D_1^{x_j}(x_\mu) D_2^{y_j}(y_\nu) (pf - f)(x_\mu, y_\nu) = 0.$$ 

It can be shown [8,9] that this problem has a unique solution

$$pf = \sum_{(\mu, \nu) \in N} [x_1, \ldots, x_\mu][y_1, \ldots, y_\nu] f \cdot (r_{\mu-1} u_\mu)(r_{\nu-1} v_\nu),$$

(26)

where $N := ((\mu, \nu) : (x_\mu, y_\nu) \in X \otimes Y)$ and

$$r_{\mu-1} u_\mu(z) = u_\mu(z) - p_{\mu-1} u_\mu(z) = \frac{\prod_{j=1}^{\mu-1} (B_\mu - B_j) \prod_{j=1}^{\mu-1} (z - x_j)}{\prod_{j=1}^{\mu-1} (B_\mu - x_j) \prod_{j=1}^{\mu-1} (z - B_j)},$$

$$r_{\nu-1} v_\nu(w) = v_\nu(w) - p_{\nu-1} v_\nu(w) = \frac{\prod_{j=1}^{\nu-1} (D_\nu - D_j) \prod_{j=1}^{\nu-1} (w - y_j)}{\prod_{j=1}^{\nu-1} (D_\nu - y_j) \prod_{j=1}^{\nu-1} (w - D_j)}.$$ 

Here $p_{\mu-1} h$, respectively $p_{\nu-1} h$, is the $h$ interpolating univariate $u$- and $v$-polynomial. The coefficients are iterated divided differences where $[x_1, \ldots, x_\mu]$ is taken with respect to $(u_1, \ldots, u_\mu)$ and acts on the first variable and $[y_1, \ldots, y_\nu]$ is with respect to $(v_1, \ldots, v_\nu)$ acting on the second
variable. They can be computed recursively according to
\[
[x_0, \ldots, x_0, y_0, \ldots, y_0] f
\]
\[
= \begin{cases}
\frac{x_0 - x_1}{(x_0 - B_p)(B_p - x_1)} \gamma_{p,0}, & \text{if } x_0 \neq x_1, \\
\frac{x_0 - y_0}{(x_0 - D_p)(D_p - y_1)} \delta_{p,0}, & \text{if } y_0 \neq y_1,
\end{cases}
\]
(27)

with
\[
\gamma_{p,0} := \frac{\prod_{j=1}^{n-1} (B_p - B_j)}{\prod_{j=2}^{n} (B_p - x_j)} \\
\delta_{p,0} := \frac{\prod_{j=1}^{n-1} (D_p - D_j)}{\prod_{j=2}^{n} (D_p - y_j)}
\]

starting from generalized Taylor's coefficients \([x, \ldots, x][y, \ldots, y]f\) which can be computed by
the univariate algorithm mentioned above. This algorithm has to be applied iteratively with
respect to the variables \(x\) and \(y\) and to the systems \(U\) and \(V\), respectively. Thus (26) can be
computed with no more than \(O(m^2n + n^2m)\) arithmetical operations where the main part of
work consists in computing the table of divided differences (27).

The computation of \(pf\) does not depend on particular orderings of the grids \(X\) and \(Y\). Next we
will extend formula (26) to certain subsystems \(Z\) of the full grid. What subsystems \(Z \subset X \otimes Y\)
are good in the sense that (26) holds with \(X \otimes Y\) replaced by \(Z\)? There is only one restriction:
the divided differences should be well-defined and (27) should apply.

**Definition.** For any positive integer \(m\) let \(m := (1, \ldots, m)\). A subsystem \(N\) of the grid
\(m \otimes n := ((\mu, \nu) : (\mu, \nu) = (1, 1), \ldots, (m, n)\) in lexicographic order)
will be called a lower index list from \(m \otimes n\) provided with respect to the partial ordering
\((\alpha, \beta) \leq (\mu, \nu) \iff \alpha \leq \mu \text{ and } \beta \leq \nu\),
there holds
\((\mu, \nu) \in N \text{ and } (\alpha, \beta) \leq (\mu, \nu) \Rightarrow (\alpha, \beta) \in N\).

Evidently, \(N\) is a lower index list from \(m \otimes n\) if and only if there exist integers \(n \geq n_1 \geq \cdots
\geq n_m \geq 0\) such that
\(N = ((\mu, \nu) \in m \otimes n : 1 \leq \mu \leq m \text{ and } 1 \leq \nu \leq n_\mu \text{ in lexicographic order})\).

**Definition.** A subsystem \(Z\) of a grid \(X \otimes Y\) will be called a lower data set from \(X \otimes Y\) or a subset
of \(X \otimes Y\) possessing the rectangular property iff there exist reorderings of the univariate grids \(X\)
and \(Y\), \(X = (x_0, \ldots, x_m)\) and \(Y = (y_1, \ldots, y_n)\) say, such that its associated index list
\(N_Z := ((\mu, \nu) : (x_\mu, y_\nu) \in Z)\)
is a lower index list from $m \otimes n$.

It is not hard to show [8] that if $Z$ is a lower data set from $X \otimes Y$ with associated index list $N_Z$, then

$$pf = \sum_{(\mu, v) \in N_Z} [x_1, \ldots, x_\mu, y_1, \ldots, y_v] f \cdot (r_{\mu-1} r_{v-1})$$

interpolates $f$ at the points of $Z$ in the sense of Hermite. Only those partial sums of (26) which correspond to lower data lists $N$ from $m \otimes n$ can be interpreted as interpolants themselves, namely with respect to $Z = ((x_\mu, y_\nu) \in X \otimes Y: (\mu, \nu) \in N)$. The extension to more than two variables is evident.

References