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Podleś' quantum sphere: dual coalgebra and classification of covariant first-order differential calculus

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Abstract

The dual coalgebra of Podleś' quantum sphere $\mathcal{O}_q(\mathbb{S}_c^2)$ is determined explicitly. This result is used to classify all finite-dimensional covariant first-order differential calculi over $\mathcal{O}_q(\mathbb{S}_c^2)$ for all but exceptional values of the parameter c.

Keywords: Quantum groups; Podleś' quantum sphere; Dual coalgebra; Differential calculus

1. Introduction

Podleś' quantum sphere $\mathcal{O}_q(\mathbb{S}_c^2)$ [10] is one of the best investigated examples of a quantum space, i.e., of a comodule algebra over the q-deformed coordinate ring of some affine algebraic group. Nevertheless, classification of covariant first-order differential calculus (FODC) over $\mathcal{O}_q(\mathbb{S}_c^2)$ in the sense of Woronowicz [13] has so far been achieved only under additional assumptions and in low dimensions. In [11] certain 2-dimensional covariant FODC over $\mathcal{O}_q(\mathbb{S}_c^2)$ which in many respects behave similarly as their classical counterparts have been classified. It turned out that only in the so-called quantum subgroup case $c = 0$ such a calculus exists and is then uniquely determined. All covariant FODC which as right modules are freely generated by the differentials of the generators e_i,

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$i = -1, 0, 1$, of $\mathcal{O}_q(\mathbb{S}_c^2)$ have been determined in [1]. It was shown by computer calculations that for all but exceptional values of c exactly one such calculus exists. Finally, in [3] a general notion of dimension of covariant FODC was introduced and all 2-dimensional covariant FODC over $\mathcal{O}_q(\mathbb{S}_c^2)$ have been classified.

In the present paper *all* finite-dimensional covariant FODC over $\mathcal{O}_q(\mathbb{S}_c^2)$ for all but exceptional values of c are classified. It turns out that for generic c there exists precisely one irreducible covariant FODC for any irreducible $\mathcal{O}_q(\mathrm{SL}(2))$-subcomodule of $\mathcal{O}_q(\mathbb{S}_c^2)$. The subcomodule $\mathbb{C} \cdot \mathbf{1}$ corresponds to the trivial calculus while in general the irreducible differential calculus has the same dimension as the corresponding $\mathcal{O}_q(\mathrm{SL}(2))$-subcomodule. For generic c any covariant FODC over $\mathcal{O}_q(\mathbb{S}_c^2)$ can be uniquely written as a direct sum of irreducible FODC. The exceptional cases include the quantum subgroup case $c = 0$.

The main tool on this way is the notion of quantum tangent space introduced for quantum groups in [13] and generalized to a large class of quantum spaces in [4]. Podleś' quantum sphere can be obtained as right K_c-invariant elements in $\mathcal{O}_q(\mathrm{SL}(2))$ where K_c denotes a left coideal subalgebra of $U_q(\mathfrak{sl}_2)$ generated by one twisted primitive element X_c. The notion of quantum tangent space allows one to identify finite-dimensional covariant FODC over $\mathcal{O}_q(\mathbb{S}_c^2)$ with finite-dimensional left subcomodules $T_\varepsilon \subset \mathcal{O}_q(\mathbb{S}_c^2)^\circ$ of the dual coalgebra which are right K_c-invariant and contain the counit ε. Thus, as a first step towards classification, the dual coalgebra $\mathcal{O}_q(\mathbb{S}_c^2)^\circ$ is determined explicitly in Theorem 4. It turns out that for all but exceptional values of c the restriction $\mathcal{O}_q(\mathrm{SL}(2))^\circ \to \mathcal{O}_q(\mathbb{S}_c^2)^\circ$ is onto.

Next, the subspace $F(\mathcal{O}_q(\mathbb{S}_c^2)^\circ, K_c)$ of elements of $\mathcal{O}_q(\mathbb{S}_c^2)^\circ$ with finite right K_c-action is determined. The action of the generator X_c induces a $U_q(\mathfrak{sl}_2)$-action on $F(\mathcal{O}_q(\mathbb{S}_c^2)^\circ, K_c)$ such that the decomposition into irreducible $U_q(\mathfrak{sl}_2)$-modules corresponds to the decomposition into right K_c-invariant left $\mathcal{O}_q(\mathbb{S}_c^2)^\circ$-comodules. To calculate $F(\mathcal{O}_q(\mathbb{S}_c^2)^\circ, K_c)$ explicit results of [9] are employed.

The quantum tangent spaces of the covariant FODC constructed in [2] are calculated. It turns out that for generic c the resulting tangent spaces cover all tangent spaces obtained in the classification. Therefore up to exceptional values of c all covariant FODC over $\mathcal{O}_q(\mathbb{S}_c^2)$ can be constructed by this method. Moreover, it is shown in Proposition 17 that these FODC are free left and right $\mathcal{O}_q(\mathbb{S}_c^2)$-modules and inner calculi.

The organization of this paper is as follows. In Section 2 the definition and some properties of $\mathcal{O}_q(\mathbb{S}_c^2)$ are recalled. Section 3 serves to give a complete description of the dual coalgebra $\mathcal{O}_q(\mathbb{S}_c^2)^\circ$. The main idea on this way is to show that all representations of $\mathcal{O}_q(\mathbb{S}_c^2)$ can be written as direct sums of representations of certain localizations of $\mathcal{O}_q(\mathbb{S}_c^2)$. These localizations are seen to be isomorphic to $U_q(\mathfrak{b}_-)^{\mathrm{op}}$ and the dual coalgebra of $U_q(\mathfrak{b}_-)^{\mathrm{op}}$ is known [6]. In Section 4 the subspace $F(\mathcal{O}_q(\mathbb{S}_c^2)^\circ, K_c)$ is determined and decomposed into $U_q(\mathfrak{sl}_2)$-modules. The notion of covariant FODC and quantum tangent space are recalled in the last section. Combination of the above steps lead to the classification result in Theorem 11.

If not stated otherwise, all notations and conventions coincide with those introduced in [8]. Throughout this paper, $q \in \mathbb{C} \setminus \{0\}$ will be assumed not to be a root of unity. For any element a of a coalgebra $\mathcal{A}$ with counit ε and a distinguished group-like element $\mathbf{1}$ define $a^+ := a - \varepsilon(a)\mathbf{1}$ and for any subset $\mathcal{B} \subset \mathcal{A}$ set $\mathcal{B}^+ := \{b^+ \mid b \in \mathcal{A}\}$.

2. Podleś' quantum sphere

Let u^i_j, $i,j=1,2$, denote the matrix coefficients of the vector representation of $U_q(\mathfrak{sl}_2)$, i.e., the generators of the quantum group $\mathcal{O}_q(\mathrm{SL}(2))$. Recall that the elements u^i_j satisfy the relations

$$\begin{gathered}u^1_1u^1_2=qu^1_2u^1_1,\qquad u^1_1u^2_1=qu^2_1u^1_1,\qquad u^1_2u^2_2=qu^2_2u^1_2,\qquad u^2_1u^2_2=qu^2_2u^2_1,\\ u^1_2u^2_1=u^2_1u^1_2,\qquad u^1_1u^2_2-u^2_2u^1_1=\big(q-q^{-1}\big)u^1_2u^2_1,\qquad u^1_1u^2_2-qu^1_2u^2_1=1\end{gathered}$$

and that the coalgebra structure of $\mathcal{O}_q(\mathrm{SL}(2))$ takes the form

$$\Delta u^i_j=\sum_k u^i_k\otimes u^k_j.$$

In the notation of [10], the matrix coefficients of the three-dimensional representation of $U_q(\mathfrak{sl}_2)$ are given by

$$\big(\pi^i_j\big)_{i,j=-1,0,1}=\begin{pmatrix} u^2_2u^2_2 & -(q^2+1)u^2_2u^2_1 & -qu^2_1u^2_1\\ -q^{-1}u^1_2u^2_2 & 1+(q+q^{-1})u^1_2u^2_1 & u^1_1u^2_1\\ -q^{-1}u^1_2u^1_2 & (q+q^{-1})u^1_2u^1_1 & u^1_1u^1_1\end{pmatrix},$$

where upper and lower indices of u and π refer to lines and columns, respectively. For any $\varepsilon_{-1},\varepsilon_0,\varepsilon_1\in\mathbb{C}$, where $\varepsilon_0\neq 0$ or $\varepsilon_{-1}\varepsilon_1\neq 0$ consider the subalgebra $\mathcal{O}_q(\mathbb{S}^2_{\varepsilon_{-1},\varepsilon_0,\varepsilon_1})\subset\mathcal{O}_q(\mathrm{SL}(2))$ generated by $e_i:=\sum_j\varepsilon_j\pi^j_i$, $i=-1,0,1$. Note that $\varepsilon(e_i)=\varepsilon_i$ and that $\mathcal{O}_q(\mathbb{S}^2_{\varepsilon_{-1},\varepsilon_0,\varepsilon_1})$ obtains the structure of a right $\mathcal{O}_q(\mathrm{SL}(2))$-comodule algebra by $\Delta(e_i)=\sum_j e_j\otimes\pi^j_i$. A complete set of defining relations of $\mathcal{O}_q(\mathbb{S}^2_{\varepsilon_{-1},\varepsilon_0,\varepsilon_1})$ is given by

$$\begin{aligned}\big(1+q^2\big)\big(e_{-1}e_1+q^{-2}e_1e_{-1}\big)+e_0^2&=\rho,\\ -q^2e_{-1}e_0+e_0e_{-1}&=\lambda e_{-1},\\ \big(1+q^2\big)(e_{-1}e_1-e_1e_{-1})+\big(1-q^2\big)e_0^2&=\lambda e_0,\\ e_1e_0-q^2e_0e_1&=\lambda e_1,\end{aligned}$$

where $\rho=q^{-2}(q^2+1)^2\varepsilon_{-1}\varepsilon_1+\varepsilon_0^2$ and $\lambda=(1-q^2)\varepsilon_0$. Up to isomorphism the comodule algebra $\mathcal{O}_q(\mathbb{S}^2_{\varepsilon_{-1},\varepsilon_0,\varepsilon_1})$ depends only on $c=(\varepsilon_{-1}\varepsilon_1:\varepsilon_0)\in\mathbb{C}P^1$ [10] and will therefore be denoted by $\mathcal{O}_q(\mathbb{S}^2_c)$.

