Van Douwen’s diagram for dense sets of rationals

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Received 10 March 2005; accepted 14 July 2005
Available online 11 July 2006

Abstract

We investigate cardinal invariants related to the structure Dense(\mathbb{Q})/nwd of dense sets of rationals modulo the nowhere dense sets. We prove that \( s_{\mathbb{Q}} \leq \min\{s, \text{add}(\mathcal{M})\} \), thus dualizing the already known \( r_{\mathbb{Q}} \geq \max\{r, \text{cof}(\mathcal{M})\} \) [B. Balcar, F. Hernández-Hernández, M. Hrušák, Combinatorics of dense subsets of the rationals, Fund. Math. 183 (2004) 59–80, Theorem 3.6]. We also show the consistency of each of \( h_{\mathbb{Q}} < s_{\mathbb{Q}} \) and \( h < h_{\mathbb{Q}} \). Our results answer four questions of Balcar, Hernández and Hrušák [B. Balcar, F. Hernández-Hernández, M. Hrušák, Combinatorics of dense subsets of the rationals, Fund. Math. 183 (2004) 59–80, Questions 3.11].

Keywords: Cardinal invariants of the continuum; van Douwen’s diagram; Splitting number; Distributivity number; Meager ideal; Dense set; Nowhere dense set; Iterated forcing; Laver forcing

0. Introduction

The structure \( \mathcal{P}(\omega)/\text{fin} \) of the power set of the natural numbers \( \omega \) modulo the finite sets plays an important role in set theory. It is naturally ordered by \( \leq \) where \([A] \leq [B]\) if \( A \subseteq^* B \). Here, \([A]\) denotes the equivalence class of a set \( A \subseteq \omega \) and, for \( A, B \subseteq \omega \), we say \( A \) is almost contained in \( B \) (\( A \subseteq^* B \) in symbols) if \( A \setminus B \) is finite. Much of the combinatorial structure of \( (\mathcal{P}(\omega)/\text{fin}, \leq) \) can be described by cardinal invariants of the continuum like, for example, the splitting number \( s \) or the distributivity number \( h \). Their relationship has been thoroughly investigated and is usually displayed in van Douwen’s diagram (see Fig. 1).

For \( A, B \in [\omega]^\omega \), say \( A \) splits \( B \) if \( A \cap B \) and \( B \setminus A \) are both infinite. \( A \subseteq [\omega]^\omega \) is a splitting family if every \( B \in [\omega]^\omega \) is split by a member of \( A \). The splitting number \( s \) is the least size of a splitting family. \( \mathcal{D} \subseteq [\omega]^\omega \) is dense if for all \( A \in [\omega]^\omega \) there is \( D \in \mathcal{D} \) almost contained in \( A \). \( \mathcal{D} \) is open if it is downward closed under \( \subseteq^* \). The distributivity number \( h \) of \( \mathcal{P}(\omega)/\text{fin} \) is the smallest size of a family of dense open sets with empty intersection. Equivalently, it is the least cardinality \( \kappa \) of a family \( \{A_\alpha : \alpha < \kappa\} \) of maximal almost disjoint families such that no \( X \in [\omega]^\omega \) is almost contained in a member of \( A_\alpha \) for all \( \alpha \). \( h \) is the least \( \kappa \) such that \( (\mathcal{P}(\omega)/\text{fin}, \leq) \) is not \( \kappa \)-distributive as a forcing notion. It is also the cardinal to which \( \epsilon \) is collapsed after forcing with \( \mathcal{P}(\omega)/\text{fin} \) [2]. The additivity of the meager ideal \( \text{add}(\mathcal{M}) \) is the least \( \kappa \) such that the meager ideal is not \( \kappa \)-additive. The covering of the meager ideal \( \text{cov}(\mathcal{M}) \) is the smallest size of a family of meager sets covering the real line. For other cardinal invariants which figure in Fig. 1

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if every B all of the ZFC -inequalities displayed in the diagram may be found in these references.

In recent years, research has increasingly focused on cardinal invariants which are defined in a similar way for structures other than \( \mathcal{P}(\omega) / \text{fin} \), the analogue of van Douwen’s diagram has been redrawn, and the connection of the new cardinals with the classical ones has been studied. As an example, let us mention the work on partitions of \( \mathcal{P}(\omega) / \text{fin} \) in [12] a n d [10]. In this context, a natural class of structures is those of the form \( \mathcal{M} \), which is more interesting from our point of view because it is but which we will not use in our work) see [7], [26], [27] or [3].

