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Matching properties in connected domination critical graphs☆

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Abstract

A dominating set of vertices S of a graph G is connected if the subgraph $G[S]$ is connected. Let $\gamma_c(G)$ denote the size of any smallest connected dominating set in G. A graph G is k-γ-connected-critical if $\gamma_c(G) = k$, but if any edge $e \in E(\bar{G})$ is added to G, then $\gamma_c(G + e) \leqslant k - 1$. This is a variation on the earlier concept of criticality of edge addition with respect to ordinary domination where a graph G was defined to be k-critical if the domination number of G is k, but if any edge is added to G, the domination number falls to $k - 1$.

A graph G is factor-critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$, bicritical if $G - u - v$ has a perfect matching for every pair of distinct vertices $u, v \in V(G)$ or, more generally, k-factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. In two previous papers [N. Ananchuen, M.D. Plummer, Matching properties in domination critical graphs, Discrete Math. 277 (2004) 1–13; N. Ananchuen, M.D. Plummer, 3-factor-criticality in domination critical graphs, Discrete Math. 2007, to appear [3].] on ordinary (i.e., not necessarily connected) domination, the first and third authors showed that under certain assumptions regarding connectivity and minimum degree, a critical graph G with (ordinary) domination number 3 will be factor-critical (if $|V(G)|$ is odd), bicritical (if $|V(G)|$ is even) or 3-factor-critical (again if $|V(G)|$ is odd). Analogous theorems for connected domination are presented here. Although domination and connected domination are similar in some ways, we will point out some interesting differences between our new results for the case of connected domination and the results in [N. Ananchuen, M.D. Plummer, Matching properties in domination critical graphs, Discrete Math. 277 (2004) 1–13; N. Ananchuen, M.D. Plummer, 3-factor-criticality in domination critical graphs, Discrete Math. 2007, to appear [3].].

Keywords: Connected domination; Critical edge; Matching; Factor-critical; Bicritical; 3-factor-critical; Claw-free

1. Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a *dominating set* for G if every vertex of G either belongs to S or is adjacent to a vertex of S. If S dominates G, we write $S \succ G$. The minimum cardinality of a dominating set in a graph G is called the *domination number* of G and is denoted by $\gamma(G)$. Graph G is said to be k-γ-*critical* if $\gamma(G) = k$, but $\gamma(G + e) = k - 1$ for each edge $e \in E(\bar{G})$.

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A dominating set $S \subseteq V(G)$ is a *connected* dominating set if the subgraph spanned by S is connected. If S is a connected dominating set for G we write $S \succ_c G$. The minimum cardinality of a connected dominating set in G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. (Note that since a graph must be connected to have a connected dominating set, henceforth in this paper, when referring to connected domination, we shall assume all graphs under consideration are connected.) A graph G is *k-γ-connected critical* if $\gamma_c(G)) = k$, but $\gamma_c(G + uv) \leqslant k - 1$, for any edge $uv \in E(\bar{G})$. Note that while the addition of an edge may reduce the ordinary domination number by at most one, edge addition may reduce the connected domination number by at most two. (See Theorem 1 of [5].) In this paper, we will be concerned only with the case $k = 3$ and will refer to a connected-critical graph with connected domination number 3 as a 3-*c-critical* graph.

The origins of the concept of connected domination are a bit hazy, although in the first published paper on the subject, Sampathkumar and Waliker [10] attribute the terminology to Hedetniemi. For a summary of their results, as well as a number of other early results on connected domination, see [7,8]. The algorithmic aspects of both domination and connected domination were first discussed by Garey and Johnson in their book [6] where it is claimed that both domination and connected domination are NP-complete, even when the graph is planar and regular of degree 4. For an excellent and more recent discussion of the computational and extremal aspects of connected domination, see [4].

More recently, Chen et al., [5] began the study of connected domination critical graphs by obtaining some results most of which have previous analogs for ordinary domination critical graphs. We will state and use several of their results below. Also following their notation, we will adopt the following. If u, v and w are vertices of G and $\{u, v\} \succ_c G - w$, but neither u nor v dominates w, we write $[u, v] \longrightarrow_c w$.

Following the work of Sumner and Blitch [11] on 3-critical graphs, Chen et al. [5] proved the following very useful result.

