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Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras

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A B S T R A C T

We construct all solvable Lie algebras with a specific n-dimensional nilradical $\mathfrak{n}_{n,3}$ which contains the previously studied filiform $(n-2)$-dimensional nilpotent algebra $\mathfrak{n}_{n-2,1}$ as a subalgebra but not as an ideal. Rather surprisingly it turns out that the classification of such solvable algebras can be deduced from the classification of solvable algebras with the nilradical $\mathfrak{n}_{n-2,1}$. Also the sets of invariants of coadjoint representation of $\mathfrak{n}_{n,3}$ and its solvable extensions are deduced from this reduction. In several cases they have polynomial bases, i.e. the invariants of the respective solvable algebra can be chosen to be Casimir invariants in its enveloping algebra.

1. Introduction

The current article belongs to a series of papers initiated by Rubin and Winternitz in [1] and continued throughout the years with his various collaborators in [2–7]. All these papers dealt with the problem of classification of all solvable Lie algebras with the given n-dimensional nilradical, e.g. Abelian, Heisenberg algebra, the algebra of strictly upper triangular matrices etc., for arbitrary finite dimension n. Other similar series have been recently investigated by different groups in [8] (naturally graded nilradicals with maximal nilindex and a Heisenberg subalgebra of codimension one) and [9] (a certain series of quasi-filiform nilradicals).

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As is well known, the problem of classification of all solvable (including nilpotent) Lie algebras in an arbitrarily large finite dimension is presently unsolved and is generally believed to be unsolvable. All known full classifications terminate at relatively low dimensions, e.g. the classification of nilpotent algebras is available at most in dimension 8 [10,11], for the solvable ones in dimension 6 [12,13]. The unifying idea behind the series [1–7] is a belief that the knowledge of full classification of all solvable extensions of certain series of nilradicals can be very useful for both theoretical considerations – e.g. testing various hypotheses about general structure of solvable Lie algebras – and practical purposes – e.g. when a generalization of a given algebra or its nilradical to higher dimensions appears in some physical theory.

In this paper we shall consider the nilradical

$$\mathfrak{n}_{n,3} = \operatorname{span}\{x_1, \ldots, x_n\}, \quad n \geqslant 5,$$

with the following nonvanishing Lie brackets

$$\begin{aligned} &[x_2, x_n] = x_1, \\ &[x_3, x_{n-1}] = x_1, \\ &[x_k, x_{n-1}] = x_{k-1}, \quad 4 \leqslant k \leqslant n-2, \\ &[x_{n-1}, x_n] = x_2. \end{aligned} \tag{1}$$

When $n = 5$, the only remaining nonvanishing Lie brackets are

$$[x_2, x_5] = [x_3, x_4] = x_1, \quad [x_4, x_5] = x_2. \tag{2}$$

The n-dimensional nilpotent Lie algebra $\mathfrak{n}_{n,3}$ is nilpotent of degree of nilpotency[1] equal to $n-3$ and with $(n-2)$-dimensional maximal Abelian ideal. It has one-dimensional center $C(\mathfrak{n}_{n,3}) = \operatorname{span}\{x_1\}$.

Later it will become important for our investigation that it contains as a subalgebra the nilpotent algebra $\mathfrak{n}_{n-2,1}$

$$[y_k, y_{n-2}] = y_{k-1}, \quad 2 \leqslant k \leqslant n-3, \tag{3}$$

whose solvable extensions were investigated in [6]. Namely, we have $\tilde{\mathfrak{n}}_{n-2,1}$ spanned by $x_1, x_3, \ldots, x_{n-1}$. Similarly, $\mathfrak{n}_{n,3}$ also contains $\tilde{\mathfrak{n}}_{6,3}$ spanned by $x_1, x_2, x_3, x_4, x_{n-1}, x_n$. Here, tildes were used to denote these particular embeddings of algebras of the type (3) and (1), respectively, into the n-dimensional nilradical $\mathfrak{n}_{n,3}$. We stress that neither $\tilde{\mathfrak{n}}_{n-2,1}$ nor $\tilde{\mathfrak{n}}_{6,3}$ are ideals.

In general, the knowledge of solvable extensions of a subalgebra of the given nilradical does not provide much help in the classification of all solvable extensions of the nilradical. That is because the outer derivations of the nilradical need not to leave the subalgebra invariant – indeed, it is not invariant even with respect to inner derivations. However, in the particular case of the nilradical $\mathfrak{n}_{n,3}$ considered here all the classification can be reduced to the cases of $\mathfrak{n}_{n-2,1}$ already investigated in [6] and $\mathfrak{n}_{6,3}$.

In the following we shall firstly find out the general form of an automorphism and a derivation of $\mathfrak{n}_{n,3}$. Next, we use this knowledge in the construction of all solvable extensions of the nilradical $\mathfrak{n}_{n,3}$. Finally, we deduce generalized Casimir invariants of both $\mathfrak{n}_{n,3}$ and its solvable extensions.

Throughout the paper we shall use the same notation as in [7]. We have attempted to make the present paper self-contained but if any doubts arise about chosen conventions, etc. the reader may consult [7] as a suitable reference. Also, if the reader desires to get a more general background information about the classification of solvable Lie algebras, the construction of Casimir invariants and so on, we refer him to the review parts of [7] and the literature cited there.

2. Automorphisms and derivations of the nilradical $\mathfrak{n}_{n,3}$

In the computations below we shall assume that $n \geqslant 7$. The results for $n = 5, 6$ are derived in Sections 3.1 and 3.2.

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[1] Also called the nilindex. It is the largest value of k for which the kth power $\mathfrak{g}^k = [\mathfrak{g}, [\mathfrak{g}, \ldots, [\mathfrak{g}, \mathfrak{g}] \ldots]]$ of $\mathfrak{g}$ is nonvanishing. Equivalently, it can be defined as the number of nonvanishing ideals in the lower central series (5) including $\mathfrak{g}^1 = \mathfrak{g}$.

The nilpotent algebra $\mathfrak{n} = \mathfrak{n}_{n,3}$ has the following complete flag of ideals

$$0 \subset \mathfrak{n}^{n-3} \subset \mathfrak{n}^{n-4} \subset \mathfrak{z}_2 \subset \mathfrak{z}_3 \cap \mathfrak{n}^2 \subset \cdots \subset \mathfrak{z}_{n-5} \cap \mathfrak{n}^2 \subset \mathfrak{n}^2 \subset (\mathfrak{z}_2)_{\mathfrak{n}} \subset (\mathfrak{n}^{n-4})_{\mathfrak{n}} \subset \mathfrak{n}, \qquad (4)$$

where

- $\mathfrak{n}^k$ are elements of the lower central series, defined recursively by:

$$\mathfrak{n}^1 = \mathfrak{n}, \quad \mathfrak{n}^k = [\mathfrak{n}^{k-1}, \mathfrak{n}], \quad k \geqslant 2, \qquad (5)$$

- $\mathfrak{z}_k$ are elements of the upper central series - that means that $\mathfrak{z}_k$ is the unique ideal in $\mathfrak{n}$ such that $\mathfrak{z}_k/\mathfrak{z}_{k-1}$ is the center of $\mathfrak{n}/\mathfrak{z}_{k-1}$; the recursion starts from the center of $\mathfrak{n}$, i.e. $\mathfrak{z}_1 = C(\mathfrak{n})$,
- and $(\mathfrak{n}^{n-4})_{\mathfrak{n}}$ is the centralizer of $\mathfrak{n}^{n-4}$ in $\mathfrak{n}$, i.e.

$$(\mathfrak{n}^{n-4})_{\mathfrak{n}} = \{x \in \mathfrak{n} | [x, y] = 0, \ \forall y \in \mathfrak{n}^{n-4}\}.$$

By construction, the flag (4) is invariant with respect to any automorphism of the Lie algebra $\mathfrak{n}$, i.e. in the basis respecting the flag any automorphism will be represented by an upper triangular matrix. Because derivations of $\mathfrak{n}$ can be viewed as infinitesimal automorphisms (i.e. elements of the Lie algebra of the matrix Lie group of automorphisms of $\mathfrak{n}$), the same triangular form holds also for them.

