Galois theory and a general notion of central extension

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Abstract

We propose a theory of central extensions for universal algebras, and more generally for objects in an exact category \( \mathcal{C} \), centrality being defined relatively to an “admissible” full subcategory \( \mathcal{X} \) of \( \mathcal{C} \). This includes not only the classical notions of central extensions for groups and for algebras, but also their generalization by Fröhlich to a pair consisting of a variety \( \Omega \) of \( \Omega \)-groups and a subvariety \( \mathcal{X} \). Our notion of central extension is adapted to the generalized Galois theory developed by the first author, the use of which enables us to classify completely the central extensions of a given object \( B \), in terms of the actions of an “internal Galois pregoupoid”.

1. Introduction

1.1. Exact categories are those finitely-complete ones in which the surjections (the morphisms that factorize through no proper subobject of the codomain) are well-behaved in the following sense: (i) every morphism \( f \) factorizes as \( i p \) where \( i \) is a monomorphism and \( p \) is a surjection, (ii) every pullback of a surjection is a surjection, and (iii) every equivalence relation on an object \( A \) is the kernel-congruence of some surjection \( p: A \to B \). We recall in Section 2 the basic facts about exact categories, as well as some recent results of Carboni, Kelly, and Pedicchio [1] on those exact categories possessing the Maltsev property (that is, the permutability \( RS = SR \) of congruences on any object \( A \)) or the weaker Goursat property (the condition \( RSR = SRS \) for congruences).

Every variety, in the sense of universal algebra, is an exact category, the monomorphisms and the surjections in which are just the injective and the surjective

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homomorphisms. As is well known, a variety has the Maltsev property (or “is Maltsev”) precisely when its theory contains a ternary operation $m$ satisfying $mxx = y$ and $mxyy = x$—such as the operation $xy^{-1}z$ in the theory of groups; and there is a similar characterization, in terms of pairs of ternary operations, of the varieties which are Goursat—see, for instance, [1, Section 1]. Certainly all varieties in which part of the structure on an object is that of a group are Maltsev and a fortiori Goursat.

For an object $B$ in the exact category $\mathcal{C}$, we have the slice category $\mathcal{C}/B$, an object of which is a morphism $f: A \to B$ in $\mathcal{C}$, and a morphism $f \to f'$ in which is a morphism $g: A \to A'$ in $\mathcal{C}$ with $f'g = f$. MacLane [12] uses the notation $\mathcal{C} \downarrow B$ for $\mathcal{C}/B$: we however shall write $\mathcal{C} \downarrow B$ instead for the full subcategory of $\mathcal{C}/B$ whose objects are the extensions of $B$; by which we mean the surjections $f: A \to B$. (For greater clarity, such an extension will often be denoted by $(A, f)$, rather than just $f$.) So $\mathcal{C} \downarrow B$, which we might call Ext B when $\mathcal{C}$ is understood, is the category of extensions of $B$.

1.2.

It is, however, often easier to study a full subcategory Centr $B$ of Ext $B$, called the category of central extensions of $B$; and this is our concern in the present paper. We define “central extension” in a context much wider than any considered in the past, and furthermore show how to describe Centr $B$ using the generalized Galois theory developed by the first author in [3], [4], and [5].

In fact, our notion of centrality for an extension $f: A \to B$ in $\mathcal{C}$ is not absolute, but depends on the choice of a full replete subcategory $\mathcal{X}$ of $\mathcal{C}$; different choices for $\mathcal{X}$ may give different notions of centrality. This $\mathcal{X}$ is required to have certain properties. First, it is to be a reflective subcategory of $\mathcal{C}$, closed in $\mathcal{C}$ under subobjects and quotient objects; call such an $\mathcal{X}$, by analogy with the case of varieties, a Birkhoff subcategory of $\mathcal{C}$. The largest Birkhoff subcategory of $\mathcal{C}$ is $\mathcal{C}$ itself; the smallest is Sub 1, the full subcategory determined by those $X$ in $\mathcal{C}$ for which the unique morphism $X \to 1$ into the terminal object is monomorphic. When $\mathcal{C}$ is a variety, a Birkhoff subcategory $\mathcal{X}$ is of course the same thing as a subvariety.

Suppose now that $\mathcal{X}$ is a Birkhoff subcategory of $\mathcal{C}$ and let $I: \mathcal{C} \to \mathcal{X}$ be the left adjoint of the inclusion $H: \mathcal{X} \to \mathcal{C}$. It is easy to see that $I$ sends surjections to surjections, thus inducing for each $B$ in $\mathcal{C}$ a functor $I^B: \mathcal{C} \downarrow B \to \mathcal{X} \downarrow IB$, which has a right adjoint $H^B: \mathcal{X} \downarrow IB \to \mathcal{C} \downarrow B$. The Birkhoff subcategory $\mathcal{X}$ is said to be admissible if each $H^B$ is fully faithful: it is to each admissible $\mathcal{X}$ that we attach a notion of centrality for extensions. We study admissibility in Section 3, observing that the two extreme Birkhoff subcategories $\mathcal{C}$ itself and Sub 1 are always admissible, and showing in Theorem 3.4 that every Birkhoff $\mathcal{X}$ is admissible when $\mathcal{C}$ is Goursat—or more generally when $\mathcal{C}$ is such that the lattice of congruences on each object is modular. So a non-varietal example is given by taking for $\mathcal{C}$ the dual $\text{Ab}^{\text{op}}$ of the category of abelian groups, which is abelian and hence Maltsev, and taking for $\mathcal{X}$ the dual $\text{Tor}^{\text{op}}$ of the torsion groups. In contrast, a subvariety of a variety need not be admissible.
1.3.

In Section 4 we introduce and study central extensions in \( \mathcal{C} \) with respect to an admissible \( \mathcal{X} \). First, we call an extension \( f: A \to B \) trivial (with respect to \( \mathcal{X} \)) if it lies in the image of the fully-faithful \( H^B: \mathcal{X} \downarrow IB \to \mathcal{C} \downarrow B \); the idea is that, if \( \mathcal{X} \) is thought of as simpler than \( \mathcal{C} \) or better known, these trivial extensions in \( \mathcal{C} \) are “really nothing more” than extensions in \( \mathcal{X} \). Clearly every extension is trivial if \( \mathcal{X} \) is \( \mathbf{1} \)-and in fact only then, as we see on taking \( B = \mathbf{1} \) and observing that \( \mathbf{1} \), being a quotient of 1, is itself 1; at the other extreme, if \( \mathcal{X} \) is \( \mathbf{Sub} \mathbf{1} \) (and, once again, only then) there are no trivial extensions other than the isomorphisms \( f: A \to B \). The trivial extensions of \( B \) form a full subcategory \( \text{Triv}_B \), or \( \text{Triv} B \) for short, of \( \text{Ext} B = \mathcal{C} \downarrow B \). Note that pulling back extensions of \( B \) along a morphism \( g: B' \to B \) gives a functor \( g^*: \mathcal{C} \downarrow B \to \mathcal{C} \downarrow B' \); we show that this takes trivial extensions of \( B \) to trivial extensions of \( B' \).

Consider now a surjection \( p: E \to B \); that is, an extension \( (E, p) \) of \( B \). The extension \( f: A \to B \) of \( B \) is said to be \((E, p)\)-split (with respect to \( \mathcal{X} \)) when \( p^*(A, f) \) is a trivial extension of \( E \). Such extensions of \( B \) form a full subcategory \( \text{Spl}(E, p) \), or \( \text{Spl}(E, p) \) for short, of \( \mathcal{C} \downarrow B \), containing \( \text{Triv} B \). It follows from the remark above about \( g^* \) that \( \text{Spl}(E, p) \subseteq \text{Spl}(E', p') \) whenever there is a map \( g: (E', p') \to (E, p) \) in \( \mathcal{C} \downarrow B \). If there is a surjection \( \bar{p}: \bar{E} \to B \) with \( \bar{E} \) projective (with respect to surjections), we have \( \text{Spl}(E, p) \subseteq \text{Spl}(\bar{E}, \bar{p}) \) for all \( (E, p) \); for example, when \( \mathcal{C} \) is a variety, we may take for \( \bar{E} \) the free algebra on the underlying set of \( B \).

We shall call an extension \( f: A \to B \) central (with respect to \( \mathcal{X} \)) if it belongs to \( \text{Spl}(E, p) \) for some extension \( p: E \to B \). The central extensions form a full subcategory \( \text{Centr}_B \), or \( \text{Centr} B \) for short, of \( \mathcal{C} \downarrow B \), containing \( \text{Triv} B \); it is the union of all the \( \text{Spl}(E, p) \), and is \( \text{Spl}(\bar{E}, \bar{p}) \) if a projective extension \( (\bar{E}, \bar{p}) \) exists as above. It follows easily that the \( g^*: \mathcal{C} \downarrow B \to \mathcal{C} \downarrow B' \) induced by \( g: B' \to B \) takes central extensions of \( B \) to central extensions of \( B' \).

There are close analogies between, on the one hand, central extensions in our sense and, on the other, covering spaces in algebraic topology, étale coverings in algebraic geometry, and in particular separable algebras in classical Galois theory—all of which in turn (see [3–5]) are cases of coverings (that is, locally-constant objects) in a topos. It would be inappropriate to pursue here the details of these analogies, many of which are made clear by a perusal of the articles just cited; we note only the consequence that a central extension \( f: A \to B \) might, with equal propriety, be called a covering of \( B \). At the same time, the classical central extensions of groups or of algebras, and more generally those extensions of \( \Omega \)-groups that are central in the sense of Fröhlich [2] and Luc [10], are—as we shall show in Section 5—precisely the central extensions in our sense for an appropriate variety \( \mathcal{C} \) and subvariety \( \mathcal{X} \).

This last being so, the classical case of group extensions, where \( \mathcal{C} \) is the variety of groups and \( \mathcal{X} \) that of abelian groups, shows that the inclusions \( \text{Triv} B \subseteq \text{Centr} B \subseteq \text{Ext} B = \mathcal{C} \downarrow B \) are proper in general. The equality \( \text{Triv} B = \mathcal{C} \downarrow B \) occurs, as we have seen, only when \( \mathcal{X} = \mathcal{C} \). There are other cases, however, in which
every extension is central; it follows from classical Galois theory that this is the case when \( \mathcal{C} \) is the variety \( G-\text{Set} \) of sets on which a group \( G \) acts, and \( \mathcal{X} \) is the subvariety \( \text{Set} \) given by the objects with trivial action. In the extreme case when \( \mathcal{X} \) is \( \text{Sub} 1 \), every central extension is trivial, and is just an isomorphism. There are also other cases—of limited practical interest, of course—in which every central extension is trivial: we provide some analysis of this possibility.