Except for the recent \( \text{cof}(\mathcal{M}) \leq i \) [1, Theorem 3.6], all of the ZFC -inequalities displayed in the diagram may be found in these references.

For example, letting \( \text{nwd} \) stand for the ideal of nowhere dense sets of rationals, ordered by \( \subseteq \) if \( A \subseteq \text{nwd} \) B if \( A \setminus B \) is nowhere dense. Cardinal invariants of the latter, which is more interesting from our point of view because it is \( \sigma \)-closed under \( \subseteq \text{nwd} \), have been studied by Balcar, Hernández and Hrušák [1]. In Section 1, we dualize one result of theirs, \( t_\mathbb{Q} \geq \max(\text{cof}(\mathcal{M}), t) \), to obtain \( s_\mathbb{Q} \leq \min(\text{add}(\mathcal{M}), s) \) (Theorem 1.4). Thus the corresponding van Douwen diagram looks as in Fig. 2.

For \( A, B \in \text{Dense}(\mathbb{Q}) \), say \( A \text{ Q-splits } B \) if \( A \cap B \) and \( B \setminus A \) are both dense. \( A \subseteq \text{Dense}(\mathbb{Q}) \) is a \( \mathbb{Q} \)-splitting family if every \( B \in \text{Dense}(\mathbb{Q}) \) is \( \mathbb{Q} \)-split by a member of \( A \). The \( \mathbb{Q} \)-splitting number \( s_\mathbb{Q} \) is the least size of a \( \mathbb{Q} \)-splitting family. \( \mathcal{D} \subseteq \text{Dense}(\mathbb{Q}) \) is \( \mathbb{Q} \)-dense if for all \( A \in \text{Dense}(\mathbb{Q}) \) there is \( D \in \mathcal{D} \) with \( D \subseteq \text{nwd} A \). The distributivity number \( h_\mathbb{Q} \) of Dense(\( \mathbb{Q} \))/nwd is the smallest size of a family of \( \mathbb{Q} \)-dense open sets with empty intersection. \( h_\mathbb{Q} \) is the least \( \kappa \) such that (Dense(\( \mathbb{Q} \))/nwd, \( \leq \)) is not \( \kappa \)-distributive as a forcing notion. It is also the cardinal to which \( t \) is collapsed after forcing with Dense(\( \mathbb{Q} \))/nwd [1, Theorem 3.3].

One natural question is whether all inequalities in the diagram are consistently strict. In this direction we prove the consistency of \( h_\mathbb{Q} < s_\mathbb{Q} \) (Theorem 2.2), thus answering [1, Questions 3.11 (4)]. Still open are the consistency of \( p_\mathbb{Q} < t_\mathbb{Q} \) (the well known p versus t problem) and of \( t_\mathbb{Q} < i_\mathbb{Q} \) [1, Questions 3.11 (5)]. Another natural problem is to relate the cardinals in the two diagrams. With respect to this, Balcar, Hernández and Hrušák have shown \( p_\mathbb{Q} = p \), \( t_\mathbb{Q} = t \) and \( i_\mathbb{Q} = i \), as well as the consistency of \( r < t_\mathbb{Q} \) and \( h_\mathbb{Q} < h \), while the consistency of \( s_\mathbb{Q} < s_2 \) follows from our results in Section 1 (Corollary 1.5). In Sections 3 and 4 we prove the consistency of \( h < h_\mathbb{Q} \) (Theorem 3.1), thus answering [1, Questions 3.11 (1)]. In Section 5, we discuss some open problems.

Our terminology and notation are standard. For prerequisites in set theory in general and forcing theory in particular see [15], [18] or [3]. For convenience, we often identify the rationals \( \mathbb{Q} \) with the collection of finite binary sequences \( 2^{<\omega} \).