Therefore, we find it convenient to change the basis (x_k) of $\mathfrak{n}$ defined in Eq. (1) to a seemingly less natural (i.e. Lie brackets appear more cumbersome) basis (e_k) whose essential advantage over the original one is that it respects the flag (4), i.e. the kth subspace in the flag is $\text{span}\{e_1, \ldots, e_k\}$ for all k. Namely, we take

$$e_1 = x_1, \ e_2 = x_3, \ e_3 = x_2, \ e_4 = x_4, \ \ldots, e_{n-2} = x_{n-2}, \ e_{n-1} = x_n, \ e_n = x_{n-1}. \qquad (6)$$

The nonvanishing Lie brackets now become

$$\begin{aligned} &[e_2, e_n] = e_1, \\ &[e_3, e_{n-1}] = e_1, \\ &[e_4, e_n] = e_2, \\ &[e_k, e_n] = e_{k-1}, \quad 5 \leqslant k \leqslant n-2, \\ &[e_{n-1}, e_n] = -e_3. \end{aligned} \qquad (7)$$

The important subalgebras isomorphic to $\mathfrak{n}_{n-2,1}, \mathfrak{n}_{6,3}$ are now expressed as

$$\tilde{\mathfrak{n}}_{n-2,1} = \text{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}, \quad \tilde{\mathfrak{n}}_{6,3} = \text{span}\{e_1, e_2, e_3, e_4, e_{n-1}, e_n\},$$

respectively. The ideals in the derived,[2] lower central and upper central series are

$$\begin{aligned} &\mathfrak{n}^2 = \mathfrak{n}^{(1)} = \text{span}\{e_1, \ldots, e_{n-3}\}, \quad \mathfrak{n}^{(2)} = 0, \\ &\mathfrak{n}^k = \text{span}\{e_1, e_2, e_4, \ldots, e_{n-k-1}\}, \quad 3 \leqslant k \leqslant n-5, \\ &\mathfrak{n}^{n-4} = \text{span}\{e_1, e_2\}, \quad \mathfrak{n}^{n-3} = \text{span}\{e_1\}, \quad \mathfrak{n}^{n-2} = 0, \\ &\mathfrak{z}_1 = \mathfrak{n}^{n-3}, \quad \mathfrak{z}_2 = \text{span}\{e_1, e_2, e_3\}, \\ &\mathfrak{z}_k = \text{span}\{e_1, \ldots, e_{k+1}, e_{n-1}\}, \quad 3 \leqslant k \leqslant n-4, \quad \mathfrak{z}_{n-3} = \mathfrak{n}. \end{aligned}$$

In order to find the structure of an arbitrary automorphism of $\mathfrak{n}_{n,3}$ we consider its matrix in the basis (6)

$$\Phi(e_k) = e_j \Phi_{jk} \qquad (8)$$

[2] The elements $\mathfrak{n}^{(k)}$ of the derived series are defined recursively by:

$$\mathfrak{n}^{(0)} = \mathfrak{n}, \quad \mathfrak{n}^{(k)} = [\mathfrak{n}^{(k-1)}, \mathfrak{n}^{(k-1)}], \ k \geqslant 1.$$

(summation over repeated indices applies throughout the paper unless otherwise stated). As mentioned above, such a matrix must be necessarily upper triangular because the flag (4) is preserved. It is also obvious that the knowledge of its last three columns, i.e. of $\Phi(e_{n-2}), \Phi(e_{n-1})$ and $\Phi(e_n)$, is sufficient for the knowledge of the whole matrix Φ due to the definition of an automorphism

$$\Phi([x,y]) = [\Phi(x), \Phi(y)], \quad \forall x, y \in \mathfrak{n}$$

and the Lie brackets (7) – we can recover all $\Phi(e_k),\ 1 \leqslant k \leqslant n-3$ through multiple brackets of $\Phi(e_{n-2})$, $\Phi(e_{n-1})$ and $\Phi(e_n)$. A natural question is the following: Under which conditions do the relations

$$\Phi(e_{n-2}) = \alpha e_{n-2} + \sum_{k=1}^{n-3} \phi_k e_k,$$

$$\Phi(e_{n-1}) = \beta e_{n-1} + \gamma e_{n-2} + \sum_{k=1}^{n-3} \psi_k e_k,$$

$$\Phi(e_n) = \kappa e_n + \lambda e_{n-1} + \mu e_{n-2} + \sum_{k=1}^{n-3} \rho_k e_k$$

give rise to an automorphism of $\mathfrak{n}_{n,3}$?

Obviously, we must have $\alpha\beta\kappa \neq 0$ to have an invertible map. The preservation of $\mathfrak{z}_3$ implies $\gamma = 0,\ \psi_k = 0, k = 5, \ldots, n-3$. The remaining conditions are found as follows

- $0 = \Phi([e_{n-2}, e_{n-1}])$ implies $\phi_3 = 0$,
- $0 = \Phi([[e_{n-1}, e_n], e_n])$ leads to $\psi_4 = \frac{\lambda}{\kappa}\beta$,
- $0 = \Phi([[e_{n-1}, e_n], e_{n-1}]) + \Phi((-ad_{e_n})^{n-4} e_{n-2})$ leads to $\alpha = \beta^2\kappa^{5-n}$.

All other Lie brackets are either used to define $\Phi(e_k),\ 1 \leqslant k \leqslant n-3$ or are preserved trivially. Therefore, we conclude that any automorphism Φ of $\mathfrak{n}_{n,3}$ is defined in terms of $2n$ parameters which have been denoted by $\beta, \kappa, \lambda, \psi_1, \psi_2, \psi_3, \phi_1, \phi_2, \phi_4, \ldots, \phi_{n-3}, \rho_1, \ldots, \rho_{n-3}$. It acts on the generators of the Lie algebra $\mathfrak{n}_{n,3}$ in the following way:

$$\begin{aligned}
\Phi(e_{n-2}) &= \beta^2\kappa^{5-n} e_{n-2} + \sum_{k=4}^{n-3} \phi_k e_k + \phi_2 e_2 + \phi_1 e_1, \\
\Phi(e_{n-1}) &= \beta e_{n-1} + \frac{\lambda}{\kappa}\beta e_4 + \sum_{k=1}^{3} \psi_k e_k, \\
\Phi(e_n) &= \kappa e_n + \lambda e_{n-1} + \mu e_{n-2} + \sum_{k=1}^{n-3} \rho_k e_k.
\end{aligned} \tag{9}$$

Taking automorphisms infinitesimally close to the unity, i.e. constructing the Lie algebra of the group of automorphisms, we find the algebra of derivations $\mathfrak{Der}(\mathfrak{n}_{n,3})$. It consists of all linear maps D which act on the generators e_{n-2}, e_{n-1}, e_n as follows:

$$\begin{aligned}
D(e_{n-2}) &= (2c_{n-1} + (5-n)d_n)e_{n-2} + \sum_{k=4}^{n-3} b_k e_k + b_2 e_2 + b_1 e_1, \\
D(e_{n-1}) &= c_{n-1} e_{n-1} + d_{n-1} e_4 + \sum_{k=1}^{3} c_k e_k, \\
D(e_n) &= \sum_{k=1}^{n} d_k e_k;
\end{aligned} \tag{10}$$

the action of D on the remaining basis elements $e_1, \ldots, e_{n-3}$ is uniquely determined using multiple brackets and the Leibniz's law

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

The $2n$-dimensional algebra of derivations $\mathfrak{Der}(\mathfrak{n}_{n,3})$ contains a $(n-1)$-dimensional ideal of inner derivations $\mathfrak{Inn}(\mathfrak{n}_{n,3})$ having the form

$$\begin{aligned} D(e_{n-2}) &= -c_3 e_{n-3}, \\ D(e_{n-1}) &= c_3 e_3 + c_1 e_1, \\ D(e_n) &= \sum_{k=1}^{n-3} d_k e_k. \end{aligned} \tag{11}$$

Indeed, such a derivation D can be expressed as

$$D = \mathrm{ad}\left(d_1 e_2 + c_1 e_3 + d_2 e_4 + \sum_{k=4}^{n-3} d_k e_{k+1} - d_3 e_{n-1} + c_3 e_n \right). \tag{12}$$

Because e_1 spans the kernel of ad, i.e. the center of $\mathfrak{n}_{n,3}$, derivations of the form (11) exhaust all inner derivations.