Lemma 1.1. *Let G be a 3-c-critical graph and let S be an independent set of $n \geqslant 3$ vertices in $V(G)$.*

(i) *then the vertices of S can be ordered as $a_1, a_2, \ldots, a_n$ in such a way that there exists a path of distinct vertices $x_1, x_2, \ldots, x_{n-1}$ in $G - S$ so that $[a_i, x_i] \longrightarrow_c a_{i+1}$ for $i = 1, 2, \ldots, n - 1$, and*
(ii) $\operatorname{diam}(G) \leqslant 3$.

The following lemma, may be viewed as being related to toughness. Proof of part (i) may be found in [5]. Part (ii) was later proved by the first author [1].

Lemma 1.2. *Let G be a 3-c-critical graph. Then*:

(i) *if T is a cutset of vertices for G, it follows that $G - T$ has at most $|T| + 1$ components, and moreover*;
(ii) *if the cutset T has at least two vertices, $G - T$ has at most $|T|$ components.*

Throughout the rest of this paper, $c(G)$ (respectively, $c_o(G)$) will denote the number of components (respectively, *odd* components) of a graph G. Also if G is a graph and $H \subseteq V(G)$, then $G[H]$ will denote the subgraph induced by H.

A *perfect* (respectively, *near-perfect*) matching in a graph G is a matching which covers all (respectively, all but one) of the vertices of G.

Lemma 1.3. *Let G be a 3-c-critical graph. Then*:

(i) *if $|V(G)|$ is even, G contains a perfect matching, while*;
(ii) *if $|V(G)|$ is odd, G contains a near-perfect matching.*

Proof. Part (i) is proved in [5]. We prove only part (ii). Suppose G is a 3-c-critical graph with an odd number of vertices and suppose G does not contain a near-perfect matching. Consider the Gallai–Edmonds decomposition of G. (See [9].) That is, let $D(G)$ denote the set of all vertices $v \in V(G)$ such that some maximum matching of G does not cover v. Let $A(G)$ denote the set of all neighbors of vertices of $D(G)$ which are not themselves in $D(G)$ and finally, let $C(G) = V(G) - (D(G) \cup A(G))$. Since G contains no near-perfect matching, then by Tutte's Theorem and parity, the number of odd components of $G - A(G)$ is at least two larger than $|A(G)|$. If $A(G) = \emptyset$, then G is disconnected, a contradiction. So $A(G) \neq \emptyset$ and hence is a vertex cutset of G. But $c(G - A(G)) \geqslant |A(G)| + 2$, a contradiction of Lemma 1.2. $\square$

A *factor-critical* graph G is one for which $G - v$ contains a perfect matching for every vertex $v \in V(G)$ and a graph G is said to be *bicritical* if $G - u - v$ contains a perfect matching for every choice of two distinct vertices u and $v \in V(G)$. More generally, a graph G is k-factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. Factor-critical and bicritical graphs play important roles in a canonical decomposition theory for arbitrary graphs in terms of their matchings. The interested reader is referred to [9] for much more on this subject.

Our purpose is to prove several new theorems which say that under certain assumptions on connectivity and minimum degree, a 3-c-critical graph G either is factor-critical (when $|V(G)|$ is odd), bicritical (when $|V(G)|$ is even) or 3-factor-critical (again when $|V(G)|$ is odd).

2. 3-c-criticality and bicriticality

Our first main result shows that if the connectivity and minimum degree are sufficiently high in a 3-c-critical graph of even order, then the graph must be bicritical.

Theorem 2.1. *Suppose $n \geqslant 4$ and G is a 3-connected 3-c-critical graph of order $2n$. Then if $\delta(G) \geqslant n-1$, G is bicritical.*

Proof. Suppose, to the contrary, that G is not bicritical. Then there exist vertices x and y in $V(G)$ such that $G' = G - x - y$ has no perfect matching. By Tutte's Theorem, there is a subset $S' \subseteq V(G')$ such that $c_o(G' - S') > |S'|$. By parity, $c_o(G' - S') \geqslant |S'| + 2$. Set $S = S' \cup \{x, y\}$. Since G contains a perfect matching by Lemma 1.3(i) above, we have

$$c_o(G' - S') = c_o(G - S) \leqslant |S| = |S'| + 2.$$

Thus $c_o(G - S) = |S|$.

