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Essential Components of the Set of Weakly Pareto-Nash Equilibrium Points

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Abstract—In this paper, we first give a generalization of Ky Fan's inequality to vector-valued functions. We prove that, for every vector-valued function (satisfying some continuity and convexity conditions), there exists at least one essential component of the set of its Ky Fan's points. As applications, we show that, for every multiobjective game (satisfying some continuity and convexity conditions), there exists at least one essential component of the set of its weakly Pareto-Nash equilibrium points.

Keywords—Ky Fan's point, Multiobjective game, Equilibrium point, Essential component.

1. INTRODUCTION AND PRELIMINARIES

Games with multiple noncommensurable criteria are called multicriteria games or games with vector payoffs. Blackwell's [1] was the first paper which investigated zero-sum games with vector payoffs as a generalization of the scalar criterion games. In 1959, Shapley [2] introduced the concept of equilibrium points in games with vector payoffs. Recently, much attention has been attracted to such multicriteria models, since they can be better applied to real-world situations. As stated in [2] by Shapley, the payoff of a game sometimes most naturally takes the form of a vector having numerical components that represent commodities (such as men, ship, money, etc.) whose relative values cannot be ascertained. (Note that in our paper, the term *component* is adopted both for vector space and for topological space.) Besides Shapley's notion, many other concepts of solutions for multicriteria games have been proposed and many results on existence of such solutions have been obtained, see [3–5] and references therein.

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In game theory, the stability and perfection of Nash equilibrium points have become important topics, see [6] and references therein. Among those works, Kohlberg and Mertens' work has been influential and significant, since they applied axiomatic methods to deal with such issues and put forward new concepts of strategic stability of Nash equilibrium points for finite noncooperative games (briefly, finite games). In 1986, Kohlberg and Mertens [7] proposed a list of requirements which satisfactory solution concepts for finite games should satisfy and they introduced hyper-stable set, fully stable set, and stable set of Nash equilibrium points, each of which satisfies most of the requirements. Besides, they proved that every finite game has finite components of its Nash equilibrium points and at least one component is essential in the sense that the component is robust to perturbations of the game's payoffs, i.e., every game nearby has a Nash equilibrium nearby. Hillas [8] also proposed other versions of strategic stability, i.e., he considered the subset of Nash equilibrium points robust against perturbations of the best response correspondence. On the other hand, in 1963, Jiang [9] has proved that every finite game has at least one essential component of its Nash equilibrium points. Jiang's work was motivated by [10] in which Kinoshita introduced the notion of essential components of fixed points and proved that for any continuous mapping of the Hilbert cube into itself, there exists at least one essential component of the set of its fixed points. In recent years, some existence results for essential components of the solution sets of nonlinear problems have been obtained, see [11–13]. As for applications, Wilson [14] gave an algorithm for computing essential components of Nash equilibrium points. Govindan and Wilson [15,16] studied intensively the properties of essential components of Nash equilibrium points for finite games.

The aim of this paper is to establish the existence of essential components of the set of weakly Pareto-Nash equilibrium points for multiobjective games. Our approach can be stated as follows. Recall that in 1972, Ky Fan [17] gave a minimax inequality for real-valued functions, which is fundamental in proving many existence theorems in nonlinear analysis. There have been numerous generalizations of Ky Fan's minimax inequality. For our purpose, we give a generalization of Ky Fan's inequality to vector-valued functions (Theorem 1.1). Then we introduce the notion of vector Ky Fan's points and of essential components of vector Ky Fan's points. We prove the existence of essential components of vector Ky Fan's points. As applications, we show that every multiobjective game (satisfying some continuity and convexity conditions) possesses at least one essential component of its weakly Pareto-Nash equilibrium points. Our results include corresponding results in the literature as special cases.

Let H be a real Banach space and C be a cone of H. A cone C is *convex* if and only if $C + C = C$, and *pointed* if and only if $C \cap (-C) = \{\theta\}$, where θ denotes the zero element of H. Let A be a subset of H, we denote $\operatorname{int} A$ the topological interior of A in H. Z^+ denotes the set of all positive integers. In the following context, E denotes a real Banach space and H denotes a real Banach space with a closed, convex, and pointed cone C with $\operatorname{int} C \neq \emptyset$.