It much more commonly happens that every central extension that is a retraction is trivial. It is easy to see that this is so precisely when every central extension is normal: here, the extension \((A, f)\) of \( B \) is said to be normal if \((A, f) \in \text{Spl}(A, f)\)—that is, if \( f^*(A, f) \) is trivial. This corresponds to the use of "normal" in classical Galois theory, where "normal extension" coincides with "Galois extension". It follows that, in the case where \((\mathcal{C}, \mathcal{X}) = (G-\text{Set}, \text{Set})\) and \( G \) has a non-normal subgroup, not all central extensions are normal. We show in Theorem 4.8, however, that all central extensions are normal when \( \mathcal{C} \) is Goursat.

1.4.

We show in Section 5 that our notion of central extension includes, as a special case, that of Fröhlich for \( \Omega \)-groups. Recall that by a variety of \( \Omega \)-groups is meant a variety \( \mathcal{C} \) among whose operations and identities are those of the theory of groups, and each of whose \( n \)-ary operations \( \omega \in \Omega \) satisfies the "idempotence" identity \( \omega(e, e, \ldots, e) = e \), where \( e \) is the unit for the group structure. In such a variety the kernel-congruence of \( f: A \to B \) is determined by the classical kernel \( K = f^{-1}(e) \) of \( f \), since \( f\alpha = f\alpha' \) if and only if \( \alpha'\alpha^{-1} \in K \). Because of the idempotence identities above this subgroup \( K \) of \( A \) is a subalgebra; the subalgebras of \( A \) that arise thus are called \( \Omega \)-ideals.

Now let \( \mathcal{X} \) be a subvariety of a variety \( \mathcal{C} \) of \( \Omega \)-groups, write \( \eta_A: A \to IA \) for the unit of the reflection of \( \mathcal{C} \) onto \( \mathcal{X} \), and write \( RA \) for the \( \Omega \)-ideal of \( A \) which is the kernel of \( \eta_A \); note that \( R \) is functorial, each \( x: C \to A \) in \( \mathcal{C} \) restricting to an \( Rx: RC \to RA \). Since \( \mathcal{C} \) is Maltsev, \( \mathcal{X} \) is, by the remarks in Section 1.2, an admissible subcategory of \( \mathcal{C} \). An extension \( f: A \to B \) is said by Fröhlich to be central (see [10]) if, for any \( C \) and any morphisms \( x, y: C \to A \), we have \( Rx = Ry \) whenever \( fx = fy \). We show in Theorem 5.2 that this coincides with centrality in our sense, for this \( \mathcal{C} \) and \( \mathcal{X} \).

In turn, Fröhlich's notion of central extension for \( \Omega \)-groups includes many classical notions as special cases; we recall some important ones. For instance, when \( \mathcal{C} \) is the variety of groups and \( \mathcal{X} \) is that of abelian groups, the extension \( f: A \to B \) is central if and only if its kernel \( K \) lies in the centre of \( A \)—so that we are dealing with central extensions of groups in the usual sense. By an entirely similar proof, when \( \mathcal{C} \) is a variety of algebras and \( \mathcal{X} \) is the subvariety given by the commutative ones, the extension \( f: A \to B \) with kernel \( K \) is central precisely when \( K \) lies in the centre of \( A \)—that is, when \( ka = ak \) for all \( k \in K \) and \( a \in A \). Again, when \( \mathcal{C} \) is a variety of algebras and \( \mathcal{X} \) is the subvariety given by those in which every product \( xy \) is 0, the extension \( f: A \to B \) with kernel \( K \) is central precisely when we have \( ka = ak = 0 \) for all \( k \in K \) and \( a \in A \); an important example is that of Lie algebras.
In these classical cases, centrality of an extension may be expressed as above in terms of elements and identities. The same is true for any pair \((\mathcal{C}, \mathcal{X})\) of varieties—but in general much less simply. There is, however, a notable simplification when the variety \(\mathcal{C}\) is Maltsev, which we give in Theorem 5.5.

1.5.

We turn finally in Section 6 to the matter of classifying the central extensions of \(B\)—in the strong sense of describing the category \(\text{Centr } B\), to within equivalence, in terms that might be called algebraic.

We first recall something of what has hitherto been known. In such classical cases as the central extensions of groups, of associative algebras, and of Lie algebras, there is an appropriate cohomology theory that provides detailed information. For example, when \(B\) is a group and \(K\) is an abelian group, the set of isomorphism classes of central extensions of \(B\) with the kernel \(K\) is isomorphic—see, for instance, [11, Chapter IV, Theorem 4.1]—to \(H^2(B, K)\), where \(K\) is seen as a \(B\)-module with trivial action. In fact the usual proof of this shows that the category of central extensions of \(B\) with kernel \(K\) is equivalent to the category \(Z^2(B, K)\) of 2-cocycles \(B \to K\); and the naturality in \(K\) of these equivalences allows us to organize these categories of cocycles into a description of the category \(\text{Centr } B\), the \(Z^2(B, K)\) being the fibres of a functor \(\text{Centr } B \to \text{Ab}\) sending the extension \(f' : A \to B\) to its kernel \(K\).

Recall that \(H_1(B, \mathbb{Z})\), where \(\mathbb{Z}\) is the group of integers with trivial \(B\)-action, is the reflexion \(B/[B, B]\) into \(\text{Ab}\) of the group \(B\), and is trivial precisely when \(B\) is perfect. Since the sequence

\[
0 \to \text{Ext}(H_1 B, K) \to H^2(B, K) \to \text{Hom}(H_2 B, K) \to 0,
\]

wherein \(H_1 B = H_1(B, \mathbb{Z})\) and \(H_2 B = H_2(B, \mathbb{Z})\), is exact by the universal coefficient formula, we have for a perfect \(B\) the isomorphism \(H^2(B, K) \cong \text{Hom}(H_2 B, K)\). When \(K\) here is allowed to vary, one finds that \(\text{Centr } B\) is equivalent to the “co-slice category” \(H_2 B/\text{Ab}\), an object of which is an abelian group \(K\) and a morphism \(v : H_2 B \to K\).

There is another way of looking at this. When the group \(B\) is perfect, it admits a universal central extension \(w : \hat{B} \to B\); that is, a central extension of \(B\) such that, if \(f : A \to B\) is any other, there is a unique \(u : \hat{B} \to A\) with \(fu = w\). In fact, if \(0 \to H \to F \to B \to 0\) is a free resolution of \(B\), we have \(\hat{B} = [F, F]/[H, F]\), and the kernel of \(w\) is \((H \cap [F, F])/[H, F]\), which is Hopf’s formula for the Schur multiplier and hence for \(H_2 B\). The equivalence \(\text{Centr } B \simeq H_2 B/\text{Ab}\) follows easily, the \(v : H_2 B \to K\) of the last paragraph being the restriction to the kernels of the \(u : \hat{B} \to A\) above.

This generalizes to the case of a variety \(\mathcal{C}\) of \(\Omega\)-groups and a subvariety \(\mathcal{X}\), as in Section 1.4. Now the object \(B\) of \(\mathcal{C}\) is called perfect if \(RB = B\), so that \(IB\) is trivial; and Fröhlich [2] proved the existence for a perfect \(B\) of a universal central extension \(w : \hat{B} \to B\). Writing \(\pi_1 B\) for the kernel of \(w\), we find that \(\text{Centr } B \simeq \pi_1 B/\mathcal{X}\).
The first to go beyond the case of a perfect $B$ in describing $\text{Centr } B$ for $\Omega$-groups was Janelidze [3], who used his generalized Galois theory to deal with the “next after perfect” case where $RRB = RB$. His Galois theory was generalized further still in [4] and then in [5]; the former of these is strong enough to describe $\text{Centr } B$ for all $B$ in the $\Omega$-groups context, although it contains the details only for group extensions; the latter, at the cost of having in general not a Galois groupoid but a “Galois pregroupoid”, gives us a corresponding result in the full generality of an exact $\mathcal{C}$ and an admissible $\mathcal{X}$ as above. This result has the following form.

To each extension $p: E \rightarrow B$ there is associated an internal pregroupoid in $\mathcal{X}$, the Galois pregroupoid $\Gal(E, p)$ of the extension; it is an internal groupoid whenever the extension $(E, p)$ is normal. The results of [5] give an equivalence of categories $\text{Spl}(E, p) \simeq \{\Gal(E, p), \mathcal{X}\}$, where the right side is a certain full subcategory (described in Section 6.1 below) of the category of internal actions of $\Gal(E, p)$. So, whenever there is an extension $p: E \rightarrow B$ with $E$ projective, we have $\text{Centr } B \simeq \{\Gal(E, p), \mathcal{X}\}$.

Note that this does indeed contain the result $\text{Centr } B \simeq H_2 B/\text{Ab}$ for the central extensions of a perfect group $B$. For every central extension is split over the universal central extension $w: \tilde{B} \rightarrow B$, and $\Gal(\tilde{B}, w)$ is simply the abelian group $H_2 B$ which is the kernel of $w$. Thus the general result gives $\text{Centr } B \simeq \{H_2 B, \text{Ab}\}$, which is easily seen to be just $H_2 B/\text{Ab}$.

2. Revision of exact categories, and the Maltsev and Goursat properties

2.1.

We recall from the recent paper [1] of Carboni, Kelly, and Pedicchio the basic facts about exact categories, and the results we need on the Maltsev and Goursat properties—we omit almost all the proofs, which are to be found in [1].

Consider a category $\mathcal{C}$ with finite limits. Recall that a subobject of $A \in \mathcal{C}$ is an isomorphism-class of monomorphisms with codomain $A$, and that these subobjects form an ordered set $\text{Sub } A$ with finite meets. A morphism $f: A \rightarrow B$ is a surjection if it factorizes through no proper subobject of $B$—that is, if a monomorphism $i$ with $f = ig$ is necessarily invertible. Every surjection is an epimorphism, although the converse is false in general. A quotient object of $A$ is an isomorphism-class of surjections with domain $A$; these form an ordered set $\text{Quot } A$, in which $[s] \leq [r]$ if $s = tr$ for some $t$. Surjections compose; $p$ is a surjection if some $pf$ is so; and every pushout of a surjection is a surjection.

The category $\mathcal{C}$ is said to admit images (or equally to admit factorizations) if every morphism $f: A \rightarrow B$ can be written as $f = ip$ with $i$ monomorphic and $p$ surjective; this factorization of $f$ is essentially unique, and the subobject $[i]$ of $B$ is called the image of $f$. When $\mathcal{C}$ admits images, the ordered set Quot $A$ has finite joins; it has a greatest element represented by $1_A: A \rightarrow A$, and it has the binary meet $[r] \wedge [s]$ precisely when the pushout in $\mathcal{C}$ of $r$ and $s$ exists. (Henceforth, for a surjection $r$, we usually write
r rather than \([r]\) for the corresponding quotient object, and write \(r \land s\) and so on.)

The finitely-complete category \(\mathcal{C}\) is said to be regular\(^1\) if it admits images and if, moreover, factorizations are stable under pulling-back. Since every pullback of a monomorphism is a monomorphism, to assert this stability is to require that every pullback of a surjection be a surjection.

