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Ideals with larger projective dimension and regularity

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ABSTRACT

We define a family of homogeneous ideals with large projective dimension and regularity relative to the number of generators and their common degree. This family subsumes and improves upon constructions given by Caviglia (2004) and McCullough (2011). In particular, we describe a family of three-generated homogeneous ideals, in arbitrary characteristic, whose projective dimension grows asymptotically as a power of the degree of the generators.
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1. Introduction

Throughout this paper, let K be a field of any characteristic and set $R = K[x_1, \ldots, x_n]$. We consider the following question of Stillman:

Question 1.1 (*Stillman, Peeva and Stillman (2009, Problem 3.14)*)**.** *Fix a sequence of natural numbers $d_1, \ldots, d_N$. Does there exist a number p (independent of n) such that*

$$\mathrm{pd}(R/I) \leq p$$

for all graded ideals I with a minimal system of homogeneous generators of degrees $d_1, \ldots, d_N$?

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This question is open in all but low degree cases. In Zhang (2011), Zhang's work on local cohomology modules in characteristic p suggested that $\sum_{i=1}^{N} d_i$ was an upper bound for $\mathrm{pd}(R/I)$. In McCullough (2011), the second author showed this was false by producing a family of ideals whose projective dimensions far exceeded this bound. However, in the three-generated ideal case, these ideals had projective dimension of only $d+2$ where d is the common degree of the generators. To the best of our knowledge there were no known ideals with three degree d generators with larger projective dimension. Clearly then $d+2$ is a lower bound for any answer to the three-generated case of Stillman's Conjecture. We note that by the work of Burch (1968) and later Bruns (1976), it is natural to ask whether any three-generated ideals in degree d have larger projective dimension than this.

In this paper we generalize the family of ideals given in McCullough (2011) to a larger family with much larger projective dimension. In the three-generated case, we produce a family of ideals with generators of degree d and projective dimension larger than $\sqrt{d}^{\sqrt{d}-1}$. We therefore give a new lower bound for any answer to Stillman's question.

The paper is organized as follows. In Section 2 we recall some previous results and definitions. In Section 3 we define our family of ideals and compute its projective dimension. In Section 4 we compute some specific examples and show that this family subsumes two interesting families of ideals previously defined. We conclude with some computations and questions regarding the Castelnuovo–Mumford regularity of these ideals.

2. Preliminaries

Let $R = K[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_N)$ be a homogeneous ideal and set $d_i = \deg(f_i)$. Let $F_\bullet$ be the minimal graded free resolution of R/I. Then we may write

$$F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}},$$

where $R(-j)$ denotes a rank one free module with generator in degree j. In this case $F_0 = R$ and $F_1 = \bigoplus_{j=1}^{N} R(-d_j)$. The exponents $\beta_{i,j}$ are called the Betti numbers of R/I. We can define the projective dimension of R/I as

$$\mathrm{pd}(R/I) = \max\{i \mid \beta_{i,j} \neq 0 \text{ for some } j\}.$$

Thus, Stillman's question can be rephrased by asking if $\mathrm{pd}(R/I)$ is bounded by a formula dependent only on $\beta_{1,j}$.

The Castelnuovo–Mumford regularity of R/I is defined as

$$\mathrm{reg}(R/I) = \max\{j - i \mid \beta_{i,j} \neq 0 \text{ for some } i\}.$$

The Betti numbers are often displayed in matrix form called a Betti table. In the (i, j) entry we put $\beta_{i,j-i}$. Thus, we can view the projective dimension of R/I as the index of the last nonzero column in the Betti table and the regularity of R/I as the index of the last nonzero row.