DEFINITION 1.1. *Let X be a nonempty subset of E and $\varphi : X \times X \to H$ be a vector-valued function. An element $x^* \in X$ is called a vector Ky Fan's point of φ (briefly, Ky Fan's point of φ) if $\varphi(x^*, y) \notin \operatorname{int} C$ for all $y \in X$.*

If $H = R$ and $C = [0, +\infty)$, then Ky Fan's points of a vector-valued function reduce to the Ky Fan's points of a real-valued function, defined by Tan, Yu and Yuan in [18].

In order to establish the existence theorem of vector Ky Fan's points, we first recall some notions. The following two definitions can be found in [19].

DEFINITION 1.2. *Let X be a nonempty subset of E and $f : X \to H$ be a vector-valued function. f is said to be C-continuous at $x_0 \in X$ if, for any open neighborhood V of θ in H, there exists an open neighborhood U of x_0 in X such that, for all $x \in U$,*

$$f(x) \in f(x_0) + V + C,$$

and C-continuous on X if it is C-continuous at any point of X.

DEFINITION 1.3. *Let X be a nonempty convex subset of E and $f : X \to H$ be a vector-valued function. f is said to be C-concave if, for any $x_1, x_2 \in X$ and any $\lambda \in [0,1]$,*

$$f(\lambda x_1 + (1-\lambda)x_2) - (\lambda f(x_1) + (1-\lambda)f(x_2)) \in C,$$

and C-convex if $-f$ is C-concave.

LEMMA 1.1. *Let H be a Banach space with a closed, convex, and pointed cone C with $\operatorname{int} C \neq \emptyset$. Then we have $\operatorname{int} C + C \subset \operatorname{int} C$.*

PROOF. We only need to show that, for any $x \in \operatorname{int} C$ and any $y \in C, x + y \in \operatorname{int} C$. Since $x \in \operatorname{int} C$, there exists an open neighborhood V of θ in H, such that $x + V \subset C$. Since $y \in C, x + V \subset C$ and C is a convex cone of H, we have $y + x + V \subset C + C = C$. Hence, $y + x \in \operatorname{int} C$ and the proof is complete.

LEMMA 1.2. *If f is C-concave, then the set $D := \{x \in X : f(x) \in \operatorname{int} C\}$ is convex.*

PROOF. Let $x_1, x_2 \in D$, then $f(x_1) \in \operatorname{int} C$ and $f(x_2) \in \operatorname{int} C$. Since $\operatorname{int} C$ is convex, we have $\lambda f(x_1) + (1-\lambda) f(x_2) \in \operatorname{int} C$ for any $\lambda \in [0,1]$. It follows from the C-concavity of f that $f(\lambda x_1 + (1-\lambda)x_2) \in \lambda f(x_1) + (1-\lambda)f(x_2) + C$. By Lemma 1.1, $f(\lambda x_1 + (1-\lambda)x_2) \in \operatorname{int} C$, i.e., $\lambda x_1 + (1-\lambda)x_2 \in D$. Hence, D is convex and our proof is complete.

The following lemma is the well-known Ky Fan's Section Theorem, see [20].

LEMMA 1.3. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and A be a subset of $X \times X$ such that:*

(i) *for each $y \in X$, the set $\{x \in X : (x,y) \in A\}$ is closed in X;*
(ii) *for each $x \in X$, the set $\{y \in X : (x,y) \notin A\}$ is convex or empty; and*
(iii) *for each $x \in X$, $(x,x) \in A$.*

Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset A$.

Now we can give our existence theorem of Ky Fan's points of a vector-valued function.