For $1 \leqslant i \leqslant |S|$, let C_i denote an odd component of $G - S$. Set $s = |S|$. Clearly, $s \geqslant 3$. For $1 \leqslant i \leqslant s$, choose $y_i \in V(C_i)$. Then $T = \{y_1, y_2, \ldots, y_s\}$ is an independent set of size $s \geqslant 3$. By Lemma 1.1(i), the vertices in T may be ordered as $a_1, a_2, \ldots, a_s$ in such a way that there exists a path $x_1 x_2 \cdots x_{s-1}$ in $G - T$ such that $[a_i, x_i] \longrightarrow_c a_{i+1}$, for $1 \leqslant i \leqslant s-1$. Clearly then, $x_i \in S$ and $a_i x_i \in E(G)$, but $a_{i+1} x_i \notin E(G)$ for $1 \leqslant i \leqslant s-1$. Moreover, for $1 \leqslant j \leqslant s-1$, $a_1 x_j \in E(G)$ and $a_i x_j \in E(G)$ for $2 \leqslant i \leqslant s$ and $j \neq i-1$. Let $\{x_s\} = S - \{x_1, x_2, \ldots, x_{s-1}\}$.

Claim 1. $s \geqslant n - 1$.

Since $\delta(G) \geqslant n-1$, $|V(C_i)| \geqslant n-s+1$ for $2 \leqslant i \leqslant s$ and $|V(C_1)| \geqslant n-s$. So $2n \geqslant |S| + \sum_{i=1}^{s} |V(C_i)| \geqslant s + (n-s) + (s-1)(n-s+1) = -s^2 + ns + 2s - 1$. Thus $s^2 - (n+2)s + (2n+1) \geqslant 0$. It then follows that $s \geqslant (n+2+\sqrt{n^2-4n})/2$ or $s \leqslant (n+2-\sqrt{n^2-4n})/2$.

For $n = 4$, $(n+2+\sqrt{n^2-4n})/2 = (n+2-\sqrt{n^2-4n})/2 = 3$. Thus $s = 3 = n-1$.

For $n \geqslant 5$, if $s \leqslant (n+2-\sqrt{n^2-4n})/2$, then $3 \leqslant s \leqslant (n+2-\sqrt{n^2-4n})/2 < (n+2-\sqrt{n^2-8n+16})/2 = 3$, a contradiction. Hence $s \geqslant (n+2+\sqrt{n^2-4n})/2$. But then since $(n+2+\sqrt{n^2-4n})/2 > (n+2+\sqrt{n^2-8n+16})/2 = n-1$, $s \geqslant n-1$, as claimed.

Since G has $2n$ vertices and $|S| = s = c_o(G - S)$, it follows that $s \leqslant n$. Hence $n - 1 \leqslant s \leqslant n$.

We distinguish two cases.

Case 1. Suppose $s = n$.

Then each component of $G - S$ is a singleton and $G - S$ has no even components. Thus let us set $V(C_i) = \{y_i\}$, $1 \leqslant i \leqslant s$.

Since $\delta(G) \geqslant n-1$, $a_i x_s \in E(G)$ for $2 \leqslant i \leqslant s$. If $a_1 x_s \in E(G)$, then $\{a_1, x_s\} \succ_c G$, a contradiction. Hence $a_1 x_s \notin E(G)$.

Claim 2. *For $2 \leqslant i \leqslant s = n$, $x_{i-1} x_s \in E(G)$.*

Consider $G + a_1a_i$. Since $G - S$ contains exactly $n \geqslant 4$ components, $\{a_1, a_i\}$ is not a connected dominating set for $G + a_1a_i$. Since G is 3-c-critical, there exists a vertex $z \in V(G) - \{a_1, a_i\}$ such that either $[a_1, z] \longrightarrow_c a_i$ or $[a_i, z] \longrightarrow_c a_1$. Suppose first that $[a_1, z] \longrightarrow_c a_i$. Then $z \in S$ and $za_i \notin E(G)$. Thus $z = x_{i-1}$. Since $a_1x_s \notin E(G)$ and $[a_1, x_{i-1}] \longrightarrow_c a_i$, it follows that $x_{i-1}x_s \in E(G)$.

Now consider the case when $[a_i, z] \longrightarrow_c a_1$. Then $z \in S$ and $za_1 \notin E(G)$. Thus $z = x_s$. Since $a_ix_{i-1} \notin E(G)$ and $[a_i, x_s] \longrightarrow_c a_1$, it follows that $x_{i-1}x_s \in E(G)$. Hence in either case, $x_{i-1}x_s \in E(G)$ for $2 \leqslant i \leqslant s = n$ as claimed.

Note that $N_G[x_s] = S \cup \{a_2, a_3, \ldots, a_s\}$. Hence $\{x_1, x_s\} \succ_c G$, a contradiction. This proves that $s \neq n$.

Case 2. Suppose $s = n - 1$.