For $c\neq\infty=(1:0)$ one can choose $\varepsilon_0=1$, $\varepsilon_{-1}=\varepsilon_1$. Then $\lambda=1-q^2$ and $\rho=(q+q^{-1})^2c+1$. Defining $A=(1+q^2)^{-1}(1-e_0)$, the above relations can be rewritten as

$$e_{-1}e_1=A-A^2+c, \tag{1}$$

$$e_1e_{-1}=q^2A-q^4A^2+c, \tag{2}$$

$$e_1 A = q^2 A e_1, \tag{3}$$
$$e_{-1} A = q^{-2} A e_{-1}. \tag{4}$$

Similarly, for $c = \infty = (1:0)$ choose $\varepsilon_0 = 0$ and $\varepsilon_{-1} = \varepsilon_1 = 1$, i.e., $\lambda = 0$ and $\rho = (q + q^{-1})^2$. Defining $A = -(1 + q^2)^{-1} e_0$, the above relations are equivalent to

$$e_{-1} e_1 = -A^2 + 1, \tag{5}$$
$$e_1 e_{-1} = -q^4 A^2 + 1, \tag{6}$$
$$e_1 A = q^2 A e_1, \tag{7}$$
$$e_{-1} A = q^{-2} A e_{-1}. \tag{8}$$

If in the sequel it is necessary to fix an explicit realization of $\mathcal{O}_q(\mathbb{S}^2_c) \subset \mathcal{O}_q(\mathrm{SL}(2))$, the coefficients $\varepsilon_{-1}, \varepsilon_0, \varepsilon_1$ will be chosen as above.

Define linear functionals f_λ, $\lambda \in \mathbb{C} \setminus \{0\}$, and g in the dual Hopf algebra $\mathcal{O}_q(\mathrm{SL}(2))^\circ$ of $\mathcal{O}_q(\mathrm{SL}(2))$ by

$$f_\lambda\big((u^i_j)\big) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad g\big((u^i_j)\big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9}$$

where as above upper and lower indices refer to lines and columns, respectively, and

$$\Delta f_\lambda = f_\lambda \otimes f_\lambda, \qquad \Delta g = g \otimes \varepsilon + \varepsilon \otimes g. \tag{10}$$

Note that (9) and (10) imply in particular $f(1) = 1$ and $g(1) = 0$.

Recall that the dual pairing [8] between $U_q(\mathfrak{sl}_2)$ and $\mathcal{O}_q(\mathrm{SL}(2))$ induces linear functionals E and F in $\mathcal{O}_q(\mathrm{SL}(2))^\circ$ satisfying

$$E\big((u^i_j)\big) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad F\big((u^i_j)\big) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{11}$$

and

$$\Delta E = E \otimes K + \varepsilon \otimes E, \qquad \Delta F = F \otimes \varepsilon + K^{-1} \otimes F \tag{12}$$

where $K = f_{q^{-1}}$. Let $\mathcal{U} \subset \mathcal{O}_q(\mathrm{SL}(2))^\circ$ denote the algebra generated by the functionals f_λ, $\lambda \in \mathbb{C} \setminus \{0\}$, E, F, and g. For transcendental q the Hopf algebra $\mathcal{U}$ is isomorphic to $\mathcal{O}_q(\mathrm{SL}(2))^\circ$ [6, 9.4.9]. The above functionals satisfy the relations:

$$f_\lambda f_\mu = f_{\lambda\mu}, \qquad f_\lambda E = \lambda^{-2} E f_\lambda, \qquad f_\lambda F = \lambda^2 F f_\lambda,$$
$$f_\lambda g = g f_\lambda, \qquad Eg = (g + 2)E, \qquad Fg = (g - 2)F,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{13}$$

Note that the subalgebra of $\mathcal{O}_q(\mathrm{SL}(2))^\circ$ generated by E, F, K, and K^{-1} is isomorphic to $U_q(\mathfrak{sl}_2)$, [8, 4.4.1]. Evaluating the functionals f_λ, g, E, and F on the matrix coefficients π^i_j, one obtains

$$\begin{aligned} f_\lambda\big((\pi^i_j)\big) &= \lambda^{-2}E^{-1}_{-1} + E^0_0 + \lambda^2 E^1_1, \\ g\big((\pi^i_j)\big) &= -2E^{-1}_{-1} + 2E^1_1, \\ E\big((\pi^i_j)\big) &= -\big(q^2+1\big)E^{-1}_0 + E^0_1, \\ F\big((\pi^i_j)\big) &= -q^{-1}E^0_{-1} + \big(q+q^{-1}\big)E^1_0, \end{aligned} \tag{14}$$

where E^i_j, $i, j = -1, 0, 1$, denotes the (3×3)-matrix with entry 1 at position (i, j) and zero elsewhere.

Many interesting examples of quantum homogeneous spaces can be defined by infinitesimal invariants [9, Section 2]. The method of classification of differential calculi developed in [4] and used in Section 5 applies precisely to this class of comodule algebras. Podleś' quantum sphere fits into this scheme as follows. Fix a square root $q^{1/2}$ of q. For $n \in \mathbb{N}_0/2$ set $c(n) = -1/(q^n + q^{-n})^2$. Since q is not a root of unity, $c(n) \neq c(m)$ for all $n, m \in \mathbb{N}_0/2$, $n \neq m$. Define subsets of $\mathbb{C}P^1$ by

$$\begin{aligned} J_1 &:= \big\{c \in \mathbb{C}P^1 \bigm| c \neq c(n)\ \forall n \in \mathbb{N}/2 \setminus \mathbb{N}\big\}, \\ J_2 &:= \big\{c \in \mathbb{C}P^1 \bigm| c \neq c(n)\ \forall n \in \mathbb{N}_0/2\big\}. \end{aligned}$$

It is known [9, Remark 4.5.3] that the following statements are equivalent:

(1) $c \in J_1$,
(2) $\mathcal{O}_q(\mathbb{S}^2_c) \cong \{b \in \mathcal{O}_q(\mathrm{SL}(2)) \mid X(b_{(1)})b_{(2)} = 0\}$ for a twisted primitive element

$$X = \alpha\big(K^{-1} - 1\big) + \beta K^{-1}E + \gamma F \in \mathcal{U} \quad \text{and}$$
$$c = \begin{cases} \dfrac{\beta\gamma q^{-1}}{\alpha^2(q-q^{-1})^2} & \text{if } \alpha \neq 0, \\ \infty & \text{if } \alpha = 0 \text{ and } \beta\gamma \neq 0. \end{cases}$$

Calculating the pairing between X and the explicit generators $e_i \in \mathcal{O}_q(\mathrm{SL}(2))$ chosen above, one obtains $-\varepsilon_1(q - q^{-1})\alpha = \gamma$ and $\beta = q\gamma$ in the case $c \neq \infty$. Similarly, for $c = \infty$ one obtains $\alpha = 0$ and $\beta = q\gamma$. Thus the embeddings from above are realized by

$$X_c = \begin{cases} qK^{-1}E + F & \text{if } c = \infty, \\ K^{-1} - 1 & \text{if } c = 0, \\ -(c^{1/2}(q-q^{-1}))^{-1}(K^{-1} - 1) + qK^{-1}E + F & \text{else} \end{cases} \tag{15}$$

for any square root $\varepsilon_1 = c^{1/2}$ of c. Define $K_c = \mathbb{C}[X_c] \subset U_q(\mathfrak{sl}_2)$. If $c \in J_2$ then any finite-dimensional $U_q(\mathfrak{sl}_2)$-module is a direct sum of irreducible K_c-modules and therefore $\mathcal{O}_q(\mathrm{SL}(2))$ is a faithfully flat left (and right) $\mathcal{O}_q(\mathbb{S}^2_c)$-module [9, Theorem 5.2].

3. The dual coalgebra $\mathcal{B}^\circ = \mathcal{O}_q(\mathbb{S}^2_c)^\circ$

To understand the dual coalgebra [12, Section 6.0] $\mathcal{O}_q(\mathbb{S}^2_c)^\circ$ of Podleś' quantum sphere, it is useful to consider first the dual Hopf algebra $(U_q(\mathfrak{b}_-)^{\mathrm{op}})^\circ$ where $U_q(\mathfrak{b}_-) \subset U_q(\mathfrak{sl}_2)$ denotes the subalgebra generated by F, K, and K^{-1}. Further, let U_0, $U_q(\mathfrak{n}_+)$, $U_q(\mathfrak{n}_-)$, and $U_q(\mathfrak{b}_+)$ denote the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $\{K, K^{-1}\}$, E, F, and $\{E, K, K^{-1}\}$, respectively. By [6, Theorem 2.1.8] the dual Hopf algebra $(U_0)^\circ$ is isomorphic to the commutative Hopf algebra

$$\mathbb{C}\big[\gamma, \chi_\lambda \,\big|\, \lambda \in \mathbb{C} \setminus \{0\}\big]\big/(\chi_\lambda \chi_\mu = \chi_{\lambda\mu},\ \chi_1 = 1),$$

where $\gamma(K) = 1$, $\chi_\lambda(K) = \lambda$, and the coalgebra structure is given by

$$\Delta\gamma = \gamma \otimes 1 + 1 \otimes \gamma, \qquad \Delta\chi_\lambda = \chi_\lambda \otimes \chi_\lambda. \tag{16}$$

The subalgebra $U_q(\mathfrak{n}_+) \subset U_q(\mathfrak{sl}_2)$ is a right U_0-comodule algebra with coaction

$$\delta_R\big(E^i\big) = E^i \otimes K^{-i}$$

and therefore has a left $(U_0)^\circ$-module structure. The corresponding left crossed product algebra $U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ$ is a Hopf algebra with $\Delta E = 1 \otimes E + E \otimes \chi_{q^{-2}}$ containing $U_q(\mathfrak{b}_+)$ where $K \in U_q(\mathfrak{b}_+)$ corresponds to $\chi_{q^{-2}}$. The dual pairing of Hopf algebras (in the conventions of [8, 6.3.1])

$$\langle\cdot,\cdot\rangle : U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)^{\mathrm{op}} \to \mathbb{C} \tag{17}$$

given by $\langle K, K\rangle = q^{-2}$, $\langle K, F\rangle = \langle E, K\rangle = 0$, and $\langle E, F\rangle = 1/(q^{-1} - q)$ extends to a pairing of Hopf algebras

$$\langle\cdot,\cdot\rangle : \big(U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ\big) \otimes U_q(\mathfrak{b}_-)^{\mathrm{op}} \to \mathbb{C} \tag{18}$$

such that

$$\langle\gamma, K\rangle = 1, \qquad \langle\chi_\lambda, K\rangle = \lambda, \qquad \langle\gamma, F\rangle = \langle\chi_\lambda, F\rangle = 0.$$

Lemma 1. *For $a \in U_q(\mathfrak{n}_+)$, $u \in U_0$, $b \in U_q(\mathfrak{n}_-)$, and $f \in (U_0)^\circ$ one has*

$$\langle af, bu\rangle = f(u)\langle a, b\rangle.$$

In particular, the pairing (18) *is non-degenerate.*

By the above lemma the map of Hopf algebras

$$\Phi : \big(U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ\big) \to \big(U_q(\mathfrak{b}_-)^{\mathrm{op}}\big)^\circ \tag{19}$$

induced by (18) is injective. The following result is proven in [6, 9.4.8] for transcendental q. Yet it also holds for $q \in \mathbb{C} \setminus \{0\}$ not a root of unity and is reproduced here in our setting for the convenience of the reader.