THEOREM 1.1. *Let X be a nonempty convex compact subset of E. Suppose that $\varphi : X \times X \to H$ satisfies the following conditions:*

(i) *for each fixed $y \in X, x \to \varphi(x,y)$ is C-continuous;*
(ii) *for each fixed $x \in X, y \to \varphi(x,y)$ is C-concave; and*
(iii) *for each $x \in X, \varphi(x,x) \notin \operatorname{int} C$.*

Then there exists $x^* \in X$ such that $\varphi(x^*, y) \notin \operatorname{int} C$ for all $y \in X$.

PROOF. Consider the set

$$A = \{(y,x) \in X \times X : \varphi(x,y) \notin \operatorname{int} C\}.$$

By (iii), for any $x \in X$, we have $(x,x) \in A$. For each fixed $x \in X$, by (ii) and Lemma 1.2, the set

$$A(x) = \{y \in X : (y,x) \notin A\} = \{y \in X : \varphi(x,y) \in \operatorname{int} C\}$$

is convex. Furthermore, we show that, for any $y \in X$, the set

$$A(y) = \{x \in X : (x,y) \in A\} = \{x \in X : \varphi(x,y) \notin \operatorname{int} C\}$$

is closed. Indeed, let $\{x_n\}$ be any sequence in $A(y)$ with $x_n \to x \in X$. Suppose that $x \notin A(y)$, then $\varphi(x,y) \in \operatorname{int} C$. Since $\operatorname{int} C$ is open, there exists an open neighborhood V of θ such that $\varphi(x,y) + V \subset \operatorname{int} C$, and thus, by Lemma 1.1, $\varphi(x,y) + V + C \subset \operatorname{int} C + C \subset \operatorname{int} C$. By the C-continuity of $\varphi(\cdot, y)$, we have $\varphi(x_n, y) \in \varphi(x,y) + V + C \subset \operatorname{int} C$ for sufficiently large n, which contradicts that $\varphi(x_n, y) \notin \operatorname{int} C$. Hence, $x \in A(y)$.

By Ky Fan's Section Theorem (Lemma 1.3), there exists $x^* \in X$ such that $X \times \{x^*\} \subset A$, i.e., $\varphi(x^*, y) \notin \operatorname{int} C$ for all $y \in X$. The proof is finished.

2. ESSENTIAL COMPONENTS OF THE SET OF KY FAN'S POINTS

In this section, we establish the existence of essential components of Ky Fan's points of vector-valued functions. First, we recall some notions of continuity for set-valued mappings. Let X be a Hausdorff topological space and 2^X be the family of all nonempty subsets of X. Denote by $K(X)$ the collection of all nonempty compact subsets of X.

DEFINITION 2.1. *If X and Y are two Hausdorff topological spaces and $T : Y \to 2^X$ is a set-valued mapping, then*

(i) *T is upper semicontinuous at $y_0 \in Y$ if for each open set U in X with $U \supset T(y_0)$, there exists an open neighborhood $O(y_0)$ of y_0 such that $U \supset T(y)$ for any $y \in O(y_0)$;*
(ii) *T is upper semicontinuous on Y if T is upper semicontinuous at every point $y_0 \in Y$; and*
(iii) *T is an usco mapping if T is upper semicontinuous on Y and $T(y)$ is compact for every $y \in Y$.*

Now let X be a nonempty convex compact subset of a Banach space E and H be a Banach space with a closed, convex, and pointed cone C with $\operatorname{int} C \neq \emptyset$. Let M be the collection of all vector-valued functions $\varphi : X \times X \to H$ such that

(i) for each $y \in X$, $x \to \varphi(x, y)$ is C-continuous;
(ii) for each $x \in X$, $y \to \varphi(x, y)$ is C-concave;
(iii) for each $x \in X$, $\varphi(x, x) = \theta$; and
(iv) $\sup_{(x,y)\in X\times X} \|\varphi(x, y)\| < +\infty$.

For each $\varphi, \psi \in M$, define

$$\rho(\varphi, \psi) = \sup_{(x,y)\in X\times X} \|\varphi(x, y) - \psi(x, y)\|.$$

Clearly, (M, ρ) is a complete metric space. For each $\varphi \in M$, denote by $F(\varphi)$ the set of all Ky Fan's points of φ. Then F defines a set-valued mapping from M into X and, by Theorem 1.1, $F(\varphi) \neq \emptyset$ for any $\varphi \in M$.