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1 This answers [1, Questions 3.11 (2)].
2 This answers [1, Questions 3.11 (3)].
1. A characterization of $\text{add}(\mathcal{M})$

We start with an alternative characterization of $\text{add}(\mathcal{M})$ which is a close relative of a characterization obtained by Keremedis [17]. $F = (F_n : n \in \omega)$ is a dedi (dense and disjoint) sequence if all $F_n \subseteq \mathbb{Q}$ are nowhere dense, pairwise disjoint, and for all $s \in 2^{<\omega}$ there is $n$ with $F_n \subseteq \{t : s \subseteq t\}$.

**Lemma 1.1.** $\text{add}(\mathcal{M}) = \min\{|F| : F$ is a family of dedi sequences and for all $D \in \text{Dense}(\mathbb{Q})$ there is $F = (F_n : n \in \omega) \in F$ such that $D \cap F_n$ is infinite for all $n$.}$

**Proof.** Call the cardinal on the right-hand side $\kappa$. Recall Keremedis’s characterization of $\text{add}(\mathcal{M})$ [17] (see also [1, Theorem 1.4] and, for related results, [3, Theorem 2.2.6]).

$$\text{add}(\mathcal{M}) = \min\{|F| : F \subseteq \text{nwd and } \forall D \in \text{Dense}(\mathbb{Q}) \exists F \in F |D \cap F| = \aleph_0\}.$$ This immediately entails $\text{add}(\mathcal{M}) \leq \kappa$. So it suffices to show $\kappa \leq \text{add}(\mathcal{M})$.

Let $\{N_\alpha : \alpha < \text{add}(\mathcal{M})\}$ be a witness for $\text{add}(\mathcal{M})$, i.e., all $N_\alpha$ are closed nowhere dense and $\bigcup_\alpha N_\alpha$ is not meager. Let $T_\alpha = \{f | n : f \in N_\alpha, n \in \omega\}$ be the tree associated with $N_\alpha$. For each $t \in 2^{<\omega}$, let $T_\alpha^t = \{t \cap s : s \in T_\alpha\}$ be the copy of $T_\alpha$ below $t$. Clearly $T_\alpha^0 = T_\alpha$ and $T_\alpha^t \in \text{nwd}$ for all $t$. Also it is easy to find $(t_n^\alpha : n \in \omega)$ such that

- the $T_\alpha^t$ are pairwise disjoint
- for all $s \in 2^{<\omega}$ there is $n$ such that $s \subseteq t_n^\alpha$ (so $T_\alpha^{t_n^\alpha} \subseteq \{t : s \subseteq t\}$).

So it suffices to show that $\{(T_\alpha^{t_n^\alpha} : n \in \omega) : \alpha < \text{add}(\mathcal{M})\}$ is a witness for $\kappa$.

Assume not, and let $D \in \text{Dense}(\mathbb{Q})$ be a counterexample. So for all $\alpha < \text{add}(\mathcal{M})$ there is $n_\alpha$ such that

$$|D \cap T_\alpha^{t_n^\alpha}| < \aleph_0.$$ (**)

Let $M_\alpha = \{ f \in 2^\omega : \forall k \geq |t^\alpha_{\eta_{\alpha}}| f|k| \in T^\alpha_{\eta_{\alpha}} \}$. Then $M_\alpha$ is closed nowhere dense. Also let $G_D = \{ f \in 2^\omega : \exists^{\infty} k f|k| \in D \}$, the $G_\delta$-closure of $D$. $G_D$ is a dense $G_\delta$-set. By (**) we immediately get

$$G_D \cap M_\alpha = \emptyset.$$  

(****)

For $t \in 2^{<\omega}$, let $D^t = \{ s \in 2^{<\omega} : t \subseteq s \in D \}$. So $D^t \in \text{Dense}(\mathbb{Q})$ and $D^\emptyset = D$. Clearly (*) and (****) imply the corresponding results with respect to $t^\alpha_{\eta_{\alpha}}$,

$$|D^\alpha_{\eta_{\alpha}} \cap T_\alpha| < \aleph_0$$

(++)

and

$$G_{D^\alpha_{\eta_{\alpha}}} \cap N_\alpha = \emptyset.$$  

Let $H = \bigcap_t G_{D^t}$. This is a countable intersection of dense $G_\delta$-sets and thus still a dense $G_\delta$. By (++) we see

$$H \cap N_\alpha = \emptyset$$

for all $\alpha < \text{add}(\mathcal{M})$. Thus $\bigcup_\alpha N_\alpha$ is contained in a meager $F_\sigma$-set, a contradiction. This finishes the proof of the lemma. \hfill \Box