Since $c_o(G - S) = s = n - 1$ and G is of order $2n$, it follows that $G - S$ contains either $n - 2$ singleton components and exactly one odd component of order 3 or $n - 1$ singleton components and exactly one even component of order 2.

Suppose first that $G - S$ contains $n - 2$ singleton components and exactly one odd component of order 3. Without loss of generality, we may assume that $C_1, C_2, \ldots, C_{s-1}$ are singletons and C_s is the odd component of order 3. Then set $V(C_i) = \{y_i\}$ for $1 \leqslant i \leqslant s - 1$. Also set $V(C_s) = \{y_s, w_1, w_2\}$. Since $\{y_1, y_2, \ldots, y_s\} = \{a_1, a_2, \ldots, a_s\}$, either $a_2 \neq y_s$ or $a_3 \neq y_s$. Then $d_G(a_2) \leqslant n - 2$ or $d_G(a_3) \leqslant n - 2$. But this contradicts the minimum degree assumption.

Hence $G - S$ must contain $n - 1$ singleton components and exactly one even component of order 2. By a similar argument, G contains a vertex of degree less than $n - 1$, again a contradiction. Hence G must be bicritical as claimed. □

Remark 1. It is not difficult to show directly that there is no 3-c-critical graph on six or fewer vertices which is also bicritical.

Remark 2. Let us now consider the sharpness of the above result. For integers $k \geqslant 1$ and $s \geqslant 2$, we construct a graph $H_{k,s}$ as follows. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$. Set $V(H_{k,s}) = X \cup Y \cup \{a, b\}$, a set of $k + s + 2$ distinct vertices. Form complete graphs on X and on Y. Join a to each vertex of $X \cup \{y_1\}$ and join b to each vertex of $X \cup (Y - y_1)$.

It is not difficult to show that the graph $H_{k,s}$ is 3-c-critical and 2-connected. Clearly, the graph $H_{2r+1,2s+1}$ is not bicritical for any choice of positive integers r and s. Note that the graph $H_{2r+1,2s+1}$ shows that the bound on connectivity in Theorem 2.1 is best possible.

(Fig. 1 displays the graph $H_{3,5}$.)

Remark 3. We can "inflate" the graph $H_{k,s}$ to a graph $H_{k,s,r,t}$ as follows. Replace the vertices a and b with complete graphs $K(a)$ and $K(b)$ on $r \geqslant 1$ and $t \geqslant 1$ vertices, respectively, and join each vertex of $K(a)$ to every neighbor of a and every vertex of $K(b)$ to every neighbor of b. It is easy to check that the resulting graph $H_{k,s,r,t}$ on $k + s + r + t$ vertices is also 3-c-critical. Note that for $n \geqslant 4$, the graph $H_{n-2,n-1,1,2}$ is a graph on $2n \geqslant 8$ vertices which is 3-c-critical, 3-connected and has minimum degree $n - 1$. Hence the graph $H_{n-2,n-1,1,2}$ is bicritical by Theorem 2.1. (Fig. 2 shows the graph $H_{3,4,1,2}$.)

Remark 4. One might expect that the bound on minimum degree in Theorem 2.1 can be lowered if the connectivity is increased, but this is not the case. For each integer $n \geqslant 3$, let $X = \{x_1, x_2, \ldots, x_{n-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$. Now set $V(G_n) = X \cup Y \cup \{a, b\}$, thus yielding a set of $2n$ distinct vertices. Form a complete graph on X.

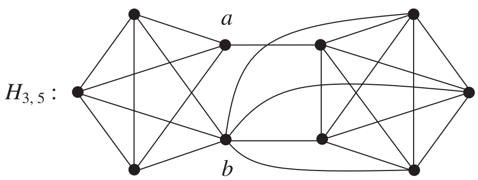

Fig. 1.

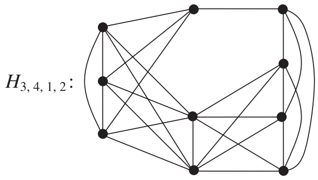

Fig. 2.

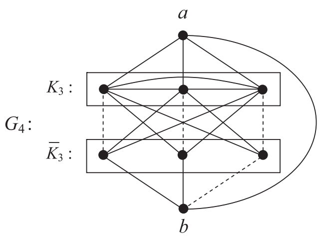

Fig. 3.

Join each x_i to each vertex of $(Y - y_i) \cup \{a\}$ and join b to each vertex of $(Y - y_{n-1}) \cup \{a\}$. Note that G_n is 3-c-critical and $(n-2)$-connected with minimum degree $n-2$. But G_n is not bicritical since $G - \{x_1, x_2\}$ has no perfect matching. (Fig. 3 shows graph G_4.)