LEMMA 2.1. *$F : M \to 2^X$ is an usco mapping.*

PROOF. Since X is compact, by Theorem 7.1.16 of [21], it suffices to show that F is a closed mapping, i.e., the graph Graph(F) of F is closed in $M \times X$, where

$$\operatorname{Graph}(F) = \{(\varphi, x) \in M \times X : x \in F(\varphi)\}.$$

Let $\{(\varphi_n, x_n)\}_{n\in Z^+}$ be any sequence in Graph(F) with $(\varphi_n, x_n) \to (\varphi, x^*) \in M \times X$. Then $\varphi_n(x_n, y) \notin \operatorname{int} C$ for all $n \in Z^+$ and all $y \in X$. Suppose that $(\varphi, x^*) \notin$ Graph(F), then there exists some $y^* \in X$ such that $\varphi(x^*, y^*) \in \operatorname{int} C$. Since $\operatorname{int} C$ is an open set, there exists an open neighborhood V of θ such that $\varphi(x^*, y^*) + V \subset \operatorname{int} C$. Since $\varphi(\cdot, y^*)$ is C-continuous and $x_n \to x^*$, there exists $N_0 \in Z^+$ such that, for any $n \geq N_0$,

$$\varphi(x_n, y^*) \in \varphi(x^*, y^*) + \frac{1}{2}V + C.$$

Moreover, since $\varphi_n \to \varphi$, there exists $N_1 \in Z^+$ with $N_1 \geq N_0$ such that, for any $n \geq N_1$, $\varphi_n(x, y) \in \varphi(x, y) + (1/2)V$ for all $(x, y) \in X \times X$. Hence, we have, for any $n \geq N_1$,

$$\varphi_n(x_n, y^*) \in \varphi(x_n, y^*) + \frac{1}{2}V \subset \varphi(x^*, y^*) + V + C \subset \operatorname{int} C + C \subset \operatorname{int} C,$$

which is a contradiction. Therefore, $(\varphi, x^*) \in$ Graph(F), and thus, Graph(F) is closed. The proof is complete.

For each $\varphi \in M$, the *component of a point* $x \in F(\varphi)$ is the union of all the connected subsets of $F(\varphi)$ containing x. Note that the components are connected closed subsets of $F(\varphi)$, and thus, are connected compact, see [22, p. 356]. It is easy to see that the components of two distinct points of $F(\varphi)$ either coincide or are disjoint, so that all components constitute a decomposition of $F(\varphi)$ into connected pairwise disjoint compact subsets, i.e.,

$$F(\varphi) = \bigcup_{\alpha \in \Lambda} F_\alpha(\varphi),$$

where Λ is an index set, for each $\alpha \in \Lambda$, $F_\alpha(\varphi)$ is a nonempty connected compact subset of $F(\varphi)$ and, for any $\alpha, \beta \in \Lambda (\alpha \neq \beta)$, $F_\alpha(\varphi) \cap F_\beta(\varphi) = \emptyset$.

DEFINITION 2.2. *Let $\varphi \in M$ and S be a nonempty closed subset of $F(\varphi)$. S is said to be an essential set of $F(\varphi)$ if, for each open set $O \supset S$, there exists $\delta > 0$ such that for any $\varphi' \in M$ with $\rho(\varphi, \varphi') < \delta, F(\varphi') \cap O \neq \emptyset$. If a component $F_\alpha(\varphi)$ of $F(\varphi)$ is an essential set, then $F_\alpha(\varphi)$ is said to be an essential component of $F(\varphi)$. An essential set S of $F(\varphi)$ is said to be a minimal essential set of $F(\varphi)$ if S is a minimal element of the family of essential sets in $F(\varphi)$ ordered by set inclusion.*

LEMMA 2.2. *For each $\varphi \in M$, there exists at least one minimal essential set of $F(\varphi)$.*

PROOF. By Lemma 2.1, the set-valued mapping $F : M \to K(X)$ is upper semicontinuous. It follows that $F(\varphi)$ is an essential set of itself. Denote by Φ the family of all essential sets of $F(\varphi)$ ordered by set inclusion relation. Then Φ is nonempty and every decreasing chain of elements in Φ has a lower bound as the intersection is still in Φ due to the compactness. By Zorn's Lemma, Φ has a minimal element S which is a minimal essential set of $F(\varphi)$.