Proposition 2. *The map Φ is an isomorphism.*

Proof. Recall that there is a canonical isomorphism $U_q(\mathfrak{b}_-)^{\mathrm{op}} \cong U_q(\mathfrak{n}_-) \otimes U_0$ of vector spaces. Let $J \subset U_q(\mathfrak{b}_-)^{\mathrm{op}}$ denote any two-sided ideal of finite codimension. As J is $\mathbb{N}_0$-graded via the adjoint action of U_0, it contains some ideal $I \subset U_0$ of finite codimension and $(U_q(\mathfrak{n}_-)^+)^n$ for some $n \in \mathbb{N}$. Therefore J contains the left ideal

$$\bigl(U_q(\mathfrak{n}_-)^+\bigr)^n \otimes U_0 + U_q(\mathfrak{n}_-) \otimes I \subset U_q(\mathfrak{n}_-) \otimes U_0$$

of finite codimension. Thus

$$\begin{aligned}
\bigl(U_q(\mathfrak{b}_-)^{\mathrm{op}}/J\bigr)^* &\subset \Bigl(\bigl(U_q(\mathfrak{n}_-)\big/\bigl(U_q(\mathfrak{n}_-)^+\bigr)^n\bigr) \otimes (U_0/I)\Bigr)^* \\
&= \bigl(U_q(\mathfrak{n}_-)\big/\bigl(U_q(\mathfrak{n}_-)^+\bigr)^n\bigr)^* \otimes (U_0/I)^* \\
&\subset U_q(\mathfrak{n}_+) \otimes (U_0)^\circ,
\end{aligned}$$

where in the last inclusion one uses that $U_q(\mathfrak{n}_+)$ is the graded dual of $U_q(\mathfrak{n}_-)$ via the pairing (17). By Lemma 1 one obtains $U_q(\mathfrak{n}_+) \otimes (U_0)^\circ \subset \operatorname{Im}\Phi$ and therefore Φ is onto. $\square$

For the computation of $\mathcal{O}_q(\mathbb{S}_c^2)^\circ$, some results about the representation theory of the algebra $\mathcal{O}_q(\mathbb{S}_c^2)$ are collected.

Lemma 3. *Any finite-dimensional representation $\mu : \mathcal{O}_q(\mathbb{S}_c^2) \to \mathrm{End}(V)$ is a direct sum $\mu = \mu_0 \oplus \mu_{\neq 0}$ where $\mu_0(A)$ is nilpotent and $\mu_{\neq 0}(A)$ is invertible. In particular, the coalgebra $\mathcal{O}_q(\mathbb{S}_c^2)^\circ$ is a direct sum $C_0 \oplus C_{\neq 0}$ where C_0 and $C_{\neq 0}$ denote the coalgebras of matrix coefficients of finite-dimensional representations of $\mathcal{O}_q(\mathbb{S}_c^2)$ with nilpotent and invertible A action, respectively. In addition,*

(i) *If $c \neq c(n)$ for all $n \in \mathbb{N}$ then $\mu_{\neq 0} = 0$.*
(ii) *If $c \neq 0$ then $\mu_0(e_{\pm 1})$ are isomorphisms.*
(iii) *If $c = c(n)$ for some $n \in \mathbb{N}$ then there exists exactly one indecomposable representation $\mu_n : \mathcal{O}_q(\mathbb{S}_c^2) \to \mathrm{End}(V)$ such that $\mu_n(A)$ is invertible. This representation is n-dimensional.*
(iv) *If $c = 0$ then $C_0 = C_{0+} \oplus C_{00} \oplus C_{0-}$ where C_{0+}, C_{0-}, and C_{00} denote the coalgebras of matrix coefficients of finite-dimensional representations with the following properties*:
- C_{0+}: *the action of e_1 is invertible,*
- C_{0-}: *the action of e_{-1} is invertible,*
- C_{00}: *the action of both e_1 and e_{-1} is nilpotent.*

Proof. Relations (3) and (4) imply that e_1 and e_{-1} transform the generalized eigenspace V_λ of A with corresponding eigenvalue λ to the generalized eigenspace $V_{q^{-2}\lambda}$ and $V_{q^2\lambda}$, respectively. Set $V_{\neq 0} := \bigoplus_{\lambda\neq 0} V_\lambda$. Then $V = V_0 \oplus V_{\neq 0}$ is a direct sum of representations of $\mathcal{O}_q(\mathbb{S}^2_c)$.

Since q is not a root of unity e_1 and e_{-1} act nilpotently on $V_{\neq 0}$. Assume that $v \in V_{\neq 0}$ is an eigenvector of A with eigenvalue λ such that $e_{-1}v = 0$, $e_1^n v = 0$, and $w := e_1^{n-1}v \neq 0$. Then relations (2), (6) and (1), (5) applied to v and w, respectively, imply

$$\left.\begin{aligned} 0 &= q^2\lambda - q^4\lambda^2 + c \\ 0 &= q^{-2(n-1)}\lambda - q^{-4(n-1)}\lambda^2 + c \end{aligned}\right\} \quad \text{for } c \neq \infty, \tag{20}$$

$$\left.\begin{aligned} 0 &= -q^4\lambda^2 + 1 \\ 0 &= -q^{-4(n-1)}\lambda^2 + 1 \end{aligned}\right\} \quad \text{for } c = \infty. \tag{21}$$

The second set of equations cannot be fulfilled as q is not a root of unity. The first set of equations implies $c = c(n)$ and therefore proves (i).

Since $\mu_0(A)$ is nilpotent the second statement follows from (1) and (5).

To prove the third statement assume first that there exists $u \in V_{\neq 0}$ such that $(A-\nu)^2 u = 0$ but $(A-\nu)u \neq 0$ for some $\nu \in \mathbb{C} \setminus \{0\}$. Applying e_{-1} several times, we may assume, using the notations from above, that $e_{-1}(A-\nu)u = 0$ and hence $\nu = \lambda$ and $(A-\nu)u = v$. Then (20) implies $\lambda = q^{n-2}/(q^n + q^{-n})$. The relation $e_{-1}v = 0$ implies that $e_{-1}u$ is an eigenvector of A with corresponding eigenvalue $q^2\lambda$ or $e_{-1}u = 0$. Suppose that $e_{-1}^k u = 0$ for some $k > 1$ and $e_{-1}^{k-1}u \neq 0$. Then, on the one hand, the eigenvalue of A corresponding to $e_{-1}^{k-1}u$ coincides with $q^{2k-2}\lambda$. On the other hand, e_{-1}^{k-1} fulfills the properties of v considered above (20) and hence is an eigenvector of A corresponding to the eigenvalue $q^{n-2}/(q^n + q^{-n}) = \lambda$. Therefore $k = 1$ and $e_{-1}u = 0$. By Eq. (2) and $(A-\lambda)^2 u = 0$, one now obtains

$$\bigl(q^2 - 2q^4\lambda\bigr)Au + \bigl(c + q^4\lambda^2\bigr)u = -q^2\frac{q^n - q^{-n}}{q^n + q^{-n}}Au + \bigl(c + q^4\lambda^2\bigr)u = 0.$$

As $n \geqslant 1$ and $q^{2n} \neq 1$, this is a contradiction to the assumption that u is not an eigenvector of A. Thus A is diagonalizable. The relations (20) imply that all eigenvalues of A lie in the set $\{q^{n-2k}/(q^n + q^{-n}) \mid k = 1, 2, \ldots, n\}$. In view of (1) and (2), the eigenspaces for different eigenvalues are isomorphic and $V_{\neq 0}$ is the direct sum $\bigoplus_{i\in I} \mathcal{O}_q(\mathbb{S}^2_c)b_i$ where $\{b_i \mid i \in I\}$ is an arbitrary basis of V_λ. By construction, $\dim \mathcal{O}_q(\mathbb{S}^2_c)b_i = n$ for all $i \in I$.

To validate the last statement, note first that any finite-dimensional representation $\mu : \mathcal{O}_q(\mathbb{S}^2_0) \to \mathrm{End}(V)$ is a direct sum $\mu = \mu_+ \oplus \mu'$, $V = V_+ \oplus V'$, where $\mu_+(e_1)$ is invertible and $\mu'(e_1)$ is nilpotent. Indeed, (3) implies that $AV_+ \subset V_+$ and $AV' \subset V'$. On the other hand, (1) leads to

$$e_{-1}V_+ = e_{-1}e_1V_+ = \bigl(A - A^2\bigr)V_+ \subset V_+$$

and $e_1^k V' = 0$ yields

$$e_1^{k+1} e_{-1} V' = e_1^k \big(q^2 A - q^4 A^2\big) V' \subset e_1^k V' = 0.$$

Note then that (1), (3), and the nilpotency of $\mu(A)$ imply that $\mu_+(e_{-1})$ is nilpotent. Similarly, $\mu' = \mu_0' \oplus \mu_-$ where $\mu_0'(e_{-1})$ is nilpotent and $\mu_-(e_{-1})$ is invertible. $\square$

The inclusions $\mathcal{O}_q(\mathbb{S}^2_c) \cong \mathcal{O}_q(\mathbb{S}^2_{c^{1/2},1,c^{1/2}}) \subset \mathcal{O}_q(\mathrm{SL}(2))$ where $c^{1/2}$ is a fixed square root of $c \in \mathbb{C}$ and $\mathcal{O}_q(\mathbb{S}^2_\infty) \cong \mathcal{O}_q(\mathbb{S}^2_{1,0,1}) \subset \mathcal{O}_q(\mathrm{SL}(2))$ of right $\mathcal{O}_q(\mathrm{SL}(2))$-comodule algebras induce homomorphisms of right $\mathcal{O}_q(\mathrm{SL}(2))^\circ$-module coalgebras $\mathcal{O}_q(\mathrm{SL}(2))^\circ \to \mathcal{O}_q(\mathbb{S}^2_c)^\circ$ for any $c \in \mathbb{C}P^1$. For $m, l \in \mathbb{N}_0$ and $\lambda \in \mathbb{C} \setminus \{0\}$, let $\psi^{ml}_{\lambda^2}$ denote the image of $f_\lambda g^m E^l$ under this projection. It follows from (14) that $f_\lambda = f_{-\lambda}$ on $\mathcal{O}_q(\mathbb{S}^2_c)$ and therefore the definition of ψ^{ml}_μ does not depend on the choice of a root of μ.