Let $\mathfrak{m}$ be the graded maximal ideal of R. We also denote the length of the maximal regular sequence on R/I contained in $\mathfrak{m}$ by $\mathrm{depth}(R/I)$. Finally, we let $\mathrm{socle}(R/I)$ denote $\{x \in R/I \mid x\mathfrak{m} = 0\}$. To compute projective dimension, we make use of the graded version of the Auslander–Buchsbaum Theorem (see, e.g., Eisenbud (1995, Theorem 19.9)), so in order to show that R/I has maximal projective dimension, we need only show that $\mathrm{socle}(R/I) \neq 0$.

Further motivating Question 1.1 is Problem 3.15 of Peeva and Stillman (2009) is an analog of Stillman's question for regularity: Is there a bound for $\mathrm{reg}(R/I)$ dependent only on $d_1, \ldots, d_N$? Caviglia showed that this question is equivalent to Question 1.1. See Engheta (2005), pages 11–14 for a nice explanation of this argument.

It is clear that there is an affirmative answer to Stillman's question when $N \leq 2$ or when $d_1 = \cdots = d_N = 1$. Eisenbud and Huneke (in unpublished work) verified the case $N = 3$ and $d_1 = d_2 = d_3 = 2$ by showing that for ideals I generated by three quadrics, $\mathrm{pd}(R/I) \leq 4$. In Engheta (2010), Engheta verified the case $N = 3$ and $d_1 = d_2 = d_3 = 3$ giving $\mathrm{pd}(R/I) \leq 36$ for this case. This bound is likely not tight as the largest known projective dimension of R/I for an ideal I generated by three cubics is just 5. The first such example was found by Engheta (2010). A simpler example is given in McCullough (2011).

Few other special cases of Stillman's question are known. However, in McCullough (2011), the second author defined a family of homogeneous ideals whose projective dimension grows quickly relative to the number and degrees of the generators. These ideals were defined as follows:

Definition 2.1. Fix integers m, n, d such that $m \geq 1, n \geq 0$ and $d \geq 2$. Order the $M_{m,d-1} = \frac{(m+d-2)!}{(m-1)!(d-1)!}$ monomials of degree $d-1$ over the variables $x_1, \dots, x_m$ and denote the ith such monomial by Z_i. Let $p = M_{m,d-1}$ and let $R = K[x_1, \dots, x_m, y_{1,1}, \dots, y_{p,n}]$ be a polynomial ring in $m + pn$ variables over K. We define $I_{m,n,d}$ to be the ideal generated by the following $m+n$ degree d homogeneous polynomials:

$$\left\{x_i^d \mid 1 \leq i \leq m\right\} \cup \left\{\sum_{j=1}^{p} Z_j y_{j,k} \mid 1 \leq k \leq n\right\}.$$

It was shown that the projective dimension of R/I is

$$\operatorname{pd}(R/I) = m + np = m + n\frac{(m+d-2)!}{(m-1)!(d-1)!}.$$

In the three-generated degree d case ($m = 2, n = 1$), the projective dimension of R/I is $d + 2$. In the general case with N degree d generators ($m = 2, n = N - 2$), the projective dimension of R/I grows asymptotically as a polynomial in d of degree $N - 2$. In the following section we generalize this example and define a new family of ideals with projective dimension far exceeding both of these.

3. A new family of ideals

A generalization of Definition 2.1 is given by the following:

Definition 3.1. Let K be a field and fix $n, n \geq 1$. Further, fix integers $g, m_1, \dots, m_n$ such that $g \geq 2$, $m_n \geq 0$, $m_{n-1} \geq 1$ and $m_i \geq 2$ for $1 \leq i \leq n-2$. Set:

- $M_n = m_n$,
- $M_k = m_k - 1$ for $k < n$,
- $d_k = m_k + \cdots + m_n + 1$,
- $d = d_1$.

Unless explicit bounds are given, we will use j or j' for an arbitrary integer in $\{1, 2, \dots, g\}$ and k or k' for an arbitrary integer in $\{1, 2, \dots, n\}$.