3. Construction of solvable Lie algebras with the nilradical $\mathfrak{n}_{n,3}$

Firstly, we recall how the knowledge of automorphisms and derivations of a given nilpotent Lie algebra $\mathfrak{n}$ can be employed in the construction of all solvable Lie algebras $\mathfrak{s}$ with the nilradical $\mathfrak{n}$.

Let us consider a basis of $\mathfrak{s}$ in the form $(e_1, \ldots, e_n, f_1, \ldots, f_p)$ where $(e_1, \ldots, e_n)$ is a basis of $\mathfrak{n}$ with prescribed Lie brackets. Since $\mathfrak{n}$ is an ideal in $\mathfrak{s}$ and the derived algebra of $\mathfrak{s}$ falls into $\mathfrak{n}$ we necessarily have Lie brackets of the form

$$[f_a, e_j] = (A_a)_j^k e_k, \quad [f_a, f_b] = \gamma_{ab}^j e_j. \tag{13}$$

Furthermore, $\mathfrak{n}$ must be the maximal nilpotent ideal of $\mathfrak{s}$, i.e. any nonvanishing linear combination of the matrices A_a must be non-nilpotent.

The algebra $\mathfrak{s}$ does not change if we transform its basis. Since the structure of $\mathfrak{n}$ is fixed we allow only such transformations that the Lie brackets in $\mathfrak{n}$ are not altered, i.e.

$$e_k \to \tilde{e}_k = e_j \Phi_{jk}, \quad f_a \to \tilde{f}_a = f_b \Xi_{ba} + e_k \Psi_{ka}, \tag{14}$$

where Φ is a matrix of an automorphism of $\mathfrak{n}$ in the original basis $(e_1, \ldots, e_n)$, Ξ is a regular matrix and Ψ is arbitrary.

We represent all non-nilpotent elements f_a in the basis of $\mathfrak{s}$ by the corresponding operators in $\mathfrak{Der}(\mathfrak{n}) \subset \mathfrak{gl}(\mathfrak{n})$,

$$f_a \in \mathfrak{s} \to D_a = \mathrm{ad}_{f_a}|_{\mathfrak{n}} \in \mathfrak{Der}(\mathfrak{n}). \tag{15}$$

We note that under this mapping of f_a's to outer derivations we lose some information – from the knowledge of D_a, D_b we can reconstruct the Lie bracket $[f_a, f_b]$ only modulo the kernel of this map, i.e. modulo elements in the center of $\mathfrak{n}$. Nevertheless, the construction of all non-equivalent sets of $(D_1, \ldots, D_p)$ is crucial in the construction of all solvable Lie algebras $\mathfrak{s}$ with the nilradical $\mathfrak{n}$.

Because Eq. (15) defines a homomorphism of $\mathfrak{s}$ into $\mathfrak{Der}(\mathfrak{n})$ we can translate properties of f_a's to D_a's. In particular, a commutator of any D_a, D_b must be an inner derivation and no nontrivial linear combination of D_a's can be nilpotent. That means that $(D_1, \ldots, D_p)$ must span an Abelian subalgebra $\mathfrak{a}$ in the factor algebra $\mathfrak{Der}(\mathfrak{n})/\mathfrak{Inn}(\mathfrak{n})$ such that no nonvanishing element of $\mathfrak{a}$ is nilpotent. The subalgebras conjugated under any automorphism of $\mathfrak{n}$ are equivalent. Therefore, in an abstract formulation we can say that the Lie brackets of solvable extensions of $\mathfrak{n}$ are determined modulo elements in the center of $\mathfrak{n}$ by conjugacy classes of Abelian subalgebras $\mathfrak{a}$ of the factor algebra $\mathfrak{Der}(\mathfrak{n})/\mathfrak{Inn}(\mathfrak{n})$ such that no element of $\mathfrak{a}$ is represented by a nilpotent operator on $\mathfrak{n}$. Now the practical issue is how one can conveniently construct these classes for particular $\mathfrak{n} = \mathfrak{n}_{n,3}$?

Let us start by considering one additional basis element $f_1 \equiv f$, i.e. one derivation D. The elements of $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ can be uniquely represented by outer derivations of the form

$$\begin{aligned} D(e_{n-2}) &= (2c_{n-1} + (5-n)d_n)e_{n-2} + \sum_{k=4}^{n-4} b_k e_k + b_2 e_2 + b_1 e_1, \\ D(e_{n-1}) &= c_{n-1}e_{n-1} + d_{n-1}e_4 + c_3 e_3 + c_2 e_2, \\ D(e_n) &= d_n e_n + d_{n-1}e_{n-1} + d_{n-2}e_{n-2} \end{aligned} \tag{16}$$

(the action on $e_1, \ldots, e_{n-3}$ follows from the Leibniz's law). Above, a suitable inner derivation (11) was added to an arbitrary derivation, eliminating $n-1$ parameters. We mention that the form (16) of the representative of the coset $[D]$ is not invariant under conjugation by an automorphism

$$D \to D_\Phi = \Phi^{-1} \circ D \circ \Phi,$$

so that we may be forced to use a representative $\Phi(D)'$ of the coset $[\Phi(D)]$ different from $\Phi(D)$. Such a change of representative amounts to an addition of an inner derivation and is understood in all simplifications below whenever we employ an automorphism. Due to the triangular shape of D we see that the sought-after Abelian subalgebras are at most two-dimensional since any higher dimensional subalgebra in $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ will necessarily involve nonvanishing nilpotent elements.

Next, we find all possible canonical forms of the coset (16) up to conjugation by automorphisms and rescaling. In order to reduce the problem to the one already investigated in [6] we realize that the derivation of the form (16) leaves

$$\tilde{\mathfrak{n}}_{n-2,1} = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}$$

invariant if and only if $d_{n-1} = 0$. We conjugate a given derivation D by the automorphism defined by

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1} + \frac{d_{n-1}}{d_n - c_{n-1}} e_4, \quad \Phi(e_n) = e_n + \frac{d_{n-1}}{d_n - c_{n-1}} e_{n-1},$$

whenever possible, i.e. when $d_n \neq c_{n-1}$. Now we have $\hat{d}_{n-1} = 0$, i.e. $D_\Phi \equiv \widehat{D}$ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant. The case when none of the conjugate derivations D_Φ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant, which necessarily means that $d_n = c_{n-1},\ d_{n-1} \neq 0$, will be dealt with later on.

Provided we set $d_{n-1} = 0$, the outer derivation (16) restricted to $\tilde{\mathfrak{n}}_{n-2,1}$ has the same structure as in [6, Eq. (25)]. Consequently, we may consider all solvable extensions of $\tilde{\mathfrak{n}}_{n-2,1}$ and then extend these to solvable extensions of $\mathfrak{n}_{n,3}$, i.e. determine the parameters c_{n-1}, c_3, c_2. In this way we obtain all solvable extensions of $\mathfrak{n}_{n,3}$ except the case $d_n = c_{n-1},\ d_{n-1} \neq 0$.

The value of the parameter c_{n-1} is fixed by the structure of the solvable extension of $\tilde{\mathfrak{n}}_{n-2,1}$. Namely, in relation to parameters α, β introduced below in Theorem 1 we have

$$c_{n-1} = \frac{1}{2}\left(\beta + (n-5)\alpha\right), \quad d_n = \alpha.$$

When $c_{n-1} \neq 0$ any derivation D can be brought to D_ϕ with $c_2 = 0$ using an automorphism Φ specified by

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1} - \frac{c_2}{c_{n-1}} e_2, \quad \Phi(e_n) = e_n.$$

When $c_{n-1} = 0$ we cannot eliminate nonvanishing c_2 by any automorphism but we can bring it to 1 by rescaling of e_k's provided such scaling remains available by the structure of the solvable extension of the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$. It turns out that for $c_{n-1} = 0$ two non-conjugate extensions of a derivation of $\tilde{\mathfrak{n}}_{n-2,1}$ exist, namely those determined by $c_2 = 0, 1$.

