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Sufficient criteria are necessary for monotone control volume methods

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ABSTRACT

Control volume methods are prevailing for solving the potential equation arising in porous media flow. The continuous form of this equation is known to satisfy a maximum principle, and it is desirable that the numerical approximation shares this quality. Recently, sufficient criteria were derived guaranteeing a discrete maximum principle for a class of control volume methods. We show that most of these criteria are also necessary. An implication of our work is that no linear nine-point control volume method can be constructed for quadrilateral grids in 2D that is exact for linear solutions while remaining monotone for general problems.

1. Introduction

In this work, we consider control volume discretizations to the elliptic equation

$$-\nabla \cdot (\mathbf{K}\nabla u) = q \quad \text{in } \Omega. \tag{1}$$

Here, $\mathbf{K}$ is a conductivity (or diffusion) tensor, u is a potential, q is a source or sink term, and Ω is the domain of definition in $\mathbb{R}^2$. Equations of this type appear frequently in applications, and of particular interest in the continuation is when the divergence is derived from a conservation law.

Eq. (1) satisfies a maximum principle, which can be stated as follows: Let $G_\omega(q)$ be the solution operator of Eq. (1) on any subdomain $\omega \subset \Omega$, with zero Dirichlet boundary conditions, $G_\omega(q) = 0$ on $\partial\omega$. Then for all $q \geq 0$, the solution also satisfies $G_\omega(q) \geq 0$.

Discrete approximations to Eq. (1) are often sought that are of a conservative (control volume) form, are exact for linear solutions, and have a local approximation (using adjacent cells) to the flux $-K\nabla u$ in terms of a linear combination of potentials. For quadrilateral grids in 2D, this leads to nine-point stencils, as illustrated in Fig. 1. We refer to such discretizations simply as control volume discretizations, and examples can be found in [1–3].

A discrete maximum principle: For any subgrid bounded by a closed Jordan curve, with homogeneous Dirichlet conditions, the discretization must yield a system matrix whose inverse has no negative elements.

Remarks. The requirement of non-negativity on all subgrids makes the discrete maximum principle stronger than requiring non-negativity of the matrix on the main grid. However, it is not as restrictive as the classical M-matrix conditions [8]. The definition here is more precise than that given in [4], which did not specify restrictions on the subgrids: The boundary conditions are interpreted as implemented using ghost cells from the original grid, and the boundary of the subgrid is defined as the linear interpolation between ghost cell centers. This has the implication that the boundary of the union of active cells in the subgrid is also a Jordan curve, and thus contains no coinciding corners.

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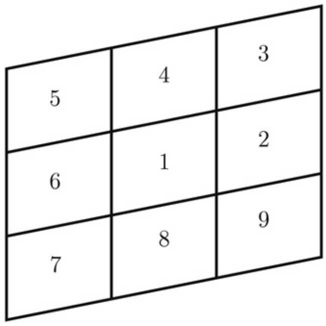

Fig. 1. Local cell numbering in a nine-point stencil.

In recent years, there have been efforts to give conditions under which control volume methods for Eq. (1) satisfy this discrete maximum principle; see for instance [4,5]. Similar investigations were done for parabolic equations in [6]. In [4], a splitting of the system matrix was used to find sufficient criteria for monotonicity of a quadrilateral grid. A form on these criteria is

$$\begin{array}{llll}
\text{(A0):} & m_1^{i,j} > 0, & \text{(A2):} & m_1^{i,j} + m_2^{i,j} + m_6^{i,j} > 0,\\
\text{(A1a):} & m_2^{i,j} < 0, & \text{(A3a):} & m_2^{i,j} m_4^{i,j-1} - m_3^{i,j-1} m_1^{i,j} > 0,\\
\text{(A1b):} & m_4^{i,j} < 0, & \text{(A3b):} & m_6^{i,j} m_4^{i,j-1} - m_5^{i,j-1} m_1^{i,j} > 0,\\
\text{(A1c):} & m_6^{i,j} < 0, & \text{(A3c):} & m_2^{i,j} m_8^{i,j+1} - m_9^{i,j+1} m_1^{i,j} > 0,\\
\text{(A1d):} & m_8^{i,j} < 0, & \text{(A3d):} & m_6^{i,j} m_8^{i,j+1} - m_7^{i,j+1} m_1^{i,j} > 0.
\end{array}$$

In these criteria, $m_k^{i,j}$ are elements of the system matrix, where the superscripts refer to a logical cell numbering (i, j), starting in the lower left corner of the grid. The first and second indices refer to the grid column and row number, respectively. The subscripts refer to the relative position of a neighbor cell; see Fig. 1. As an example, if $k = 2$, then $m_k^{i,j}$ is the weight of cell $(i + 1, j)$ in the discrete form of the operator in Eq. (1) as evaluated in cell (i, j). Thus, $m_2^{i,j}$ will be placed in the matrix row corresponding to cell (i, j), and the column of cell number $(i + 1, j)$.