Finally, for $0 \leq k \leq n$ let

$$\mathscr{A}_k = \left\{ (a_{j,k'}) \;\middle|\; \begin{array}{l} 0 \leq a_{j,k'} \leq M_{k'} \text{ and } \sum_{j=1}^{g} a_{j,k'} = m_{k'} \text{ for} \\ 1 \leq k' \leq k, \text{ and } a_{j,k'} = 0 \text{ for } k < k' \leq n \end{array} \right\},$$

$$R = K[\mathsf{X}, y_{\mathsf{A}} \mid \mathsf{X} = (x_{j,k}), \mathsf{A} \in \mathscr{A}_n],$$

$$I_{g,(m_1,\dots,m_n)} = (x_{1,1}^d, \dots, x_{g,1}^d, f),$$

where

$$f = \sum_{k=1}^{n-1} \sum_{\mathsf{A} \in \mathscr{A}_{k-1}} \sum_{j=1}^{g} \mathsf{X}^{\mathsf{A}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}} + \sum_{\mathsf{B} \in \mathscr{A}_n} \mathsf{X}^{\mathsf{B}} y_{\mathsf{B}}.$$

By X^{A} we mean $\prod_{j,k} x_{j,k}^{a_{j,k}}$, where $\mathsf{A} = (a_{i,j})$.

The notation above was chosen so that the monomial terms in the generator f are all of the form X^{A} or $\mathsf{X}^{\mathsf{A}} y_{\mathsf{A}}$ for some $g \times n$ matrix A. We note that the restrictions on g and the m_i guarantee that $\mathscr{A}_i \neq \varnothing$ for all $0 \leq i \leq n-2$. Further, both $\mathscr{A}_{n-1}$ and $\mathscr{A}_n$ are non-empty if and only if $m_{n-1} \geq 2$. Before computing the projective dimension of these ideals, we give an example in detail.

Example 3.2. Consider the ideal $I = I_{2,(2,2,2)}$. Then $d = d_1 = 2+2+2+1 = 7$, $d_2 = 2+2+1 = 5$, and $d_3 = 2+1 = 3$. $M_1 = M_2 = 1$ and $M_3 = 2$. We then have that

$$\begin{aligned}
\mathcal{A}_0 &= \left\{\left(\begin{smallmatrix} 0&0&0\\0&0&0 \end{smallmatrix}\right)\right\},\\
\mathcal{A}_1 &= \left\{\left(\begin{smallmatrix} 1&0&0\\1&0&0 \end{smallmatrix}\right)\right\},\\
\mathcal{A}_2 &= \left\{\left(\begin{smallmatrix} 1&1&0\\1&1&0 \end{smallmatrix}\right)\right\},\\
\mathcal{A}_3 &= \left\{\left(\begin{smallmatrix} 1&1&2\\1&1&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&1&1\\1&1&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&1&0\\1&1&2 \end{smallmatrix}\right)\right\}.
\end{aligned}$$