**Lemma 1.2.** $s_\mathbb{Q} \leq \text{add}(\mathcal{M})$.

**Proof.** We use the characterization of Lemma 1.1.

Given a dedi sequence $F = \langle F_n : n \in \omega \rangle$ we can easily find a co-infinite $A \in [\omega]^{\omega}$ such that for all $s \in 2^{<\omega}$ there is $n \in A$ with $F_n \subseteq \{ t : s \subseteq t \}$ and $n \in \omega \setminus A$ with $F_n \subseteq \{ t : s \subseteq t \}$. Let $D_0 = \bigcup_{n \in A} F_n$ and $D_1 = \bigcup_{n \in \omega \setminus A} F_n$. Clearly $D_0$ and $D_1$ are pairwise disjoint dense subsets of $\mathbb{Q}$.

Let $\mathcal{F} = \{ F^\alpha = \langle F^\alpha_n : n \in \omega \rangle : \alpha < \text{add}(\mathcal{M}) \}$ be a family of dedi sequences which are a witness for $\text{add}(\mathcal{M})$ according to Lemma 1.1. Let $A^\alpha$, $D^\alpha_0$, $D^\alpha_1$ be as in the previous paragraph. We claim that $\{ D^\alpha_0 : \alpha < \text{add}(\mathcal{M}) \}$ is a $\mathbb{Q}$-splitting family.

For indeed, let $D \in \text{Dense}(\mathbb{Q})$. By assumption there is $\alpha < \text{add}(\mathcal{M})$ such that $D \cap F^\alpha_n$ is infinite for all $n$. Fix $s \in 2^{<\omega}$. There is $n \in A^\alpha$ such that $F^\alpha_n \subseteq \{ t : s \subseteq t \}$. Thus there is $t \supseteq s$ with $t \in D^\alpha_0 \cap D$. An analogous argument shows that given $s \in 2^{<\omega}$ there is $t \supseteq s$ with $t \in D^\alpha_1 \cap D$. Thus $D \cap D^\alpha_0$ and $D \cap D^\alpha_1$ are both dense. A fortiori, $D \setminus D^\alpha_0$ is dense as well. Thus $D^\alpha_0 \mathbb{Q}$-splits $D$. This proves the lemma. \hfill \Box

**Lemma 1.3.** $s_\mathbb{Q} \leq s_0$.

**Proof.** Cichoń [11] has proved $s_\mathbb{Q} \leq s_{\mathcal{N}_0}$ where $s_{\mathcal{N}_0}$ is the $\mathcal{N}_0$-splitting number (the size of the least $\mathcal{F} \subseteq [\omega]^{\omega}$ such that for every countable $A \subseteq [\omega]^{\omega}$ there is $F \in \mathcal{F}$ splitting all members of $A$). A result of Kamburelis [16] says that $\min(\text{cov}(\mathcal{M}), s_{\mathcal{N}_0}) \leq s$ (see also [9] for this and related results). By Lemma 1.2, $s_\mathbb{Q} \leq \text{add}(\mathcal{M}) \leq \text{cov}(\mathcal{M})$. Therefore $s_\mathbb{Q} \leq \min(\text{cov}(\mathcal{M}), s_{\mathcal{N}_0}) \leq s$. \hfill \Box

Dualizing $\tau_\mathbb{Q} \geq \max(\text{cof}(\mathcal{M}), \tau)$ from [1, Theorem 3.6], we proved

**Theorem 1.4.** $s_\mathbb{Q} \leq \text{min}(\text{add}(\mathcal{M}), s)$.

Our result also improves $h_\mathbb{Q} \leq \text{add}(\mathcal{M})$ of [1, Theorem 3.2].

As a consequence we also obtain

**Corollary 1.5.** $s_\mathbb{Q} < h$ is consistent. A fortiori, $s_\mathbb{Q} < s$ is consistent.

Indeed, this holds in the Mathias model which satisfies $\text{add}(\mathcal{M}) < h$.