We would point out the rather dramatic difference in the required minimum degree in Theorem 2.1 where it is $n - 1$ and the corresponding Theorem 2.1 in [2] where one requires only minimum degree 4 to guarantee bicriticality in the case of ordinary domination.

In the case when the 3-c-critical even graph is claw-free, however, we can dispense with any minimum degree condition.

Theorem 2.2. *Suppose $n \geqslant 4$ and G is a 3-connected 3-c-critical claw-free graph of order $2n$. Then G is bicritical.*

Proof. Suppose, to the contrary, that G is not bicritical. By applying an argument similar to that at the beginning of the proof of Theorem 2.1, again we have that G contains a subset S of s vertices where $c_o(G - S) = |S| = s$. Since G is 3-connected, $s \geqslant 3$.

Suppose first that $s = 3$. Then S is a minimum cutset and therefore each vertex of S is adjacent to some vertex in each component of $G - S$. Therefore G contains a claw, a contradiction. Hence $s \geqslant 4$.

For $1 \leqslant i \leqslant s$, choose $y_i \in V(C_i)$ where again we denote the odd components of $G - S$ by $C_1, C_2, \ldots, C_s$. Then $T = \{y_1, y_2, \ldots, y_s\}$ is independent. Thus by Lemma 1.1(i), the vertices in T may be ordered as $a_1, a_2, \ldots, a_s$ in such a way that there exists a path $x_1, x_2, \ldots, x_{s-1}$ in $G - T$ where $[a_i, x_i] \longrightarrow_c a_{i+1}$, for $1 \leqslant i \leqslant s - 1$. Clearly $x_i a_i \in E(G)$ for $i = 1, 2, \ldots, s - 1$. But then $G[\{x_1; a_1, a_3, a_4\}]$ is a claw centered at vertex x_1. This contradiction completes the proof.

As an infinite family of graphs satisfying the hypotheses of Theorem 2.2, we offer the infinite family $\{H_{2n-6,2,2,2,2} | n \geqslant 4\}$ defined in Remark 3. Note that the minimum degree of the graph $H_{2n-6,2,2,2,2}$ is 3 for any $n \geqslant 4$.

3. 3-c-criticality and factor-criticality

In the case of odd graphs, the minimum degree requirement necessary to guarantee factor-criticality is much weaker than the minimum degree requirement given in Theorem 2.1.

Theorem 3.1. *Suppose $n \geqslant 2$ and G is a 3-c-critical graph of order $2n + 1$. Then if $\delta(G) \geqslant 2$, G is factor-critical.*

Proof. Suppose to the contrary that G is not factor-critical. Then there exists a vertex x in $V(G)$ such that $G' = G - x$ has no perfect matching. By Tutte's Theorem, there is a subset $S' \subseteq V(G')$ such that $c_o(G' - S') > |S'|$. Set $S = S' \cup \{x\}$. By Lemma 1.2 and parity,

$$|S'| + 2 \leqslant c_o(G' - S') = c_o(G - S) \leqslant |S| + 1 = |S'| + 2.$$

Thus $c_o(G - S) = |S| + 1$. By part (ii) of Lemma 1.2, $|S| = 1$. In [1, Theorem 3.5], the first author gave a characterization of all 3-c-critical graphs having a cutvertex. It follows from that characterization that G must contain exactly one vertex of degree one. But this contradicts our minimum degree hypothesis and hence the theorem is proved. □

For an infinite family of graphs satisfying the hypotheses of Theorem 3.1 we offer $\{H_{1,2n-2,1,1} | n \geqslant 2\}$ defined in Remark 3. We also point out that the hypothesis in Theorem 3.1 stating that $\delta(G) \geqslant 2$ is a necessary one, for every factor-critical graph trivially has minimum degree at least 2.

We conclude with a result concerning 3-factor-criticality.

Theorem 3.2. *Suppose G is a 3-c-critical 4-connected $K_{1,4}$-free graph of odd order. Then G is 3-factor-critical.*

Proof. Suppose to the contrary that G is not 3-factor-critical. Then there exist vertices x, y, w in $V(G)$ such that $G' = G - \{x, y, w\}$ has no perfect matching. By Tutte's Theorem, there is a subset $S' \subseteq V(G')$ such that $c_o(G' - S') > |S'|$. Set $S = S' \cup \{x, y, w\}$ and $|S| = s$. By Theorem 3.1 and parity,

$$|S| - 1 = |S'| + 2 \leqslant c_o(G' - S') = c_o(G - S) \leqslant |S| - 1.$$

Thus $c_o(G - S) = s - 1$. Since G is 4-connected, $s \geqslant 4$. Thus, $c_o(G - S) = s - 1 \geqslant 3$. For $1 \leqslant i \leqslant s - 1$, let C_i denote an odd component of $G - S$. For $1 \leqslant i \leqslant s - 1$, choose $y_i \in V(C_i)$. Then $T = \{y_1, y_2, \ldots, y_{s-1}\}$ is an independent set of size $s - 1 \geqslant 3$. By Lemma 1.1(i), the vertices in T may be ordered as $a_1, a_2, \ldots, a_{s-1}$ in such a way that there exists a path $x_1 x_2, \cdots, x_{s-2}$ in $G - T$ such that $[a_i, x_i] \longrightarrow_c a_{i+1}$, for $1 \leqslant i \leqslant s - 2$. Clearly then, $x_i \in S$ and $a_i x_i \in E(G)$, but $a_{i+1} x_i \notin E(G)$ for $1 \leqslant i \leqslant s - 2$. Moreover, for $1 \leqslant j \leqslant s - 2$, $a_1 x_j \in E(G)$ and $a_i x_j \in E(G)$ for $2 \leqslant i \leqslant s - 1$ and $j \neq i - 1$. Let $\{u, v\} = S - \{x_1, x_2, \ldots, x_{s-2}\}$. Without loss of generality, we may renumber the odd components of $G - S$ in such a way that $a_i \in V(C_i)$.

Claim 1. $|S| = 4$.

Clearly, $|S| \leqslant 5$ as otherwise $G[\{x_1; a_1, a_3, a_4, a_5\}]$ is a $K_{1,4}$ centered at x_1. Suppose to the contrary that $|S| = 5$. Since $[a_i, x_i] \longrightarrow_c a_{i+1}$ and G is $K_{1,4}$-free, it follows that $|V(C_2)| = |V(C_3)| = |V(C_4)| = 1$. Because $\delta(G) \geqslant 4$ and for $2 \leqslant i \leqslant 4$, $a_i x_{i-1} \notin E(G)$, it follows that each a_i, $i = 2, 3, 4$, must be adjacent to both u and v. Then u and v are not adjacent to a_1 since G is $K_{1,4}$-free. Because $[a_1, x_1] \longrightarrow_c a_2$, x_1 is adjacent to both u and v. But then $\{x_1, x_2\} \succ_c G$, a contradiction. This proves our claim.

By Claim 1 and the fact that $a_2 x_1 \notin E(G)$ and $a_3 x_2 \notin E(G)$, it follows that $|V(C_2)| \geqslant 3$ and $|V(C_3)| \geqslant 3$ since $\delta(G) \geqslant 4$. Hence, $G - S$ has no even components, otherwise G contains $K_{1,4}$ as a subgraph.

Claim 2. *Suppose a_1 is adjacent to both u and v. For each vertex $\alpha \in V(C_2) \cup V(C_3)$, there is no vertex $z \in V(G) - \{a_1, \alpha\}$ such that $[\alpha, z] \longrightarrow_c a_1$.*

Suppose to the contrary that there exists a vertex $z_1 \in V(G) - \{a_1, \alpha\}$ such that $[\alpha, z_1] \longrightarrow_c a_1$. Clearly, $\alpha z_1 \in E(G)$. Since $G - S$ has three odd components, $z_1 \in S \subseteq N_G(a_1)$, a contradiction. This settles the claim.

Claim 3. *The vertex a_1 is adjacent to exactly one of $\{u, v\}$.*

If a_1 is adjacent to neither vertex of $\{u, v\}$, then x_1 is adjacent to both u and v since $[a_1, x_1] \longrightarrow_c a_2$. But then $\{x_1, x_2\} \succ_c G$, a contradiction. Hence a_1 is adjacent to at least one of u and v.

Suppose now that a_1 is adjacent to both u and v. Choose $b_2 \in V(C_2) - a_2$ and consider $G + a_1 b_2$. Since G is 3-c-critical, there is a vertex $z \in V(G) - \{a_1, b_2\}$ such that $[a_1, z] \longrightarrow_c b_2$ or $[b_2, z] \longrightarrow_c a_1$. By Claim 2, we have

$[a_1, z] \longrightarrow_c b_2$. Clearly $z \in S$ since $G - S$ has three odd components. Furthermore, $z \notin N_G(b_2)$. Thus $z \neq x_1$. If $z = x_2$, then no vertex of $\{a_1, z\}$ is adjacent to a_3, a contradiction. Hence $z \neq x_2$.

