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Stability of Surjectivity

Jacek Tabor

Institute of Mathematics, Jagiellonian University, Reymonta 4 st., 30-059 Kraków, Poland
E-mail: tabor@im.uj.edu.pl

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We investigate the stability of surjective functions in Banach spaces. As a corollary to the main theorem we obtain a new stability result for isometries. © 2000 Academic Press

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1. INTRODUCTION

About 1940 S. Ulam [7] posed the question of stability of additive functions. This problem was solved by D. H. Hyers in [4]. Since then the stability of various functional equations has been studied. For some further information on this subject we refer the reader to the survey papers [3, 5].

In this paper we propose a slightly different approach. Instead of studying the stability of functional equations, we propose to study the stability of a property of a function—surjectivity. Our idea is based on the fact that real-world observations have always some minimal error. Thus it seems natural to ask if a function which "looks like" a surjection to within some small possible error can be approximated by a surjection. We characterize those Banach spaces for which the answer to the above problem is positive.

2. APPROXIMATE SURJECTIONS

The notion of an approximate surjection introduced by R. Bourgin in [2] has proved to be useful in investigating the stability of isometries.

Definition. Let S be a set, let X be a Banach space, and let $\varepsilon \geqslant 0$ be arbitrary. We say that a function $f\colon S \to X$ is ε-onto if

$$\forall x \in X \quad \exists s \in S, \qquad \|x - f(s)\| \leqslant \varepsilon.$$

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Clearly a given function is 0-onto function if and only if it is a surjection. A natural question of stability of ε-onto functions arises.

DEFINITION 2. A Banach space X is said to have property (D) if every dense subset of X has the same cardinality as X.

As show Theorems 1 and 2 property (D) is crucial in characterization of those Banach spaces for which approximately surjective functions are stable. Moreover, many (possibly all) nonseparable Banach spaces have this property (see Problem 1 and Proposition 1).

By $B(a, r)$ we denote the closed ball with center at a and radius r.

THEOREM 1. *Let X be a Banach space which has property (D) and let $\varepsilon > 0$. Then for every set S and every ε-onto function $f: S \to X$ there exists a surjective function $F: S \to X$ such that*

$$\|f(s) - F(s)\| \leqslant 7\varepsilon \qquad \textit{for} \quad s \in S.$$

Moreover, given a countable subset C of S we can choose F in such a way that $F|_C = f|_C$.

Proof. By the Kuratowski–Zorn Lemma there exists a maximal with respect to inclusion subset A of X such that

$$B(a, \tfrac{7}{3}\varepsilon) \cap B(b, \tfrac{7}{3}\varepsilon) = \varnothing \qquad \text{for} \quad a, b \in A, \quad a \neq b. \tag{2}$$

The fact that A is maximal implies that

$$X = \bigcup_{a \in A} B(a, \tfrac{14}{3}\varepsilon).$$

For arbitrary $a \in A$ we now define

$$A_a := f(S) \cap B(a, \tfrac{7}{3}\varepsilon).$$

We will now show in Steps 1 and 2 that for every $a \in A$ the set A_a has the same cardinality as X. So let $a \in A$ be arbitrary.

Step 1. We put

$$X_a := \text{lin}_{\mathbf{Q}}(A_a \cup \{a\})$$

(by $\text{lin}_{\mathbf{Q}}(W) \subset X$ we denote the vector space over the field $\mathbf{Q}$ generated by the set W). In the first step we show that X_a is a dense subset of X.

For an indirect proof let us suppose that there exists an $x \in X$ such that

$$d(x, X_a) = \tfrac{9}{8}\varepsilon. \tag{3}$$

This implies the existence of $x_1 \in X_a$ satisfying

$$\|x - x_a\| \leqslant \tfrac{8}{7}\,\varepsilon. \tag{4}$$

Let $z := (x - x_a) + a$. Then (3) and (4) are equivalent to

$$d(z, X_a) = \tfrac{9}{8}\,\varepsilon, \tag{5}$$

$$\|z - a\| \leqslant \tfrac{8}{7}\,\varepsilon. \tag{6}$$

Let $s \in S$ be arbitrary. If $f(s) \in A_a$ then $f(s) \in X_a$, so by (5)