*Added to the revised version.* By recent, still unpublished, work of James Hirschorn, there is an alternative proof of Lemma 1.2: let $s_\mathcal{C}$ denote the value of $s$ after adding one Cohen real, i.e., $s_\mathcal{C}$ is the (unique) cardinal $\kappa$ such that $\mathbb{C}_* \kappa = \kappa$. Hirschorn (private communication) has shown $s_\mathbb{Q} = s_\mathcal{C}$. On the other hand, Cichoń and Pawlikowski [13] proved that $\text{non}(\mathcal{M})_\mathcal{C} = \text{add}(\mathcal{M})_\mathcal{C} = \text{add}(\mathcal{M})$ (see also [3, Theorem 3.3.23]). Since $s \leq \text{non}(\mathcal{M})$ holds in $ZFC$, $s_\mathcal{C} \leq \text{add}(\mathcal{M})$ follows. A fortiori, $s_\mathbb{Q} \leq \text{add}(\mathcal{M})$. 


2. Matrices of iterations

Let $\mathcal{F} \subseteq [\omega]^{\omega}$ be a filter. Laver forcing $L_\mathcal{F}$ with respect to the filter $\mathcal{F}$ consists of all trees $T \subseteq \omega^{<\omega}$ such that for all $\sigma \supseteq \text{stem}(T)$ belonging to $T$, the set of all immediate successors $\text{succ}_T(\sigma) = \{n : \sigma \upharpoonright n \in T\}$ belongs to $\mathcal{F}$. We order $L_\mathcal{F}$ by inclusion. $L_\mathcal{F}$ is a $\sigma$-centered forcing notion which adds a dominating real $\ell_\mathcal{F}$ which diagonalizes the filter $\mathcal{F}$ (that is, the range of $\ell_\mathcal{F}$ is almost contained in all members of $\mathcal{F}$). For a tree $T \in L_\mathcal{F}$ and $\sigma \in T$ with $\text{stem}(T) \subseteq \sigma$, let $T_\sigma = \{\tau \in T : \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}$ be the subtree of $T$ determined by $\sigma$.

$\mathcal{F} \subseteq [\mathcal{Q}]^{\omega}$ is a maximal $\mathcal{Q}$-filter if $\mathcal{F} \subseteq \text{Dense}(\mathcal{Q})$ is a filter and it is maximal. The latter means that for all $A \in \text{Dense}(\mathcal{Q})$ which do not belong to $\mathcal{F}$ there is $B \in \mathcal{F}$ such that $A \cap B$ is not dense. In particular, if $A \in \mathcal{F}$ and $A \subseteq \text{ndd}$, then $B \in \mathcal{F}$. Maximal $\mathcal{Q}$-filters are the analogue of ultrafilters for the structure $(\text{Dense}(\mathcal{Q}), \subseteq_{\text{ndd}})$, but note that unlike for ultrafilters, $A \notin \mathcal{F}$ does not imply $\mathcal{Q} \setminus A \in \mathcal{F}$ for maximal $\mathcal{Q}$-filters $\mathcal{F}$, even if both sets are dense.

Laver forcing with maximal $\mathcal{Q}$-filters plays a prominent role in our work. We shall use it both for the consistency of $\text{h}_\mathcal{Q} < \text{a}_\mathcal{Q}$ in this section as well as for the consistency of $\text{h} < \text{h}_\mathcal{Q}$ in the two subsequent sections. If $\mathcal{F}$ is a maximal $\mathcal{Q}$-filter, the range of $\ell_\mathcal{F}$ is dense.

The following lemma has been proved for ultrafilters by Shelah [21].

**Lemma 2.1.** Assume $V \subseteq W$ are models of ZFC, $\mathcal{F} \in V$ and $\mathcal{G} \in W$ are filters with $\mathcal{F} \subseteq \mathcal{G}$. Then the following are equivalent.

(i) Any open dense subset of $L_\mathcal{F}$ belonging to the model $V$ is predense in $L_\mathcal{G}$ in $W$.

(ii) Any $\mathcal{F}$-positive set belonging to $V$ is still $\mathcal{G}$-positive in $W$.