In any finitely-complete \(\mathcal{C}\), pulling back a morphism \(f : A \to B\) along itself gives a pair of morphisms \(r_1, r_2 : R \to A\), universal among pairs \(x_1, x_2 : X \to A\) with \(fx_1 = fx_2\); the (clearly jointly-monomorphic) pair \(r_1, r_2\) is often called the kernel-pair of \(f\). If the factorization of \(f\) is \(ip\), the pair \(r_1, r_2\) is equally the kernel-pair of \(p\). When \(\mathcal{C}\) is regular, this surjection \(p\) is the coequalizer of \(r_1\) and \(r_2\); accordingly, in a regular category, the surjections are precisely the coequalizers—see \([1, \text{Section 2}]\).

Regular categories are the ones which admit a good calculus of relations. A relation \(R\) from \(A\) to \(B\), written as \(R : A \to B\), is a subobject \(\langle r_1, r_2 \rangle : R \subseteq A \times B\) of \(A \times B\); as subobjects, the relations from \(A\) to \(B\) form an ordered set with finite meets. To each relation \(R : A \to B\) there is an opposite relation \(R^\circ : B \to A\), namely the subobject \(\langle r_2, r_1 \rangle : R \subseteq B \times A\). Each morphism \(f : A \to B\) in \(\mathcal{C}\) may be considered as a relation, by identifying it with its graph \(\langle 1_A, f \rangle : A \to A \times B\). To relations \(R : A \to B\) and \(S : B \to C\) is assigned a composite \(SR : A \to C\), defined by pullbacks and images; and the stability of factorizations under pulling-back is exactly what is needed to make this composition associative. For further basic properties of relations in a regular category, see again \([1, \text{Section 2}]\).

A relation \(R : A \to A\) in the regular \(\mathcal{C}\) is said to be an equivalence relation if it is reflexive, symmetric, and transitive; that is to say, if \(1_A \leq R\), \(R^\circ \leq R\), and \(RR \leq R\). Note that any intersection of equivalence relations is an equivalence relation. The kernel-pair \(r_1, r_2 : R \to A\) of \(f : A \to B\), seen as the relation \(R : A \to A\) given by \(\langle r_1, r_2 \rangle : R \subseteq A \times A\), is an equivalence relation, which we may call simply the kernel of \(f\), where confusion is unlikely. Those equivalence relations that arise thus as kernels are called congruences; and the regular category \(\mathcal{C}\) is said to be exact if every equivalence relation in \(\mathcal{C}\) is a congruence.

Every variety, in the sense of universal algebra, is an exact category—and one in which the terms “subobject”, “surjective”, “congruence”, and so on have just their classical meanings. There are however many non-varietal examples—see \([1, \text{Section 1}]\). Among these are all the categories monadic over the category \(\text{Set}\) of sets; besides varieties, these include the dual \(\text{Set}^{\text{op}}\) of the category of sets, the category of compact hausdorff spaces, the dual of this latter category, and many others. The functor-category \([\mathcal{F}, \mathcal{C}]\) is exact when \(\mathcal{C}\) is so, as is the subcategory \(\text{Mod}[\mathcal{F}, \mathcal{C}]\) of finite-product-preserving functors when \(\mathcal{F}\) admits finite products; so, in the light of Lawvere's thesis \([9]\), the category of compact hausdorff groups is exact. Any topos is exact, as is its dual. Every abelian category is exact: non-varietal examples are the

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\(^1\) This is Joyal's formulation of the concept of "regular category" introduced by Barr in Lecture Notes in Mathematics, Vol. 236; it is equivalent to the latter in the presence of finite limits.
duals $\text{Ab}^{\text{op}}$ and $\text{Tor}^{\text{op}}$ of the categories of abelian groups and of abelian torsion groups. And if $\mathcal{C}$ is exact so are the slice category $\mathcal{C}/A$ and the co-slice category $A/\mathcal{C}$ for $A \in \mathcal{C}$.

2.2.

We suppose for this section that our category $\mathcal{C}$ is exact. Writing $\text{Cong } A$ for the set of congruences on $A$, ordered as subobjects of $A \times A$, we have an isomorphism $(\text{Quot } A)^{\text{op}} \cong \text{Cong } A$ of ordered sets: a quotient object given by the surjection $r : A \to B$ determines the congruence $R$, or $\langle r_1, r_2 \rangle : R \to A \times A$, that is its kernel; and $r$ in turn is the coequalizer of $r_1$ and $r_2$. The $B$ here may be written as $A/R$. If the congruence $S$ on $A$ corresponds similarly to $s : A \to A/S$, the intersection $R \wedge S$ as subobjects of $A \times A$ is again a congruence, and is thus the meet in $\text{Cong } A$; it is (see [1, Section 2]) the kernel of $r \vee s$. The join $R \vee S$ in $\text{Cong } A$ need not exist; it does so, of course, precisely when the meet $r \wedge s$ in $\text{Quot } A$ does so—that is, when the pushout of $r$ and $s$ exists in $\mathcal{C}$; then $R \vee S$ is the kernel of the diagonal of this pushout. When $\mathcal{C}$ is a variety, $R \vee S$ always exists.

Since $1 \leq R$ and $1 \leq S$ here, we have $R \leq RS$ and $S \leq RS$, while $RS \leq RSR$ and $SR \leq RSR$, and so on; moreover every such term is less than or equal to $R \vee S$, if this last exists. From this we easily get the following—restating the first two cases, and the most important ones, of [1, Theorem 3.1]:

**Proposition 2.1.** For congruences $R$ and $S$ on $A \in \mathcal{C}$, the composite $RS$ is a congruence—and is then necessarily $R \vee S$—if and only if $SR \leq RS$, which is in fact equivalent to $SR = RS$; while $RSR$ is a congruence—and is then necessarily $R \vee S$—if and only if $SRS \leq RSR$. □

We moreover have the following, which is [1, Proposition 3.2]:

**Proposition 2.2.** If the congruences $R$ and $S$ on $A$ satisfy $SRS \leq RSR$, so that $R \vee S = RSR$ (which is certainly the case if $SR = RS$), and if $R \leq T$ in $\text{Cong } A$, we have $(R \vee S) \wedge T = R \vee (S \wedge T)$. □

The exact category $\mathcal{C}$ is said to have the Maltsev property, or simply to be Maltsev, if the permutability $SR = RS$ holds for all congruences on any object $A$; and to be Goursat if we have only the weaker $SRS = RSR$ for congruences. That the Goursat property is strictly weaker than the Maltsev one, even for varieties, was shown by Mitschke [14]; for more details and other references, see [1, Section 1]. (The Maltsev and Goursat properties were studied in [1] for general regular categories, not necessarily exact, $R$ and $S$ then being equivalence relations rather than congruences; we restrict ourselves here to the exact case, because our theory of extensions needs the exactness of $\mathcal{C}$.) Proposition 2.2 gives what is [1, Proposition 3.3]:
Proposition 2.3. In a Goursat exact category \( \mathcal{C} \), and hence in a Maltsev one, each Cong \( A \) admits joins \( R \lor S \) and is a modular lattice. \( \square \)

Remark 2.4. A variety \( \mathcal{V} \) need not be Goursat when each Cong \( A \) is a modular lattice—or even a distributive one; a counter-example is the variety of lattices. See [1, Remark 3.4] for references and details.

By a classical result of Maltsev [13], a variety is Maltsev precisely when its theory contains a ternary operation \( m \) satisfying \( mxy = y \) and \( mxv = x \)—such as the operation \( xy^{-1}z \) in the theory of groups. So among the Maltsev varieties are those of groups, abelian groups, modules over some fixed ring, rings, commutative rings, associative algebras, Lie algebras, and so on. There is also (see [6, p. 9]) such an operation in the variety of Heyting algebras, so that this and the variety of boolean algebras are Maltsev. Then there are non-varietal examples: if the exact \( \mathcal{C} \) is Maltsev, so are the categories \( \mathcal{C}/A \) and \( A/\mathcal{C} \), any functor-category \( \mathcal{F}, \mathcal{C} \), and the subcategory \( \text{Mod}[\mathcal{F}, \mathcal{C}] \) of this given by the models in \( \mathcal{C} \) of a one-sorted algebraic theory \( \mathcal{F} \), provided that this theory contains a Maltsev operation \( m \). (For the proof of this last statement, see [1, Proposition 3.7].) Thus, for instance, the category of compact hausdorff groups is Maltsev. So, too, is any abelian category \( \mathcal{C} \); for the category of abelian groups in \( \mathcal{C} \) is \( \mathcal{C} \) itself. Further examples still of Maltsev exact categories, such as the duals of the categories of sets and of compact hausdorff spaces, come from [1, Theorem 5.7].

The examples in [1, Section 1] of varieties that are Goursat but not Maltsev are rather exotic; we have no “good” examples provided by varieties which are clearly of mathematical importance. Yet, curiously, for several of our chief results below, the Goursat condition on \( \mathcal{C} \) is just what we need; which is why we mention it separately from the “simpler” Maltsev condition.

2.3.

Here we give several technical results needed below. Since the first three are not in [1], we discuss them in their natural setting of a merely regular \( \mathcal{C} \).

A morphism \( s: A \to C \) in any category \( \mathcal{C} \) induces a functor \( s_*: \mathcal{C}/A \to \mathcal{C}/C \) sending \( f: X \to A \) to \( sf: X \to C \). When \( \mathcal{C} \) admits pullbacks, this has a right adjoint \( s^*: \mathcal{C}/C \to \mathcal{C}/A \) sending \( g: Y \to C \) to \( f: X \to A \), where

\[
\begin{array}{ccc}
X & \xrightarrow{k} & Y \\
\downarrow{f} & & \downarrow{g} \\
A & \xrightarrow{s} & C
\end{array}
\]

is a pullback; what we called \( s_*: \mathcal{C} \downarrow C \to \mathcal{C} \downarrow A \) in Section 1 is of course just the restriction of \( s_* \). The counit \( s_*g \to 1 \) of this adjunction has \( k: sf \to g \) as its \( g \)-component.
When $\mathcal{C}$ is regular and $s$ is a surjection, the pullback $k$ of $s$ is a surjection in $\mathcal{C}$ and hence in $\mathcal{C}/C$. By a well-known result (see, for instance, [8, p. 84]), we have:

**Proposition 2.5.** When $\mathcal{C}$ is regular and $s$ is a surjection, the functor $s^*: \mathcal{C}/C \to \mathcal{C}/A$ reflects isomorphisms, as does its restriction $s^*: \mathcal{C} \downarrow C \to \mathcal{C} \downarrow A$. □

**Corollary 2.6.** In a regular category $\mathcal{C}$, if $s$ is surjective and a pullback $s^*(g)$ is invertible, then $g$ is invertible. □

We know that the following is quite well known—but not where to find it in print:

**Proposition 2.7.** In a regular category $\mathcal{C}$, suppose that the exterior and the left square of the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B & \xrightarrow{f} & C \\
\downarrow{h} & & \downarrow{g} & & \downarrow{f} \\
D & \xrightarrow{p} & E & \xrightarrow{i} & F
\end{array}
$$

are pullbacks. Then the right square is a pullback if $p$ is surjective.