Finally, the ideal I is

$$\left(x_{1,1}^7, x_{2,1}^7, f\right),$$

where

$$\begin{aligned}
f &= \mathsf{X}^{\left(\begin{smallmatrix} 0&0&0\\0&0&0 \end{smallmatrix}\right)} x_{1,1}^2 x_{1,2}^5 + \mathsf{X}^{\left(\begin{smallmatrix} 0&0&0\\0&0&0 \end{smallmatrix}\right)} x_{2,1}^2 x_{2,2}^5 + \mathsf{X}^{\left(\begin{smallmatrix} 1&0&0\\1&0&0 \end{smallmatrix}\right)} x_{1,2}^2 x_{1,3}^3 + \mathsf{X}^{\left(\begin{smallmatrix} 1&0&0\\1&0&0 \end{smallmatrix}\right)} x_{2,2}^2 x_{2,3}^3\\
&\quad + \mathsf{X}^{\left(\begin{smallmatrix} 1&1&2\\1&1&0 \end{smallmatrix}\right)} y_{\left(\begin{smallmatrix} 1&1&2\\1&1&0 \end{smallmatrix}\right)} + \mathsf{X}^{\left(\begin{smallmatrix} 1&1&1\\1&1&1 \end{smallmatrix}\right)} y_{\left(\begin{smallmatrix} 1&1&1\\1&1&1 \end{smallmatrix}\right)} + \mathsf{X}^{\left(\begin{smallmatrix} 1&1&0\\1&1&2 \end{smallmatrix}\right)} y_{\left(\begin{smallmatrix} 1&1&0\\1&1&2 \end{smallmatrix}\right)}\\
&= x_{1,1}^2 x_{1,2}^5 + x_{2,1}^2 x_{2,2}^5 + x_{1,1}x_{2,1}x_{1,2}^2 x_{1,3}^3 + x_{1,1}x_{2,1}x_{2,2}^2 x_{2,3}^3\\
&\quad + x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{1,3}^2 y_{\left(\begin{smallmatrix} 1&1&2\\1&1&0 \end{smallmatrix}\right)} + x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{1,3}x_{2,3} y_{\left(\begin{smallmatrix} 1&1&1\\1&1&1 \end{smallmatrix}\right)}\\
&\quad + x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{2,3}^2 y_{\left(\begin{smallmatrix} 1&1&0\\1&1&2 \end{smallmatrix}\right)}.
\end{aligned}$$

We note that $\mathcal{A}_2$ is not used in the definition of I, and in general $\mathcal{A}_{n-1}$ is not used in the definition of $I_{g,(m_1,\ldots,m_n)}$. Moreover, I is an ideal with 3 degree 7 generators in a polynomial ring R with 9 variables and R/I has projective dimension 9 by the following theorem.

Theorem 3.3. *Using the notation above with* $I = I_{g,(m_1,\ldots,m_n)}$, $\operatorname{depth}(R/I) = 0$.

In the following proofs, we say that $\mathsf{A} = (a_{j,i}) \in \mathcal{A}_k$ and $\mathsf{B} = (b_{j,i}) \in \mathcal{A}_{k'}$ *start the same* if $a_{j,i} = b_{j,i}$ for all $i \le \min(k, k')$ and all j with $1 \le j \le g$. Note that if $\mathsf{A} \in \mathcal{A}_0$, then A and B start the same for all $\mathsf{B} \in \mathcal{A}_k$, $0 \le k \le n$.

To prove the theorem, we first need the following lemma:

Lemma 1. *For each* k, $0 \le k \le n-1$, *let* $\mathsf{E}_k = (e_{j',k'})$ *be a* $g \times n$ *matrix where* $e_{j',k'} = d_{k'} - 1$ *for* $1 \le j' \le g$, $1 \le k' \le k$ *and zero elsewhere. Then*

$$\mathsf{X}^{\mathsf{E}_k} x_{j,k+1}^{d_{k+1}} \in I$$

for all j *such that* $1 \le j \le g$ *(interpret* $\mathsf{E}_0 = 0$*).*

Proof. Induct on k. When $k = 0$, this says $x_{j,1}^d \in I$ and indeed these are the first g generators of I. Assume $k \ge 1$, and choose any $\mathsf{A} \in \mathcal{A}_{k-1}$. Note that $\mathsf{A} \le \mathsf{E}_k$, so

$$\mathsf{X}^{\mathsf{A}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}} \mathsf{X}^{\mathsf{C}} = \mathsf{X}^{\mathsf{E}_k} x_{j,k+1}^{d_{k+1}}$$

for some matrix C with nonnegative integer entries. Notice that the matrix C is of the form