LEMMA 2.3. *For each $\varphi \in M$, each minimal essential set in $F(\varphi)$ is connected.*

PROOF. By Lemma 2.2, there exists at least one minimal essential set of $F(\varphi)$. Let $m(\varphi)$ be a minimal essential set of $F(\varphi)$. Suppose otherwise that $m(\varphi)$ were not connected. Then there exist two nonempty closed subsets $c_1(\varphi)$ and $c_2(\varphi)$ of $F(\varphi)$ satisfying $m(\varphi) = c_1(\varphi) \cup c_2(\varphi)$ and two open sets V_1 and V_2 in X satisfying $V_1 \cap V_2 = \emptyset$ such that $V_1 \supset c_1(\varphi)$, $V_2 \supset c_2(\varphi)$. Since $m(\varphi)$ is minimal, neither $c_1(\varphi)$, nor $c_2(\varphi)$ is essential. Thus, there exist two open sets $O_1 \supset c_1(\varphi)$ and $O_2 \supset c_2(\varphi)$ such that, for any $\delta > 0$, there exists $\varphi_1, \varphi_2 \in M$ with $\rho(\varphi, \varphi_1) < \delta, \rho(\varphi, \varphi_2) < \delta$, but $F(\varphi_1) \cap O_1 = \emptyset$, $F(\varphi_2) \cap O_2 = \emptyset$. Letting $W_1 := V_1 \cap O_1, W_2 := V_2 \cap O_2$, then both W_1 and W_2 are open and $W_1 \supset c_1(\varphi)$ and $W_2 \supset c_2(\varphi)$. Since $c_1(\varphi)$ and $c_2(\varphi)$ are compact, there exist two open sets U_1 and U_2 such that $c_1(\varphi) \subset U_1 \subset \bar{U}_1 \subset W_1$, $c_2(\varphi) \subset U_2 \subset \bar{U}_2 \subset W_2$. Since $U_1 \cup U_2 \supset m(\varphi)$ and $m(\varphi)$ is essential, there exists $\delta' > 0$ such that, for each $\varphi' \in M$ with $\rho(\varphi, \varphi') < \delta'$, $F(\varphi') \cap (U_1 \cup U_2) \neq \emptyset$. Moreover, since $U_1 \subset O_1$ and $U_2 \subset O_2$, there exist ψ_1, $\psi_2 \in M$ with $\rho(\varphi, \psi_1) < (1/3)\delta', \rho(\varphi, \psi_2) < (1/3)\delta'$, but $F(\psi_1) \cap U_1 = \emptyset$, $F(\psi_2) \cap U_2 = \emptyset$.

Now we define a function $\psi : X \times X \to H$ by

$$\psi(x, y) = \lambda(x)\psi_1(x, y) + \mu(x)\psi_2(x, y), \qquad \forall (x, y) \in X \times X,$$

where

$$\lambda(x) = \frac{d(x, \bar{U}_2)}{d(x, \bar{U}_1) + d(x, \bar{U}_2)}, \qquad \mu(x) = \frac{d(x, \bar{U}_1)}{d(x, \bar{U}_1) + d(x, \bar{U}_2)}.$$

Note that $\lambda(x)$ and $\mu(x)$ are continuous, $\lambda(x) \geq 0$, $\mu(x) \geq 0$, and $\lambda(x) + \mu(x) = 1$ for any $x \in X$.

It can be checked that

(1) for each fixed $y \in X$, $x \to \psi(x, y)$ is C-continuous;
(2) for each fixed $x \in X$, $y \to \psi(x, y)$ is C-concave;
(3) for each $x \in X$, $\psi(x, x) = \theta$; and
(4) $\sup_{(x,y) \in X \times X} \|\psi(x, y)\| < +\infty$.