**Proof.** (i) $\implies$ (ii). Assume there is $A \in V$ which is $\mathcal{F}$-positive such that $\omega \setminus A$ belongs to $\mathcal{G}$ in $W$. In $V$ consider the collection $D$ of all $T \in L_\mathcal{F}$ such that $\text{stem}(T)(n) \in A$ for some $n < |\text{stem}(T)|$. This set is clearly open. It is dense because given any $T \in L_\mathcal{F}$ there is $m \in A$ such that $\sigma \supseteq m \in T$. Then $T_{\sigma \upharpoonright m} \in D$ and $T_{\sigma \upharpoonright m} \leq T$. On the other hand, if $T \in L_\mathcal{G}$ is defined by $\text{stem}(T) = \emptyset$ and $\text{succ}_T(\sigma) = \omega \setminus A$ for all $\sigma \in T$, then $T$ is incompatible with all members of $D$. Thus $D$ is not predense in $W$.

(ii) $\implies$ (i). This is a rank argument. Let $D \subseteq L_\mathcal{F}$ be open dense, $D \in V$. Define $\text{rk} : \omega^{<\omega} \to \omega$ on by recursion on the ordinals.

- $\text{rk}(\sigma) = 0$ iff there is $T \in D$ with $\text{stem}(T) = \sigma$.
- For $\alpha > 0$, $\text{rk}(\sigma) = \alpha$ iff $\text{rk}(\sigma) = n$ is less than $\alpha$ and $\{n : \text{rk}(\sigma \upharpoonright n) < \alpha\}$ is positive modulo $\mathcal{F}$.

Since the second clause is the same for all rank functions, we shall omit it in future. First notice that $\text{rk}(\sigma)$ is defined (and thus less than $\omega_1$) for all $\sigma \in \omega^{<\omega}$. Indeed, suppose $\text{rk}(\sigma)$ was undefined. Using the definition of $\text{rk}$, we could then define by recursion a tree $T \in L_\mathcal{F}$ with $\text{stem}(T) = \sigma$ and such that $\text{rk}(\tau)$ is undefined for all $\tau \in T$ extending $\sigma$. Since $D$ is open dense, there is $S \in D$ below $T$. Thus $\text{rk}(\text{stem}(S)) = 0$ by definition on $\text{rk}$, a contradiction.

We need to check $D$ is predense in $L_\mathcal{G}$ in $W$. So fix $T \in L_\mathcal{G}$, and let $\sigma = \text{stem}(T)$. By induction of $\text{rk}(\sigma)$, we prove that there is $S \in D$ compatible with $T$. If $\text{rk}(\sigma) = 0$, then there is $S \in D$ with $\text{stem}(S) = \sigma$, and $S$ and $T$ are clearly compatible. If $\text{rk}(\sigma) > 0$, consider $A = \{n : \text{rk}(\sigma \upharpoonright n) < \text{rk}(\sigma)\} \in V$. By definition of $\text{rk}$, $A$ is $\mathcal{F}$-positive in $V$. By assumption $A$ is still $\mathcal{G}$-positive in $W$. Thus there is $n \in A$ such that $\sigma \upharpoonright n \in T$. Now, $T_{\sigma \upharpoonright n}$ has stem $\sigma \upharpoonright n$ which has rank $< \text{rk}(\sigma)$. Thus, by the inductive hypothesis, there is $S \in D$ compatible with $T_{\sigma \upharpoonright n}$. Since $T_{\sigma \upharpoonright n}$ is a subtree of $T$, the latter is also compatible with $S$, and we are done. □

We observe that if $\mathcal{F}$ is a maximal $\mathcal{Q}$-filter in $V$ and $\mathcal{G} \supseteq \mathcal{F}$ is a maximal $\mathcal{Q}$-filter in $W$, then every $\mathcal{F}$-positive set from $V$ is still $\mathcal{G}$-positive in $W$. Indeed, if $A \in V$ is positive modulo $\mathcal{F}$, then $\mathcal{Q} \setminus A \notin \mathcal{F}$ which means by maximality of $\mathcal{F}$ that there is $B \in \mathcal{F}$ such that $(\mathcal{Q} \setminus A) \cap B \notin \text{Dense}(\mathcal{Q})$. Since $\mathcal{F} \subseteq \mathcal{G}$, we get $B \in \mathcal{G}$, and $\mathcal{Q} \setminus A$ cannot belong to $\mathcal{G}$ either. So $A$ is positive modulo $\mathcal{G}$.

In particular, the above lemma holds for maximal $\mathcal{Q}$-filters.