Therefore, $z \in \{u, v\}$. Without loss of generality, we may suppose that $z = u$; that is, $[a_1, u] \longrightarrow_c b_2$. Then u dominates $(V(C_2) \cup V(C_3)) - b_2$. Next, choose $b_3 \in V(C_3) - a_3$ and consider $G + a_1b_3$. By an argument similar to that above, there exists a vertex $z_1 \in S - N_G[b_3]$ such that $[a_1, z_1] \longrightarrow_c b_3$. Thus $z_1 \notin \{x_1, x_2, u\}$ and hence $z_1 = v$; that is, $[a_1, v] \longrightarrow_c b_3$. Then v dominates $(V(C_2) \cup V(C_3)) - b_3$. Finally, choose $c_3 \in V(C_3) - \{a_3, b_3\}$. Note that $S \subseteq N_G[c_3]$. Consider now $G + a_1c_3$. Again by an argument similar to that above, there must exist a vertex $z_2 \in S - N_G[c_3]$ such that $[a_1, z_2] \longrightarrow_c c_3$. But this is impossible since $S \subseteq N_G[c_3]$. Hence a_1 is adjacent to exactly one of u and v as claimed.

By Claim 3, we may assume without loss of generality that $a_1u \notin E(G)$, but $a_1v \in E(G)$. Since $[a_1, x_1] \longrightarrow_c a_2$, x_1 is adjacent to u. Thus $x_1v \notin E(G)$ and $x_2v \notin E(G)$, otherwise $\{x_1, x_2\} \succ_c G$. Since $[a_2, x_2] \longrightarrow_c a_3$, $a_2v \in E(G)$. Recall that $|V(C_2)| \geqslant 3$ and $|V(C_3)| \geqslant 3$. Let $b_2 \in V(C_2) - a_2$ and $b_3 \in V(C_3) - a_3$. Consider $G + b_2b_3$. Clearly, $\{b_2, b_3\}$ is not a connected dominating set for $G + b_2b_3$. Since G is 3-c-critical, there exists a vertex $z \in V(G) - \{b_2, b_3\}$ such that either $[b_2, z] \longrightarrow_c b_3$ or $[b_3, z] \longrightarrow_c b_2$. In both cases, $z \in S$ since $G - S$ has three odd components and $|V(C_i)| \geqslant 3$ for $2 \leqslant i \leqslant 3$. Furthermore, in both cases $z \neq u$, otherwise no vertex of $\{b_i, z\}$ is adjacent to a_1 for $2 \leqslant i \leqslant 3$. Hence, $z \in S - u$. We distinguish two cases.

Case 1. $[b_2, z] \longrightarrow_c b_3$.

Then $z \notin N_G[b_3]$. Thus $z \neq x_1$ and $z \neq x_2$. Hence, $z = v$. That is $[b_2, v] \longrightarrow_c b_3$. Thus v dominates $(V(C_1) \cup V(C_3)) - b_3$ and $vb_3 \notin E(G)$. Now consider $G + a_2b_3$. Clearly, $\{a_2, b_3\}$ is not a connected dominating set for $G + a_2b_3$. Since G is 3-c-critical, by a similar argument as above there exists a vertex $z_1 \in S - u$ such that either $[a_2, z_1] \longrightarrow_c b_3$ or $[b_3, z_1] \longrightarrow_c a_2$. Suppose first that $[a_2, z_1] \longrightarrow_c b_3$. Then $z_1 \notin N_G[b_3]$. Thus $z_1 \notin \{x_1, x_2\}$. Then $z_1 = v$. But then no vertex of $\{a_2, z_1\}$ is adjacent to x_1, a contradiction. Hence, $\{a_2, z_1\}$ does not dominate $G + a_2b_3$. Therefore, $[b_3, z_1] \longrightarrow_c a_2$. Then $z_1 \notin N_G[a_2]$. Thus $z_1 \neq x_2$ and $z_1 \neq v$. Hence, $z_1 = x_1$. But then no vertex of $\{b_3, z_1\}$ is adjacent to v, a contradiction. Hence, $\gamma_c(G + a_2b_3) > 2$, a contradiction. Therefore, Case 1 cannot occur.

Case 2. $[b_3, z] \longrightarrow_c b_2$.

Then $z \notin N_G[b_2]$. Thus $z \neq x_1$. Hence, $z = x_2$ or $z = v$. Suppose first that $z = x_2$. That is $[b_3, x_2] \longrightarrow_c b_2$. Then x_2 dominates $(V(C_1) \cup V(C_2)) - b_2$ and $x_2b_2 \notin E(G)$. Now consider $G + b_2a_3$. Clearly, $\{b_2, a_3\}$ is not a connected dominating set for $G + b_2a_3$. Since G is 3-c-critical, by a similar argument as above there exists a vertex $z_1 \in S - u$ such that either $[b_2, z_1] \longrightarrow_c a_3$ or $[a_3, z_1] \longrightarrow_c b_2$. Suppose first that $[a_3, z_1] \longrightarrow_c b_2$. Then $z_1 \notin N_G[b_2]$. Thus $z_1 \neq x_1$. Further, $z_1 \neq x_2$ since $x_2a_3 \notin E(G)$. Hence, $z_1 = v$. But then no vertex of $\{a_3, z_1\}$ is adjacent to x_2, a contradiction. Hence, $\{a_3, z_1\}$ does not dominate $V(G) - b_2$. Therefore, $[b_2, z_1] \longrightarrow_c a_3$. Then $z_1 \notin N_G[a_3]$and hence $z_1 \neq x_1$. Further, $z_1 \neq x_2$ since $x_2b_2 \notin E(G)$. Thus $z_1 = v$. But then no vertex of $\{b_2, z_1\}$ is adjacent to x_2, a contradiction. Hence, $\{b_2, z_1\}$ does not dominate $V(G) - a_3$. Thus $\gamma_c(G + b_2a_3) > 2$, a contradiction. Therefore, $z \neq x_2$. Hence, $z = v$. That is $[b_3, v] \longrightarrow_c b_2$. Then v dominates $(V(C_1) \cup V(C_2)) - b_2$ and $b_2v \notin E(G)$. Now consider $G + b_2a_3$. By applying an argument similar to that given in the subcase when $z = x_2$ above, together with the facts that $x_2v \notin E(G)$ and $b_2v \notin E(G)$, it follows that $\gamma_c(G + b_2a_3) > 2$. This contradiction proves that Case 2 cannot occur. Hence, $\gamma_c(G + b_2b_3) > 2$, a contradiction.

Therefore, G must be 3-factor-critical as claimed. $\square$

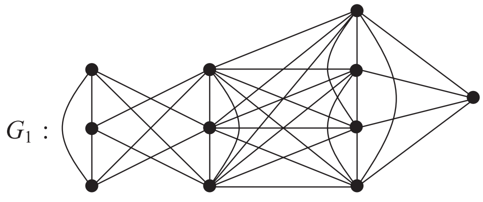

Fig. 4.

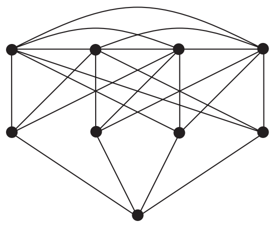

Fig. 5.

Remark 5. The graphs G_1 in Fig. 4 and G_2 in Fig. 5 are both 3-c-critical of odd order, but neither is 3-factor-critical. Note that G_1 is 3-connected and $K_{1,4}$-free and G_2 is 4-connected, but contains $K_{1,4}$ as an induced subgraph. Hence, our assumptions on connectivity and $K_{1,4}$-freedom in Theorem 3.2 are best possible.

Remark 6. For integers $k \geqslant 2$ and $t \geqslant 1$, let us construct a graph $G_{k,t}$ as follows. Let $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and $Z = \{z_1, z_2, \ldots, z_t\}$. Set $V(G_{k,t}) = X \cup Y \cup Z \cup \{a\}$, a set of $2k + t + 1$ distinct vertices. Form complete graphs on X, Y and Z, respectively. Join a to every vertex of Z and for $1 \leqslant i \leqslant k$, join y_i to every vertex of $(Z \cup X) - x_i$.

It is easy to see that $G_{k,t}$ is 3-c-critical and $K_{1,4}$-free. If $k \geqslant 4$, $t \geqslant 4$ and t is even, then $G_{k,t}$ is also 4-connected of odd order and hence is 3-factor-critical by Theorem 3.2. Note also that for $n \geqslant 5$, the graph $H_{n-2,n-1,1,3}$ defined in Remark 3 also satisfies the assumptions of Theorem 3.2 and hence is 3-factor-critical.

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