Hence, $\psi \in M$. Note that

$$\rho(\varphi, \psi) \leq \rho(\varphi, \psi_1) + \rho(\psi_1, \psi) < \delta',$$

and thus, $F(\psi) \cap (U_1 \cup U_2) \neq \emptyset$. Without loss of generality, we may assume that $F(\psi) \cap U_1 \neq \emptyset$, i.e., there exists $x^* \in F(\psi) \cap U_1$. Since $x^* \in U_1$, we have $\lambda(x^*) = 1$, $\mu(x^*) = 0$, and thus, $\psi(x^*, y) = \psi_1(x^*, y)$. Therefore, $\psi_1(x^*, y) \notin \operatorname{int} C$ for all $y \in X$ since $x^* \in F(\psi)$, i.e., $x^* \in F(\psi_1)$, which contradicts that $F(\psi_1) \cap U_1 = \emptyset$. Hence, $m(\varphi)$ must be connected.

THEOREM 2.1. *For each $\varphi \in M$, there exists at least one essential component of $F(\varphi)$.*

PROOF. By Lemmas 2.2 and 2.3, there exists at least one minimal essential connected subset S of $F(\varphi)$. By Theorem 3.2 of [23, p. 112], there is a component $F_\alpha(\varphi)$ of $F(\varphi)$ such that $S \subset F_\alpha(\varphi)$. It is obvious that $F_\alpha(\varphi)$ is essential.

3. ESSENTIAL COMPONENTS OF THE SET OF WEAKLY PARETO-NASH EQUILIBRIUM POINTS

A finite-player noncooperative multiobjective game in its strategic form (also called a normal form) $G := (X_i, F^i)_{i\in N}$, where $N := \{1, 2, \ldots, n\}$, is defined as follows: for each $i \in N$, X_i is the set of player i's strategies; each F^i, the payoff of player i, is a vector-valued function from $X := \prod_{i\in N} X_i$ into R^{k_i}, where k_i is a positive integer.

For each positive integer m, denote

$$R^m_+ := \left\{ (u^1, \ldots, u^m) \in R^m : u^j \geq 0, \text{ for all } j = 1, \ldots, m \right\},$$

and

$$\operatorname{int} R^m_+ := \left\{ (u^1, \ldots, u^m) \in R^m : u^j > 0, \text{ for all } j = 1, \ldots, m \right\}.$$

For R^m, we take the norm $\|r\| = \sum_{i=1}^m |r_i|$ where $r = (r_1, \ldots, r_m) \in R^m$. For each $i \in N$, denote $\hat{i} = N \backslash \{i\}$, $X_{\hat{i}} = \prod_{j \in N\backslash\{i\}} X_j$, $x_{\hat{i}} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{\hat{i}}$, and $x := (x_i, x_{\hat{i}}) \in X$. Without loss of generality, we assume that $k_1 \leq k_2 \leq \cdots \leq k_n$.

DEFINITION 3.1. *A strategy profile $x^* \in X$ is called a weakly Pareto-Nash equilibrium point of a multiobjective game $G := (X_i, F^i)_{i\in N}$ if, for each $i \in N$,*

$$F^i\left(y_i, x^*_{\hat{i}}\right) - F^i\left(x^*_i, x^*_{\hat{i}}\right) \notin \operatorname{int} R^{k_i}_+, \qquad \text{for all } y_i \in X_i.$$

If $k_i = 1$, for all $i = 1, \ldots, n$, then the noncooperative multiobjective game $G := (X_i, F^i)_{i\in N}$ is just a noncooperative game in the literature and the weakly Pareto-Nash equilibrium points reduce to Nash equilibrium points of usual noncooperative games.

In order to prove our existence theorem of weakly Pareto-Nash equilibrium points for multiobjective games, we give the following lemmas which can be proved easily, and hence, the proofs are omitted.