**Proof.** Let the pullback of $f$ along $i$ be

$$
\begin{array}{ccc}
P & \xrightarrow{v} & C \\
\downarrow{u} & & \downarrow{f} \\
E & \xrightarrow{i} & F
\end{array}
$$

and let $w: B \to P$ be the unique morphism with $vw = j$ and $uw = g$. Since the exterior of (2.1) and the diagram (2.2) are pullbacks, so too (by a classical and simple result) is

$$
\begin{array}{ccc}
A & \xrightarrow{u} & P \\
\downarrow{h} & & \downarrow{u} \\
D & \xrightarrow{p} & E
\end{array}
$$

Accordingly $p^*(w)$ is invertible in $\mathcal{C}/D$; whence, by Proposition 2.5, $w$ is invertible in $\mathcal{C}/E$, and hence in $\mathcal{C}$, as desired. □

From now on we again suppose that $\mathcal{C}$ is exact. A morphism $s: A \to C$ gives $s \times s: A \times A \to C \times C$, and we have the pulling-back functor $(s \times s)^* : \mathcal{C}/(C \times C) \to \mathcal{C}/(A \times A)$; this takes monomorphisms to monomorphisms, thus inducing a functor $s^*: \text{Sub}(C \times C) \to \text{Sub}(A \times A)$, the value $s^* P$ at the relation $P: C \to C$ is $s^* P$s—see [1, Section 6]. It is convenient to say that a pair of morphisms $h, k : X \to C$ belongs to the relation $P$, and to write $(h, k) \in P$, if $(h, k): X \to C \times C$ factorizes through the subobject $P$ of $C \times C$. One sees immediately that a pair $(f, g): X \to A$ belongs to

...
If and only if \((sf, sg) : X \to C\) belongs to \(P\). As is shown in [1, Section 6], \(s^\circ\) has the left adjoint \(s_0\) sending the relation \(R : A \to A\) to \(sRs^\circ\), this subobject of \(C \times C\) being in fact the image of \(\langle sr_1, sr_2 \rangle : R \to C \times C\); accordingly \((sf, sg) : X \to C\) certainly belongs to \(s \circ R\) if \((f, g) : X \to A\) belongs to \(R\). Our concern below is only with the case of a surjective \(s\); then, as is shown in [1, Proposition 6.2], \(s^\circ\) is fully faithful, and it preserves composition of relations:

\[ s^\circ(PP') = (s^\circ P)(s^\circ P') \quad \text{for } s \text{ surjective.} \tag{2.3} \]

As is further shown in [1, Section 6], \(s^\circ\) takes congruences to congruences, thus giving by restriction a functor \(s^\#: \text{Cong } C \to \text{Cong } A\) where

\[ s^\# P = s^\circ P \] for a congruence \(P\). \tag{2.4}

Since \(\text{Cong } C \cong (\text{Quot } C)^{op}\) and so on, there is a corresponding functor \(s_1 : \text{Quot } C \to \text{Quot } A\); when, as we henceforth suppose, \(s\) is surjective, this simply sends a surjection \(p : C \to E\) to the surjection \(ps : A \to E\)—see [1, (6.4)]. Accordingly the fully-faithful \(s\) maps \(\text{Quot } C\) isomorphically onto the sub-ordered-set of \(\text{Quot } A\) given by those quotient objects less than or equal to \(s\). It follows, as in [1, Proposition 6.4], that a right adjoint \(s_1^\#: \text{Quot } A \to \text{Quot } C\) for \(s\) “exists locally at \(r \in \text{Quot } A\)” if and only if we have the meet \(r \land s\), given by the diagonal of the pushout

\[
\begin{array}{ccc}
A & \overset{s}{\longrightarrow} & C \\
\downarrow r & & \downarrow v \\
B & \underset{u}{\longrightarrow} & D
\end{array}
\] \tag{2.5}

whereupon \(s^r = v\), while \(s_1s^r = vs = r \land s\).

Let us translate this back into the language of congruences. Write \(\langle s_1, s_2 \rangle : S \to A \times A\) for the kernel of \(s\), with similar notations for the kernels of \(r\) and of \(v\); from the diagram (2.5) we get an induced morphism \(k\) in

\[
\begin{array}{ccc}
R & \overset{k}{\longrightarrow} & V \\
\downarrow r_1 & \quad & \downarrow v_1 \\
A & \underset{s}{\longrightarrow} & C
\end{array}
\] \tag{2.6}

As in [1, Proposition 6.6] the translation gives:

**Proposition 2.8.** For the surjective \(s\) with kernel \(S\), the fully-faithful \(s^\#: \text{Cong } C \to \text{Cong } A\) maps \(\text{Cong } C\) isomorphically onto the sub-ordered-set of \(\text{Cong } A\) given by those congruences greater than or equal to \(S\). A left adjoint \(s^\#: \text{Cong } A \to \text{Cong } C\) for \(s^\#\) exists locally at \(R \in \text{Cong } A\) if and only if the join \(R \lor S\) exists in \(\text{Cong } A\); whereupon we have \(s^\# R = V\) and \(s^\# s^R = R \lor S\). \(\Box\)

It is further shown in [1, Section 6] that \(s^\# R\) is the smallest congruence containing \(s \circ R\), existing precisely where this does so; so that \(s^\# R = s \circ R\) exactly when \(s \circ R\) is
already a congruence. It also follows from the description there and above of $s \circ R$ as an image that this is the case precisely when the $k$ of (2.6) is a surjection. It is not in general so; [1, Proposition 6.5] gives (cf. Proposition 2.1):

**Proposition 2.9.** For a surjective $s$ with kernel $S$ and any $R \in \text{Cong } A$, the following are equivalent:

(i) $s \circ R$ is a congruence, and is therefore $s \not\sim R$;
(ii) $RSR \leq SRS$;
(iii) $SRS$ is a congruence, and is therefore $R \not\sim S$;
(iv) $s \not\sim R$ exists, and the $k$ of (2.6) is surjective.

In a Goursat exact category, these always hold; they also hold a fortiori if $RS = SR$, and certainly in any Maltsev exact category. □

Finally, we also need the following, which is an immediate consequence of [1, Theorem 5.2 and Proposition 5.4]:

**Proposition 2.10.** Let (2.5) be a commutative diagram of surjections in the exact $\mathcal{C}$, and let $R$ and $S$ denote the congruences which are the kernels of $r$ and $s$. Then if the diagram is a pullback, it is necessarily also a pushout; and if it is a pushout, it is a pullback if and only if $RS = SR$ and $R \cap S = 1_A$. Thus when $\mathcal{C}$ is Maltsev and the diagram is a pushout, it is a pullback if and only if $R \cap S = 1_A$. □

### 3. Admissible subcategories

**3.1.**

We develop now the notion of admissible subcategory outlined in Section 1.2. We consider a reflective full subcategory $\mathcal{X}$ of the exact category $\mathcal{C}$, further supposing for simplicity that $\mathcal{X}$ is replete, meaning that each isomorph in $\mathcal{C}$ of an object of $\mathcal{X}$ lies in $\mathcal{X}$. We write $H : \mathcal{X} \to \mathcal{C}$ for the inclusion, $I : \mathcal{C} \to \mathcal{X}$ for its left adjoint, and $\eta : 1 \to HI$ for the unit of the adjunction; we can always suppose the counit to be an equality $IH = 1$.

We often suppress $H$ from the notation, writing $\eta_A : A \to IA$ for the component of $\eta$. Of course $\mathcal{X}$ has finite limits, formed as in $\mathcal{C}$; in particular, a morphism in $\mathcal{X}$ is monomorphic in $\mathcal{X}$ if and only if it is so in $\mathcal{C}$.

It is a simple and well-known fact that $\mathcal{X}$ is closed in $\mathcal{C}$ under subobjects—in the sense that every subobject in $\mathcal{C}$ of an object of $\mathcal{X}$ lies in $\mathcal{X}$—if and only if each $\eta_A$ is surjective; we recall the argument. Suppose each $\eta_A$ is surjective, and consider a monomorphism $i : A \to X$ with $X \in \mathcal{X}$. By the universal property, we have $i = f \eta_A$ for some $f$; so the surjection $\eta_A$ is monomorphic and thus invertible, whence $A \in \mathcal{X}$. Suppose conversely that $\mathcal{X}$ is closed under subobjects, and let $\eta_A$ factorize as $ip$, its image being $i : B \to IA$. Then, by the universal property, $p : A \to B$ is $f \eta_A$ for some $f$. 


whence \( \eta_A = ip = iff \eta_A \); now if \( i = 1 \) by the uniqueness clause, so that the monomorphism \( i \) is invertible, and \( \eta_A \) is surjective.

Suppose henceforth that \( \mathcal{X} \) is closed in \( \mathcal{C} \) under subobjects. Then the image in \( \mathcal{C} \) of a morphism \( f: X \to Y \) in \( \mathcal{X} \) lies in \( \mathcal{X} \); it follows that \( f \) is surjective in \( \mathcal{X} \) if and only if it is so in \( \mathcal{C} \); it then follows, more generally, that the canonical factorization of \( f \) in \( \mathcal{C} \) as \( f = ip \) with \( i \) monomorphic and \( p \) surjective is also its canonical factorization in \( \mathcal{X} \); finally, since pullbacks in \( \mathcal{X} \) are formed as in \( \mathcal{C} \), it follows that \( \mathcal{X} \) like \( \mathcal{C} \) is a regular-category. The example where \( \mathcal{C} = \text{Ab} \) and \( \mathcal{X} \) consists of the torsion-free groups shows that, in general, \( \mathcal{X} \) is not exact.

The naturality of \( \eta \) is expressed by the commutativity of

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & IA \\
\downarrow f & & \downarrow \eta f \\
B & \xrightarrow{\eta_B} & IB
\end{array}
\]

(3.1)

for each \( f: A \to B \) in \( \mathcal{C} \); note that, since the \( \eta_A \) are surjective, we have

\[
\text{If is surjective (in } \mathcal{C} \text{, and hence in } \mathcal{X} \text{) for each surjective } f. \tag{3.2}
\]

**Proposition 3.1.** In order that \( \mathcal{X} \), reflective and closed in \( \mathcal{C} \) under subobjects, be also closed in \( \mathcal{C} \) under quotient objects, it is necessary and sufficient that (3.1) be a pushout whenever \( f \) is surjective.

**Proof.** Let \( \mathcal{X} \) be closed in \( \mathcal{C} \) under quotient objects. To show (3.1) to be a pushout it suffices—because \( \mathcal{C} \) admits factorizations—to consider surjections \( u: IA \to X \) and \( v: B \to X \) with \( u\eta_A = vf \). But then \( X \) like \( IA \) is in \( \mathcal{X} \), so that \( v = w\eta_B \) for a unique \( w \); whence \( u\eta_A = vf = w\eta_B f = w \). If \( \eta_A \), giving \( u = w \). If. For the converse, suppose that (3.1) is a pushout with \( A \in \mathcal{X} \); then \( \eta_A \) is invertible, so that its pushout \( \eta_B \) is also invertible, whence \( B \in \mathcal{X} \). \( \square \)

As we said in Section 1.2, we shall call the reflective full subcategory \( \mathcal{X} \) of \( \mathcal{C} \) a Birkhoff subcategory when it is closed in \( \mathcal{C} \) under both subobjects and quotient objects. Clearly \( \mathcal{C} \) itself is a Birkhoff subcategory of \( \mathcal{C} \), and the biggest such. On the other hand, any Birkhoff \( \mathcal{X} \) must contain the terminal object \( 1 \) of \( \mathcal{C} \), and hence the full subcategory \( \text{Sub } 1 \) of \( \mathcal{C} \) described in Section 1.2, which is of course reflective; moreover \( \text{Sub } 1 \) is Birkhoff, since if \( X \to 1 \) is monomorphic, any surjection \( X \to Y \) is monomorphic and hence invertible.

Note that a Birkhoff subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is necessarily exact. We have seen that it is regular. Now if \( \langle r_1, r_2 \rangle: R \to X \times X \) is an equivalence relation in \( \mathcal{X} \) on \( X \in \mathcal{X} \), it is also an equivalence relation in \( \mathcal{C} \), so that \( R \) is the kernel in \( \mathcal{C} \) of the coequalizer \( X \to X/R \); but here \( X/R \in \mathcal{X} \), so that \( R \) is a congruence in \( \mathcal{X} \). Note that the torsion-free abelian groups do not form a Birkhoff subcategory of \( \text{Ab} \).

When \( \mathcal{C} \) is a variety, the full subcategory \( \mathcal{X} \) given by those algebras satisfying a set of identities is called a subvariety; it follows from Birkhoff's classical theorem that the
Birkhoff subcategories of the variety $\mathcal{C}$ are precisely its subvarieties. A non-varietal example of a Birkhoff subcategory, already mentioned in Section 1.2, is that where $(\mathcal{C}, \mathcal{X}) = (\text{Ab}^{\text{op}}, \text{Tor}^{\text{op}})$.

We suppose henceforth that $\mathcal{X}$ is a Birkhoff subcategory of the exact $\mathcal{C}$, and write $\langle r_A^1, r_A^2 \rangle : R_A \rightarrow A \times A$ for the kernel of the surjection $\eta_A : A \rightarrow IA$. Moreover, for $f : A \rightarrow B$ in $\mathcal{C}$ we write $R_f : R_A \rightarrow R_B$ for the restriction of $f \times f : A \times A \rightarrow B \times B$, which gives a commutative diagram

$$
\begin{array}{ccc}
R_A & \longrightarrow & A \\
\downarrow{r_f} & & \downarrow{f} \\
R_B & \longrightarrow & B
\end{array}
$$

Since (3.1) is by Proposition 3.1 a pushout when $f$ is surjective, comparing (3.1) with (2.5) and (3.3) with (2.6) shows, using Proposition 2.8, that

$$f \# R_A = R_B \quad \text{for a surjective } f : A \rightarrow B,$$

while $R_f$ is a case of the induced morphism $k$ in (2.6). Now Proposition 2.9 gives:

**Proposition 3.2.** For a surjective $f : A \rightarrow B$, with kernel $F : A \rightarrow A$, the following are equivalent:

1. $R_f : R_A \rightarrow R_B$ is surjective;
2. $R_A F R_A \leq F R_A F$;
3. $F R_A F$ is a congruence, and is therefore $F \cap R_A$.

These certainly hold whenever $F R_A = R_A F$, and whenever $\mathcal{C}$ is Goursat (or Mal'tsev). □

3.2.

We continue to suppose that $\mathcal{X}$ is a Birkhoff subcategory of the exact $\mathcal{C}$. Recall from Section 1.1 the meaning we attach to $\mathcal{C} \downarrow B$. It follows from (3.2) that, as we said in Section 1.2, $I : \mathcal{C} \rightarrow \mathcal{X}$ induces for each $B \in \mathcal{C}$ a functor $I^B : \mathcal{C} \downarrow B \rightarrow \mathcal{X} \downarrow IB$ sending the surjection $f : A \rightarrow B$ to the surjection $f : IA \rightarrow IB$.

It is easy to see that $I^B$ has a right adjoint $H^B : \mathcal{X} \downarrow IB \rightarrow \mathcal{C} \downarrow B$. For, given surjections $f : A \rightarrow B$ and $\phi : X \rightarrow IB$, to give a morphism $If \rightarrow \phi$ in $\mathcal{X} \downarrow IB$ is equally to give a morphism $If \eta_A \rightarrow \phi$ in $\mathcal{C} \downarrow IB$, which is the same thing as a morphism $\eta_B f \rightarrow \phi$ in $\mathcal{C} \downarrow IB$. To give this, however, is just to give a morphism $f \rightarrow H^B \phi$ in $\mathcal{C} \downarrow B$, where $H^B \phi$ is the surjection $s : C \rightarrow B$ given by the pullback

$$
\begin{array}{ccc}
C & \longrightarrow & X \\
\downarrow{s} & & \downarrow{\phi} \\
B & \longrightarrow & IB
\end{array}
$$
If \( \beta : IC \to X \) is the unique morphism with \( \beta \eta_C = t \), we have \( \phi \beta \eta_C = \phi t = \eta_B s = Is \eta_C \), whence \( \phi \beta = Is \). Thus (3.5) is the pasting composite

\[
\begin{array}{ccc}
C & \xrightarrow{\eta_C} & IC \\
| & & | \\
\downarrow s & \beta & \downarrow \phi \\
B & \xrightarrow{\eta_a} & IB & \xrightarrow{\iota} & IB
\end{array}
\]  
(3.6)

and clearly the \( \phi \)-component of the counit \( \varepsilon^B : 1^BH^B \to 1 \) of the adjunction \( 1^B \vdash H^B \) is the morphism \( \beta : Is \to \phi \) of \( \mathcal{X} \downarrow IB \).

We say that the Birkhoff subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is admissible when each \( H^B : \mathcal{X} \downarrow IB \to \mathcal{C} \downarrow B \) is fully faithful. This is of course so precisely when, for each \( B \), the counit \( \varepsilon^B \) is invertible; equivalently, when each of its components \( \beta \) as above is invertible. Admissibility may be seen as a kind of exactness condition on \( I \)—the preservation by \( I \) of some pullbacks (but not of all, which would make \( \mathcal{X} \) a localization of \( \mathcal{C} \)). For we have:

**Proposition 3.3.** The following are equivalent:

(i) the Birkhoff subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is admissible;

(ii) \( I \) preserves all pullbacks of the form (3.5) with \( X \in \mathcal{X} \) and \( \phi \) surjective;

(iii) \( I \) preserves all pullbacks of the form

\[
\begin{array}{ccc}
C & \xrightarrow{u} & Z \\
\downarrow s & & \downarrow \phi \\
B & \xrightarrow{v} & Y
\end{array}
\]  
(3.7)

where \( Z, Y \in \mathcal{X} \) and \( \psi \) is surjective.

**Proof.** Clearly \( I \) of (3.5) is the right square of (3.6), which is a pullback precisely when \( \beta \) is invertible; thus (i) and (ii) are equivalent. Since (ii) is a special case of (iii), it remains to show that (ii) implies (iii). The \( \psi \) of (3.7) has the form \( w \eta_B \) for a unique \( w : IB \to Y \), so that (3.7) can be written as the pasting composite of two pullbacks:

\[
\begin{array}{ccc}
C & \xrightarrow{t} & X & \xrightarrow{x} & Z \\
\downarrow s & & \downarrow \phi & & \downarrow \psi \\
B & \xrightarrow{\eta_a} & IB & \xrightarrow{w} & Y
\end{array}
\]

Here \( X \in \mathcal{X} \), since \( \mathcal{X} \) is closed under limits in \( \mathcal{C} \); and the pullback \( \phi \) of \( \psi \) is a surjection. So \( I \) preserves the left pullback by (ii), and the right pullback because it already lies in \( \mathcal{X} \); thus \( I \) preserves the pullback (3.7). □

In fact it is more immediately fruitful to analyze in another way the invertibility of \( \beta \), in terms of congruences. Write \( S \) and \( T \) for the congruences that are the kernels of the surjections \( s \) and \( t \) in (3.5), observing that \( R_C \leq T \) since \( \beta \eta_C = t \). Since the left square of
(3.6) is a pushout by Proposition 3.1, its diagonal is the intersection \( \eta_c \land s \) in Quot \( C \); this being \( \phi t \), we have \( \eta_c \land s \leq t \), and so \( R_c \lor S \geq T \). We now further have

\[
R_c = R_c \lor 1_c = R_c \lor (S \land T) \leq (R_c \lor S) \land T = T;
\]

here the first equality is trivial; the second expresses that, (3.5) being a pullback, \( S \land T = 1_c \) since \( s \) and \( t \) are jointly monomorphic (cf. Proposition 2.1); the inequality holds because \( R_c \leq T \); and the final equality because \( R_c \lor S \geq T \). However, \( \beta \) is invertible precisely when \([\eta_c] = [t]\), or equivalently \( R_c \lor (S \land T) = (R_c \lor S) \land T \). (3.8)

Taking account of Propositions 2.2, 2.3 and 2.9, we get:

**Theorem 3.4.** (a) The Birkhoff subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is admissible if, for each \( C \in \mathcal{C} \) and each pair \( S, T \) of congruences on \( C \) with \( R_c \leq T \), we have \( R_c \lor (S \land T) = (R_c \lor S) \land T \).

(b) This is certainly the case if, for each \( C \) and each congruence \( S \) on \( C \), we have \( SR_c S \leq R_c SR_c \); or, equivalently, if \( (\eta_c) \circ S \) is a congruence.

(c) It is a fortiori the case if, for each \( C \) and \( S \), we have \( SR_c = R_c S \).

(d) Every Birkhoff subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is admissible if \( \text{Cong} \mathcal{A} \) is a modular lattice for each \( C \in \mathcal{C} \); this is so in particular whenever \( \mathcal{C} \) is Goursat, and hence whenever \( \mathcal{C} \) is Maltsev.

We comment on some examples. Of the extreme Birkhoff subcategories given by \( \mathcal{C} \) itself and by \( \text{Sub} \mathcal{C} \), it is trivial that the first is admissible, and nearly trivial that the second is so—since every surjection in \( \text{Sub} \mathcal{C} \) is invertible. In fact each of these cases falls under (c) of the theorem above: in the first of them \( R_c = 1_c \), while in the second \( R_c = C \times C \), so that each satisfies \( SR_c = R_c S \).

Because a variety of \( \Omega \)-groups, as defined in Section 1.4, is Maltsev, every Birkhoff subcategory \( \mathcal{X} \)—that is, every subvariety—is admissible. The same is true for the non-Goursat variety of lattices, since there each \( \text{Cong} \mathcal{A} \) is distributive (see Remark 2.4). Again, since the abelian category \( \text{Ab}^{\text{op}} \) is Maltsev, the Birkhoff subcategory \( \text{Tor}^{\text{op}} \) is admissible. (Note that, for any Birkhoff \( \mathcal{X} \) and for \( C \in \mathcal{X} \), the set of congruences on \( C \) in \( \mathcal{X} \) is just the set \( \text{Cong} \mathcal{C} \) of congruences on \( C \) in \( \mathcal{C} \), with the same composition; so if \( \mathcal{C} \) is Maltsev or Goursat, or has each \( \text{Cong} \mathcal{C} \) modular, the same is true of \( \mathcal{X} \).)

In the example \((\mathcal{C}, \mathcal{X}) = (\text{G-Set}, \text{Set})\) of Section 1.3, it is easy to check directly that the Birkhoff subcategory \( \mathcal{X} \) of \( \mathcal{C} \) is admissible, although here Theorem 3.4 does not apply.

Not every subvariety \( \mathcal{X} \) of a variety \( \mathcal{C} \) is admissible; we give three examples.

**Example 3.5.** Let an object of \( \mathcal{C} \) be a set \( A \) with two distinguished elements \( a_1 \) and \( a_2 \), and let \( \mathcal{X} \) consist of those objects \( A \) having \( a_1 = a_2 \). In (3.5) take \( B = \{b_1, b_2\} \), so that \( IB = 1 \), and take \( X = \{u, x_1 = x_2\} \). Then \( C = B \times X \) has four elements, \( IC \) has three elements, and the \( \beta \) of (3.6) is not invertible.
Example 3.6. Let $\mathcal{C}$ be the variety of semigroups and $\mathcal{A}$ that of abelian semigroups. In (3.5) take $B = \{b_1, b_2\}$ with the multiplication given by $xy = x$, so that $IB = 1$; and take $X = \{u, 0\}$ with $u^2 = u0 = 0u = 00 = 0$. Then $C = B \times X$ has four elements, $IC$ has three elements since commutativity forces $(b_1, 0) \sim (b_2, 0)$, and $\beta$ is not invertible.

Example 3.7. Let $\mathcal{C}$ be the variety of monoids, and $\mathcal{A}$ that of abelian monoids. In (3.5) take for $B$ and $X$ the monoids obtained by freely adding an identity to the semigroups $B$ and $X$ of Example 3.6; again $\beta$ is not invertible.

4. Central extensions

4.1.

We now establish and amplify the assertions of Section 1.3 on trivial, split, central, and normal extensions. Recall from Section 1.1 that, by an extension of $B$, we mean a surjection $f: A \to B$; and from Section 1.3 that this is said to be trivial, with respect to the admissible subcategory $\mathcal{A}$ of $\mathcal{C}$, when it lies in the image of the fully-faithful $H^B: \mathcal{A} \downarrow IB \to \mathcal{C} \downarrow B$. Recall, too, the name $\text{Triv } B = \text{Triv}_\mathcal{A} B$ for the category of trivial extensions of $B$.

Clearly the $(A, f)$-component of the unit $\eta_B: 1 \to H^B IB$ of the adjunction $IB \dashv H^B$ is the comparison map between the diagram (3.1), which we repeat here for convenience as

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & IA \\
\downarrow f & & \downarrow \eta_f \\
B & \xrightarrow{\eta_B} & IB
\end{array}
\end{equation}

(4.1)

and the pullback along $\eta_B$ of $I\eta_f$, which is $H^B \eta f$. Accordingly

the extension $(A, f)$ is trivial if and only if (4.1) is a pullback

(4.2)

Recall too the notation $g^* : \mathcal{C} \downarrow B \to \mathcal{C} \downarrow D$ for the restriction of the pulling-back functor $g^*: \mathcal{C}/B \to \mathcal{C}/D$ induced by a morphism $g: D \to B$.

Proposition 4.1. $g^*$ takes trivial extensions of $B$ to trivial extensions of $D$.

Proof. Let the left square below be the pullback along $g$ of the trivial extension $(A, f)$:

\begin{equation}
\begin{array}{ccc}
C & \xrightarrow{h} & A \\
\downarrow k & \xrightarrow{f} & \downarrow \eta_f \\
D & \xrightarrow{g} & B \\
\downarrow \eta_B & & \downarrow IB
\end{array}
\end{equation}

(4.3)
Since both the squares here are pullbacks, so too is the exterior rectangle. By the naturality of $\eta$, the latter is equal to the exterior rectangle in

$$
\begin{array}{ccc}
C & \xrightarrow{\eta_{c}} & IC & \xrightarrow{h} & IA \\
\downarrow k & & \downarrow i_k & & \downarrow if \\
D & \xrightarrow{\eta_{d}} & ID & \xrightarrow{i_g} & IB
\end{array}
$$

(4.4)

which is accordingly a pullback. Now applying $I$ to the exterior rectangle of (4.3) gives the right square of (4.4); so that this last too is a pullback, by Proposition 3.3. So by a classical result—see for example [12, p. 72, Exercise 9]—the left square of (4.4) is a pullback, and $k: C \to D$ is a trivial extension.

In the light of (4.2) and Proposition 3.1, we get from Proposition 2.10 a criterion for the triviality of $(A, f)$:

**Proposition 4.2.** Let the congruence $F$ be the kernel of the extension $(A, f)$. Then $(A, f)$ is trivial if and only if $FR_A = R_A F$ and $F \wedge R_A = 1_A$. If $\mathcal{C}$ is Malteev, $(A, f)$ is trivial precisely when $F \wedge R_A = 1_A$. $\square$

As we indicated in Section 1.3, every extension in $\mathcal{C}$ is trivial when $\mathcal{X}$ is all of $\mathcal{C}$, and only then. At the other extreme, when $\mathcal{X}$ is $\text{Sub} 1$, every surjection in $\mathcal{X}$ is invertible; so that the only trivial extensions are the isomorphisms. Conversely, if the only trivial extensions are the isomorphisms, all the surjections in $\mathcal{X}$, being trivial extensions, must be invertible—so that every morphism in $\mathcal{X}$ is a monomorphism, and $\mathcal{X}$ is necessarily $\text{Sub} 1$.

4.2

Recall now from Section 1.3 that an extension $f: A \to B$ of $B$ is said to be $(E, p)$-split, where $p: E \to B$ is itself an extension of $B$, when the extension $p^*(A, f)$ of $E$ is trivial; and that the category of such extensions is called $\text{Spl}(E, p) = \text{Spl}_{\mathcal{X}}(E, p)$. The basic properties of the $\text{Spl}(E, p)$ given in Section 1.3 need no further proof. Recall, too, that an extension $f: A \to B$ is called central when it belongs to $\text{Spl}(E, p)$ for some extension $p: E \to B$; and that the category of central extensions is written $\text{Centr} B = \text{Centr}_{\mathcal{X}} B$. We now prove the following assertion of Section 1.3:

**Proposition 4.3.** For any $g: D \to B$, the functor $g^*: \mathcal{C} \downarrow B \to \mathcal{C} \downarrow D$ takes central extensions of $B$ to central extensions of $D$.

**Proof.** Let $(A, f)$ belong to $\text{Spl}(E, p)$, and form the pullback

$$
\begin{array}{ccc}
G & \xrightarrow{g} & D \\
\downarrow h & & \downarrow a \\
E & \xrightarrow{p} & B
\end{array}
$$
Now $q^*g^*(A, f) = h^*p^*(A, f)$, which is trivial by Proposition 3.1; so $g^*(A, f)$ belongs to $\text{Spl}(G, q)$ and is central. \[\square\]

We saw in Section 1.3 that the inclusions $\text{Triv} B \subset \text{Centr} B \subset \mathcal{C} \downarrow B$ are proper in general, although there are some important cases, other than the extreme one $\mathcal{X} = \mathcal{C}$, in which every extension is central. We further promised to give some analysis of those cases in which every central extension is trivial.

One such is the case $\mathcal{X} = \text{Sub} 1$, where the only trivial extensions are the isomorphisms; for if $p : E \to B$ is surjective and $p^*(A, f)$ is invertible, then $f$ itself is invertible by Corollary 2.6. There are however other cases.

**Proposition 4.4.** The following are equivalent:

(i) every central extension is trivial;

(ii) $I$ preserves the pullback

\[
\begin{array}{c}
C \\
\downarrow k
\end{array}
\begin{array}{c}
\to \\
\downarrow f
\end{array}
\begin{array}{c}
A \\
\downarrow g
\end{array}
\begin{array}{c}
\to \\
\downarrow g
\end{array}
\begin{array}{c}
D \\
\to B
\end{array}
\] (4.5)

whenever $(A, f)$ is a central extension;

(iii) $I$ preserves the pullback (4.5) whenever $(A, f)$ is a central extension and $g$ is a surjection.

**Proof.** In the proof of Proposition 4.1, we used Proposition 3.3 to show that $I$ preserves the pullback (4.5) whenever $(A, f)$ is a trivial extension. So (i) implies (ii), of which (iii) is a special case, and it remains to show that (iii) implies (i). Suppose then that $f$ and $g$ in the pullback (4.5) are surjections, and that the extension $(C, k)$ is trivial; we are to prove the extension $(A, f)$ trivial. In the diagram (4.4), the left square is a pullback by (4.2), since $(C, k)$ is trivial, while the right square is a pullback by the hypothesis (iii); so the exterior rectangle of (4.4), which is also the exterior rectangle of (4.3), is a pullback. However the left square of (4.3) is the pullback (4.5), in which $g$ is surjective; by Proposition 2.7, therefore, the right square of (4.3) is a pullback, so that $(A, f)$ is trivial by (4.2) \[\square\]