$$\begin{pmatrix}
d_1 - 1 - a_{1,1} & \cdots & d_{k-1} - 1 - a_{1,k-1} & d_k - 1 & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & & \vdots\\
d_1 - 1 - a_{j-1,1} & \cdots & d_{k-1} - 1 - a_{j-1,k-1} & d_k - 1 & 0 & \cdots & 0\\
d_1 - 1 - a_{j,1} & \cdots & d_{k-1} - 1 - a_{j,k-1} & d_k - 1 - m_k & 0 & \cdots & 0\\
d_1 - 1 - a_{j+1,1} & \cdots & d_{k-1} - 1 - a_{j+1,k-1} & d_k - 1 & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & & \vdots\\
d_1 - 1 - a_{g,1} & \cdots & d_{k-1} - 1 - a_{g,k-1} & d_k - 1 & 0 & \cdots & 0
\end{pmatrix}.$$

It is enough to show $h\mathsf{X}^{\mathsf{C}} \in I$ for all terms h of f such that $h \neq \mathsf{X}^{\mathsf{A}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}}$. The remaining terms of f are of the form

$$h = \mathsf{X}^{\mathsf{B}} x_{j',k'}^{m_{k'}} x_{j',k'+1}^{d_{k'+1}}$$

for some $\mathsf{B} \in \mathcal{A}_{k'-1}$ with $1 \leq k' \leq n-1$ and some $j' \leq g$ such that $\mathsf{A} \neq \mathsf{B}$ or $\mathsf{A} = \mathsf{B}$ and $j' \neq j$ or of the form

$$h = \mathsf{X}^{\mathsf{B}} y_{\mathsf{B}}$$

for some $\mathsf{B} \in \mathcal{A}_{k'}$ with $k' = n$. Assume h is one of these terms and let $M = h\mathsf{X}^{\mathsf{C}}$.

If A and B do not start the same, then consider the first index $t \leq \min(k-1, k'-1)$ where they disagree. Then the exponent of $x_{s,t}$ in M will be at least d_t for some s, and the exponents of $x_{s',t'}$ will be $d_{t'} - 1$ for all $t' < t$ (since A and B agree here), so by the inductive hypothesis, M is in I.

Now assume that A and B start the same. We will break this up according to cases:

Case $k' < k$: The exponent of $x_{j',k'}$ in M is at least $d_{k'}$. This is true since we added $d_{k'} - 1 - a_{j',k'}$ (the (j', k') entry of C) to $m_{k'}$ and $m_{k'} \geq a_{j',k'} + 1$. Because A and B start the same, we can write

$$M = \mathsf{X}^{\mathsf{E}_{k'}} x_{j',k'}^{d_{k'}} \mathsf{X}^{\mathsf{D}}$$

where D is some $g \times n$ matrix with nonnegative integral entries. The inductive hypothesis again implies this term is in I.

Case $k' = k$: Then $\mathsf{A} = \mathsf{B}$. Recall that $k \leq n-1$ and thus $m_{k'} \geq 1$. Since the terms defined by A and B are distinct, $j \neq j'$. So the exponent of $x_{j',k'}$ in M is at least $d_{k'}$, and this term is in I.

Case $k' > k$: Notice that at least two terms in each column k of B are nonzero. This is follows when $k \leq n-2$ because $m_i \geq 2$ for $1 \leq i \leq n-2$. If $k = n-1$ then $k' = n$ and $\mathsf{B} \in \mathcal{A}_n$. This forces $m_{n-1} > 2$ (if $m_{n-1} = 1$ then $M_{n-1} = 0$ and $\mathcal{A}_{n-1} = \mathcal{A}_n = \varnothing$) and thus at least two terms in column $n-1$ of B are nonzero. Now there exists some $j' \neq j$ such that $b_{j',k}$ is positive, and hence the exponent of $x_{j',k}$ in M is at least d_k, so this term is in I. □

Proof of Theorem 3.3. We will show that R/I has depth zero by showing that the element

$$\mathsf{S} = \mathsf{X}^{\mathsf{T}} \in (I : \mathfrak{m}) - I,$$

where $\mathsf{T} = (t_{j,k})$ and $t_{j,k} = d_k - 1$; that is, the image of S in R/I is in socle(R/I).