We note that a set of sufficient criteria analogous to the set A, denoted as B, is also given in [4]. The results given herein apply equally to these two sets of sufficient criteria.

In [4], it was shown that for a class of grids, it is impossible to construct control volume methods that satisfy all the monotonicity criteria. Here, we show that Criteria A0, A1 and A3 are not only sufficient, but also necessary, for ensuring the maximum principle. These are the strictest criteria for the case of uniform grid and homogeneous medium (see Section 5.2 of [4]), and thus, it is impossible to construct control volume methods that satisfy the maximum principle for all cases.

We also note that the Criteria A3 allow for positive contribution from the cells 3, 5, 7, and 9, referring to Fig. 1. Thus the conditions retain the property of the discrete maximum principle while being less restrictive than the classical M-matrix conditions.

Remark. Methods with more than nine cells in the stencil have more degrees of freedom, which can be utilized to obtain larger regions where the methods are monotone.

2. Most sufficient criteria are necessary

In this section, we go systematically through Criteria A0, A1 and A3 and show subdomains for which these criteria are necessary criteria for monotonicity. For all subdomains, zero Dirichlet boundary conditions are imposed in ghost cells.

2.1. Criterion A0

To show that Criterion A0 is necessary, consider a grid containing a single cell. Then the local matrix for this problem is

$$m_1^{i,j} u^{i,j} = q^{i,j} \quad \forall\, (i, j).$$

It follows that Criterion A0 is necessary for a discrete maximum principle.

2.2. Criteria A1

To show that Criterion A1b is necessary, consider a grid containing two cells, placed adjacent vertically. We refer to these cells as cell (i, j) and $(i, j+1)$. Let $q^{i,j} = 0$, while $q^{i,j+1}$ is positive, with a magnitude such that $u^{i,j+1} = 1$. Then a simple calculation shows that

$$u^{i,j} = -\frac{m_4^{i,j}}{m_1^{i,j}}.$$

Thus, if Criterion A0 holds, it is necessary for the maximum principle to hold for this subdomain that also Criterion A1b holds.

Analogous arguments show the necessity of the remaining Criteria A1.

2.3. Criteria A3

To show the necessity of Criterion A3c, consider a domain consisting of three cells and shaped like a capital L. We refer to these cells as cells (i, j), $(i+1, j)$ and $(i, j+1)$. Let $q^{i,j} = 0$, $q^{i,j+1} = 0$, while $q^{i+1,j}$ is positive, with a magnitude such that $u^{i+1,j} = 1$. Then

$$\begin{bmatrix} m_1^{i,j} & m_4^{i,j} \\ m_8^{i,j+1} & m_1^{i,j+1} \end{bmatrix} \begin{bmatrix} u^{i,j} \\ u^{i,j+1} \end{bmatrix} = -\begin{bmatrix} m_2^{i,j} \\ m_9^{i,j+1} \end{bmatrix}. \tag{2}$$

We invert the left hand side to obtain

$$\begin{bmatrix} u^{i,j} \\ u^{i,j+1} \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} m_1^{i,j+1} & -m_4^{i,j} \\ -m_8^{i,j+1} & m_1^{i,j} \end{bmatrix} \begin{bmatrix} m_2^{i,j} \\ m_9^{i,j+1} \end{bmatrix}, \tag{3}$$

where $D = m_1^{i,j} m_1^{i,j+1} - m_4^{i,j} m_8^{i,j+1}$. The positivity of the determinant D is immediate from Criteria A0 and A1. To ensure that $u^{i,j+1}$ is positive, Criterion A3c is necessary.

Criteria A3a, A3b and A3d are necessary by reflections of the domain considered above.

3. Implications

We list three major implications of the results from Section 2.

(1) It is known that no linear nine-point control volume method exact for linear fields and with a local flux expression can be constructed that satisfies Criteria A0 to A3 for general grids (see Section 5.2 of [4]). In that argument, Criterion A2 was trivial, and did not impose any constraints. It follows from this work that nine-point control volume methods cannot be constructed that satisfy a discrete maximum principle for all media and quadrilateral grids.

(2) Multiscale control volume methods (see e.g. [7]) lead to nine-point discretizations on the coarse scale. Hence, it also follows from this work that multiscale control volume methods cannot be constructed that satisfy a discrete maximum principle for the coarse scale solution for all media and quadrilateral grids.

(3) Our results further explain the discrepancy between the numerical and analytical monotonicity regions for homogeneous problems given in [4]. These appear as a consequence of the numerical monotonicity regions being generated by considering all $k \times l$ type subdomains, where k and l are integers. Thus, only convex subdomains were investigated numerically. We see that the strictness of Criteria A3 appears as a consequence of concave subdomains.

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