A similar consideration can be applied also to the parameter c_3. When $d_n \neq 0$ any derivation D can be brought to D_ϕ with $c_3 = 0$ using the automorphism Φ specified by

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1} - \frac{c_3}{d_n} e_3, \quad \Phi(e_n) = e_n.$$

When $d_n = 0$ we cannot eliminate nonvanishing c_3 by any automorphism. Whether or not c_3 can be rescaled depends on the residual automorphisms still available – if the diagonal part of automorphisms is completely fixed by the structure of the solvable extension of the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$ nothing can be done, otherwise we can scale c_3 to 1 using an automorphism

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1}, \quad \Phi(e_n) = \frac{1}{c_3} e_n.$$

To sum up, the extension to a derivation of the nilradical $\mathfrak{n}_{n,3}$ is unique up to a conjugation when $d_n \neq 0$ and $c_{n-1} \neq 0$; otherwise, several non-equivalent extensions do exist.

We recall the main classification theorem of [6]:

Theorem 1. *Let $\mathbb{F}$ be the field of real or complex numbers. Any solvable Lie algebra $\tilde{\mathfrak{s}}$ over the field $\mathbb{F}$ with the nilradical $\mathfrak{n}_{m,1}$ has dimension $\dim \tilde{\mathfrak{s}} = m+1$, or $\dim \tilde{\mathfrak{s}} = m+2$. Three types of solvable Lie algebras of dimension $\dim \tilde{\mathfrak{s}} = m+1$ exist for any $m \geqslant 4$. They are represented by the following:*

1. *$[\tilde{f}, \tilde{e}_k] = ((m-k-1)\alpha + \beta)\tilde{e}_k,\ k \leqslant m-1, [\tilde{f}, \tilde{e}_m] = \alpha \tilde{e}_m$. The classes of mutually nonisomorphic algebras of this type are*

$$\begin{aligned}
&\tilde{\mathfrak{s}}_{m+1,1}(\beta): \quad \alpha = 1, \quad \beta \in \mathbb{F}\backslash\{0, m-2\},\\
&\quad DS = [m+1, m, m-2, 0], \quad CS = [m+1, m], \quad US = [0],\\
&\tilde{\mathfrak{s}}_{m+1,2}: \quad \alpha = 1, \quad \beta = 0,\\
&\quad DS = [m+1, m-1, m-3, 0], \quad CS = [m+1, m-1], \quad US = [0],\\
&\tilde{\mathfrak{s}}_{m+1,3}: \quad \alpha = 1, \quad \beta = 2-m,\\
&\quad DS = [m+1, m, m-2, 0], \quad CS = [m+1, m], \quad US = [1],\\
&\tilde{\mathfrak{s}}_{m+1,4}: \quad \alpha = 0, \quad \beta = 1,\\
&\quad DS = [m+1, m-1, 0], \quad CS = [m+1, m-1], \quad US = [0].
\end{aligned}$$

2. *$\tilde{\mathfrak{s}}_{m+1,5}: \quad [\tilde{f}, \tilde{e}_k] = (m-k)\tilde{e}_k,\ k \leqslant m-1, [\tilde{f}, \tilde{e}_m] = \tilde{e}_m + \tilde{e}_{m-1}$.*

   $DS = [m+1, m, m-2, 0], \quad CS = [m+1, m], \quad US = [0].$

3. *$\tilde{\mathfrak{s}}_{m+1,6}(a_3, \ldots, a_{m-1}): [\tilde{f}, \tilde{e}_k] = \tilde{e}_k + \sum_{l=1}^{k-2} a_{k-l+1}\tilde{e}_l,\ k \leqslant m-1, [f, \tilde{e}_m] = 0, a_j \in \mathbb{F}$, at least one a_j satisfies $a_j \neq 0$.*
   Over $\mathbb{C}$: the first nonzero a_j satisfies $a_j = 1$.
   Over $\mathbb{R}$: the first nonzero a_j for even j satisfies $a_j = 1$. If all $a_j = 0$ for j even, then the first nonzero a_j (j odd) satisfies $a_j = \pm 1$. We have

   $DS = [m+1, m-1, 0], \quad CS = [m+1, m-1], \quad US = [0].$

For each $m \geqslant 4$ precisely one solvable Lie algebra $\tilde{\mathfrak{s}}_{m+2}$ of $\dim \tilde{\mathfrak{s}} = m+2$ with the nilradical $\mathfrak{n}_{m,1}$ exists. It is represented by a basis $(\tilde{e}_1, \ldots, \tilde{e}_m, \tilde{f}_1, \tilde{f}_2)$ and the Lie brackets involving f_1 and f_2 are

$$\begin{aligned}
&[\tilde{f}_1, \tilde{e}_k] = (m-1-k)\tilde{e}_k, \quad 1 \leqslant k \leqslant m-1, \quad [\tilde{f}_1, \tilde{e}_m] = \tilde{e}_m,\\
&[\tilde{f}_2, \tilde{e}_k] = \tilde{e}_k, \quad 1 \leqslant k \leqslant m-1, \quad [\tilde{f}_2, \tilde{e}_m] = 0, \quad [\tilde{f}_1, \tilde{f}_2] = 0.
\end{aligned}$$

For this algebra we have

$$DS = [m+2, m, m-2, 0], \quad CS = [m+2, m], \quad US = [0].$$

Above, we used the abbreviations *DS*, *CS* and *US* for (ordered) lists of integers denoting the dimensions of subalgebras in the derived, lower central and upper central series, respectively. We listed the last (then repeated) entry only once (e.g. we write $CS = [m, m-1]$ rather than $CS = [m, m-1, m-1, m-1, \ldots]$).

We must point out, however, that there is a caveat in the presented theorem. If we work over the field $\mathbb{R}$ the group of automorphisms of $\mathfrak{n}_{n-2,1}$ used in the derivation of Theorem 1 in [6] is slightly larger than the one we have available for the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$, i.e. inherited from automorphisms of $\mathfrak{n}_{n,3}$. In other words, the available automorphisms form a group only locally isomorphic to the group of automorphisms of $\mathfrak{n}_{n-2,1}$. Namely, the sign of $\alpha = \beta^2\kappa^{5-n}$ in Eq. (9) is restricted – for given n we have $\mathrm{sgn}\alpha = (\mathrm{sgn}\kappa)^{n-5}$. As a consequence, for our purposes we must for n even consider $[\tilde{f}, \tilde{e}_m] = \tilde{e}_m \pm \tilde{e}_{m-1}$ in $\tilde{\mathfrak{s}}_{m+1,5} (m = n-2)$. All other results in Theorem 1 hold irrespective of this constraint on allowed automorphisms.

The corresponding solvable extensions of the nilradical $\mathfrak{n}_{n,3}$ are summarized in Theorem 2 below.

Coming back to the case $d_n = c_{n-1}, d_{n-1} \neq 0$, we first rescale D to get $d_n = c_{n-1} = 1$ and by scaling of e_k's we set $d_{n-1} = 1$. Using the automorphism

$$\Phi(e_{n-2}) = e_{n-2}, \quad \Phi(e_{n-1}) = e_{n-1}, \quad \Phi(e_n) = e_n + \frac{d_{n-2}}{n-6} e_{n-2},$$

we get rid of d_{n-2}; it is possible since $n \neq 6$. We get D which preserves the subalgebra $\tilde{\mathfrak{n}}_{6,3}$. Therefore, it is enough to consider its solvable extensions (with $d_n = c_{n-1} = 1$) and then extend these to solvable algebras with the nilradical $\mathfrak{n}_{n,3}$. It turns out that such an enlargement is unique up to conjugation, i.e. fully determined by $d_n = c_{n-1} = 1, d_{n-1} = 1, d_{n-2} = 0$, the remaining parameters in Eq. (16) vanish.