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Let $\kappa = \kappa^{\aleph_0}$ be a regular cardinal. Build finite support iterations $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}^{\alpha}_\kappa : \alpha < \kappa)$ and $(\mathbb{P}^\gamma_{\alpha}, \dot{\mathbb{Q}}^{\gamma}_{\alpha} : \alpha < \kappa)$ for $\gamma < \omega_1$ such that

(i) $\mathbb{P}^\gamma_\kappa \prec \mathbb{P}_\kappa \prec \mathbb{P}_{\alpha}$ for $\gamma < \delta$, and $\mathbb{P}_\alpha$ is the direct limit of the $\mathbb{P}^\gamma_\alpha$. 

We produce the $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta'$ by recursion on $\alpha$.

For $\alpha = 0$, let $\mathbb{P}_0$ be $\mathbb{C}_{\omega_1}$, the forcing which adds $\omega_1$ Cohen reals, and let $\mathbb{P}_0'$ be $\mathbb{C}_\gamma$, which adds the first $\gamma$ Cohen reals. Then (i) and (iv) hold trivially, and (ii), (iii) and (v) do not apply.

If $\alpha = \beta + 1$ is a successor, we assume $\mathbb{P}_\beta$ and the $\mathbb{P}_\beta'$ have been produced as required. Let $G_\beta$ be $\mathbb{P}_\beta$-generic over $V$ and $V_\beta = V[G_\beta]$. By (i) for $\beta$, $G_\beta' = G_\beta \cap \mathbb{P}_\beta'$ is $\mathbb{P}_\beta$-generic over $V$. Let $V_\beta' = V[G_\beta']$. By recursion on $\gamma$ build a maximal $\mathbb{Q}$-filter $\mathcal{F}_\beta'$ in the model $V_\beta'$ such that $\mathcal{F}_\beta' \subseteq \mathcal{F}_\beta$ for $\gamma < \delta$. Let $\mathcal{F}_\beta = \bigcup \mathcal{F}_\beta'$ for $\gamma < \delta$. Let $\mathcal{F}_\beta = \bigcup \mathcal{F}_\beta'$ for $\gamma < \delta$. Let $\mathcal{F}_\beta' = \bigcup \mathcal{F}_\beta'$ for $\gamma < \delta$. Let $\mathcal{F}_\beta' = \bigcup \mathcal{F}_\beta'$ for $\gamma < \delta$. Let $\mathcal{F}_\beta' = \bigcup \mathcal{F}_\beta'$ for $\gamma < \delta$. Let $\mathcal{F}_\beta' = \bigcup \mathcal{F}_\beta'$ for $\gamma < \delta$. Thus there is $S = \check{S}[G_\beta'] \in \mathbb{P}_\beta$ compatible with $T = \check{T}[G_\beta']$ with common extension $U \in \mathbb{Q}_\beta'$. Hence we find $r \in G_\beta'$ forcing $\check{U} \leq \check{S}, \check{T}$. Let $g \in G_\beta'$ be such that $(q, S) \in D$. Without loss of generality, $r \leq p, q$. This means, however, that $(r, U) \leq (p, \check{T}), (q, S)$, as required. Thus we get complete embeddability.

To see that $\mathbb{P}_\alpha$ is the direct limit of the $\mathbb{P}_\alpha'$, take $(\alpha, \check{T}) \in \mathbb{P}_\alpha$. By the inductive hypothesis for $\beta$, there is $\check{\delta}$ such that $p \in \mathbb{P}_\check{\delta}$. Since $\check{Q}_\beta$ is forced to be the union of the $\check{Q}_\beta'$, there is $\gamma \geq \check{\delta}$ such that $\check{T}$ is a $\check{Q}_\gamma'$-name. Thus $(p, \check{T}) \in \mathbb{P}_\alpha'$.

For (v), work in $V_\beta$, and fix $\delta < \gamma$ and $x \in 2^\omega$ from $V_\beta \setminus V_\beta'$. Let $\check{x} \in V_\beta'$ be a $\check{Q}_\beta'$-name for a real. We need to show $\check{x}$ is forced to be distinct from $x$. In $V_\beta'$, for each $\sigma \in \omega^{<\omega}$ find $x_\sigma \in 2^\omega$ such that for all $n$, no $T \in \mathbb{Q}_\beta'$ with stem $\sigma$ forces $\check{x} \upharpoonright n \neq x_\sigma \upharpoonright n$. This is clearly possible because the conditions with stem $\sigma$ form a centered set. Also note that this still holds for $T' \in \mathbb{Q}_\beta'$ with stem $\sigma$ in the larger model $V_\beta''$.

(Indeed, in $V_\beta''$, we can define the rank function $r_\alpha$ for $\tau \supseteq \sigma$ by recursion on the ordinals by
\begin{itemize}
  \item $r_\alpha(\tau) = 0$ if there is $T \in \mathbb{Q}_\alpha'$ with stem $\tau$ forcing $\check{x} \upharpoonright n = x_\sigma \upharpoonright n$.
\end{itemize}

A standard rank argument as in Lemma 2.1 then shows that $r_\alpha(\sigma) < \infty$. Stepping back into $V_\beta''$, induction on rank $\sigma$ provides a condition stronger than the given $T'$ which forces $\check{x} \upharpoonright n = x_\sigma \upharpoonright n$, as required.)

By the induction hypothesis, $x \neq x_\sigma$ for all $\sigma \in \omega^{<\omega}$. Assume there was $T' \in \mathbb{Q}_\beta'$ such that $T' \Vdash \check{x} = x$. Let $\sigma = \text{stem}(T')$. Find $n$ such that $x \upharpoonright n \neq x_\sigma \upharpoonright n$. By the above there is $S' \leq T'$ in $\mathbb{Q}_\beta'$ forcing $\check{x} \upharpoonright n = x_\sigma \upharpoonright n$. This is a contradiction.

Finally, (iv) is immediate from the induction hypothesis, (v), and the fact that $\mathbb{P}_\alpha$ is ccc and the direct limit of the $\mathbb{P}_\alpha'$.

If $\alpha$ is a limit ordinal, (ii) and (iii) do not apply and (i) is in fact well-known [8]. We include the argument. Let $\gamma < \delta$ and assume $D \subseteq \mathbb{P}_\alpha'$ is predense. Fix $p \in \mathbb{P}_\alpha$. Since we are dealing with finite support iterations, there is $\beta < \alpha$ such that $p \in \mathbb{P}_\beta$. Since $D/\beta = \{q \upharpoonright \gamma : q \in D\}$ is predense in $\mathbb{P}_\beta$, the induction hypothesis for $\beta$ gives us $q \in D$ such that $p$ and $q/\beta'$ are compatible with common extension $r_0 \in \mathbb{P}_\beta$. Then $r = r_0 \check{q} \upharpoonright [\beta, \alpha)$ is a common extension of $p$ and $q$ in $\mathbb{P}_\beta$. That $\mathbb{P}_\alpha$ is the direct limit of the $\mathbb{P}_\alpha'$ follows trivially.

(v) is trivial unless $c(\alpha) = \omega$. Proceed as in the successor case, fix $\beta < \alpha$, $\delta < \gamma$ and $x \in 2^\omega$ from $V_\beta \setminus V_\beta'$. Work in $V_\beta$. Let $\check{x} \in V_\beta'$ be a $\check{P}_{[\beta, \alpha]}$-name for a real. Find $\alpha_k$ strictly increasing such that $\alpha_0 = 0$ and $\bigcup \alpha_k = \alpha$. In $V_{\alpha_k}$ find $x_k$ such that for all $n$, the trivial condition of $\mathbb{P}_{\alpha_k}$ does not force $\check{x} \upharpoonright n \neq x_k \upharpoonright n$. By the induction hypothesis we know that $x \neq x_k$ for all $k$. Assume there was $p \in \mathbb{P}_{[\beta, \alpha]}$ forcing $\check{x} = x$. Find $k$ such that $p \in \mathbb{P}_{[\beta, \alpha]}$. Step into $V_{\alpha_k}$ such that $p$ belongs to the generic $G_{[\beta, \alpha]}$. Find $n$ such that $x \upharpoonright n \neq x_k \upharpoonright n$. Then there is $q \in \mathbb{P}_{[\alpha_k, \alpha]}$ forcing $\check{x} \upharpoonright n = x_k \upharpoonright n$, a contradiction. (iv) now follows trivially. This completes the recursive construction.

We have done most of the work towards proving
Theorem 3.1. