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Detecting Flat Normal Cones Using Segre Classes

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Given a flat, projective morphism $Y \to T$ from an equidimensional scheme to a nonsingular curve and a subscheme Z of Y, we give conditions under which specialization of the Segre class $s(N_Z Y)$ of the normal cone of Z in Y implies flatness of the normal cone. We apply this result to study when the relative tangent star cone of a flat family is flat.

Key Words: associated graded algebra; blowup; Segre class; specialization; tangent cones.

1. INTRODUCTION

This paper, situated on the cusp of commutative algebra and algebraic geometry, is concerned with the geometry of families of blowups. Recall that the *blowup* of an affine scheme Spec A along a subscheme Spec A/I is $\operatorname{Proj}(R_I(A))$, where $R_I(A)$ is the *Rees algebra* $A \oplus I \oplus I^2 \oplus \cdots$, and that its *exceptional divisor* is $\operatorname{Proj}(\operatorname{gr}_I(A))$, where $\operatorname{gr}_I(A)$ is the *associated graded ring* $A/I \oplus I/I^2 \oplus \cdots$. The exceptional divisor can also be regarded as the projectivization of the *normal cone* $\operatorname{Spec}(\operatorname{gr}_I(A))$. In algebraic geometry the *center* Spec A/I is often a nonsingular variety, but the definitions make

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sense in general. They also make sense for an arbitrary scheme X, if one uses the structure sheaf $\mathcal{O}_X$ in place of the ring A and if I is now regarded as the ideal sheaf of a subscheme.

Using Segre classes, we will examine how the normal cone varies in a one-parameter family. Suppose that $Y \to T$ is a flat, projective morphism from an equidimensional scheme to a nonsingular curve; let Z be a subscheme of Y. We show that, under a certain hypothesis, if the Segre class of the normal cone of Z in Y specializes to the corresponding class for the fiber Z_t in Y_t over a point of $t \in T$, then the normal cone is flat above this point. (See Theorem 4.1 for a precise statement.)

Modern intersection theory (as developed by Fulton and MacPherson—see [3]) makes Segre classes a central notion. These classes play the same role for cone bundles that Chern classes do for vector bundles: they measure twisting over the base. Moreover, Segre classes are sensitive to flatness. If C is a bundle of cones over the T-scheme X and if C is flat over T above a point t, then the Segre class of C over X (an element of A_*X) specializes to the Segre class of C_t over X_t. In a very rough sense, our Theorem 4.1 reverses this implication in the case of normal cones.

Because we hope that our results will be of interest to both algebraic geometers and commutative algebraists, we need to review some fundamental notions. Thus in Section 2 we briefly recall basic definitions about cones and their Segre classes. In Section 3 we describe our algebraic notion of "internal flatness" (which follows either from flatness or from Serre's S_1 condition) and provide a computational criterion for detecting it. In Section 4 we present our main theorem. Finally, in Section 5, we examine various examples in the case where Y is a fiber product $X \times_T X$ and the subscheme Z is X embedded as the diagonal. The normal cone in this case is known as the (*relative*) *tangent star cone*. These examples suggest that Theorem 4.1 is probably optimal.

To be fully useful, our Theorem 4.1 requires a "front end" guaranteeing that certain families of normal cones satisfy the internal flatness condition. We know of one result in this direction: the tangent star cone of a hypersurface is Cohen–Macaulay [6]. Other such results would be desirable. On the other hand, for tangent star cones the Segre class specialization condition is readily checked, since—as explained at the beginning of Section 4 of [7]—the Segre classes of a projective scheme can be computed from the double point classes of generic linear projections.

2. CONES AND SEGRE CLASSES

All schemes considered in this paper are over a field of characteristic zero. If S is a graded sheaf of $\mathcal{O}_X$-algebras on a scheme X, we say that

$C = \mathrm{Spec}(S)$ is a *cone*, and we call the schemes $\mathbf{P}(C) = \mathrm{Proj}(S)$ and $\mathbf{P}(C \oplus 1) = \mathrm{Proj}(S \oplus \mathcal{O}_X)$ its *projectivization* and *projective completion*, respectively. Following Chapter 4 of [3] or Section 2.3 of [2], we define the *Segre class* of C to be

$$s(C) = \sum_{i \geq 0} q_*(c_1(\mathcal{O}(1))^i \cap [\mathbf{P}(C \oplus 1)]),$$

an element of the Chow group of algebraic cycles on X. (Here q denotes the projection map from $\mathbf{P}(C \oplus 1)$ to X.)

Among the basic properties of the Segre class, we wish to take special note of its behavior under specialization (Example 10.1.10 of [3]): if X is a scheme over T, with fiber X_t over a particular point t, and if C is flat over T, then $s(C)$ specializes to $s(C_t)$. Our Theorem 4.1 can perhaps be regarded as a kind of converse, in the case of normal cones.

3. INTERNAL FLATNESS

If $(A, \mathfrak{m})$ is a discrete valuation ring, we call $T = \mathrm{Spec}(A)$ a *nonsingular curve germ*; its *closed point* is $t = \mathrm{Spec}(A/\mathfrak{m}A)$. For a morphism $Y \to T$, the fiber over t will be denoted by Y_t. If Y is affine, say $Y = \mathrm{Spec}(B)$, we say that the morphism is *internally flat* if each embedded prime of B contracts to zero in A. To say this geometrically, the morphism is internally flat if each embedded component of Y dominates T. More generally (for Y not necessarily affine), we say that a morphism $Y \to T$ is internally flat if for each affine open subset of Y the induced morphism to T is internally flat. Note that if Y has no embedded components (in particular, if it is Cohen–Macaulay) or if the morphism $Y \to T$ is flat, then it is internally flat. Here is a computational criterion:

PROPOSITION 3.1. *Suppose that $Y \to T$ is a morphism of affine varieties corresponding to the ring homomorphism $A \to B$, where A is a discrete valuation ring with uniformizing parameter τ and B is $A[x_1, \ldots, x_n]/I$ with I equicodimensional* (*i.e., all minimal associated primes of I have the same height*). *Let $J \subset I$ be an ideal generated by a regular sequence whose length is height*(I). *Then*

1. *Y has no embedded components $\Longleftrightarrow J : (J : I) \subset I$.*
2. *$Y \to T$ is flat $\Longleftrightarrow I : \tau \subset I$.*
3. *$Y \to T$ is internally flat $\Longleftrightarrow (I : \tau) \cap (J : (J : I)) \subset I$.*

Proof. Given an irredundant primary decomposition of I, the ideal $J : (J : I)$ is the intersection of the minimal primary components. (See Proposition 3.3.1 of [10]; cf. Section 2.1 of [11].) The ideal $I : \tau$ is the intersection of the primary components contracting to zero. (See Proposition 9.7 of [4] and its proof.) Thus $(I : \tau) \cap (J : (J : I))$ is the intersection of the minimal primary components and those contracting to zero. ∎

The following proposition is similar to the Corollary to Theorem 23.1 in [8].

PROPOSITION 3.2. *Suppose that $f\colon Y \to T$ is an internally flat morphism from an equidimensional scheme to a nonsingular curve germ. Suppose that the fiber Y_t has the expected dimension, namely,* $\dim Y - 1$. *Then f is flat.*

Proof. We may assume that Y is affine. Thus $f\colon Y \to T$ corresponds to a homomorphism $\varphi\colon A \to B$ from a discrete valuation ring to an equidimensional ring. Let $F = B \otimes_A A/\mathfrak{m}A$ be the fiber ring and let $\tau \in A$ be a uniformizing parameter. If φ is not flat, then $\varphi(\tau)$ is a zero divisor and thus is an element of some associated prime $\mathfrak{q}$ of B. Since $\mathfrak{q}$ does not contract to zero, it must be minimal. Thus $\dim F \geq \dim(B/\mathfrak{q}) = \dim B$. ∎

4. DETECTING FLATNESS

Now we state our main theorem.

THEOREM 4.1. *Suppose that*:

1. *$Y \to T$ is a flat, projective morphism from an equidimensional scheme to a nonsingular curve germ (with closed point t) and Z is a subscheme of Y.*
2. *The normal cone $N_Z Y$ is internally flat over T.*
3. *The Segre class $s(N_Z Y)$ specializes to $s(N_{Z_t} Y_t)$.*

Then $N_Z Y$ is flat over T and $(N_Z Y)_t = N_{Z_t} Y_t$.

We first prove a weak version of Theorem 4.1, with an additional hypothesis and a weaker conclusion.

LEMMA 4.1. *Suppose that*:

1. *$Y \to T$ is a flat, projective morphism from an equidimensional scheme to a nonsingular curve germ and Z is a subscheme of Y.*
2. *$N_Z Y$ is internally flat over T.*
3. *$s(N_Z Y)$ specializes to $s(N_{Z_t} Y_t)$.*
4. *Z is nowhere dense in Y.*

Then $\mathbf{P}N_Z Y$ is flat over T and $(\mathbf{P}N_Z Y)_t = \mathbf{P}N_{Z_t} Y_t$.

Proof. Assume hypotheses 1, 2, and 4 of Lemma 4.1, but that $\mathbf{P}N_Z Y$ is not flat over T. We will first show that $\mathbf{P}N_Z Y$ does not specialize to $\mathbf{P}N_{Zt} Y_t$, and subsequently that the Segre class fails to specialize (i.e., that hypothesis 3 does not hold).

Since Y is equidimensional, so is $\mathbf{P}N_Z Y$, and since $N_Z Y \to T$ is internally flat, so is the morphism $\mathbf{P}N_Z Y \to T$. Thus by Proposition 3.2,

$$\dim(\mathbf{P}N_Z Y)_t > \dim \mathbf{P}N_Z Y - 1$$

and thus the dimension of $(\mathbf{P}N_Z Y)_t$ exceeds that of $\mathbf{P}N_{Z_t} Y_t$. This tells us that $\mathbf{P}N_{Zt} Y_t$ is properly contained in $(\mathbf{P}N_Z Y)_t$.

Now we turn to Segre classes. In view of hypothesis 4 and Example 4.1.2 of [3], the Segre class of the normal cone may be calculated using

$$s(N_Z Y) = \sum_{i \geq 0} p_*(c_1(\mathscr{O}(1))^i \cap [\mathbf{P}N_Z Y]), \tag{1}$$

where p: $\mathbf{P}N_Z Y \to Z$ denotes projection. Because $\mathbf{P}N_{Z_t} Y_t$ is a proper subscheme of $(\mathbf{P}N_Z Y)_t$, the blowup $\mathrm{Bl}_{Z_t} Y_t$ is likewise a proper subscheme of $\mathrm{Bl}_Z Y)_t$. Both of these schemes are equidimensional and have the same dimension, namely, $\dim Y - 1$. Hence the difference of fundamental classes

$$D = [(\mathrm{Bl}_Z Y)_t] - [\mathrm{Bl}_{Z_t} Y_t]$$

must be a positive cycle of dimension $\dim Y - 1$. (As usual, the fundamental class of a scheme ignores embedded components.) Moreover, since the two cycles comprising D can only disagree over the center of the blowup, D must be supported on $(\mathrm{P}N_Z Y)_t$.

The exceptional (Cartier) divisor $\mathscr{E}$ on $\mathrm{Bl}_Z Y$ restricts to $c_1(\mathscr{O}(-1))$ on $\mathbf{P}N_Z Y$. Thus

$$-[(\mathbf{P}N_Z Y)_t] + [\mathbf{P}N_{Z_t} Y_t] = -\mathscr{E} \cap D = c_1(\mathscr{O}(1)) \cap D.$$

(Note that the first cap product takes place on the blowup, and the second in $(\mathbf{P}N_Z Y)_t$.) Thus, by (1), the difference between the Segre classes of $N_{Z_t} Y_t$ and $(N_Z Y)_t$ is given by

$$s(N_{Z_t} Y_t) - s((N_Z Y)_t) = \sum_{i \geq 0} \pi_* \left(c_1(\mathscr{O}(1))^{i+1} \cap D\right),$$

where π: $(\mathbf{P}N_Z Y)_t \to Z_t$ is projection. To show that this cycle class is nonzero, we consider an arbitrary component V of D. Set $d = \dim V - \dim \pi(V)$. Then

$$\pi_*(c_1(\mathscr{O}(1))^j \cap V) = 0$$

if $j < d$, while

$$\pi_*(c_1(\mathscr{O}(1))^d \cap V) = m[\pi(V)]$$

for some positive integer m. (Note that the dimension of V is $\dim Y - 1$ and that $\dim \pi(V) \leq \dim Z_t = \dim Z - 1 \leq \dim Y - 2$, so that $d \geq 1$.) Now suppose $V_1, V_2, \ldots, V_n$ are those components of D for which d achieves its minimum value d_0. Then $s(N_{Z_t}Y_t)$ and $s((N_Z Y)_t)$ will differ in dimension d_0 by a *positive* linear combination of $[\pi(V_1)], [\pi(V_2)], \ldots, [\pi(V_n)]$. Since Z_t is projective, this linear combination cannot be the zero class. ∎

Proof of Theorem 4.1. Let C be an elliptic curve. Then the morphism $Y := Y \times C \times C \to T$ (projection onto the first factor followed by $Y \to T$) is a flat projective morphism from an equidimensional scheme. The subscheme $\widetilde{Z} := Z \times C$, embedded in $\widetilde{Y}$ via $(z, c) \mapsto (z, c, c)$, is nowhere dense in $\widetilde{Y}$, and its normal cone

$$N_{\widetilde{Z}}\widetilde{Y} \cong N_Z Y \times N_C(C \times C) = N_Z Y \times TC$$

is internally flat over T since $N_Z Y$ is. The Segre class of $N_{\widetilde{Z}}\widetilde{Y}$ is obtained from that of $N_Z Y$ by pullback via the projection of $Z \times C \to Z$. Similarly, $s(N_{\widetilde{Z}_t}\widetilde{Y}_t)$ is obtained from $s(N_{Z_t}Y_t)$ by pullback via the morphism $Z_t \times C \to Z_t$. Therefore, the Segre class $s(N_{\widetilde{Z}}\widetilde{Y})$ specializes to $s(N_{\widetilde{Z}_t}\widetilde{Y}_t)$.

By Lemma 4.1, we see that $\mathbf{P}N_{\widetilde{Z}}\widetilde{Y}$ is flat over T and $(\mathbf{P}N_{\widetilde{Z}})\widetilde{Y}_t = \mathbf{P}N_{\widetilde{Z}_t}\widetilde{Y}_t$. Now $\mathbf{P}N_{\widetilde{Z}}\widetilde{Y}$ is naturally isomorphic to the product of the projective completion of $N_Z Y$ with C. Hence $\mathbf{P}(N_Z Y \oplus 1)$ is flat over T and $(\mathbf{P}N_Z Y)_t = \mathbf{P}N_{Z_t}Y_t$. Thus $N_Z Y$ is flat over T and $(N_Z Y)_t = N_{Z_t}Y_t$. ∎

5. EXAMPLES: TANGENT STAR CONES

As mentioned in the Introduction, the *tangent star cone* of a scheme X is the normal cone $N_X(X \times X)$, where X is embedded in $X \times X$ as the diagonal. In particular, if I is the ideal sheaf of X in $X \times X$, then

$$\mathrm{TS}(X) = \mathrm{Spec}\left(\bigoplus_{j \geq 0} I^j/I^{j+1}\right).$$

The *projectivized tangent star cone*,

$$\mathbf{P}\,\mathrm{TS}(X) = \mathrm{Proj}\left(\bigoplus_{j \geq 0} I^j/I^{j+1}\right),$$

is the exceptional divisor of the blowup $\mathrm{Bl}_X(X \times X)$. When X is a reduced subscheme of $\mathbf{A}^n$, we can understand this blowup as the closure of the image of the map $(X \times X)\backslash X \to \mathbf{A}^n \times \mathbf{A}^n \times \mathbf{P}^{n-1}$ defined by

$$(p_1, p_2) \mapsto (p_1, p_2, \overline{p_1 p_2}),$$

where $\overline{p_1 p_2}$ denotes the line through the origin parallel to the line through the (distinct) points p_1 and p_2. Thus, $\mathbf{P}\,\mathrm{TS}\,(X)$ is the subscheme of the blowup that lies over the diagonal. The fiber of $\mathbf{P}\,\mathrm{TS}\,(X)$ over a point of X consists, set-theoretically, of limiting secants, called the *tangent star* by Johnson [5]. For an extensive study of tangent star cones, see [9].

The tangent star construction above carries over entirely analogously to the relative case: if X is a scheme over a nonsingular variety T, the *relative tangent star cone*, denoted $\mathrm{TS}\,(X/T)$, is $N_X(X \times_T X)$. As above, we have $\mathrm{TS}\,(X/T) = \mathrm{Spec}(\bigoplus_{j\geq 0} I^j/I^{j+1})$, where I denotes the ideal sheaf of the diagonal copy of X in $X \times_T X$.

If we apply Theorem 4.1 using $X \times_T X \to T$ and letting Z be the diagonal copy of X, we immediately obtain the following result for tangent star cones.

COROLLARY 5.1. *Suppose that*:

1. *$X \to T$ is a flat, projective morphism from an equidimensional scheme to a nonsingular curve germ* (*with closed point t*).
2. *The tangent star cone* $\mathrm{TS}\,(X/T)$ *is internally flat over T.*
3. *The Segre class $s(\mathrm{TS}\,(X/T))$ specializes to $s(\mathrm{TS}\,(X_t))$.*

Then $\mathrm{TS}\,(X/T)$ *is flat over T and its fiber over t is* $\mathrm{TS}\,(X_t)$.

The Segre classes of relative tangent star cones were studied in [7]. We quote Theorem 3 from that paper, using the notation s_k for the codimension k component of the Segre class.

THEOREM 5.1. *Suppose that $Y \to T$ is a smooth morphism to a nonsingular variety. Suppose that X is a purely codimension r subscheme of Y, and that the composite $X \to T$ is proper and flat. For a closed point t of T, let X_t be the fiber. If the Segre classes $s_0(\mathrm{TS}\,(X/T)), \ldots, s_{r-1}(\mathrm{TS}\,(X/T))$ specialize to the corresponding classes $s_0(\mathrm{TS}\,(X_t)), \ldots, s_{r-1}(\mathrm{TS}\,(X_t))$, then the same is true of the higher codimension Segre classes $s_r(\mathrm{TS}\,(X/T))$, $s_{r+1}(\mathrm{TS}(X/T))$, etc.*

Combining Theorem 5.1 with Corollary 5.1, we can strengthen that corollary in many instances.

COROLLARY 5.2. *Suppose that*:

1. *$Y \to T$ is a smooth morphism to a nonsingular curve germ.*
2. *X is a purely codimension r subscheme of Y, and the composite $X \to T$ is projective and flat.*
3. *The tangent star cone* $\mathrm{TS}\,(X/T)$ *is internally flat over T.*

4. *The Segre classes* $s_0(\mathrm{TS}\,(X/T)), \ldots, s_{r-1}(\mathrm{TS}\,(X/T))$ *specialize to the corresponding classes* $s_0(\mathrm{TS}\,(X_t)), \ldots, s_{r-1}(\mathrm{TS}\,(X_t))$.

Then $\mathrm{TS}\,(X/T)$ *is flat over* T *and its fiber over* t *is* $\mathrm{TS}\,(X_t)$.

If X is a hypersurface in Y, then there are two especially pleasant features. First, by Corollary 5.2, one needs to investigate only the top Segre class $s_0(\mathrm{TS}\,(X/T))$. If X has irreducible components $X_1, X_2, \ldots$ with corresponding geometric multiplicities $m_1, m_2, \ldots,$ then

$$s_0(\mathrm{TS}\,(X/T)) = \sum (m_k)^2 [X_k].$$

(See Theorem 4 of [7].) Thus $s_0(\mathrm{TS}\,(X/T))$ specializes to $s_0(\mathrm{TS}\,(X_t))$ if and only if the following two criteria are satisfied:

1. Each component X_k specializes to a reduced subscheme of the fiber X_t.

2. The specializations of distinct components have no components in common.

Following the terminology of [6], we say that the components of X *do not coalesce* when criteria 1 and 2 hold.

The second pleasant feature is that $\mathrm{TS}\,(X/T)$ is automatically Cohen–Macaulay. Thus we have the following corollary.

COROLLARY 5.3. *Suppose that*:

1. $Y \to T$ *is a smooth morphism to a nonsingular curve germ*.

2. X *is a hypersurface in* Y, *and the composite* $X \to T$ *is projective and flat*.

3. *The components of* X *do not coalesce under specialization*.

Then $\mathrm{TS}\,(X/T)$ *is flat over* T *and its fiber over* t *is* $\mathrm{TS}\,(X_t)$.

In [6] the proof that $\mathrm{TS}\,(X/T)$ is Cohen–Macaulay is entangled with the proof of the Corollary 5.3, and both proofs employ the following explicit local description of the tangent star cone. Suppose Y is a subvariety of $\mathbf{A}^n \times T$. Let $x_1, \ldots, x_n$ be coordinates on $\mathbf{A}^n$ and let $u_1, \ldots, u_n$ be coordinates for the tangent bundle with respect to $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Suppose that X is defined in Y by the equation $f(x_1, \ldots, x_n, t) = 0$ and write this polynomial as

$$f = \prod_{k=1}^{s} f_k^{r_k},$$

where the f_k's are reduced, irreducible, and distinct. Define the *polarization operator* P by

$$P = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}$$

and let P^d denote its dth iterate. Let

$$S_m f = \Big(\prod_{r_k<m} f_k^{r_k}\Big)^2 P^{2m-1}\Big(\prod_{r_k\geq m} f_k^{r_k+m-1}\Big).$$

(Note that $S_1 f = Pf$ and $S_m f = 0$ for m sufficiently large.) Then the tangent star cone is defined inside the tangent bundle of Y by the equations $S_1 f = \cdots = S_m f = 0$ for m sufficiently large. We would like to find a more conceptual argument that $\mathrm{TS}\,(X/T)$ is Cohen–Macaulay, thus disentangling the two proofs and, we hope, giving insight into other situations where one should expect the relative tangent star cone to be Cohen–Macaulay (or, more weakly, to satisfy Serre's S_1 condition).

In the remainder of this section we look at various other examples of tangent star cones.

EXAMPLE 5.1. In view of the hypersurface case, it is tempting to conjecture that if $X \subset Y$ is a local complete intersection, then $\mathrm{TS}\,(X)$ is Cohen–Macaulay. Such a conjecture is false, however. For example, let X be the complete intersection in $\mathbf{A}^3$ defined by the equations $xy = z(z - x) = 0$. Then X is a union of three lines, one of which is thickened. Using a, b, c as coordinates with respect to $\partial/\partial x, \partial/\partial y, \partial/\partial z$ and calculating using CoCoA [1], we find that the ideal I of the tangent star cone in $R = k[x, y, z, a, b, c]$ is

$$I = \langle xy, z(z-x), za + xc - 2zc, ya + xb, c^2(2xb - zb + yc), \\ bc^2(a^2 - 2ac + c^2)\rangle.$$

Then a resolution calculation for R/I shows that its projective dimension is 5. But the codimension of $\mathrm{TS}\,(X)$ inside the tangent bundle of $\mathbf{A}^3$ is 4. Thus the tangent star cone is not Cohen–Macaulay.

EXAMPLE 5.2. Let X be the flat family in $\mathbf{A}^3$ defined over the affine line T by $xy = z(z - tx) = 0$; all fibers except the central one are isomorphic to the scheme in Example 5.1. The ideal of the relative tangent star cone in $R = k[x, y, z, a, b, c, t]$ is

$$I = \langle xy, z(z - tx), zat + xct - 2zc, ya + xb, \\ c^2(2xbt - zb + yc), bc^2(a^2t^2 - 2act + c^2)\rangle.$$

This family fails to be internally flat over T (and hence cannot be Cohen–Macaulay either) by the criterion given in Proposition 3.1: use

$$J = \langle xy, z(z - tx), ya + xb, y^2c^3 + a^2bc^2t^2 - 2abc^3t \\ + bc^4 + x^2ct + z^2a - 2xzc\rangle$$

for "test ideal" and compare $(I : t) \cap (J : (J : I))$ and I.

EXAMPLE 5.3. This example shows that the internal flatness hypothesis cannot be omitted from Theorem 4.1. Let X be the flat family in $\mathbf{A}^3$ defined over the affine line T by the ideal

$$\begin{aligned}&\langle x, z\rangle \cap \langle y, z\rangle \cap \langle x-y, z-tx\rangle \\ &\qquad = \langle z(x-y), xy(x-y), z(z-ty), y(z-tx)\rangle.\end{aligned}$$

The general member of X is the union of three concurrent lines; the special member has planar reduction. The ideal of $\mathrm{TS}\,(X/T)$ is

$$\begin{aligned}I = \langle & z(x-y), xy(x-y), z(z-ty), y(z-tx), \\ & za - zb + xc - yc, zbt + yct - 2zc, yat + xbt - zb - yc, \\ & 2xya - y^2a + x^2b - 2xyb, c(abt^2 - act - bct + c^2)\rangle.\end{aligned}$$

This family fails to be internally flat over T by the criterion given in Proposition 3.1: we use

$$\begin{aligned}J = \langle & z(x-y), y(z-tx), za - zb + xc - yc, \\ & (zbt + yct - 2zc) + (yat + xbt - zb - yc) \\ & + \left(xya - \tfrac{1}{2}y^2a + \tfrac{1}{2}x^2b - xyb\right) + (abct^2 - ac^2t - bc^2t - c^3)\rangle\end{aligned}$$

for test ideal.

We now show that $s(\mathrm{TS}\,(X/T))$ specializes to $s(\mathrm{TS}\,(X_t))$. Writing the general member X_t of the family as the union $X_1 \cup X_2 \cup X_3$ of three lines and using Theorem 4 of [7], we obtain

$$\begin{aligned}s(\mathrm{TS}\,(X_t)) = {} & s(\mathrm{TS}\,(X_1)) + s(\mathrm{TS}\,(X_2)) + s(\mathrm{TS}\,(X_3)) \\ & + 2s(X_1 \cap X_2, X_1 \times X_2) + 2s(X_1 \cap X_3, X_1 \times X_3) \\ & + 2s(X_2 \cap X_3, X_2 \times X_3).\end{aligned}$$

Now $X_i \times X_j \cong \mathbf{A}^2$ and $X_i \cap X_j$ is the origin. Hence $s(X_i \cap X_j, X_i \times X_j) = [p]$, the class of a point. Therefore,

$$\begin{aligned}s(\mathrm{TS}\,(X_t)) &= [X_1] - 2[p] + [X_2] - 2[p] + [X_3] - 2[p] + 6[p] \\ &= [X_1] + [X_2] + [X_3] = [X_t].\end{aligned}$$

Exactly the same calculation applies when $t = 0$, so that $s(\mathrm{TS}\,(X_0)) = [X_0]$. Thus the Segre class specializes.

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