Finally, the two-dimensional Abelian subalgebras $\mathfrak{a}$ in $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ are easily obtained using the results of the previous analysis. Such subalgebras must contain two linearly independent elements D'_1, D'_2, whose diagonal parameters can be chosen to have the values $c_{n-1} = 1, d_n = -1$ and $c_{n-1} = 1, d_n = 0$, respectively. Due to the chosen values for D_1 we can always go over to $\tilde{D}_1 = (D'_1)_\Phi, \tilde{D}_2 = (D'_2)_\Phi$ where $\tilde{D}_1$ was diagonalized by a suitable automorphism Φ. The restriction $[\tilde{D}_1, \tilde{D}_2] \in \mathfrak{Inn}(\mathfrak{n}_{n,3})$ now restricts $\tilde{D}_2$ to be also diagonal. Therefore, all elements of $\mathfrak{a}$ act diagonally on $\mathfrak{n}_{n,3}$ in the chosen basis and can be expressed e.g. in the basis defined by D_1 $(c_{n-1} = 0, d_n = 1)$ and D_2 $(c_{n-1} = 1, d_n = 0)$. The corresponding non-nilpotent elements f_1, f_2 in $\mathfrak{s}$ in general satisfy

$$[f_1, f_2] = \alpha e_1 \in C(\mathfrak{n})$$

but a simple redefinition $f_1 \to f_1 + \frac{\alpha}{2} e_1$ gives an isomorphic solvable algebra $\mathfrak{s}$ with $[f_1, f_2] = 0$.

To sum up, we have the following theorem.

Theorem 2. *Let $\mathbb{F}$ be the field of real or complex numbers and n be an integer number greater or equal to 7. Any solvable Lie algebra $\mathfrak{s}$ over the field $\mathbb{F}$ with the nilradical $\mathfrak{n}_{n,3}$ has dimension* $\dim \mathfrak{s} = n+1$ *or* $\dim \mathfrak{s} = n+2$.

Five types of solvable Lie algebras of dimension $\dim \mathfrak{s} = n+1$ *with the nilradical $\mathfrak{n}_{n,3}$ exist. They are represented by the following*:

1. $[f, e_1] = (\alpha + 2\beta)e_1, \ [f, e_2] = 2\beta e_2, \ [f, e_3] = (\alpha + \beta)e_3,$
   $[f, e_k] = ((3-k)\alpha + 2\beta)e_k, \ 4 \leqslant k \leqslant n-2,$
   $[f, e_{n-1}] = \beta e_{n-1}, \ [f, e_n] = \alpha e_n.$
   The classes of mutually nonisomorphic algebras of this type are

$$\mathfrak{s}_{n+1,1}(\beta): \quad \alpha = 1, \quad \beta \in \mathbb{F} \backslash \left\{0, -\frac{1}{2}, \frac{n-5}{2}\right\},$$
$$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [0],$$
$$\mathfrak{s}_{n+1,2}: \quad \alpha = 1, \quad \beta = \frac{n-5}{2},$$
$$DS = [n+1, n-1, n-4, 0], \quad CS = [n+1, n-1], \quad US = [0],$$
$$\mathfrak{s}_{n+1,3}: \quad \alpha = 1, \quad \beta = 0,$$
$$DS = [n+1, n-1, n-4, 0], \quad CS = [n+1, n-1], \quad US = [0],$$
$$\mathfrak{s}_{n+1,4}: \quad \alpha = 1, \quad \beta = -\frac{1}{2},$$

$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [1],$

$\mathfrak{s}_{n+1,5}: \quad \alpha = 0, \quad \beta = 1,$

$DS = [n+1, n-1, 1, 0], \quad CS = [n+1, n-1], \quad US = [0].$

2. $\mathfrak{s}_{n+1,6}(\epsilon)$:
$[f, e_1] = (n-3)e_1,\ [f, e_2] = (n-4)e_2,\ [f, e_3] = \left(\frac{n}{2} - 1\right) e_3,$
$[f, e_k] = (n-1-k)e_k,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = \frac{n-4}{2} e_{n-1},\ [f, e_n] = e_n + \epsilon e_{n-2},$
where $\epsilon = 1$ *over* $\mathbb{C}$, *whereas over* $\mathbb{R}$ $\epsilon = 1$ *for n odd*, $\epsilon = \pm 1$ *for n even*.

$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [0].$

3. $\mathfrak{s}_{n+1,7}$:
$[f, e_1] = e_1,\ [f, e_2] = 0,\ [f, e_3] = e_3 - e_1,$
$[f, e_k] = (3-k)e_k,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = e_2,\ [f, e_n] = e_n.$

$DS = [n+1, n-1, n-4, 0], \quad CS = [n+1, n-1], \quad US = [0].$

4. $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$:
$[f, e_1] = e_1,\ [f, e_2] = e_2,\ [f, e_3] = \frac{1}{2} e_3,$
$[f, e_k] = e_k + \sum_{l=4}^{k-2} a_{k-l+1} e_l + a_{k-2} e_2 + a_{k-1} e_1,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = \frac{1}{2} e_{n-1} + a_2 e_3,\ [f, e_n] = 0,$
$a_j \in \mathbb{F}$, *at least one* a_j *satisfies* $a_j \neq 0$ *and*:

- *when* $\mathbb{F} = \mathbb{C}$ *the first nonzero* a_j *satisfies* $a_j = 1$.
- *when* $\mathbb{F} = \mathbb{R}$ *the first nonzero* a_j *for even j satisfies* $a_j = 1$. *If all* $a_j = 0$ *for j even, then the first nonzero* a_j *(j odd) satisfies* $a_j = \pm 1$.

$DS = [n+1, n-1, 1, 0], \quad CS = [n+1, n-1], \quad US = [0].$

5. $\mathfrak{s}_{n+1,9}$:
$[f, e_1] = 3e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = 2e_3 - e_2,$
$[f, e_k] = (5-k)e_k,\ 4 \leqslant k \leqslant n-2,$
$[f, e_{n-1}] = e_{n-1} + e_4,\ [f, e_n] = e_n + e_{n-1}.$

$DS = [n+1, n, n-3, 0], \quad CS = [n+1, n], \quad US = [0].$

Exactly one solvable Lie algebra $\mathfrak{s}_{n+2}$ *of* $\dim \mathfrak{s} = n+2$ *with the nilradical* $\mathfrak{n}_{n,3}$ *exists. It is presented in a basis* $(e_1, \ldots, e_n, f_1, f_2)$ *where the Lie brackets involving* f_1 *and* f_2 *are*

$$\begin{aligned}
&[f_1, e_1] = e_1, \quad [f_2, e_1] = 2e_1,\\
&[f_1, e_2] = 0, \quad [f_2, e_2] = 2e_2,\\
&[f_1, e_3] = e_3, \quad [f_2, e_3] = e_3,\\
&[f_1, e_k] = (3-k)e_k, \quad [f_2, e_k] = 2e_k, \quad 4 \leqslant k \leqslant n-2,\\
&[f_1, e_{n-1}] = 0, \quad [f_2, e_{n-1}] = e_{n-1},\\
&[f_1, e_n] = e_n, \quad [f_2, e_n] = 0, \quad [f_1, f_2] = 0.
\end{aligned}$$

For this algebra we have

$DS = [n+2, n, n-3, 0], \quad CS = [n+2, n], \quad US = [0].$

We note that the class $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$ encompasses both extensions of $\tilde{\mathfrak{s}}_{m+1,7}(a_3, \ldots, a_{m-1})$ and an extension of $\tilde{\mathfrak{s}}_{m+1,4}$ with $c_3 \neq 0$ in Eq. (16). The parameter brought to ± 1 was selected in the most convenient form for presentation and consequently is equivalent but slightly different from a direct extension of $\tilde{\mathfrak{s}}_{m+1,7}(a_3,\ldots,a_{m-1})$ to the nilradical $\mathfrak{n}_{n,3}$ – for that choice the non-equivalent values of parameters would be more cumbersome to write down.

Next, we investigate the classification of solvable extensions of $\mathfrak{n}_{n,3}$ in low dimensions $n=6,5$. Results in these dimensions somewhat differ from the general ones presented in Theorem 2.

3.1. Dimension $n = 6$

When $n=6$ the results are as follows: all the algebras presented in Theorem 2 exist (with $e_{n-2}\equiv e_4$) but they do not exhaust all the possibilities. The reason for this is that in this particular dimension we have $[f,e_{n-2}]=(2c_5-d_6)e_{n-2}+\ldots$ Therefore, if $d_6=c_5$ then also $[f,e_{n-2}]=d_6e_{n-2}+\cdots$ That implies that if we have $d_6=c_5\to 1,\ d_5\neq 0,\ d_4\neq 0$ in the derivation (16) then we can set to zero neither d_5 nor d_4 by any choice of automorphism Φ and we are left with only one scaling available - preferably used to set $d_5\to 1$.

That means that for the 6-dimensional nilradical $\mathfrak{n}_{6,3}$ we have solvable extensions $\mathfrak{s}_{7,1}(\beta)$, $\mathfrak{s}_{7,2}$, $\mathfrak{s}_{7,3}$, $\mathfrak{s}_{7,4}$, $\mathfrak{s}_{7,5}$, $\mathfrak{s}_{7,6}(\epsilon)$, $\mathfrak{s}_{7,7}$, $\mathfrak{s}_{7,8}(1,a_3)$, $\mathfrak{s}_{7,8}(0,\epsilon)$, $\mathfrak{s}_{7,9}$, $\mathfrak{s}_8$ where $\epsilon=1$ over $\mathbb{C}$ and $\epsilon=\pm 1$ over $\mathbb{R}$, whose structure is as described in Theorem 2 above and one additional class of algebras, differing from $\mathfrak{s}_{7,9}$ by one additional nonvanishing parameter α

- $\mathfrak{s}_{7,10}(\alpha),\ \alpha\neq 0:$
  $[f,e_1]=3e_1,\ [f,e_2]=2e_2,\ [f,e_3]=2e_3-e_2,$
  $[f,e_4]=e_4,\ [f,e_5]=e_5+e_4,\ [f,e_6]=e_6+e_5+\alpha e_4,$
  $$DS=[7,6,3,0],\quad CS=[7,6],\quad US=[0].$$

3.2. Dimension $n = 5$

When $n=5$, the investigation must be performed in a different way. Namely, there is no $\tilde{\mathfrak{n}}_{3,1}$ subalgebra – it has collapsed to the Heisenberg algebra which has different properties. Nevertheless, by a rather straightforward, if repetitive, computation (essentially linear algebra of 5×5 matrices) one can construct all solvable extensions of $\mathfrak{n}_{5,3}$. Since this was done already in [12] for one non-nilpotent element and for two elements the result can be derived from the previous one, we shall only list the results and compare them to their higher dimensional analogues. In order to make our comparison as simple as possible we work in a basis analogous to Eq. (6), namely

$$e_1=x_1,\quad e_2=x_3,\quad e_3=x_2,\quad e_4=x_5,\quad e_5=x_4. \tag{17}$$

The nonvanishing Lie brackets are

$$[e_2,e_5]=e_1,\quad [e_3,e_4]=e_1,\quad [e_4,e_5]=-e_3. \tag{18}$$

Although the structure of the nilradical is quite different from the other elements of the series, the set of solvable extensions is rather similar. We get analogues of all solvable algebras in Theorem 2 with some changes in the structure of $\mathfrak{s}_{n+1,6}$, $\mathfrak{s}_{n+1,8}$, $\mathfrak{s}_{n+1,9}$; in addition, the two algebras $\mathfrak{s}_{n+1,2}$ and $\mathfrak{s}_{n+1,3}$ become identical when $n=5$. The fact that the algebras $\mathfrak{s}_{n+1,6}$, $\mathfrak{s}_{n+1,8}$, $\mathfrak{s}_{n+1,9}$ must be modified when $n=5$ can be inferred already from Theorem 2 since the Lie brackets as presented there cannot be made sense of if $n=5$. These structurally different analogues are distinguished by primes below.

Explicitly, assuming the structure of $\mathfrak{n}_{5,3}$ in the form (18), we have the following Lie brackets with non-nilpotent element(s) and dimensions of the characteristic series

- $\mathfrak{s}_{6,1}(\beta),\ \beta\in\mathbb{F}\backslash\left\{0,-\frac{1}{2}\right\}:$
  $[f,e_1]=(1+2\beta)e_1,\ [f,e_2]=2\beta e_2,\ [f,e_3]=(\beta+1)e_3,\ [f,e_4]=\beta e_4, [f,e_5]=e_5,$
  $$DS=[6,5,2,0],\quad CS=[6,5],\quad US=[0].$$
- $\mathfrak{s}_{6,2}:[f,e_1]=e_1,\ [f,e_2]=0,\ [f,e_3]=e_3,\ [f,e_4]=0,\ [f,e_5]=e_5,$

$$DS = [6, 3, 0], \quad CS = [6, 3], \quad US = [0].$$

- $\mathfrak{s}_{6,4} : [f, e_1] = 0,\ [f, e_2] = -e_2,\ [f, e_3] = \frac{1}{2}e_3,\ [f, e_4] = -\frac{1}{2}e_4,\ [f, e_5] = e_5,$
  $$DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [1].$$
- $\mathfrak{s}_{6,5} : [f, e_1] = 2e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = e_3,\ [f, e_4] = e_4,\ [f, e_5] = 0,$
  $$DS = [6, 4, 1, 0], \quad CS = [6, 4], \quad US = [0].$$
- $\mathfrak{s}'_{6,6} : [f, e_1] = 2e_1,\ [f, e_2] = e_2, [f, e_3] = \frac{3}{2}e_3,\ [f, e_4] = \frac{1}{2}e_4, [f, e_5] = e_5 + e_2,$
  $$DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [0].$$
- $\mathfrak{s}_{6,7} : [f, e_1] = e_1,\ [f, e_2] = 0,\ [f, e_3] = e_3 - e_1,\ [f, e_4] = e_2,\ [f, e_5] = e_5,$
  $$DS = [6, 4, 1, 0], \quad CS = [6, 4, 3], \quad US = [0].$$
- $\mathfrak{s}'_{6,8} : [f, e_1] = 2e_1,\ [f, e_2] = 2e_2,\ [f, e_3] = e_3,\ [f, e_4] = -e_3 + e_4,\ [f, e_5] = 0,$
  $$DS = [6, 4, 1, 0], \quad CS = [6, 4], \quad US = [0].$$
- $\mathfrak{s}'_{6,9} : [f, e_1] = 3e_1, [f, e_2] = 2e_2 - e_3, [f, e_3] = 2e_3, [f, e_4] = e_4 + e_5, [f, e_5] = e_5,$
  $$DS = [6, 5, 2, 0], \quad CS = [6, 5], \quad US = [0].$$
- $\mathfrak{s}_7 : [f_1, e_1] = e_1,\ [f_1, e_2] = 0,\ [f_1, e_3] = e_3,\ [f_1, e_4] = 0,\ [f_1, e_5] = e_5,$
  $[f_2, e_1] = 2e_1,\ [f_2, e_2] = 2e_2,\ [f_2, e_3] = e_3,\ [f_2, e_4] = e_4,\ [f_2, e_5] = 0,$
  $[f_1, f_2] = 0,$
  $$DS = [7, 5, 2, 0], \quad CS = [7, 5], \quad US = [0].$$

We note that in several cases the characteristic series are different from the ones in Theorem 2. This difference in behavior is due to the structural difference between $\mathfrak{n}_{n-2,1}$ and the Heisenberg algebra.

4. Generalized Casimir invariants

We proceed to construct generalized Casimir invariants, i.e. invariants of the coadjoint representation, of the nilpotent algebra $\mathfrak{n}_{n,3}$ and its solvable extensions. We recall that a basis for the coadjoint representation of the Lie algebra $\mathfrak{g}$ is given by the first order differential operators

$$\widehat{X}_k = \mathrm{x}_a c^a_{kb} \frac{\partial}{\partial \mathrm{x}_b} \tag{19}$$

acting on functions on the vector space $\mathfrak{g}^*$. Here, c^k_{ij} are the structure constants of the Lie algebra $\mathfrak{g}$ in the given basis $(x_1, \ldots, x_N)$ and the quantities x_a are coordinates in the basis of the space $\mathfrak{g}^*$ dual to the basis $(x_1, \ldots, x_N)$ of the algebra $\mathfrak{g}$. That means that x_a are linear functionals on $\mathfrak{g}^*$, i.e. $\mathrm{x}_a \in (\mathfrak{g}^*)^*$, and through the canonical isomorphism of vector spaces $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ one can identify $\mathrm{x}_a \simeq x_a$. In what follows we shall not typographically distinguish between x_a and x_a, the meaning - vector in algebra vs. linear functional on the dual space - shall be clear from the context.

Invariants of the coadjoint representation, i.e. generalized Casimir invariants, are functions I on $\mathfrak{g}^*$ which satisfy the following system of partial differential equations

$$\widehat{X}_k I(x_1, \ldots, x_N) = 0, \quad k = 1, \ldots, N. \tag{20}$$

Several methods exist for construction of invariants of the coadjoint representation, most widely used ones are direct solution of Eq. (20) by the method of characteristics (see e.g. [14–17]) and the method of moving frames (see [18–23]).

However, we shall use a different approach. We reduce Eq. (20) to the ones encountered and solved in [6] for the subalgebra $\tilde{\mathfrak{n}}_{n-2,1}$ and its solvable extensions.

Considering first the nilpotent algebra $\mathfrak{n}_{n,3}$ we have the operators (19) in the form

$$\begin{aligned} &\widehat{E}_1 = 0, \quad \widehat{E}_2 = e_1 \frac{\partial}{\partial e_n}, \quad \widehat{E}_3 = e_1 \frac{\partial}{\partial e_{n-1}}, \quad \widehat{E}_4 = e_2 \frac{\partial}{\partial e_n}, \\ &\widehat{E}_k = e_{k-1} \frac{\partial}{\partial e_n}, \quad 5 \leqslant k \leqslant n-2, \quad \widehat{E}_{n-1} = -e_1 \frac{\partial}{\partial e_3} - e_3 \frac{\partial}{\partial e_n}, \\ &\widehat{E}_n = -e_1 \frac{\partial}{\partial e_2} - e_2 \frac{\partial}{\partial e_4} - \sum_{k=5}^{n-2} e_{k-1} \frac{\partial}{\partial e_k} + e_3 \frac{\partial}{\partial e_{n-1}}. \end{aligned} \tag{21}$$

It is evident that any solution I of Eq. (20) cannot depend[3] on e_3, e_{n-1} because of $\widehat{E}_{n-1} I = \widehat{E}_3 I = \widehat{E}_2 I = 0$. Consequently, all considered operators $\widehat{E}_j$ can be truncated to act on functions of $\tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \tilde{e}_3 = e_4, \ldots, \tilde{e}_{n-3} = e_{n-2}, \tilde{e}_{n-2} = e_n$ only. Then $\widehat{E}_{3T}, \widehat{E}_{n-1T}$ vanish and the remaining operators are exactly those present in the investigation of invariants of $\mathfrak{n}_{n-2,1}$ in [6]. Therefore, the generalized Casimir invariants of $\mathfrak{n}_{n,3}$ are the same as the ones for $\mathfrak{n}_{n-2,1}$ once written in an appropriate basis.

Similarly, when we consider the solvable extensions of $\mathfrak{n}_{n,3}$, the operators $\widehat{E}_j$ in (21) get additional $\frac{\partial}{\partial f}$ or $\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}$ terms and one $(\widehat{F})$ or two $(\widehat{F}_1, \widehat{F}_2)$ additional operators are present in Eq. (20).

Let us first consider the case with $\widehat{F}$ only. When the derivation D defining f is such that $2c_{n-1} + d_n \neq 0$, we have $\widehat{E}_1 = (2c_{n-1} + d_n) e_1 \frac{\partial}{\partial f}$ which excludes the dependence of I on f. When $2c_{n-1} + d_n = 0$ the situation is only slightly more complicated – the operators $\widehat{E}_2, \widehat{E}_4$ together again exclude the dependence of I on both f and e_n. In both cases, we can restrict all operators (21) and $\widehat{F}$ to $\mathfrak{n}_{n,3}$ and then to $\mathfrak{n}_{n-2,1}$, reducing the computation to the corresponding solvable extension of $\mathfrak{n}_{n-2,1}$.

In the second case we have two additional operators $\widehat{F}_1, \widehat{F}_2$ and $\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}$ terms in $\widehat{E}_j$. Now the operators $\widehat{E}_1, \widehat{E}_2, \widehat{E}_3, \widehat{E}_4$ are used in the same way to show that any invariant I cannot depend on f_1, f_2.

Altogether, the construction of generalized Casimir invariants was fully reduced to the one for the nilradical $\mathfrak{n}_{n-2,1}$.

As proved in [6], invariants of the Lie algebra $\mathfrak{n}_{m,1}$ and its solvable extensions are as follows:

Theorem 3. *The nilpotent Lie algebra $\mathfrak{n}_{m,1}$ has $m-2$ functionally independent invariants. They can be chosen to be the following polynomials:*

$$\begin{aligned} &\tilde{\xi}_0 = \tilde{e}_1, \\ &\tilde{\xi}_k = \frac{(-1)^k k}{(k+1)!} \tilde{e}_2^{k+1} + \sum_{j=0}^{k-1} (-1)^j \frac{\tilde{e}_2^j \, \tilde{e}_{k+2-j} \tilde{e}_1^{k-j}}{j!}, \quad 1 \leqslant k \leqslant m-3. \end{aligned} \tag{22}$$

The algebras $\tilde{\mathfrak{s}}_{m+1,1}(\beta), \ldots, \tilde{\mathfrak{s}}_{m+1,5}$ have $m-3$ invariants each. Their form is

1. $\tilde{\mathfrak{s}}_{m+1,1}(\beta), \tilde{\mathfrak{s}}_{m+1,2}$ *and* $\tilde{\mathfrak{s}}_{m+1,5}$:

$$\tilde{\chi}_k = \frac{\tilde{\xi}_k}{\tilde{\xi}_0^{(k+1)\frac{m-3+\beta}{m-2+\beta}}}, \quad 1 \leqslant k \leqslant m-3. \tag{23}$$

   For $\tilde{\mathfrak{s}}_{m+1,2}$ and $\tilde{\mathfrak{s}}_{m+1,5}$ we have $\beta = 0$ and $\beta = 1$, respectively in Eq. (23).

2. $\tilde{\mathfrak{s}}_{m+1,3}$:

$$\tilde{\chi}_1 = \tilde{\xi}_0, \quad \tilde{\chi}_k = \frac{\tilde{\xi}_k^2}{\tilde{\xi}_1^{k+1}}, \quad 2 \leqslant k \leqslant m-3. \tag{24}$$

[3] Neither can I depend on e_n.

3. $\tilde{\mathfrak{s}}_{m+1,4}$:

$$\tilde{\chi}_k = \frac{\tilde{\xi}_k}{\tilde{\xi}_0^{k+1}}, \quad 1 \leqslant k \leqslant m-3. \tag{25}$$

4. $\tilde{\mathfrak{s}}_{m+1,7}(a_3, \ldots, a_{m-1})$:

$$\tilde{\chi}_k = \sum_{q=0}^{\left[\frac{k+1}{2}\right]} (-1)^q \frac{(\ln \tilde{\xi}_0)^q}{q!} \left(\sum_{i_1+\cdots+i_q=k-2q+1} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} + \sum_{j+i_1+\cdots+i_q=k-2q-1} \frac{\tilde{\xi}_{j+1}}{\tilde{\xi}_0^{j+2}} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} \right), \quad 1 \leqslant k \leqslant m-3. \tag{26}$$

The summation indices take the values $0 \leqslant j, i_1, \ldots, i_q \leqslant k+1$.