Theorem 4. *The following sets form a vector space basis of $\mathcal{O}_q(\mathbb{S}^2_c)^\circ$.*

(i) *If $c \notin \{0, c(n) \mid n \in \mathbb{N}\}$: $\{\psi^{ml}_\lambda \mid \lambda \in \mathbb{C} \setminus \{0\}, m, l \in \mathbb{N}_0\}$.*

(ii) *If $c = c(n)$, $n \in \mathbb{N}$: $\{\psi^{ml}_\lambda \mid \lambda \in \mathbb{C} \setminus \{0\},\ m, l \in \mathbb{N}_0\} \cup \mathfrak{B}_n$, where $\mathfrak{B}_n$ denotes any basis of the n^2-dimensional subspace $C_{\neq 0}$ of $\mathcal{O}_q(\mathbb{S}^2_c)^\circ$.*

(iii) *If $c = 0$: $\{E^k F^l \mid k, l \in \mathbb{N}_0\} \cup \{\chi^+_\lambda g^m F^l,\ \chi^-_\lambda g^m E^l \mid \lambda \in \mathbb{C} \setminus \{0\},\ l, m \in \mathbb{N}_0\}$ where $\chi^\pm_\lambda$ is the character on $\mathcal{O}_q(\mathbb{S}^2_0) \cong \mathcal{O}_q(\mathbb{S}^2_{0,1,0})$ defined by $\chi^\pm_\lambda(e_i) = \delta_{i0} + \delta_{i,\pm 1}\lambda^{\pm 1}$.*

Proof. Consider the Hopf subalgebra $\mathcal{O}_{q^2}(\mathrm{SO}(3)) \subset \mathcal{O}_q(\mathrm{SL}(2))$ generated by the matrix coefficients $\{\pi^i_j \mid i, j = -1, 0, 1\}$ and let J denote the intersection of the two-sided ideal $(u^1_2) \subset \mathcal{O}_q(\mathrm{SL}(2))$ with $\mathcal{O}_{q^2}(\mathrm{SO}(3))$. There is an isomorphism of Hopf algebras $\mathcal{O}_{q^2}(\mathrm{SO}(3))/J \to U_q(\mathfrak{b}_-)^{\mathrm{op}}$:

$$u^2_2 u^2_2 \mapsto K^{-1}, \qquad u^2_1 u^2_2 \mapsto \big(1 - q^2\big) F, \qquad u^1_1 u^1_1 \mapsto K$$

such that the functionals $E, f_\lambda, g \in \mathcal{O}_{q^2}(\mathrm{SO}(3))^\circ$ given by (9)–(12) correspond to $E, \chi_{\lambda^2}, 2\gamma \in U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ = (U_q(\mathfrak{b}_-)^{\mathrm{op}})^\circ$.

Let $\mathcal{O}_q(\mathbb{S}^2_c)(e_{-1})$ denote the localization of $\mathcal{O}_q(\mathbb{S}^2_c) \subset \mathcal{O}_q(\mathrm{SL}(2))$ with respect to the left and right Ore set $\{e^n_{-1} \mid n \in \mathbb{N}_0\}$. Observe that in this localization by (1) and (5) the generator e_1 can be expressed in terms of e_{-1} and A. Therefore for $c \neq 0$, the sequence

$$\mathcal{O}_q\big(\mathbb{S}^2_c\big) \hookrightarrow \mathcal{O}_{q^2}\big(\mathrm{SO}(3)\big) \to \mathcal{O}_{q^2}\big(\mathrm{SO}(3)\big)/J \to U_q(\mathfrak{b}_-)^{\mathrm{op}}$$

induces an isomorphism $\mathcal{O}_q(\mathbb{S}^2_c)(e_{-1}) \to U_q(\mathfrak{b}_-)^{\mathrm{op}}$:

$$\begin{aligned} &e_{-1} \mapsto \varepsilon_{-1} K^{-1}, \qquad e_0 \mapsto \varepsilon_{-1}\big(q^3 - q^{-1}\big) F + \varepsilon_0, \\ &e_1 \mapsto -\varepsilon_{-1}\big(q - q^{-1}\big)^2 K F^2 - \varepsilon_0\big(q - q^{-1}\big) K F + \varepsilon_1 K. \end{aligned} \tag{22}$$

Thus, by Lemma 3(ii) and Proposition 2, one obtains

$$C_0 \cong \mathcal{O}_q\big(\mathbb{S}^2_c\big)(e_{-1})^\circ \cong \big(U_q(\mathfrak{b}_-)^{\mathrm{op}}\big)^\circ \cong U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ,$$

and the basis element $\chi_\lambda \gamma^m E^l \in U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ$ corresponds to $(1/2)^m \psi^{ml}_\lambda$. This proves (i) and one obtains (ii) taking into account that for $c = c(n)$ the representation μ_n from Lemma 3(iii) is irreducible.

In the case $c = 0$, consider now an embedding different from the standard one:

$$\mathcal{O}_q\big(\mathbb{S}^2_0\big) \hookrightarrow \mathcal{O}_{q^2}\big(\mathrm{SO}(3)\big), \qquad e_i \mapsto \pi_i^{-1} + \pi_i^0.$$

Similarly to the case $c \neq 0$, this induces an isomorphism $\mathcal{O}_q(\mathbb{S}^2_0)(e_{-1}) \to U_q(\mathfrak{b}_-)^{\mathrm{op}}$ given by (22) with $\varepsilon_i = \delta_{i0} + \delta_{i,-1}$. Thus, by Lemma 3,

$$C_{0-} \cong \big(U_q(\mathfrak{b}_-)^{\mathrm{op}}\big)^\circ \cong U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ,$$

and the basis element $\chi_\lambda \gamma^m E^l \in U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ$ corresponds to $(1/2)^m \chi^-_\lambda g^m E^l$. The subcoalgebra C_{0+} is dealt analogously replacing the two-sided ideal (u^1_2) by (u^2_1), replacing $U_q(\mathfrak{b}_-)^{\mathrm{op}}$ by $U_q(\mathfrak{b}_+)^{\mathrm{cop}}$, and using the embedding $e_i \mapsto \pi_i^0 + \pi_i^1$. The component C_{00} has been shown to coincide with the coalgebra $U_q(\mathfrak{sl}_2)/(K-1)U_q(\mathfrak{sl}_2)$ in [4, Lemma 5.2, Corollary 3.8]. The elements $\{E^k F^l \mid k, l \in \mathbb{N}_0\}$ form a basis of $U_q(\mathfrak{sl}_2)/(K-1)U_q(\mathfrak{sl}_2)$. $\square$

4. Local finiteness for the K_c-action on $\mathcal{O}_q(\mathbb{S}^2_c)^\circ$

Recall that $\mathcal{U} \subset \mathcal{O}_q(\mathrm{SL}(2))^\circ$ denotes the Hopf algebra generated by the set of functionals $\{f_\lambda, E, F, g \mid \lambda \in \mathbb{C} \setminus \{0\}\}$ and that $K_c = \mathbb{C}[X_c]$, where X_c is given by (15). For the rest of this paper assume that $0 \neq c \in J_2$. For $\mathcal{B} = \mathcal{O}_q(\mathbb{S}^2_c)$ recall that $\mathcal{B}^\circ$ is a right $\mathcal{U}$-module. Define

$$F(\mathcal{B}^\circ, K_c) = \big\{ f \in \mathcal{B}^\circ \mid \dim(f K_c) < \infty \big\}.$$

If $f \in \mathcal{B}^\circ$ is the restriction of an element $f' \in \mathcal{U}$ to $\mathcal{B}$ and $k \in K_c$ then

$$fk = k_{(0)} S^{-1}(k_{(-1)}) f' k_{(-2)}|_{\mathcal{B}} = S^{-1}(k_{(0)}) f' k_{(-1)}|_{\mathcal{B}}$$

as K_c is a left $\mathcal{U}$-comodule and $k|_{\mathcal{B}} = k(1)\varepsilon$. Thus $F(\mathcal{U})|_{\mathcal{B}} \subset F(\mathcal{B}^\circ, K_c)$, where for any Hopf algebra A,

$$F(A) = \big\{ a \in A \mid \dim(\mathrm{ad}\, A)a < \infty \big\}, \quad (\mathrm{ad}\, b)a = b_{(1)} a S(b_{(2)}).$$

Lemma 5. *The vector space $F(\mathcal{B}^\circ, K_c)$ is a right $F(\mathcal{U})$-module left $\mathcal{B}^\circ$-comodule. Any element of $F(\mathcal{B}^\circ, K_c)$ is contained in a finite-dimensional right K_c-submodule left $\mathcal{B}^\circ$-subcomodule.*

Proof. For $f \in F(\mathcal{U})$, $u \in F(\mathcal{B}^\circ, K_c)$ consider $V = uK_c$ and $W = (\mathrm{ad}\,\mathcal{U})f$. Then, for any $k \in K_c$,

$$(u \cdot f)k = (uk_{(0)}) \cdot \bigl(S^{-1}(k_{(-1)})fk_{(-2)}\bigr) \in V \cdot W.$$

Therefore $F(\mathcal{B}^\circ, K_c)$ is a right $F(\mathcal{U})$-module.

Now, for any subspace $V \subset F(\mathcal{B}^\circ, K_c)$, let $\overline{V}$ denote the left $\mathcal{B}^\circ$-comodule generated by V. The vector space $\overline{V}$ is finite-dimensional. For any $u \in V$, $k \in K_c$, applying the coaction to the second factor of $k_{(-1)} \otimes uk_{(0)} \in \mathcal{U} \otimes V$, one obtains

$$k_{(-2)} \otimes u_{(1)}k_{(-1)} \otimes u_{(2)}k_{(0)} \in \mathcal{U} \otimes \mathcal{B}^\circ \otimes \overline{V}$$

and therefore

$$u_{(1)} \otimes u_{(2)}k = u_{(1)}k_{(-1)}S^{-1}(k_{(-2)}) \otimes u_{(2)}k_{(0)} \in \mathcal{B}^\circ \otimes \overline{V}.$$

Thus $F(\mathcal{B}^\circ, K_c)$ is a left $\mathcal{B}^\circ$-comodule and $u \in \overline{V} \supset \overline{V}K_c$. $\square$

Define a $(\mathbb{C} \setminus \{0\})$-graduation on the vector space $\mathcal{B}^\circ$ by $\deg(g^m f_\mu E^l) = \mu$.

Lemma 6. *Any left $\mathcal{B}^\circ$-subcomodule $W \subset \mathcal{B}^\circ$ is a $(\mathbb{C}\setminus\{0\})$-graded vector space.*

Proof. Consider an arbitrary element $u \in W \subset \mathcal{B}^\circ$. By Theorem 4(i) one can assume that $u = \sum_\mu f_\mu a^\mu$ for some a^μ which are linear combinations of basis vectors $g^m E^l$, $m, l \in \mathbb{N}_0$. By the explicit form (10), (12) of the coproduct of g and E and by Theorem 4(i), one can write

$$\Delta u = \sum_\mu f_\mu \otimes f_\mu a^\mu + \sum_i u_i' \otimes u_i'',$$

where $\{u_i', f_\mu\}$ is a set of linear independent elements in $\mathcal{B}^\circ$. As W is a left $\mathcal{B}^\circ$-comodule, $f_\mu a^\mu \in W$ for all μ.

Let $F_\mu(\mathcal{B}^\circ, K_c)$ denote the subspace of elements of degree μ in $F(\mathcal{B}^\circ, K_c)$.

Lemma 7. $F(\mathcal{B}^\circ, K_c) \subset \widetilde{F} := \mathrm{Lin}_{\mathbb{C}}\{\psi_\lambda^{0l} \mid l \in \mathbb{N}_0,\ \lambda \in \mathbb{C} \setminus \{0\}\}$.

Proof. Consider an arbitrary element $u \in F_\mu(\mathcal{B}^\circ, K_c)$. By Theorem 4(i), one can assume that $u = \sum_{i=0}^m g^i a_i$ for some $a_i \in \widetilde{F}$ such that $\deg(a_i) = \mu$ and $a_m \neq 0$. Contrary to the assertion of the lemma, suppose that $m \geqslant 1$. Applying the coaction to u, one obtains

$$\Delta u = f_\mu g^{m-1} \otimes (mga_m + a_{m-1}) + \sum_i u_i' \otimes u_i''$$

where $\{f_\mu g^{m-1}, u_i'\}$ is a linearly independent set of elements of $\mathcal{B}^\circ$. Thus, as $F(\mathcal{B}^\circ, K_c)$ is a left $\mathcal{B}^\circ$-comodule, $mga_m + a_{m-1} \in F(\mathcal{B}^\circ, K_c)$. Thus it suffices to show that $m = 1$ leads to a contradiction. By similar arguments it is sufficient to consider the case $u = gf_\mu + a_0$, $f_\mu \in F(\mathcal{B}^\circ, K_c)$.

One checks by direct computation that $(\mathrm{ad}\,\mathcal{U})E$ is a three-dimensional vector space and therefore $E \in F(\mathcal{U})$. By Lemma 5 this implies $f_\mu E^m \in F(\mathcal{B}^\circ, K_c)$ for all $m \in \mathbb{N}_0$. Thus $gf_\mu \in F(\mathcal{B}^\circ, K_c)$.

Recall that $\mathcal{B}$ can be obtained as right X_c-invariants of $\mathcal{O}_q(\mathrm{SL}(2))$ and therefore $Fu = ((c^{1/2}(q-q^{-1}))^{-1}(K^{-1}-1) - qK^{-1}E)u$ in $\mathcal{B}^\circ$ for all $u \in \mathcal{U}$. In regard of this property a direct calculation using (13) and (15) leads to

$$gf_\mu E^l X_c = q\bigl(q^{2l} - \mu^4\bigr)gf_{q\mu}E^{l+1} - 4q\mu^4 f_{q\mu}E^{l+1} + \sum_{i=0}^{l} a_i E^i \tag{23}$$

where $a_i \in \mathrm{Lin}_{\mathbb{C}}\{gf_\nu, f_\nu \mid \nu \in \mathbb{C} \setminus \{0\}\}$. Further,

$$\begin{aligned} f_\mu E^l X_c \;=\; & q\bigl(q^{2l} - \mu^4\bigr) f_{q\mu}E^{l+1} + \alpha\bigl(q^{2l} - \mu^2\bigr) f_{q\mu}E^l + \alpha\bigl(\mu^2 - 1\bigr) f_\mu E^l \\ & + [l]\frac{q^{-l+1}K - q^{l-1}K^{-1}}{q-q^{-1}} f_\mu E^{l-1} \end{aligned} \tag{24}$$

where, as in (15), $\alpha = 0$ if $c = \infty$ and $\alpha = -(c^{1/2}(q-q^{-1}))^{-1}$ else. By (23),

$$\begin{aligned} gf_\mu(X_c)^k \;=\; & q^k\left(\prod_{i=0}^{k-1}\bigl(q^{2i} - \bigl(q^i\mu\bigr)^4\bigr)\right) gf_{q^k\mu}E^k \\ & - 4\sum_{j=0}^{k-1}\bigl(q^j\mu\bigr)^4 q^k\left(\prod_{\substack{i=0\\ i\neq j}}^{k-1}\bigl(q^{2i} - \bigl(q^i\mu\bigr)^4\bigr)\right) f_{q^k\mu}E^k + \cdots, \end{aligned}$$

where $\cdots$ denotes terms containing only smaller powers of E. Therefore $gf_\mu \in F(\mathcal{B}^\circ, K_c)$ implies $\mu^4 = q^{-2(k-1)}$ for some $k \in \mathbb{N}$. Then for $l \geqslant 0$

$$gf_\mu(X_c)^{k+l} = -4\bigl(q^{k-1}\mu\bigr)^4 q^{k+l}\left(\prod_{\substack{i=0\\ i\neq k-1}}^{k+l-1}\bigl(q^{2i} - \bigl(q^i\mu\bigr)^4\bigr)\right) f_{q^k\mu}E^{k+l} + \cdots,$$

again up to expressions containing only smaller powers of E. As

$$q^{2(k+l)} - \bigl(q^{k+l}\mu\bigr)^4 = q^{2(k+l)}\bigl(1 - q^{2(l+1)}\bigr) \neq 0 \quad \text{for all } l \geqslant 0,$$

the coefficient of $f_{q^k\mu}E^{k+l}$ does not vanish. This is a contradiction to the assumption $gf_\mu \in F(\mathcal{B}^\circ, K_c)$. $\square$

To shorten notation, let ψ_λ^l denote the basis element ψ_λ^{0l} of $\widetilde{F}$. Define three maps $\phi, \varphi, \kappa : \widetilde{F} \to \widetilde{F}$ by

$$\begin{aligned}\phi(\psi_\lambda^l) &= -\frac{q^l[l]}{q-q^{-1}}\psi_{q^2\lambda}^{l-1} + \alpha q(q^{2l}-\lambda)\psi_{q^2\lambda}^l + q^2(q^{2l}-\lambda^2)\psi_{q^2\lambda}^{l+1},\\ \varphi(\psi_\lambda^l) &= \lambda^{-1}\frac{q^{1-l}[l]}{q-q^{-1}}\psi_{q^{-2}\lambda}^{l-1},\\ \kappa(\psi_\lambda^l) &= \lambda^{-1}\psi_\lambda^l. \end{aligned}\tag{25}$$

In view of (24), this means

$$\psi_\lambda^l X_c = q^{-1}\phi(\psi_\lambda^l) + \lambda\varphi(\psi_\lambda^l) + \alpha(1-\lambda^{-1})\kappa^{-1}(\psi_\lambda^l).\tag{26}$$

Note that

$$\varphi\circ\phi - \phi\circ\varphi = \frac{\kappa-\kappa^{-1}}{q-q^{-1}}, \qquad \kappa\circ\varphi = q^2\varphi\circ\kappa, \qquad \kappa\circ\phi = q^{-2}\phi\circ\kappa,$$

i.e., the operators ϕ, φ, and κ yield a representation $\rho : U_q(\mathfrak{sl}_2) \to \mathrm{End}(\widetilde{F})$, $\rho(E) = \varphi$, $\rho(F) = \phi$, $\rho(K) = \kappa$.

Lemma 8. *For any finite-dimensional subspace $V \subset \widetilde{F}$, the following statements are equivalent.*

(i) $\Delta V \subset \mathcal{B}^\circ \otimes V$ *and* $VK_c \subset V$.
(ii) V *is a left* $U_q(\mathfrak{sl}_2)$*-module via* ρ.

Proof. (i) ⇒ (ii). As in Lemma 6, one obtains that V is $(\mathbb{C}\backslash\{0\})$-graded. Then the assertion follows from (26). To verify (ii) ⇒ (i), note that

$$\Delta\psi_\lambda^l = \sum_{r=0}^{l}\begin{bmatrix} l\\ r\end{bmatrix} q^{-r(l-r)}\psi_\lambda^r \otimes \psi_{q^{-2r}\lambda}^{l-r} = \sum_{r=0}^{l} b_r\psi_\lambda^r \otimes \varphi^r(\psi_\lambda^l)$$

where $b_r \in \mathbb{C}$ depend on r and λ but not on l. □

Lemma 5 implies that $F(\mathcal{B}^\circ, K_c)$ is a ρ-invariant subspace of $\widetilde{F}$. Recall that an element $\psi \in V \setminus \{0\}$ is called a highest weight vector of a $U_q(\mathfrak{sl}_2)$-module V with highest weight λ if $K\psi = \lambda\psi$ and $E\psi = 0$.

Proposition 9. *There exists a decomposition of* $U_q(\mathfrak{sl}_2)$*-modules,*

$$F(\mathcal{B}^\circ, K_c) = \bigoplus_{\lambda\in J^c} V_\lambda,$$

such that

$$\begin{aligned} J^c &= \{q^{-l} \mid l \in 2\mathbb{N}_0\} \quad \textit{for } c \notin \{\infty, (q^r - q^{-r})^{-2} \mid r \in \mathbb{N}/2\}, \\ J^\infty &= \{\pm q^{-l} \mid l \in 2\mathbb{N}_0\}, \\ J^{(q^r-q^{-r})^{-2}} &= \{q^{-l}, -q^{-k} \mid l \in 2\mathbb{N}_0,\ k \in 2r + 2\mathbb{N}_0\}, \quad r \in \mathbb{N}/2, \end{aligned}$$

where the components $V_{\pm q^{-l}}$ *are* $(l+1)$*-dimensional irreducible* $U_q(\mathfrak{sl}_2)$*-modules of highest weight* $\pm q^l$.

Proof. Lemma 5, and Lemma 8, (i) ⇒ (ii), imply that the $U_q(\mathfrak{sl}_2)$-module $F(\mathcal{B}^\circ, K_c)$ can be written as a direct sum

$$F(\mathcal{B}^\circ, K_c) = \bigoplus_{\lambda \in J^c} V_\lambda$$

of finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules. Here $J^c \subset \mathbb{C} \setminus \{0\}$ denotes the subset of nonzero complex numbers λ such that ϕ operates nilpotently on $\psi^0_\lambda = f_{\sqrt{\lambda}}$. Indeed, by (25) the set $\{\psi^0_\lambda \mid \lambda \in J^c\}$ is a basis of all highest weight vectors of $F(\mathcal{B}^\circ, K_c)$ with respect to the $U_q(\mathfrak{sl}_2)$-module structure. It remains to show that J^c is of the form given in the proposition.

Note that $\phi^{l+1}(\psi^0_\lambda) = 0$ and $\phi^l(\psi^0_\lambda) \neq 0$, $l \in \mathbb{N}_0$, imply $\lambda = \pm q^{-l}$. In this case $\phi^l(\psi^0_\lambda) \in \mathrm{Lin}_\mathbb{C}\{\psi^k_{q^{2l}\lambda} \mid k = 0, \ldots, l\}$ and the mapping

$$\phi : \mathrm{Lin}_\mathbb{C}\{\psi^k_{q^{2l}\lambda} \mid k = 0, \ldots, l\} \to \mathrm{Lin}_\mathbb{C}\{\psi^k_{q^{2l+2}\lambda} \mid k = 0, \ldots, l\}$$

is given by the matrix

$$q \begin{pmatrix} -(\pm q^l - 1)\alpha & -\hat{q}^{-1}q^0[1] & 0 & \cdots & 0 \\ q(1 - q^{2l}) & -(\pm q^l - q^2)\alpha & -\hat{q}^{-1}q^1[2] & \ddots & \vdots \\ 0 & q(q^2 - q^{2l}) & -(\pm q^l - q^4)\alpha & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\hat{q}^{-1}q^{l-1}[l] \\ 0 & \cdots & 0 & q(q^{2(l-1)} - q^{2l}) & -(\pm q^l - q^{2l})\alpha \end{pmatrix}$$

with respect to the bases ψ^k_μ ($\mu = q^{2l}\lambda$ and $\mu = q^{2l+2}\lambda$, respectively), $\hat{q} = q - q^{-1}$. Recall that $\beta = q$ and $\gamma = 1$. Using $q(1 - q^{2k}) = -\hat{q}q^{k+1}[k]$, the map ϕ can be written with respect to the bases $\bar{\psi}_\mu^k := (-\hat{q})^k q^{-(l-k)(l-k+1)/2}\psi_\mu^k$ as

$$q^{l+1}\begin{pmatrix} (q^{-l}\mp 1)\alpha & [1]\gamma & 0 & \cdots & 0 \\ q^{-l}[l]\beta & (q^{2-l}\mp 1)\alpha & [2]\gamma & \ddots & \vdots \\ 0 & q^{2-l}[l-1]\beta & (q^{4-l}\mp 1)\alpha & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & [l]\gamma \\ 0 & \cdots & 0 & q^{l-2}[1]\beta & (q^l \mp 1)\alpha \end{pmatrix}. \tag{27}$$

In the case of minus signs in the diagonals, this matrix is, up to the overall factor, precisely the matrix M_l describing the transpose of the left action of X_c on the $(l+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module V_l [9, Section 4]. Note that any nonzero element of $\ker M_l$ is a lowest weight vector of the $(l+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module of highest weight q^l, and therefore $q^{-l}\in J^c$ if and only if $\ker M_l \neq \{0\}$. By [9, Proposition 4.2], the matrix M_l is known to have $l+1$, not necessarily distinct, eigenvalues

$$\rho_r = \frac{\alpha}{2}(q^r - q^{-r})^2 + \frac{1}{2}\big(q^{2r} - q^{-2r}\big)R, \quad r \in I_l = \{-l/2, 1-l/2, \ldots, l/2\}$$

where

$$R^2 = \alpha^2 + \frac{4\beta\gamma q^{-1}}{(q-q^{-1})^2}.$$

In particular, M_l has eigenvalue 0 if and only if l is even or

$$0 = \rho_r\rho_{-r} = -(q^r - q^{-r})^2\left(\alpha^2 + \beta\gamma q^{-1}\left(\frac{q^r + q^{-r}}{q - q^{-1}}\right)^2\right).$$

The second case is equivalent to $c = c(n)$ for some $n \in I_l$. As this case is excluded by assumption $q^{-l} \in J^c$ if and only if l is even.

Let $\mathbb{C}w$ denote the one-dimensional representation of $U_q(\mathfrak{sl}_2)$ uniquely determined by $E\cdot w = 0$, $F\cdot w = 0$, $K\cdot w = -w$. By means of a base change, the matrix (27) corresponding to $-q^{-l}$ can be transformed into the matrix of the transpose of the left X_c-action on the finite-dimensional $U_q(\mathfrak{sl}_2)$-module $\mathbb{C}w \otimes V_l$. The eigenvalues of this action can be computed by means of [9, Proposition 4.6]. In particular, the X_c-action has a nontrivial kernel if and only if

$$0 = (\rho_r + 2\alpha)(\rho_{-r} + 2\alpha) = (q^r + q^{-r})^2\left(\alpha^2 - \beta\gamma q^{-1}\left(\frac{q^r - q^{-r}}{q - q^{-1}}\right)^2\right)$$

for some $r \in I_l$. This equation is equivalent to

$$c = \frac{1}{(q^r - q^{-r})^2}, \quad r \neq 0, \quad \text{or} \quad c = \infty, \quad r = 0.$$

Notice that $(q^r - q^{-r})^2 - c(n)^{-1} = (q^{r+n}+q^{-(r+n)})(q^{r-n}+q^{-(r-n)}) \neq 0$ for all $r, n \in \mathbb{N}_0/2$ and therefore these cases are not excluded. $\square$

5. Differential calculus over $\mathcal{O}_q(\mathbb{S}^2_c)$

For the convenience of the reader, the notion of differential calculus from [13] is recalled. A *first-order differential calculus* (FODC) over an algebra $\mathcal{B}$ is a $\mathcal{B}$-bimodule Γ together with a $\mathbb{C}$-linear map

$$\mathrm{d}:\mathcal{B} \to \Gamma$$

such that $\Gamma = \mathrm{Lin}_{\mathbb{C}}\{a\,\mathrm{d}b\,c \mid a, b, c \in \mathcal{B}\}$ and d satisfies the Leibniz rule

$$\mathrm{d}(ab) = a\,\mathrm{d}b + \mathrm{d}a\,b.$$

Let, in addition, $\mathcal{A}$ denote a Hopf algebra and $\Delta_{\mathcal{B}}:\mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ a right $\mathcal{A}$-comodule algebra structure on $\mathcal{B}$. If Γ possesses the structure of a right $\mathcal{A}$-comodule

$$\Delta_\Gamma:\Gamma \to \Gamma \otimes \mathcal{A}$$

such that

$$\Delta_\Gamma(a\,\mathrm{d}b\,c) = (\Delta_{\mathcal{B}}a)\big((\mathrm{d} \otimes \mathrm{id})\Delta_{\mathcal{B}}b\big)(\Delta_{\mathcal{B}}c),$$

then Γ is called (*right*) *covariant*. A FODC $\mathrm{d}:\mathcal{B} \to \Gamma$ over $\mathcal{B}$ is called *inner* if there exists an element $\omega \in \Gamma$ such that $\mathrm{d}x = \omega x - x\omega$ for all $x \in \mathcal{B}$. For further details on first-order differential calculi, consult [8].

Let U denote a Hopf algebra with bijective antipode and $L \subset U$ a left coideal subalgebra, i.e., $\Delta_L : L \to U \otimes L$. Consider a tensor category $\mathcal{C}$ of finite-dimensional left U-modules. As in [9, Section 2], this means that $\mathcal{C}$ is a class of finite-dimensional left U-modules containing the trivial U-module via ε and satisfying

$$X, Y \in \mathcal{C} \quad \Rightarrow \quad X \oplus Y, X \otimes Y, X^* \in \mathcal{C}. \tag{28}$$

Let $\mathcal{A} := U^0_{\mathcal{C}}$ denote the dual Hopf algebra generated by the matrix coefficients of all U-modules in $\mathcal{C}$. Assume that $\mathcal{A}$ separates the elements of U and that the antipode of $\mathcal{A}$ is bijective. Define a right coideal subalgebra $\mathcal{B} \subset \mathcal{A}$ by

$$\mathcal{B} := \big\{b \in \mathcal{A} \,\big|\, \langle u, b_{(1)}\rangle b_{(2)} = \varepsilon(u)b \quad \text{for all } u \in L\big\}. \tag{29}$$

Assume L to be $\mathcal{C}$-semisimple, i.e., the restriction of any U-module in $\mathcal{C}$ to the subalgebra $L \subset U$ is isomorphic to the direct sum of irreducible L-modules. By [9, Theorem 2.2(2)] this implies that $\mathcal{A}$ is a faithfully flat $\mathcal{B}$-module.

In this situation, right covariant first-order differential calculi over $\mathcal{B}$ can be classified via certain left ideals of $\mathcal{B}^+$ [3]. More explicitly, the subspace

$$\mathcal{L}=\left\{\sum_i a_i^+\varepsilon(b_i)\,\Big|\,\sum_i \mathrm{d}a_i\, b_i=0\right\}\subset\mathcal{B}^+ \tag{30}$$

is a left ideal which determines the differential calculus uniquely. Equivalently, $\mathcal{L}=\{a\in\mathcal{B}^+ \mid \mathrm{d}a\in\Gamma\mathcal{B}^+\}$. To construct the FODC Γ corresponding to $\mathcal{L}$, consider the $\mathcal{B}$-bimodule structure on $\widetilde{\Gamma}:=(\mathcal{B}^+/\mathcal{L})\otimes\mathcal{A}$ given by

$$c\big(\bar{b}\otimes a\big)d=\overline{c_{(0)}b}\otimes c_{(1)}ad,\quad c,d\in\mathcal{B},\ b\in\mathcal{B}^+,\ a\in\mathcal{A}, \tag{31}$$

and the differential $\mathrm{d}:\mathcal{B}\to\widetilde{\Gamma}$, $\mathrm{d}b=\overline{b_{(0)}^+}\otimes b_{(1)}$. Then $\Gamma\cong\{\mathrm{d}b_1\,b_2\mid b_1,b_2\in\mathcal{B}\}$. To the left ideal $\mathcal{L}$, one associates the vector space

$$T^\varepsilon=\big\{f\in\mathcal{B}^\circ \mid f(x)=0 \text{ for all } x\in\mathcal{L}\big\}$$

and the so-called *quantum tangent space*

$$T=(T^\varepsilon)^+=\big\{f\in T^\varepsilon \mid f(1)=0\big\}.$$

The dimension of a first-order differential calculus is defined by

$$\dim\Gamma=\dim_{\mathbb{C}}\Gamma/\Gamma\mathcal{B}^+=\dim_{\mathbb{C}}\mathcal{B}^+/\mathcal{L}.$$

In the following, all FODC are assumed to be finite-dimensional.

Proposition 10 [4, Corollary 1.2]. *There is a canonical one-to-one correspondence between n-dimensional covariant FODC over $\mathcal{B}$ and $(n+1)$-dimensional subspaces $T^\varepsilon\subset\mathcal{B}^\circ$ such that*

$$\varepsilon\in T^\varepsilon,\qquad \Delta T^\varepsilon\subset\mathcal{B}^\circ\otimes T^\varepsilon,\qquad T^\varepsilon L\subset T^\varepsilon. \tag{32}$$

A covariant FODC Γ over $\mathcal{B}$ is called *irreducible* if it does not possess any nontrivial quotient (by a right covariant $\mathcal{B}$-bimodule). Note that this property is equivalent to the property that T_Γ^ε does not possess any right L-invariant left $\mathcal{B}^\circ$-subcomodule $\widetilde{T}$ such that $\mathbb{C}\cdot\varepsilon\subsetneq\widetilde{T}\subsetneq T_\Gamma^\varepsilon$.

For a family of right covariant FODC $(\Gamma_i,\mathrm{d}_i)_{i=1,\dots,k}$, define $\mathrm{d}=\bigoplus_i \mathrm{d}_i:\mathcal{B}\to\bigoplus_i\Gamma_i$. Then $\Gamma=\mathcal{B}\,\mathrm{d}\mathcal{B}\subset\bigoplus_i\Gamma_i$ is a covariant FODC with differential d which is called the *sum* of the calculi $\Gamma_1,\dots,\Gamma_k$ [5]. The left ideal corresponding to Γ is given by $\mathcal{L}_\Gamma=\bigcap_i\mathcal{L}_{\Gamma_i}$ and therefore the relation $T_\Gamma=T_{\Gamma_1}+\cdots+T_{\Gamma_k}$ of quantum tangent spaces holds. A sum of covariant differential calculi is called a *direct sum* if $\Gamma=\bigoplus_i\Gamma_i$ is a direct sum of bimodules. This condition is equivalent to $T_\Gamma=\bigoplus_i T_{\Gamma_i}$.

As an immediate consequence of Proposition 10, Lemma 8, and Proposition 9, one obtains the following classification result for differential calculi over $\mathcal{O}_q(\mathbb{S}_c^2)$.

Theorem 11. *Assume $0 \neq c \in J_2$. For $\lambda \in J^c$, let Γ_λ denote the uniquely determined covariant FODC over $\mathcal{O}_q(\mathbb{S}^2_c)$ such that $T^{\varepsilon}_{\Gamma_\lambda} = V_\lambda + \mathbb{C}\varepsilon$. Then Γ_λ is irreducible and any finite-dimensional covariant FODC Γ over $\mathcal{O}_q(\mathbb{S}^2_c)$ is isomorphic to a direct sum*

$$\Gamma = \bigoplus_{\lambda \in J} \Gamma_\lambda$$

for some finite subset $J \subset J^c$.

To compare a FODC Γ over $\mathcal{B}$ with its classical counterpart it is often instructive to know whether Γ is generated by certain differentials as a right $\mathcal{B}$-module. For the class of quantum spaces considered here, this question can be completely answered as follows. For any covariant FODC Γ with corresponding left ideal $\mathcal{L}$ and quantum tangent space T, consider the projection

$$P_r : \Gamma \otimes_{\mathcal{B}} \mathcal{A} \to \Gamma \otimes_{\mathcal{B}} \mathcal{A}, \quad \gamma \otimes a \mapsto \gamma_{(1)} \otimes S(\gamma_{(2)})\varepsilon(a),$$

onto the subspace $(\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}} \subset \Gamma \otimes_{\mathcal{B}} \mathcal{A}$ of right coinvariant elements. The relation $\mathrm{d}b \otimes a = \mathrm{d}(b_{(1)}) \otimes S(b_{(2)})b_{(3)}a$ implies that the right $\mathcal{A}$-module $\Gamma \otimes_{\mathcal{B}} \mathcal{A}$ is generated by the elements $P_r(\mathrm{d}b \otimes 1)$, $b \in \mathcal{B}$. For any $a = \sum_i a_i^+ \varepsilon(b_i) \in \mathcal{L}$ where $\sum_i \mathrm{d}a_i\, b_i = 0$, one obtains

$$P_r(\mathrm{d}a \otimes 1) = P_r\left(\sum_i \mathrm{d}a_i \otimes b_i\right) = 0.$$

Therefore P_r induces a well-defined surjection

$$\mathcal{B}^+/\mathcal{L} \to (\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}, \quad b \mapsto P_r(\mathrm{d}b \otimes 1). \tag{33}$$

Lemma 12. *The pairing*

$$(\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}} \times T \to \mathbb{C}, \quad (\mathrm{d}a \otimes b, X) \mapsto X(a)\varepsilon(b),$$

is non-degenerate. Further, $b \in \mathcal{L}$ if and only if $b \in \mathcal{B}^+$ and $P_r(\mathrm{d}b \otimes 1) = 0$.

Proof. To verify the first statement note that by construction the elements $P_r(\mathrm{d}b \otimes 1)$, $b \in \mathcal{B}$, separate T. On the other hand, (33) implies the relation $\dim_{\mathbb{C}}((\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}) \leqslant \dim_{\mathbb{C}} \mathcal{B}^+/\mathcal{L} = \dim_{\mathbb{C}} T$; thus T separates $(\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}$ and (33) is an isomorphism. □

Lemma 13. *Let $W \subset \mathcal{B}$ be a right $\mathcal{A}$-subcomodule then $\mathrm{d}W$ generates Γ as a right $\mathcal{B}$-module if and only if the elements of W separate the quantum tangent space T_Γ. If $\dim W = \dim \Gamma$ and the elements of W separate T_Γ then Γ is a free right $\mathcal{B}$-module generated by the differentials of an arbitrary basis of W.*

Proof. Let $\Gamma' \subset \Gamma$ denote the right $\mathcal{B}$-module generated by $\mathrm{d}W$. Note that Γ' is a right $\mathcal{A}$-comodule. Then as $\mathcal{A}$ is a faithfully flat left $\mathcal{B}$-module,

$$\Gamma' = \Gamma \quad \Longleftrightarrow \quad \Gamma' \otimes_{\mathcal{B}} \mathcal{A} = \Gamma \otimes_{\mathcal{B}} \mathcal{A} \quad \Longleftrightarrow \quad (\Gamma' \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}} = (\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}.$$

Now, if W separates T_Γ then $(\Gamma' \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}$ separates T_Γ and therefore, by Lemma 12, coincides with $(\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}$. Conversely, if $\Gamma' = \Gamma$ then $(\Gamma' \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}$ separates T_Γ and therefore the elements of W separate T_Γ. This proves the first statement.

To prove the second statement, let Γ'' denote the free right $\mathcal{B}$-module generated by the differentials of an arbitrary basis $e_1, \ldots, e_k$ of W. Then, as above,

$$\begin{aligned} \Gamma'' \cong \Gamma \quad &\Longleftrightarrow \quad \Gamma'' \otimes_{\mathcal{B}} \mathcal{A} \cong \Gamma \otimes_{\mathcal{B}} \mathcal{A} \\ &\Longleftrightarrow \quad P_r(\mathrm{d}e_i \otimes 1),\ i = 1, \ldots, k,\ \text{form a basis of } (\Gamma \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}. \end{aligned}$$

In view of Lemma 12, this property is equivalent to the nondegeneracy of the pairing between W and T_Γ. □

Combining the above lemma with Theorem 11, one can now classify all covariant FODC over $\mathcal{O}_q(\mathbb{S}^2_c)$ generated as right $\mathcal{O}_q(\mathbb{S}^2_c)$-modules by the differentials $\mathrm{d}e_i$, $i = -1, 0, 1$. The straightforward calculations of the pairing of the tangent spaces with the generators e_i, $i = -1, 0, 1$, are omitted.

Corollary 14. *For $c \in J_2 \setminus \{0, \infty, (q^{1/2} - q^{-1/2})^{-2}\}$, there exists exactly one covariant FODC $\Gamma_{q^{-2}}$ over $\mathcal{O}_q(\mathbb{S}^2_c)$ which is generated by $\{\mathrm{d}e_i \mid i = -1, 0, 1\}$ as a right $\mathcal{O}_q(\mathbb{S}^2_c)$-module. The elements $\{\mathrm{d}e_i \mid i = -1, 0, 1\}$ form a right $\mathcal{O}_q(\mathbb{S}^2_c)$-module basis of this calculus.*

For $c = \infty$ there exist exactly three covariant FODCs over $\mathcal{O}_q(\mathbb{S}^2_c)$ which are generated by $\{\mathrm{d}e_i \mid i = -1, 0, 1\}$ as right $\mathcal{O}_q(\mathbb{S}^2_c)$-modules. One of them, Γ_{-1}, is one-dimensional, the elements $\{\mathrm{d}e_i \mid i = -1, 0, 1\}$ form a right $\mathcal{O}_q(\mathbb{S}^2_c)$-module basis of each of the other two calculi $\Gamma_{\pm q^{-2}}$.

For $c = (q^{1/2} - q^{-1/2})^{-2}$, there exist exactly two covariant FODCs over $\mathcal{O}_q(\mathbb{S}^2_c)$ which are generated by $\{\mathrm{d}e_i \mid i = -1, 0, 1\}$ as right $\mathcal{O}_q(\mathbb{S}^2_c)$-modules. One of them, $\Gamma_{-q^{-1}}$, is two-dimensional, the elements $\{\mathrm{d}e_i \mid i = -1, 0, 1\}$ form a right $\mathcal{O}_q(\mathbb{S}^2_c)$-module basis of the other calculus $\Gamma_{q^{-2}}$.

For generic value of c, the above corollary reproduces the results obtained in [1] by means of computer calculations.

The odd-dimensional covariant FODC $\Gamma_{q^{-l}}$, $l \in 2\mathbb{N}$, for arbitrary c and $\Gamma_{-q^{-l}}$, $l \in 2\mathbb{N}$, for $c = \infty$, can be explicitly constructed by a method by U. Hermisson. To match the above conventions, the relevant lemma from [2,3] is cited in terms of right comodule algebras.

Let $\mathcal{A}$ denote a coquasitriangular Hopf algebra with universal r-form $\mathbf{r}$ and $\mathcal{B}$ a right $\mathcal{A}$-comodule algebra. Let ν be a comodule algebra endomorphism of $\mathcal{B}$. Let further $W \subset \mathcal{B}$ denote a finite-dimensional right $\mathcal{A}$-subcomodule and let $W' = \mathrm{Hom}(W, \mathbb{C})$ be the dual right $\mathcal{A}$-comodule defined by $(\Delta f)(w) = (f \otimes S^{-1})\Delta w$ for $w \in W$, $f \in W'$. More explicitly, if $\{b_1, \dots, b_N\}$ is a basis of W and $\{\gamma^1, \dots, \gamma^N\} \subset W'$ is the dual basis then $\Delta b_i = \sum_j b_j \otimes \psi_i^j$ implies $\Delta\gamma^i = \sum_j \gamma^j \otimes S^{-1}(\psi_j^i)$.

Lemma 15. *The free right $\mathcal{B}$-module $W' \otimes \mathcal{B}$ can be endowed with a right $\mathcal{A}$-covariant $\mathcal{B}$-bimodule structure by*

$$a(f \otimes b)c := f_{(0)} \otimes \nu(a_{(0)})bc\,\mathbf{r}(a_{(1)}, f_{(1)}), \quad a, b, c \in \mathcal{B},\ f \in W',$$

and will be denoted by $\Gamma_{\mathbf{r},\nu,W}$. Moreover, if $\omega = \sum_{i=1}^N \gamma^i \otimes b_i \in \Gamma_{\mathbf{r},\nu,W}$ denotes the canonical invariant element then $\mathrm{d}:\mathcal{B} \to \Gamma_{\mathbf{r},\nu,W}$, $\mathrm{d}b := \omega b - b\omega$, defines a covariant FODC $(\mathrm{d}\mathcal{B}\cdot\mathcal{B}, \mathrm{d})$ over $\mathcal{B}$.

Lemma 16. *The quantum tangent space of the differential calculus Γ described in Lemma 15 is the linear span of the functionals $\chi_i \in \mathcal{B}^\circ$, $i = 1, \dots, N$, defined by*

$$\chi_i(a) = \mathbf{r}\big(\nu(a), S^{-1}(b_i)\big) - \varepsilon(b_i)\varepsilon(a).$$

Proof. For $a \in \mathcal{B}$ one obtains

$$\begin{aligned}
-P_r(\mathrm{d}a \otimes 1) &= P_r\left(\sum_k \big(a\gamma^k b_k - \gamma^k b_k a\big) \otimes 1\right) \\
&= P_r\left(\sum_{i,k} \gamma^i \otimes \big[\nu(a_{(0)})b_k\mathbf{r}\big(a_{(1)}, S^{-1}\big(\psi_i^k\big)\big) - b_i a\big]\right) \\
&= \sum_{i,j,k} \gamma^j \otimes S^{-1}\big(\psi_j^i\big)\big[\varepsilon\big(\nu(a_{(0)})b_k\big)\mathbf{r}\big(a_{(1)}, S^{-1}\big(\psi_i^k\big)\big) - \varepsilon(b_i)\varepsilon(a)\big] \\
&= \sum_{i,j} \gamma^j \otimes S^{-1}\big(\psi_j^i\big)\big[\mathbf{r}\big(\nu(a), S^{-1}(b_i)\big) - \varepsilon(b_i)\varepsilon(a)\big].
\end{aligned}$$

The last equality follows from $\Delta b_i = \sum_k b_k \otimes \psi_i^k$ and $\Delta\nu(a) = \nu(a_{(0)}) \otimes a_{(1)}$. Therefore, by Lemma 12,

$$\begin{aligned}
a \in \mathcal{L} \quad &\Longleftrightarrow \quad a \in \mathcal{B}^+ \text{ and } P_r(\mathrm{d}a \otimes 1) = 0 \\
&\Longleftrightarrow \quad a \in \mathcal{B}^+ \text{ and } \chi_i(a) = 0\ \forall i = 1, \dots, N. \quad \square
\end{aligned}$$

Let $V(n)$, $n \geqslant 1$, denote the $(2n+1)$-dimensional $U_q(\mathfrak{sl}_2)$-submodule of $\mathcal{O}_q(\mathbb{S}_c^2)$ with highest-weight vector $b_1 = e_1^n$. For $\nu = \mathrm{id}$, the quantum tangent space T of the differential calculus Γ from Lemma 15 satisfies

$$\mathbf{r}\big(\nu(\cdot), S^{-1}\big(e_1^n\big)\big) = \chi_1(\cdot) + \varepsilon\big(e_1^n\big)\varepsilon(\cdot) \in T^\varepsilon = T \oplus \mathbb{C}\varepsilon.$$

The standard universal r-form of $\mathcal{O}_q(\mathrm{SL}(2))$ is defined by

$$\mathbf{r}\big(u^i_j, u^k_l\big) = q^{-1/2}\begin{cases} q & \text{if } i=j=k=l,\\ 1 & \text{if } i=j\neq k=l,\\ q-q^{-1} & \text{if } j=k<i=l,\\ 0 & \text{else.}\end{cases}$$

In particular, $\mathbf{r}(a, u^2_1) = 0$ for all $a \in \mathcal{O}_q(\mathrm{SL}(2))$ and therefore $\bar{\chi} : \mathcal{B} \to \mathbb{C}$,

$$\bar{\chi}(a) := \varepsilon(e_1)^{-n}\mathbf{r}\big(a, S^{-1}\big(e^n_1\big)\big) = \varepsilon(e_1)^{-n}\mathbf{r}\big(S(a), e^n_1\big) = \mathbf{r}\big(S(a), \big(u^1_1\big)^{2n}\big) = \mathbf{r}\big(a, \big(u^2_2\big)^{2n}\big),$$

is a character which satisfies

$$\bar{\chi}(e_i) = q^{-2ni}\varepsilon(e_i).$$

Since $\bar{\chi} = \psi^0_{q^{-2n}} \in V_{q^{-2n}}$ and $\dim V_{q^{-2n}} \oplus \mathbb{C}\varepsilon = 2n+2 \geqslant \dim T^\varepsilon$, one obtains $T^\varepsilon = V_{q^{-2n}} \oplus \mathbb{C}\varepsilon$. Hence the differential calculus Γ coincides with $\Gamma_{q^{-2n}}$. Similarly, the differential calculus $\Gamma_{-q^{-l}}$, $l \in 2\mathbb{N}$, over $\mathcal{O}_q(\mathbb{S}^2_\infty)$ can be realized using the comodule algebra endomorphism $\nu : e_i \mapsto -e_i$.

Note that

$$(\Gamma_{q^{-2n}} \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}} = (\Gamma_{\mathbf{r},\mathrm{Id},V(n)} \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}}, \tag{34}$$

where, as above, $V(n)$ denotes the $(2n+1)$-dimensional representation of $U_q(\mathfrak{sl}_2)$. Indeed, by the above remarks, $\Gamma = \Gamma_{q^{-2n}}$ can be considered as a right $\mathcal{B}$-submodule of $\Gamma_{\mathbf{r},\mathrm{Id},V(n)}$ and, as $\mathcal{A}$ is a flat $\mathcal{B}$-module, this implies $\Gamma_{q^{-2n}} \otimes_{\mathcal{B}} \mathcal{A} \subset \Gamma_{\mathbf{r},\mathrm{Id},V(n)} \otimes_{\mathcal{B}} \mathcal{A}$. As $\dim(\Gamma_{\mathbf{r},\mathrm{Id},V(n)} \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}} = 2n+1$ by construction and $\dim(\Gamma_{q^{-2n}} \otimes_{\mathcal{B}} \mathcal{A})_{\mathrm{inv}} = 2n+1$ by Lemma 12, the identification (34) follows. Now (34) implies $\Gamma_{q^{-2n}} \otimes_{\mathcal{B}} \mathcal{A} = \Gamma_{\mathbf{r},\mathrm{Id},V(n)} \otimes_{\mathcal{B}} \mathcal{A}$ and, by faithful flatness of $\mathcal{A}$, this in turn gives $\Gamma_{q^{-2n}} = \Gamma_{\mathbf{r},\mathrm{Id},V(n)}$. Thus one has the following proposition.

Proposition 17. *For any $0 \neq c \in J_2$, the FODC $\Gamma_{q^{-2n}}$, $n \in \mathbb{N}$, is isomorphic to $\Gamma_{\mathbf{r},\mathrm{Id},V(n)}$. For $c = \infty$, the FODC $\Gamma_{-q^{-2n}}$ is isomorphic to $\Gamma_{\mathbf{r},\nu,V(n)}$ where $\nu(e_i) = -e_i$. In particular, $\Gamma_{\pm q^{-2n}}$ are free left and right $\mathcal{B}$-modules and inner first-order differential calculi.*

Remark 18. Covariant FODCs over $\mathcal{O}_q(\mathbb{S}^2_0)$ are qualitatively different from those over $\mathcal{O}_q(\mathbb{S}^2_c)$, $0 \neq c \notin J_2$. Let $\widetilde{\Gamma}_{kl}$ denote the $(kl+k+l)$-dimensional FODC over $\mathcal{O}_q(\mathbb{S}^2_0)$ with quantum tangent space $\widetilde{T}_{kl} = \mathrm{Lin}_{\mathbb{C}}\{E^iF^j \mid 0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant l,\ (i,j) \neq (0,0)\}$. By Proposition 10 and [4, Lemma 5.3], any covariant FODC over $\mathcal{O}_q(\mathbb{S}^2_0)$ can be written as a (not necessarily direct) sum of calculi $\widetilde{\Gamma}_{kl}$ for certain k, l. In particular, the only irreducible calculi are $\widetilde{\Gamma}_{10}$ and $\widetilde{\Gamma}_{01}$, constructed in [7].

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