LEMMA 3.1. *Let $f : X \to R^m$ be a vector-valued function, where $f = (f_1, \ldots, f_m)$. Then f is R^m_+-continuous if and only if f_i is lower semicontinuous for every $i = 1, \ldots, m$.*

LEMMA 3.2. *Let X be a convex subset of a normed space and $f : X \to R^m$ be a vector function, where $f = (f_1, \ldots, f_m)$. Then f is R^m_+-concave if and only if f_i is concave for every $i = 1, \ldots, m$.*

By applying Theorem 1.1, we obtain the following existence theorem.

THEOREM 3.1. *Let $G := (X_i, F^i)_{i\in N}$ be a multiobjective game, where $F^i = (f^i_1, \ldots, f^i_{k_i})$. Suppose that G satisfies the following conditions:*

(i) *for each $i \in N, X_i$ is a nonempty convex compact subset of a normed space E_i;*
(ii) *for each $i \in N$ and each $j = 1, \ldots, k_i$, f^i_j is upper semicontinuous on X;*
(iii) *for fixed $i \in N$ and each fixed $x_i \in X_i, u_{\hat{i}} \to f^i_j(x_i, u_{\hat{i}})$ is lower semicontinuous on $X_{\hat{i}}$ for every $j = 1, \ldots, k_i$; and*
(iv) *for fixed $i \in N$ and each fixed $x_{\hat{i}} \in X_{\hat{i}}, u_i \to f^i_j(u_i, x_{\hat{i}})$ is concave for every $j = 1, \ldots, k_i$.*

Then there exists a weakly Pareto-Nash equilibrium point x^* of G.

PROOF. Define the vector-valued function $\varphi : X \times X \to R^{k_n}$ by

$$\varphi(x,y) = \sum_{i=1}^{n} \varphi_i(x,y),$$

where

$$\varphi_i(x,y) = (\underbrace{F^i(y_i,x_{\hat{i}}) - F^i(x_i,x_{\hat{i}})}_{k_i \text{ components}}; \underbrace{f_1^i(y_i,x_{\hat{i}}) - f_1^i(x_i,x_{\hat{i}}), \dots, f_1^i(y_i,x_{\hat{i}}) - f_1^i(x_i,x_{\hat{i}})}_{k_n - k_i \text{ components}}) \in R^{k_n}.$$

It is easy to check that

(1) for each fixed $y \in X, x \to \varphi(x,y)$ is $R_+^{k_n}$-continuous (by Lemma 3.1);
(2) for each fixed $x \in X, y \to \varphi(x,y)$ is $R_+^{k_n}$-concave (by Lemma 3.2); and
(3) for each $x \in X, \varphi(x,x) = 0 \notin \operatorname{int} R_+^{k_n}$.

Therefore, by Theorem 1.1, there exists $x^* \in X$ such that $\varphi(x^*,y) \notin \operatorname{int} R_+^{k_n}$ for all $y \in X$. For each $i \in N$ and each $y_i \in X_i$, set $y = (y_i, x_{\hat{i}}^*) \in X$, then $\varphi_i(x^*,y) = \varphi(x^*,y) \notin \operatorname{int} R_+^{k_n}$. If $F^i(y_i,x_{\hat{i}}^*) - F^i(x_i^*,x_{\hat{i}}^*) \in \operatorname{int} R_+^{k_i}$, then $f_j^i(y_i,x_{\hat{i}}^*) - f_j^i(x_i^*,x_{\hat{i}}^*) \in \operatorname{int} R_+$ for each $j = 1,\dots,k_i$ and $\varphi_i(x^*,y) \in \operatorname{int} R_+^{k_n}$ which contradicts that $\varphi_i(x^*,y) \notin \operatorname{int} R_+^{k_n}$. Hence, $F^i(y_i,x_{\hat{i}}^*) - F^i(x_i^*,x_{\hat{i}}^*) \notin \operatorname{int} R_+^{k_i}$ for each $i \in N$, i.e., x^* is a weakly Pareto-Nash equilibrium point of the game G. The proof is complete.

Now for each $i \in N$, let X_i be a nonempty convex compact subset of a normed space E_i and Z be the collection of all vector functions $z = (F^1,\dots,F^n)$ such that Conditions (ii), (iii), (iv) in Theorem 3.1 hold and (iv) $\sup_{x\in X} \sum_{i=1}^n \|F^i(x)\| < +\infty$. For any $z_1 = (F_1^1,\dots,F_1^n)$, $z_2 = (F_2^1,\dots,F_2^n) \in Z$, define

$$h(z_1,z_2) = \sup_{x\in X} \sum_{i=1}^{n} \left\|F_1^i(x) - F_2^i(x)\right\|.$$

Clearly (Z,h) is a complete metric space. Every $z \in Z$ determines a multiobjective game, denoted also by z. Denote by $E(z)$ the set of all weakly Pareto-Nash equilibrium points of the game z. By Theorem 3.1, for each $z \in Z$, $E(z) \neq \emptyset$. Similar to Definition 2.2, we can define the minimal essential set and essential component of $E(z)$. In order to establish the existence of essential components of $E(z)$, we need the following lemma.

LEMMA 3.3. *Let $(M,\rho), Y$ and (Z,h) be three metric spaces, $F : M \to 2^Y$ be an usco mapping and $G : Z \to 2^Y$ be a set-valued mapping. Suppose that there exists a continuous mapping $T : Z \to M$ such that $G(z) \supset F(T(z))$ for each $z \in Z$. Suppose furthermore that there exists at least one essential component of $F(\varphi)$ for each $\varphi \in M$. Then there exists at least one essential component of $G(z)$ for each $z \in Z$.*

PROOF. For any $z \in Z$, $\varphi = T(z) \in M$, $F(\varphi) = \bigcup_{\alpha\in\Lambda} c_\alpha(\varphi)$, and $c_\alpha(\varphi)$ is a component of $F(\varphi)$ for each $\alpha \in \Lambda$. Suppose that $c_{\alpha_0}(\varphi)$ is an essential component of $F(\varphi)$ for some $\alpha_0 \in \Lambda$. Since $G(z) \supset F(\varphi) = \bigcup_{\alpha\in\Lambda} c_\alpha(\varphi), c_{\alpha_0}(\varphi)$ is contained in a component S_β of $G(z)$. Then S_β is an essential component of $G(z)$. Indeed, for any open set $O \supset S_\beta$, clearly we have $O \supset c_{\alpha_0}(\varphi)$, and thus, there exists $\delta > 0$ such that, for any $\psi \in M$ with $\rho(\varphi,\psi) < \delta, F(\psi) \cap O \neq \emptyset$. Since T is continuous, there exists $\delta' > 0$ such that for any $w \in Z$ with $h(z,w) < \delta'$, $\rho(\varphi,T(w)) = \rho(T(z),T(w)) < \delta$. Hence, $F(T(w)) \cap O \neq \emptyset$. Finally, since $G(w) \supset F(T(w))$, we have $G(w) \cap O \neq \emptyset$ and our proof is complete.

THEOREM 3.2. *For each $z \in Z$, there exists at least one essential component of $E(z)$.*

PROOF. For any $z \in Z$, define $T : Z \to M$ by $T(z) = \varphi$, where $\varphi(x,y) = \sum_{i=1}^n \varphi_i(x,y)$ and, for each $i, \varphi_i(x,y)$ is the same as that in Theorem 3.1. Then T is continuous. Indeed, for any $\varepsilon > 0$, there exists $\delta = (1/2k_n)\varepsilon > 0$ such that for any $w \in Z$ with $h(z,w) < \delta$,

$$\rho(T(z),T(w)) = \sup_{(x,y)\in X\times Y} \|\varphi(x,y) - \psi(x,y)\| \leq 2k_n h(z,w) = \varepsilon,$$

where $\psi = T(w)$. The same argument as in the proof of Theorem 3.1 states that $E(z) \supset F(T(z))$. Thus, by Theorem 2.1 and Lemma 3.3, there exists at least one essential component of $E(z)$.

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