$$\|z - f(s)\| \geqslant \tfrac{9}{8}\,\varepsilon > \varepsilon.$$

If $f(s) \notin A_a$, then by the definition of A_a

$$\|f(s) - a\| \geqslant \tfrac{7}{3}\,\varepsilon,$$

so by (6),

$$\|f(s) - z\| \geqslant \|f(s) - a\| - \|z - a\| \geqslant \tfrac{7}{3}\,\varepsilon - \tfrac{8}{7}\,\varepsilon > \varepsilon.$$

Thus we have obtained that

$$\|f(s) - z\| > \varepsilon \qquad \text{for every} \quad s \in S,$$

a contradiction to the assumption that f is ε-onto. This means that X_a is dense in X.

Step 2. We are now going to show that

$$\text{card } A_a = \text{card } X_a.$$

First notice that A_a is not finite. If A_a were finite, then $X_a = \text{lin}_{\mathbf{Q}}(A_a \cup \{a\})$ would be countable, so we would have a dense countable subset of X, which by assumption would imply that the Banach space X is countable—a contradiction.

The inequality card $A_a \geqslant$ card $\mathbf{N}$ implies that

$$\begin{aligned} \text{card } \mathbf{Q} \times (A_a \cup \{a\}) &= \text{card } A_a, \\ \text{card } A_a^n &= \text{card } A_a \qquad \text{for} \quad n \in \mathbf{N}, \\ \text{card } \mathbf{N} \cdot \text{card } A_a &= \text{card } A_a. \end{aligned} \tag{7}$$

Clearly

$$\text{card lin}_{\mathbf{Q}}(A_a \cup \{a\}) \leqslant \text{card} \bigcup_{n \in \mathbf{N}} (\mathbf{Q} \times (A_a \cup \{a\}))^n. \tag{8}$$

Then by (8) and (7) we get that

$$\begin{aligned}\operatorname{card} X_a &= \operatorname{card} \bigcup_{n\in\mathbf{N}} (\mathbf{Q}\times(A\cup\{a\}))^n \\ &\leqslant \sum_{n\in\mathbf{N}} \operatorname{card} \mathbf{Q}\times(A_a\cup\{a\})^n \leqslant \sum_{n\in\mathbf{N}} \operatorname{card} A_a^n \\ &\leqslant \sum_{n\in\mathbf{N}} \operatorname{card} A_a \leqslant \operatorname{card} \mathbf{N}\cdot\operatorname{card} A_a = \operatorname{card} A_a.\end{aligned}$$

It follows from Step 1, Step 2, and the assumptions of the theorem that

$$\operatorname{card} A_a = \operatorname{card} X. \tag{9}$$

We can construct a surjective approximation F of f which satisfies the assertions of the theorem.

Step 3. Let C be a fixed countable subset of S. For $a\in A$ we define

$$B_a := A_a \setminus f(C).$$

As A_a is not countable and C is, we obtain by (9) that

$$\operatorname{card} B_a = \operatorname{card} A_a = \operatorname{card} X.$$

This implies that for every $a\in A$ there exists a surjective function h_a: $B_a \to B(a, \frac{14}{3}\varepsilon)$. We define a function F: $S\to X$ by the formula

$$F(s) := \begin{cases} f(s) & \text{if} \quad f(s)\in X \setminus \bigcup_{a\in A} B_a, \\ h_a(f(s)) & \text{if} \quad f(s)\in B_a, \quad a\in A.\end{cases}$$

First note that F is well-defined as the sets A_a (and consequently B_a) are pairwise disjoint.

Clearly $f(C)\subset X\setminus\bigcup_{a\in A} B_a$, so by the definition of F, $F|_C = f|_C$. For $a\in A$,

$$F(f^{-1}(B_a)) = h_a(f(f^{-1}(B_a))) = h_a(B_a) = B(a, \tfrac{14}{3}\varepsilon),$$

and therefore

$$\bigcup_{a\in A} B(a, \tfrac{14}{3}\varepsilon) \subset F(S).$$

By (2) this means that

$$X = F(S).$$

Thus we have shown that F is a surjection.

Now we will show that (1) holds. So let $s \in S$ be arbitrary. If $f(s) \notin \bigcup_{a \in A} B_a$ then $f(s) = F(s)$. So let $s \in S$ be such that $f(s) \in B_a \subset B(a, 2\,\frac{1}{3}\varepsilon)$ for some $a \in A$. Then

$$\begin{aligned}\|F(s) - f(s)\| &\leqslant \|F(s) - a\| + \|f(s) - a\| \\ &= \|h_a(f(s)) - a\| + \tfrac{7}{3}\,\varepsilon.\end{aligned}$$

By definition $h_a(f(s)) \in B(a, \frac{14}{3}\varepsilon)$, so $\|h_a(f(s)) - a\| \leqslant \frac{14}{3}\varepsilon$. Thus

$$\|F(s) - f(s)\| \leqslant \tfrac{14}{3}\,\varepsilon + \tfrac{7}{3}\,\varepsilon = 7\varepsilon. \quad \blacksquare$$

We now show that the assumption in Theorem 1 that X has the property (D) is essential.

THEOREM 2. *Let X be a Banach space which does not have the property (D). Then there exists a function f: $X \to X$ which is ε-onto for every $\varepsilon > 0$, but for every surjection F: $X \to X$*

$$\sup_{x \in X} \|f(x) - F(x)\| = \infty.$$

Proof. Let S be a dense subset of X with cardinality strictly smaller then cardinality of X. We define

$$f(x) := \begin{cases} 0 & \text{for} \quad x \in X \backslash S, \\ x & \text{for} \quad x \in S. \end{cases}$$

As S is dense f is obviously ε-onto for every $\varepsilon > 0$.

Suppose, on the contrary, that there exists a surjective function F: $X \to X$ such that

$$K := \sup_{x \in X} \|F(x) - f(x)\| < \infty.$$

Then for $x \in X \backslash S$,

$$\|F(x)\| = \|F(x) - f(x)\| \leqslant K,$$

which implies that $F(X\backslash S) \subset B(0, K)$. As F is surjective, $F(X) = X$, so we obtain that $X\backslash B(0, K) \subset F(S)$, and consequently that

$$\operatorname{card}(X) = \operatorname{card}(X\backslash B(0, K)) \leqslant \operatorname{card}(F(S)) \leqslant \operatorname{card}(S).$$

Thus we have obtained that $\operatorname{card}(S) = \operatorname{card}(X)$—a contradiction. ∎

In view of Theorems 1 and 2 it seems to be important to characterize Banach spaces X which have property (D).

Problem 1. Let X be a Banach space. Are the following two conditions equivalent?

- X has property (D),
- X is not separable.

Let B be a dense subset of a Banach space X. Then clearly $\operatorname{card} B \leqslant \operatorname{card} X \leqslant \operatorname{card} B^{\mathbf{N}}$, so the question whether $\operatorname{card} B = \operatorname{card} B^{\mathbf{N}}$ is relevant. Although we do not know the general answer to this question, in the following proposition we show how to construct Banach spaces in which all dense subsets have this property.

PROPOSITION 1. *Let X be a Banach space of infinite dimension and let B be a subset of X such that $\operatorname{lin}_{\mathbf{Q}}(B)$ is dense and*

$$\|b_0 - b_1\| \geqslant 1 \qquad \text{for} \quad b_0, b_1 \in B, \quad b_0 \neq b_1. \tag{10}$$

If $\operatorname{card} B = \operatorname{card} 2^{\Omega}$, for some Ω, then $\operatorname{card} B = \operatorname{card} B^{\mathbf{N}}$ and X has the property (D).

Proof. Let D be an arbitrary dense subset of X. Then for every $b \in B$ there exists $d_b \in D$ such that

$$\|d_b - b\| \leqslant \tfrac{1}{3}.$$

Then by (10) for b_0, $b_1 \in B$, $b_0 \neq b_1$,

$$\|b_{b_0} - d_{b_1}\| \geqslant \|b_0 - b_1\| - \|d_{b_0} - b_0\| - \|d_{b_1} - b_1\| \geqslant 1 - \tfrac{2}{3} = \tfrac{1}{3},$$

which implies that $d_{b_0} \neq d_{b_1}$, so $\operatorname{card} B \leqslant \operatorname{card} D$. As $\operatorname{lin}_{\mathbf{Q}}(B)$ is dense in X, for every element e of X there exists a sequence $\{b_n\}$ of elements of $\operatorname{lin}_{\mathbf{Q}}(B)$ such that $\lim_{n \to \infty} b_n = e$, which implies that

$$\operatorname{card} X \leqslant \operatorname{card} \operatorname{lin}_{\mathbf{Q}}(B)^{\mathbf{N}}.$$

However,

$$\operatorname{card}(\operatorname{lin}_{\mathbf{Q}}(B))^{\mathbf{N}} = \operatorname{card} \operatorname{lin}_{\mathbf{Q}}(B)^{\operatorname{card} \mathbf{N}} = \operatorname{card} B^{\operatorname{card} \mathbf{N}}$$
$$= \operatorname{card}(2^{\Omega})^{\mathbf{N}} = \operatorname{card} 2^{\Omega \times \mathbf{N}} = 2^{\operatorname{card} \Omega \times \mathbf{N}} = \operatorname{card} Bn$$

which implies that

$$\operatorname{card} B \leqslant \operatorname{card} X \leqslant \operatorname{card} B,$$

so

$$\operatorname{card} D = \operatorname{card} X. \quad \blacksquare$$

We immediately obtain the following corollary to the proposition.

COROLLARY 1. *Let H be an infinite dimensional Hilbert space with orthonormal base of cardinality $2^{\operatorname{card}(\Omega)}$, for certain Ω. Then H has the property (D).*

3. STABILITY OF THE ISOMETRY EQUATION

As approximate surjections appear to be important in the investigation of isometries, we now show that the results of Section 2 can easily be applied to obtain some new results from the stability of isometries.

We generalize the Main Theorem from [6]. For the convenience of the reader we will quote this result and the definition of ε-isometric functions.

DEFINITION 3. Let X, Y be Banach spaces. A function f: $X \to Y$ is said to be an ε-isometry if

$$|\|f(x) - f(y)\| - \|x - y\|| \leqslant \varepsilon$$

for $x, y \in X$.

THEOREM O-S [6]. *Let X and Y be real Banach spaces. Suppose that $\varepsilon > 0$ and that f: $X \to Y$ is a surjective ε-isometry satisfying $f(0) = 0$. Then there exists a unique linear isometry U: $X \to Y$ such that*

$$\|f(x) - U(x)\| \leqslant 2\varepsilon$$

for every $x \in X$.

Theorem O-S is important as it finally solved the question of the minimal constant of the Hyers–Ulam stability of surjective isometries. However, if we study stability, we often assume that we cannot measure the distances exactly and therefore we cannot check if the function f is surjective—the most we can do is to check if f is δ-onto, for some $\delta > 0$. Thus in the following theorem we generalize Theorem O-S to δ-onto functions.

THEOREM 3. *Let X, Y be Banach spaces, and let $f: X \to Y$ be an ε-isometry which is δ-onto and such that $f(0)=0$.*

Then there exists a unique linear isometry $U: X \to Y$ such that

$$\|f(x) - U(x)\| \leqslant 2\varepsilon + 35\delta \qquad \textit{for} \quad x \in X. \tag{11}$$

Proof. Let H be a Hilbert space with orthonormal basis of cardinality $\operatorname{card}(2^Y)$. First we show that every dense subset of $Y \times H$ has the cardinality of $Y \times H$. So let D be a dense subset of $Y \times H$ and let $p_H: Y \times H \to H$ be the canonical projection. Then clearly $C := p_H(D)$ is a dense subset of H, so by Corollary 1, card $C =$ card $H =$ card 2^Y. Then

$$\operatorname{card} D \geqslant \operatorname{card} C = \operatorname{card} 2^Y = \operatorname{card}(Y \times 2^Y) = \operatorname{card}(Y \times H)$$

which implies that $\operatorname{card} D = \operatorname{card}(Y \times H)$.

Now we define the function $f_H: X \times H \to Y \times H$ by the formula

$$f_H(x, h) := (f(x), h) \qquad \text{for} \quad (x, h) \in X \times H.$$

Note that as f is δ-onto f_H is also δ-onto. So by Theorem 1 there exists a surjective function $F_H: X \times H \to Y \times H$ such that

$$\|F_H(x, h) - f_H(x, h)\| \leqslant 7\delta \qquad \text{for} \quad (x, h) \in X \times H. \tag{12}$$

and

$$F_H(0) = f_H(0) = 0.$$

As f_H is an η-isometry

$$\begin{aligned} |\|F_H(a) - F_H(b)\| - \|a-b\|| &\leqslant |\|f_H(a) - f_H(b)\| - \|a-b\|| \\ &\quad + \|F_H(a) - f_H(a)\| + \|F_H(b) - f_H(b)\| \\ &\leqslant \varepsilon + 7\delta + 7\delta = \varepsilon + 14\delta. \end{aligned}$$

for $a, b \in X \times H$, which means that F_H is an $(\varepsilon + 14\delta)$-isometry. As F_H is surjective, we can now appeal to Theorem O-S and conclude that there exists a unique linear isometry $U_H: X \times H \to Y \times H$ such that

$$\|U_H(a) - F_H(a)\| \leqslant 2(\varepsilon + 14\delta) \qquad \text{for} \quad a \in X \times H.$$

This and (12) yields that

$$\|U_H(a) - f_H(a)\| \leqslant 2\varepsilon + 35\delta \qquad \text{for} \quad a \in X \times H. \tag{13}$$

Because U_H is linear (13) implies that

$$U_H(a) = \lim_{n \to \infty} \frac{1}{n} f_H(na) \qquad \text{for} \quad a \in X \times H. \tag{14}$$

Clearly $f_H(a) \in Y \times \{0\}$ for $a \in X \times \{0\}$, so (14) implies that

$$U_H(a) \in Y \times \{0\} \qquad \text{for} \quad a \in X \times \{0\}. \tag{15}$$

We define the function U: $X \to Y$ by the formula

$$U(x) := p_Y(U_H(x, 0)) \qquad \text{for} \quad x \in X,$$

where p_Y is the canonical projection onto Y. As U_H is a linear isometry by (15) we obtain that U is also a linear isometry. Now, by (13),

$$\|U(a) - f(a)\| \leqslant 2\varepsilon + 35\delta \qquad \text{for} \quad a \in X. \quad \blacksquare$$

We would like to mention that in our opinion it would be interesting to check whether the estimation in (11) is independent of δ. We show that in finite dimensional normed vector spaces this is the case. This result is related to that obtained in [1].

PROPOSITION 2. *Let X, Y be finite dimensional normed vector spaces. Let f: $X \to Y$ be an ε-isometry such that $f(0) = 0$. Suppose that one of the following conditions holds*

(i) $\dim X = \dim Y$,

(ii) *f is δ-onto for some $\delta > 0$.*

Then there exists a unique linear isometry U: $X \to Y$ such that

$$\|f(x) - U(x)\| \leqslant 37\varepsilon$$

for $x \in X$.

Proof. Suppose that (i) holds. R. D. Bourgin proved that then f is ε-onto (see Lemma 2.8 and Proposition 4.1 in [2]). So by Theorem 3 we obtain the assertion of the proposition.

Now suppose that (ii) holds. Then by Theorem 3 there exists a linear isometry U: $X \to Y$, such that $\|f(x) - U(x)\| \leqslant 2\varepsilon + 35\delta$ for $x \in X$. As f is δ-onto, this implies that U is $2\varepsilon + 36\delta$-onto; however, as U is linear and X

finite dimensional this implies that U is a linear bijection. Hence $\dim X = \dim Y$. Case (i) makes the proof complete. ▮

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