Since no term of any generator of I divides S, it is clear that $\mathsf{S} \notin I$. So we must show that every variable multiplies S into I. The fact that $x_{j,k}\mathsf{S} \in I$ for every j, k follows from the following preceding Lemma. We now show that $y_{\mathsf{A}}\mathsf{S} \in I$ for all $\mathsf{A} \in \mathcal{A}_n$. Notice that

$$y_{\mathsf{A}}\mathsf{S} = y_{\mathsf{A}}\mathsf{X}^{\mathsf{A}} \cdot \mathsf{X}^{\mathsf{C}}$$

where C is again some $g \times n$ matrix with nonnegative integral entries and $y_{\mathsf{A}}\mathsf{X}^{\mathsf{A}}$ is the term in f associated to y_{A}. As before, it is enough to show $h\mathsf{X}^{\mathsf{C}} \in I$ for all terms h in f such that $h \neq y_{\mathsf{A}}\mathsf{X}^{\mathsf{A}}$. Each h has an X^{B} as a factor, for some $\mathsf{B} \in \mathcal{A}_k$, $k \in \{1, 2, \ldots, n-2, n\}$.

If A and B do not start the same, let t be the first index where they differ. Then the exponent of some $x_{s,t}$ will be at least d_t for some s, so by the lemma, this term is in I.

Otherwise, A and B start the same and $k < n-1$ (if $k = n$ then they cannot start the same). In other words,

$$h = \mathsf{X}^{\mathsf{B}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}}.$$

Hence $h\mathsf{X}^{\mathsf{C}}$ has $x_{j,k+1}^{d_{k+1}}\mathsf{X}^{\mathsf{E}_k}$ as a factor and thus, by the lemma, is an element of I. □

Corollary 3.4.

$$\mathrm{pd}(R/I) = gn + \binom{m_n + g - 1}{g - 1} \prod_{i=1}^{n-1} \left(\binom{m_i + g - 1}{g - 1} - g \right).$$

Proof. This follows from the graded version of the Auslander–Buchsbaum Theorem and by counting the number of variables in the R. We get gn variables $x_{j,k}$ with $1 \leq j \leq g$ and $1 \leq k \leq n$. For each $\mathsf{A} \in \mathcal{A}_n$, we get a variable y_{A}. Note that $\mathcal{A}_n$ consists of exactly those matrices A with nonnegative integer entries such that

(1) All the entries in column k sum to m_k.
(2) For all $k < n$, there are at least two nonzero entries in column k.

In other words, the term $\prod_{j=1}^{g} x_{j,k}^{a_{j,k}}$ is a monomial of degree m_k in g variables and when $k < n$, this monomial is not a pure power. The formula for the projective dimension follows by counting all such monomials. $\square$

Example 3.5. Continuing the notation from Example 3.2, the previous theorem shows us that

$$\mathsf{S} = \mathsf{X}^{\begin{pmatrix} 6 & 4 & 2 \\ 6 & 4 & 2 \end{pmatrix}} = x_{1,1}^6 x_{2,1}^6 x_{1,2}^4 x_{2,2}^4 x_{1,3}^2 x_{2,3}^2 \in (I : \mathfrak{m}) - I.$$

So the image of S in R/I is in the socle of R/I. It follows that $\operatorname{depth}(R/I) = 0$ and hence $\operatorname{pd}(R/I) = 9$.

Corollary 3.6. *Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring R over K with three homogeneous generators in degree p^2 such that* $\operatorname{pd}(R/I) \geq p^{p-1}$.

Proof. This follows from the previous Corollary by taking the ideal

$$I = I_{2,\underbrace{(p+1,\ldots,p+1}_{p-1 \text{ times}},0)}. \quad \square$$

We note that this answers two questions posed by the second author in the negative. The following result can be viewed as an improvement to Corollary 4.7 in McCullough (2011).