The Lie algebra $\tilde{\mathfrak{s}}_{m+2}$ *has* $m-4$ *functionally independent invariants that can be chosen to be*

$$\tilde{\chi}_k = \frac{\tilde{\xi}_{k+1}}{\tilde{\xi}_1^{\frac{k+2}{2}}}, \quad 1 \leqslant k \leqslant m-4. \tag{27}$$

The results for $\mathfrak{n}_{n,3}$ and its solvable extensions are now as follows:

Theorem 4. *Let* $n \geqslant 6$. *The nilpotent Lie algebra* $\mathfrak{n}_{n,3}$ *has* $n-4$ *functionally independent invariants. They can be chosen to be the following polynomials*

$$\xi_0 = e_1,$$
$$\xi_k = \frac{(-1)^k k}{(k+1)!} e_2^{k+1} + \sum_{j=0}^{k-1} (-1)^j \frac{e_2^j \, e_{k+3-j} \, e_1^{k-j}}{j!}, \quad 1 \leqslant k \leqslant n-5. \tag{28}$$

The algebras $\mathfrak{s}_{n+1,1}(\beta), \ldots, \mathfrak{s}_{n+1,9}$ *have* $n-5$ *invariants each. Their form is*

1. $\mathfrak{s}_{n+1,1}(\beta)$, $\mathfrak{s}_{n+1,2}$, $\mathfrak{s}_{n+1,3}$, $\mathfrak{s}_{n+1,6}$, $\mathfrak{s}_{n+1,7}$ *and* $\mathfrak{s}_{n+1,9}$:

$$\chi_k = \frac{\xi_k}{\xi_0^{(k+1)\frac{2\beta}{1+2\beta}}}, \quad 1 \leqslant k \leqslant n-5. \tag{29}$$

For $\mathfrak{s}_{n+1,2}$ *is* $\beta = \frac{n-5}{2}$, *for* $\mathfrak{s}_{n+1,3}$ *and* $\mathfrak{s}_{n+1,7}$ *we have* $\beta = 0$, *for* $\mathfrak{s}_{n+1,6}(\epsilon)$ *we have* $\beta = \frac{n-4}{2}$ *and for* $\mathfrak{s}_{n+1,9}$ *is* $\beta = 1$, *respectively in Eq. (29).*

2. $\mathfrak{s}_{n+1,4}$:

$$\chi_1 = \xi_0, \quad \chi_k = \frac{\xi_k^2}{\xi_1^{k+1}}, \quad 2 \leqslant k \leqslant n-5. \tag{30}$$

3. $\mathfrak{s}_{n+1,5}$:

$$\chi_k = \frac{\xi_k}{\xi_0^{k+1}}, \quad 1 \leqslant k \leqslant n-5. \tag{31}$$

4. $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$:

$$\chi_k = \sum_{q=0}^{\left[\frac{k+1}{2}\right]} (-1)^q \frac{(\ln \xi_0)^q}{q!} \left(\sum_{i_1+\cdots+i_q=k-2q+1} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3} \right. \tag{32}$$

$$+ \sum_{j+i_1+\cdots+i_q=k-2q-1} \frac{\xi_{j+1}}{\xi_0^{j+2}} a_{i_1+3}a_{i_2+3}\ldots a_{i_q+3}\Bigg), \quad 1 \leqslant k \leqslant n-5.$$

The summation indices take the values $0 \leqslant j, i_1, \ldots, i_q \leqslant k+1$.

When $n = 6$ *the Lie algebra* $\mathfrak{s}_{7,10}(\alpha)$ *has one invariant which can be chosen in the form* $\frac{2e_4e_1-e_2^2}{e_1^{4/3}}$, *i.e. coincides with the one for* $\mathfrak{s}_{7,9}$.

The Lie algebra $\mathfrak{s}_{n+2}$ *has* $n-6$ *functionally independent invariants that can be chosen to be*

$$\chi_k = \frac{\xi_{k+1}}{\xi_1^{\frac{k+2}{2}}}, \quad 1 \leqslant k \leqslant n-6. \tag{33}$$

We point out that the algebras $\mathfrak{s}_{n+1,3}$ and $\mathfrak{s}_{n+1,7}$ are examples of solvable non-nilpotent Lie algebras with a polynomial basis of invariants, i.e. their bases of invariants can be chosen in the form of Casimir operators in the enveloping algebra of $\mathfrak{s}_{n+1,3}$ and $\mathfrak{s}_{n+1,7}$ (the same holds also for $\tilde{\mathfrak{s}}_{m+1,1}(3-m)$ of [6]). If ever a hypothesis concerning a criterion for the existence of polynomial basis of invariants of solvable algebras is presented, these examples can be easily used as simple tests of its plausibility.

For 5-dimensional nilradical $\mathfrak{n}_{5,3}$ we have solvable algebras $\mathfrak{s}_{6,1}(\beta), \mathfrak{s}_{6,2}, \mathfrak{s}_{6,5}, \mathfrak{s}'_{6,6}, \mathfrak{s}_{6,7}, \mathfrak{s}'_{6,8}, \mathfrak{s}'_{6,9}$ with no invariants and $\mathfrak{s}_{6,4}$ which has two invariants. They can be chosen in the polynomial form

$$e_1, \quad 2e_1^2 f - 2e_1e_2e_5 + e_1e_3e_4 + e_2e_3^2.$$

The algebra $\mathfrak{s}_7$ has one invariant

$$\frac{(f_2-2f_1)e_1^2 + (2e_2e_5 - e_3e_4)e_1 - e_2e_3^2}{e_1^2}.$$

We observe that invariants of the solvable Lie algebras with the nilradical $\mathfrak{n}_{5,3}$ (if nonconstant) depend on elements outside of $\mathfrak{n}_{5,3}$, i.e. f or f_1, f_2. This is related to the fact that there is no $\tilde{\mathfrak{n}}_{3,1}$ subalgebra – it degenerates to the Heisenberg algebra, the properties of which are markedly different.

5. Conclusions

We have fully classified all solvable Lie algebras with the nilradical $\mathfrak{n}_{n,3}$ in arbitrary dimension n and constructed their generalized Casimir invariants.

There are two general lessons to be learned from this computation. Firstly, it turned out that the knowledge of all solvable extensions of a suitable subalgebra $\tilde{\mathfrak{n}}$ of the given nilpotent algebra $\mathfrak{n}$ may lead to a significant simplification of the whole computation and is definitively worth investigating if such subalgebras are identified in $\mathfrak{n}$. This can hold notwithstanding the fact that not all automorphisms of $\mathfrak{n}$ preserve the subalgebra $\tilde{\mathfrak{n}}$. Of course, it was important in our investigation that the structure of the subalgebra was restrictive enough, i.e. we expect that a similar simplification can be achieved probably for subalgebras with high enough degree of nilpotency, e.g. filiform or quasi-filiform.

Secondly, it was of profound importance that (almost) all automorphisms of $\tilde{\mathfrak{n}}$ could be obtained as a restriction of automorphisms of $\mathfrak{n}$. In our case we had a local isomorphism of $Aut(\tilde{\mathfrak{n}})$ and $Aut(\mathfrak{n})|_{\tilde{\mathfrak{n}}}$; the two differ topologically by the absence of some connected components of $Aut(\tilde{\mathfrak{n}})$ in $Aut(\mathfrak{n})|_{\tilde{\mathfrak{n}}}$. This minor difference could be easily taken into account and the classification of all solvable extensions of $\tilde{\mathfrak{n}}$ with respect to this restricted group of automorphisms acting on $\tilde{\mathfrak{n}}$ was obtained by inspection from previously known results [6]. On the other hand, had the $Aut(\tilde{\mathfrak{n}})$ and $Aut(\mathfrak{n})|_{\tilde{\mathfrak{n}}}$ been locally non-isomorphic, the knowledge of solvable extensions of $\tilde{\mathfrak{n}}$ would not be of much use in the study of solvable extensions of $\mathfrak{n}$. A simple example of this is the maximal Abelian ideal $\mathfrak{a}$ of $\mathfrak{n}$. Its group of automorphisms *per se* is typically much larger than the automorphisms inherited from $\mathfrak{n}$, i.e. many transformations used in $\mathfrak{a}$ are not allowed in $\mathfrak{n}$ and, at the same time, most of solvable extensions of $\mathfrak{a}$ cannot be enlarged to solvable extensions of $\mathfrak{n}$ – the Lie brackets in $\mathfrak{n}$ simply do not allow that.

Therefore, the particular properties of the subalgebra and its immersion into the whole nilradical are of crucial importance for the whole setup to work.

Finally, we have seen that although the considered series of nilpotent algebras can be rather naturally constructed starting from dimension $n = 5$, the 5-dimensional one has substantially different properties. They reflect themselves also in possible solvable extensions and their invariants.

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