**Remark 4.5.** If $\mathcal{X}$ is a localization of $\mathcal{C}$, which is to say that $I$ preserves all pullbacks, every central extension is trivial, by Proposition 4.4. Note that by Proposition 3.3, a localization $\mathcal{X}$ of $\mathcal{C}$ is an admissible subcategory of $\mathcal{C}$ if it is a Birkhoff one. Presumably pairs $(\mathcal{C}, \mathcal{X})$ with $\mathcal{X}$ both Birkhoff and a localization are rather special—but they do exist. For instance, let $\mathcal{C}$ be the product $\mathcal{A} \times \mathcal{B}$ of varieties, where $\mathcal{B}$ is such that its terminal object 1 has no proper subobjects; and let $\mathcal{X}$ consist of the objects $(A, 1)$ of $\mathcal{A} \times \mathcal{B}$.
Remark 4.6. Suppose that \( \mathcal{C} \) is Maltsev, and consider a pullback (4.5) with \( f \) and \( g \), and hence \( h \) and \( k \), surjective. Then (4.5) is a pushout in \( \mathcal{C} \) by Proposition 2.10, whence—since \( I \) is a left adjoint—\( I \) of (4.5) is a pushout in \( \mathcal{X} \). We saw in Section 3.1 that \( \mathcal{X} \) like \( \mathcal{C} \) is exact, and in Section 3.3 that \( \mathcal{X} \) like \( \mathcal{C} \) is Maltsev; accordingly, by Proposition 2.10, \( I \) of (4.5) is a pullback in \( \mathcal{X} \)—or equally in \( \mathcal{C} \)—if and only if \( \ker((lh) \wedge \ker(ik)) = 1 \). Since \( \eta_\mathcal{C} \) is surjective, the right adjoint \( (\eta_\mathcal{C})^\circ \) (see Section 2.3) is fully faithful, and it of course preserves intersections; so that \( I \) of (4.5) is a pullback if and only if \( (\eta_\mathcal{C})^\circ \ker((lh) \wedge \ker(ik)) = 1_R \). However \( (\eta_\mathcal{C})^\circ \ker(Ih) \) is the kernel of \( Ih \eta_\mathcal{C} \)—that is, by Proposition 3.1, of the meet \( h \wedge \eta_\mathcal{C} \)—and is therefore \( H \vee R_\mathcal{C} \), where \( H \) is the kernel of \( h \). Thus \( I \) of (4.5) is a pullback if and only if \( (H \vee R_\mathcal{C}) \wedge (K \vee R_\mathcal{C}) = 1_{R_\mathcal{C}} \), where \( K \) here is the kernel of \( k \). Since \( H \vee K = 1_{R_\mathcal{C}} \) because (4.5) is a pullback, this is the case if \( \text{Cong}_C \) is a distributive lattice. We conclude that, when \( \mathcal{C} \) is a Maltsev exact category for which every \( \text{Cong}_C \) is a distributive lattice, centrality with respect to \( \mathcal{X} \) of an extension coincides with triviality with respect to \( \mathcal{X} \), for any choice of the admissible subcategory \( \mathcal{X} \). Such a \( \mathcal{C} \) is the variety of Heyting algebras—we observed in Section 2.2 that it is Maltsev; while that it has distributive lattices \( \text{Cong}_A \) follows from the fact that the variety of lattices does so, using the result of Jónsson [7] that this property too is equivalent to the existence in the theory of certain ternary operations satisfying certain identities. Here a possible admissible \( \mathcal{X} \) is the subvariety of boolean algebras.

4.3.

Although it is rare in practical cases for every central extension to be trivial, it is quite common for every central extension that is a retraction to be trivial. Recall from Section 1.3 that an extension \( (A,f) \) is said to be normal if \( (A,f) \in \text{Spl}(A,f) \)—that is, if \( f^* (A,f) \) is trivial; of course every normal extension is central and every trivial extension normal.

Proposition 4.7. Every central extension that is a retraction is trivial if and only if every central extension is normal.

Proof. If \( f: A \to B \) is a normal extension that is a retraction, with say \( fi = 1 \), the triviality of \( f^*(A,f) \) gives by Proposition 4.1 the triviality of \( i*f^*(A,f) = 1*(A,f) = (A,f) \); this proves the “if” part. For the converse, observe that \( f^*(A,f) \) is always a retraction, and that, by Proposition 4.3, it is central when \( (A,f) \) is so. \( \square \)

The example \( (\mathcal{C}, \mathcal{X}) = (G\text{-Set}, \text{Set}) \) of Section 1.3 shows that, even when \( \mathcal{C} \) is a variety and \( \mathcal{X} \) is a subvariety, not all central extensions need be normal. We have, however:

Theorem 4.8. When \( \mathcal{C} \) is Goursat, every central extension that is a retraction is trivial; equivalently, every central extension is normal.
**Proof.** Let \( f': A \to B \) be an extension with \( fi = 1 \) that lies in \( \text{Spl}(E, p) \), and form the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
\downarrow{j} & & \downarrow{i} \\
D & \xrightarrow{s} & A \\
\downarrow{t} & & \downarrow{f} \\
E & \xrightarrow{p} & B 
\end{array}
\]  

where the bottom square is a pullback and where \( j \) is determined by the equations \( tj = 1 \) and \( sj = ip \). Write \( S, T, F \) for the congruences that are the kernels of \( s, t, f \). By Proposition 4.2, we have

\[
T \wedge R_D = 1, \quad TR_D = R_D T, \tag{4.7}
\]

since \( t \) is trivial; while to prove \( f \) trivial we must show that

\[
F \wedge R_A = 1, \quad FR_A = RA F. \tag{4.8}
\]

We show first that \( F \wedge R_A = 1 \). Consider the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{v} & F \wedge R_A \\
\downarrow{R_D} & & \downarrow{R_A} \\
D & \xrightarrow{s} & A
\end{array}
\]

in which the bottom square is an instance of (3.3) and the top square is a pullback. Since \( s \) is surjective and \( C \) is Goursat, it follows from Proposition 3.2 that \( R_s \) is surjective, whence its pullback \( v \) is surjective. Write \( k, k' \) for the two morphisms \( F \wedge R_A \to A \) in (4.9), and \( d, d' \) for the two morphisms \( K \to D \). To say that \( F \wedge R_A = 1 \) is to say that \( k = k' \); which, because \( v \) is surjective, is equally to say that \( sd = sd' \). We certainly have \( fsd = fsd' \), since \( fk = fk' \) because \( F \wedge R_A \leq F \).

Set \( e = td \) and \( a = sd \), so that \( pe = fa \); the morphism \( d \) into the pullback \( D \) is determined by the pair \( (e, a) \), and we may write \( d = (e, a) \). Similarly, \( d' = (e', a') \) where \( e' = td', a' = sd' \), and \( pe' = fa' \). By the last paragraph, we have \( fa = fa' \), and we are to prove that \( a = a' \).

Since \( pe = fa = fa' \), there is a morphism \( \bar{d} = (e, a') : K \to D \). We shall prove that the pair \( (d, d') \) of morphisms \( K \to D \) belongs, in the sense of Section 2.3, to the congruence \( R_D \). Then, because \( (d, d') \in R_D \) and \( R_D \) is a congruence, it follows that \( (d, \bar{d}) \in R_D \). However \( (d, \bar{d}) \in T \), since \( td = \bar{d} = e \), thus \( (d, \bar{d}) \in T \wedge R_D \). This being 1 by (4.7), we have \( d = \bar{d} \), which gives the desired \( a = a' \).
Having \( fa = fa' \), we may set \( \bar{a} = ifa = ifa' : K \to A \). Now consider the morphism \( je : K \to D \). We have \( tje = e \) since \( tj = 1 \), and we have

\[
se = tpe = iptd = ifsd = ifa = \bar{a},
\]

so that \( je = (e, \bar{a}) \). Similarly \( je' = (e', \bar{a}) \). Since \( sje = sje' \), we have \((je, je') \in S\); and since \((je, je') = (jtd, jtd')\), we have \((je, je') \in R_D\), because

\[
\eta_D jtd = lj. It. \eta_D d = lj. It. \eta_D d' = \eta_D jtd'.
\]

Thus \((je, je') \in S \land R_D\).

Since \( e = tje \) and \( e' = tje' \), it follows—see Section 2.3—that \((e, e') \in t \land (S \land R_D)\); and this relation, by Proposition 2.9, is the congruence \( t_\times (S \land R_D)\), \( \% \) being Goursat and \( t \) surjective. But \( td = e \) and \( td' = e' \), so that—see Section 2.3 again, and in particular (2.4)—we have \((\bar{d}, \bar{d}') \in t \land (S \land R_D)\); moreover, by Proposition 2.8, this last congruence is \( T \lor (S \land R_D)\). Again, since \( sd = a' = sd' \), we have \((\bar{d}, \bar{d}') \in S\), so that in fact \((\bar{d}, \bar{d}') \in S \land (T \lor (S \land R_D))\). Cong \( D \) being a modular lattice by Proposition 2.3, this last congruence is \((S \land T) \lor (S \land R_D)\), which is just \( S \land R_D\) since (the lower square of (4.6) being a pullback) we have \( S \land T = 1 \). Thus \((\bar{d}, \bar{d}') \in S \land R_D\); a fortiori, \((\bar{d}, \bar{d}') \in R_D\), as we promised to prove.

This completes the proof that \( F \land R_A = 1 \), and when \( \% \) is Mal'tsev it completes the proof of the theorem, by Proposition 4.2. When \( \% \) is merely Goursat, however, we need also the second assertion of (4.8). We have in turn:

\[
\begin{align*}
SR_DTS &= STR_D S & \text{since } R_D T = TR_D \text{ by (4.7)}; \\
SR_DST &= TSR_D S & \text{since } ST = TS \text{ by Proposition 2.10}; \\
(SR_D S)ST &= TS(SR_D S) & \text{since } S^2 = S; \\
(s \land s_\# R_D)ST &= TS(s \land s_\# R_D) & \text{by Propositions 2.8 and 2.9}; \\
(s \land s_\# R_D)(s \land F) &= (s \land F)(s \land s_\# R_D) & \text{since } s \land F = \ker(\eta) = S \lor T = ST; \\
(s \land R_A)(s \land F) &= (s \land F)(s \land R_A) & \text{by (3.4)};
\end{align*}
\]

\[
R_A F = FR_A
\]

by Proposition 2.8.

This completes the proof. \( \square \)

**Remark 4.9.** For the reader concerned only with Mal’tsev varieties, the proof above that \((\bar{d}, \bar{d}') \in R_D\) can be replaced by the following, where \( m \) is a Mal’tsev operation. Using \( \sim \) for the relation \( R_D \) we have

\[
\bar{d} = (e, a') = m((e', \bar{a}), (e', \bar{a}), (e, a'))
\]

\[
\sim m((e', \bar{a}), (e, \bar{a}), (e, a')) \text{ since } (e, \bar{a}) = je \sim je' = (e', \bar{a})
\]

\[
= (m(e', e, e), m(\bar{a}, \bar{a}, a')) = (e', a') = d'.
\]
5. Varieties of $\Omega$-groups

5.1.

We recalled in Section 1.4 what is meant by a variety of $\Omega$-groups; given such a variety $\mathscr{C}$ and a subvariety $\mathcal{X}$, we introduced the notation $RA$ for the classical kernel $\eta^{-1}_A(e)$ of $\eta_A: A \rightarrow IA$, noting that $R$ is an endofunctor of $\mathscr{C}$, since any $x: C \rightarrow A$ restricts to an $Rx: RC \rightarrow RA$. Of course the congruence $RA$ is related to $\mathcal{X}$ by the fact that $(a', a) \in RA$ if and only if $a'a^{-1} \in RA$. We have already remarked in Section 3.3 that, each such variety $\mathcal{X}$ being Maltsev and a fortiori Goursat, every subvariety $\mathcal{Y}$ is admissible. The following, which we do not need explicitly below, is well known:

**Proposition 5.1.** When $f: A \rightarrow B$ is surjective, so is $Rf$.

**Proof.** Let $b \in RB$, so that $(b, e) \in R_B$. Since $R_f$ is surjective by Proposition 3.2, there is an $(a', a) \in RA$ with $fa' = b$ and $fa = e$; now $a'a^{-1} \in RA$ and $f(a'a^{-1}) = b$. \qed

For the following theorem concerning an extension $f: A \rightarrow B$ in $\mathscr{C}$, we introduce some notation. We form the pullback

\[
\begin{array}{ccc}
C & \xrightarrow{\pi} & A \\
\downarrow i & & \downarrow f \\
A & \xrightarrow{f} & B
\end{array}
\]

so that $(C, t) = f^*(A, f)$; and we write $i: A \rightarrow C$ for the unique morphism with $ti = si = 1$. We number not this but its consequence

\[Rt . Ri = Rs . Ri = 1.\] (5.2)

**Theorem 5.2.** For a variety $\mathscr{C}$ of $\Omega$-groups and a subvariety $\mathcal{X}$, the following properties of an extension $(A, f)$, using the notation above, are equivalent:

(i) $(A, f)$ is central;
(ii) $(A, f)$ is normal;
(iii) $Rt$ is monomorphic;
(iv) $Rt$ is invertible;
(v) $Ri$ is surjective;
(vi) $Ri$ is invertible;
(vii) $Rt = Rs$;
(viii) for any morphisms $x, y: D \rightarrow A$ with $fx = fy$, we have $Rx = Ry$.

**Proof.** Since $\mathscr{C}$ is Goursat, (i) and (ii) are equivalent by Theorem 4.8. Since $\mathscr{C}$ is Maltsev, (ii)—which asserts that the extension $(C, t)$ is trivial—is equivalent by Proposition 4.2 to the assertion $T \wedge R_C = 1$, which says that $t$ and $\eta_C$ are jointly monomorphic; this is clearly equivalent to (iii). By (5.2), (iii)–(vi) are equivalent, and
imply (vii); however (vii) implies (iii)—for if \( c \in R_C \) with \( tc = e \) (the unit for the group structure), (vii) gives \( sc = e \), so that \( c = e \) since \( t \) and \( s \) are jointly monomorphic. Finally, (vii) is clearly equivalent to (viii).

As we said in Section 1.4, Fröhlich used (viii) as his definition of centrality, in this case of \( \Omega \)-groups; accordingly, his notion is a special case of ours. It is well known that, in turn, Fröhlich's notion includes many classical ones as special cases; in fact it is easy to verify this using the theorem above.

Take for instance the case where \( \mathcal{C} \) is the variety of groups and \( \mathcal{X} \) is that of abelian groups, so that \( RA \) is the commutator subgroup \([A, A]\). Let the normal subgroup \( K \) of \( A \) be the classical kernel of the surjection \( f: A \to B \). Taking \( C \) as in (5.1) again, an element of \( C \) is a pair \((a, a')\) with \( fa = fa' \), or equally a pair of the form \((a, ak)\), where \( k \in K \); and (vii) is the assertion that we then have \( a = a' \) if \((a, a') \in RC \). For \( a \in A \) and \( k \in K \), the commutator \([([a, e], [a, k]) = (e, [a, k])\); now (vii) gives \([a, k] = e \), so that \( K \) is in the centre of \( A \). Conversely, if \( K \) is in the centre of \( A \) and \( k, k' \in K \), the commutator in \( C \) of \((a, ak)\) and \((a', a'k')\) is \(([a, a'], [a, a'])\), so that (vii) is satisfied.

5.2.

For a pair \((\mathcal{C}, \mathcal{X})\) of varieties, centrality of an extension can be expressed in terms of elements and identities, although not as simply as in the classical cases above. There is however a notable simplification when \( \mathcal{C} \) is a Maltsev variety—or, slightly more generally, when \( \mathcal{C} \) is a Goursat variety and \( \mathcal{X} \) is such that we have the condition \( SR_C = R_C S \) of Theorem 3.4(c) for all congruences \( S \) on any object \( C \). We consider this case, recalling from Proposition 4.2 that \((A, f)\) is then trivial if and only if \( F \wedge RA = 1 \), where \( F \) is the kernel-congruence of \( f \).

Write \( \Phi_n \) for the set of those pairs \((\rho, \sigma)\) of (derived) \( n \)-ary operations of \( \mathcal{C} \) such that the identity \( \rho(x_1, ..., x_n) = \sigma(x_1, ..., x_n) \) is satisfied by every algebra in \( \mathcal{X} \), and write \( \Phi \) for the disjoint union \( \sum_{n \geq 0} \Phi_n \). By Proposition 3.2, \( \mathcal{C} \) being Goursat, \( R_n \) is surjective for each surjective \( g: D \to A \). Applying this where \( D \) is the free \( \mathcal{C} \)-algebra on the underlying set of \( A \), we get:

**Lemma 5.3.** The pair \((a, a') \in A \times A\) lies in \( R_A \) if and only if there exist \((\rho, \sigma) \in \Phi \) and \( a_1, ..., a_n \in A \) with \( a = \rho(a_1, ..., a_n) \) and \( a' = \sigma(a_1, ..., a_n). \)

Accordingly:

**Lemma 5.4.** The extension \( f: A \to B \) is trivial if and only if, for each pair \((\rho, \sigma) \in \Phi_n \) and each \((a_1, ..., a_n) \in A^n \), we have

\[
\rho(fa_1, ..., fa_n) = \sigma(fa_1, ..., fa_n) \quad \text{implies} \quad \rho(a_1, ..., a_n) = \sigma(a_1, ..., a_n). \]
Now Theorem 4.8 gives:

**Theorem 5.5.** Let $\mathcal{C}$ be a Goursat variety and $\mathcal{A}$ a subvariety satisfying $SR_C = R_C S$ for each congruence $S$ on $C$—as is always the case when $\mathcal{C}$ is a Maltsev variety. Then the extension $f : A \to B$ is central if and only if, for each pair $(\rho, \sigma) \in \Phi_n$ and each pair $(a_1, \ldots, a_n), (a'_1, \ldots, a'_n) \in A^n$ with $fa_i = f'a_i$ for each $i$, we have

$$\rho(a_1, \ldots, a_n) = \sigma(a_1, \ldots, a_n) \quad \text{implies} \quad \rho(a'_1, \ldots, a'_n) = \sigma(a'_1, \ldots, a'_n).$$

---

### 6. Classification of central extensions

#### 6.1.

A pair $(\mathcal{C}, \mathcal{A})$ consisting of an exact category $\mathcal{C}$ and an admissible subcategory $\mathcal{A}$ gives a **Galois structure** in the sense of [5], where we take for the classes $E$ and $Z$ of that paper the surjections in $\mathcal{C}$ and in $\mathcal{A}$ respectively; moreover (by the admissibility) this structure satisfies the condition of [5, Corollary 6.9]. What is more, every extension $p : E \to B$ in an exact category is a **monadic** extension, in the sense of [5, Definition 6.7]. Accordingly the generalized Galois theory of [5] provides, as follows, a classification of central extensions in terms of internal actions of the "Galois pregroupoid".

We recall that an **internal precategory** $P$ in a category $\mathcal{A}$ is a diagram

$$
\begin{array}{ccc}
P_2 & \xrightarrow{a} & P_1 & \xrightarrow{0} & P_0 \\
\downarrow{m} & & \downarrow{c} & & \\
b & \xrightarrow{d} & \end{array}
$$

satisfying

$$de = 1 = ce, \quad da = cb, \quad dm = db, \quad cm = ca;$$

it is of course an **internal category** when

$$
\begin{array}{ccc}
P_2 & \xrightarrow{b} & P_1 \\
\downarrow{a} & & \downarrow{c} \\
P_1 & \xrightarrow{d} & P_0
\end{array}
$$

is a pullback and the "composition" $m$ satisfies the appropriate associativity and unit axioms. Certain internal precategories—see Remark 5.5 of [5]—are called **internal pregroupoids**; an internal category $P$ is an internal pregroupoid precisely when it is an internal groupoid in the usual sense.
An internal $P$-action (called in [5] an internal functor $P \rightarrow \mathcal{X}$) is a triple $F = (F_0, \pi, \xi)$ where $\pi: F_0 \rightarrow P_0$ and where $\xi: P_1 \times_{P_0} F_0 \rightarrow F_0$: here the domain of $\xi$ is the fibred product $P_1 \times_{(a, b)} F_0$, and these data are to make commutative the three diagrams

\[ \begin{array}{ccc}
\pi & F_0 & \pi \\
\downarrow & \downarrow & \downarrow \\
F_0 & F_0 & F_0
\end{array} \]

\[ \begin{array}{ccc}
\xi & F_0 \times_{P_0} F_0 & \xi \\
\downarrow & \downarrow & \downarrow \\
F_0 & F_0 & F_0
\end{array} \]

\[ \begin{array}{ccc}
P_1 \times_{P_0} F_0 & \xi & F_0 \times_{P_0} F_0 \\
\downarrow & \downarrow & \downarrow \\
P_0 & F_0 & F_0
\end{array} \]

There is an evident category $\mathcal{X}'$ with such actions as objects; let us write $\{P, \mathcal{X}\}$ for the full subcategory of this determined by those actions for which $\pi: F_0 \rightarrow P_0$ is a surjection.

Each extension $p: E \rightarrow B$ determines an internal groupoid

\[ \begin{array}{ccc}
E \times_B E & E \times_B E & E \\
\downarrow & \downarrow & \downarrow \\
E & E & E
\end{array} \]

in $\mathcal{X}$, which is nothing but the kernel-congruence of $p$ seen (as any ordered set may be) as a category. Applying $I: \mathcal{X} \rightarrow \mathcal{X}$ to (6.1) gives an internal pregroupoid in $\mathcal{X}$, called the Galois pregroupoid of $(E, p)$, and denoted by $\text{Gal}(E, p)$, or $\text{Gal}(E, p)$ for short. Corollary 6.9 of [5] now gives an equivalence of categories

\[ \text{Spl}(E, p) \simeq \{\text{Gal}(E, p), \mathcal{X}\}. \] (6.2)

Remark 6.1. When the extension $(E, p)$ is normal—the case studied in [4]—$\text{Gal}(E, p)$ is in fact an internal groupoid. If, moreover, $IE = 1$ (whereupon $E$ is said to be perfect, or connected), $\text{Gal}(E, p)$ is an internal group; in this case $B$, too, is necessarily connected.

In view of the remarks in Section 1.3, (6.2) gives:

**Theorem 6.2.** Given the object $B$, if there exists (as is always the case when $\mathcal{C}$ is a variety) a surjection $p: E \rightarrow B$ with $E$ projective, we have an equivalence

\[ \text{Centr } B \simeq \{\text{Gal}(E, p), \mathcal{X}\}. \]
References