Corollary 3.7. *Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring R over K with $2p+1$ homogeneous generators in degree $2p+1$ such that* $\operatorname{pd}(R/I) \geq p^{2p}$.

Proof. Take I to be the ideal

$$I_{2p,\underbrace{(2,2,2,\ldots,2)}_{p \text{ times}}}. \quad \square$$

Neither of these results gives an answer to Stillman's Question, but they impose large lower bounds on any possible answer.

4. Examples, special subfamilies and regularity questions

First we note that the family of ideals defined by the second author are a subfamily of the ideals defined above. Using the notation in Definition 2.1, we recall the definition for positive integers m, d and define the ideal

$$I_{m,1,d} = (x_1^d, \ldots, x_m^d, f),$$

with

$$f = \sum_i Z_i y_i,$$

where Z_i runs through the degree $d-1$ monomials in the variables $x_1, \ldots, x_m$. Up to relabeling of the variables, we note that

$$I_{m,1,d} = I_{m,(d-1)}$$

as in Definition 3.1 of the previous section. (We may replicate the last generator using new variables to get the full ideal $I_{m,n,d}$.) As stated earlier, the three-generated version of these ideals satisfies $\operatorname{pd}(R/I) = d+2$ when the generators were taken in degree d. Here we give a specific example of our new construction that improves upon this example.

Example 4.1. $I = I_{2,(3,1)}$.
This is an ideal with 3 quintic generators such that $\operatorname{pd}(R/I) = 8$.

Let R be the following polynomial ring

$$R = K\left[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y_{\binom{2\ 1}{1\ 0}}, y_{\binom{1\ 0}{2\ 1}}, y_{\binom{1\ 1}{2\ 0}}, y_{\binom{2\ 0}{1\ 1}}\right].$$

Then the ideal I is given by

$$I = \Big(x_{1,1}^5, x_{2,1}^5, x_{1,1}^3x_{1,2}^2 + x_{2,1}^3x_{2,2}^2 + x_{1,1}^2x_{1,2}x_{2,1}y_{\binom{2\ 1}{1\ 0}} + x_{1,1}x_{2,1}^2x_{2,2}y_{\binom{1\ 0}{2\ 1}} + x_{1,1}x_{1,2}x_{2,1}^2y_{\binom{1\ 1}{2\ 0}} + x_{1,1}^2x_{2,1}x_{2,2}y_{\binom{2\ 0}{1\ 1}}\Big)$$

and has Betti table

	0	1	2	3	4	5	6	7	8
Total:	1	3	53	184	287	248	124	34	4
0:	1	–	–	–	–	–	–	–	–
1:	–	–	–	–	–	–	–	–	–
2:	–	–	–	–	–	–	–	–	–
3:	–	–	–	–	–	–	–	–	–
4:	–	3	–	–	–	–	–	–	–
5:	–	–	–	–	–	–	–	–	–
6:	–	–	–	–	–	–	–	–	–
7:	–	–	–	–	–	–	–	–	–
8:	–	–	3	–	–	–	–	–	–
9:	–	–	3	4	–	–	–	–	–
10:	–	–	13	46	68	56	28	8	1
11:	–	–	33	132	218	192	96	26	3
12:	–	–	1	2	1	–	–	–	–

We also note that our family of ideals subsumes another family of ideals studied by Caviglia in Caviglia (2004). Let $R = K[w, x, y, z]$ and let $d \geq 2$. Then set

$$C_d = (x^d, y^d, xw^{d-1} - yz^{d-1}).$$

Caviglia showed that $\operatorname{reg}(R/C_d) = d^2 - 2$. To our knowledge, this family has the fastest growing regularity relative to the degree of the generators in the three-generated case. We note that these ideals are also a subfamily of the ideals defined in the previous section. In fact, up to a relabeling of the variables,

$$C_d = I_{2,(1,d-2)}.$$

In the following example, we show that some of our ideals have larger regularity than Caviglia's examples.

Example 4.2. $I = I_{2,(2,1,2)}$.
This is an ideal with 3 degree 6 generators such that $\operatorname{pd}(R/I) = 6$ and $\operatorname{reg}(R/I) = 41$. Its Betti table is displayed at the end of this section.

$$R = K\left[x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}\right]$$
$$I = (x_{1,1}^6, x_{2,1}^6, x_{1,1}^2x_{1,2}^4 + x_{2,1}^2x_{2,2}^4 + x_{1,1}x_{2,1}x_{1,2}x_{1,3}^3 + x_{1,1}x_{2,1}x_{2,2}x_{2,3}^3).$$

Calculations with Macaulay2 Grayson and Stillman (0000) indicate that many of the ideals defined in the previous section have much larger regularity than even this example. Specifically, we believe that the regularity of

$$I = I_{2,(\underbrace{2,2,2,\ldots,2}_{p\text{ times}},1,i)}$$

has regularity that grows asymptotically as i^{p+2}. When $p = 0$, this agrees with Caviglia's result. However his methods do not extend to the ideals above. Further, it has been verified using Macaulay2 Grayson and Stillman (0000) that for $0 \le d \le 12$,

$$\operatorname{reg}(R/I_{2,(2,1,d)}) = \begin{cases} \frac{1}{2}d^3 - 3d^2 + 8d - 7 & \text{if } d \text{ is even,} \\ \frac{1}{2}d^3 - \frac{5}{2}d^2 + 5d - 3 & \text{if } d \text{ is odd.} \end{cases}$$

We note that the regularity of R/I is bounded below by the degrees of the socle elements. However, the socle elements we computed above only grow linearly with the degrees of the generators. Computing the regularity of the ideals above would provide interesting computational examples and also give some insight into the regularity version of Stillman's question.

Betti Table of $R/I_{2,(2,1,2)}$:

	0	1	2	3	4	5	6
Total:	1	3	75	247	320	188	42
0:	1	–	–	–	–	–	–
1:	–	–	–	–	–	–	–
2:	–	–	–	–	–	–	–
3:	–	–	–	–	–	–	–
4:	–	–	–	–	–	–	–
5:	–	3	–	–	–	–	–
6:	–	–	–	–	–	–	–
7:	–	–	–	–	–	–	–
8:	–	–	–	–	–	–	–
9:	–	–	–	–	–	–	–
10:	–	–	3	–	–	–	–
11:	–	–	–	–	–	–	–
12:	–	–	–	–	–	–	–
13:	–	–	2	3	–	–	–
14:	–	–	–	–	–	–	–
15:	–	–	–	–	–	–	–
16:	–	–	3	6	3	–	–
17:	–	–	–	–	–	–	–
18:	–	–	1	4	5	2	–
19:	–	–	4	8	4	–	–
20:	–	–	1	4	6	4	1
21:	–	–	2	8	10	4	–
22:	–	–	6	14	11	4	1
23:	–	–	2	8	12	8	2
24:	–	–	4	16	21	10	1
25:	–	–	8	20	18	8	2
26:	–	–	3	12	18	12	3
27:	–	–	6	24	32	16	2
28:	–	–	3	12	18	12	3
29:	–	–	4	16	24	16	4
30:	–	–	3	12	18	12	3
31:	–	–	4	16	24	16	4
32:	–	–	1	4	6	4	1
33:	–	–	4	16	24	16	4
34:	–	–	1	4	6	4	1
35:	–	–	2	8	12	8	2
36:	–	–	1	4	6	4	1
37:	–	–	2	8	12	8	2
38:	–	–	1	4	6	4	1
39:	–	–	2	8	12	8	2
40:	–	–	–	–	–	–	–
41:	–	–	2	8	12	8	2

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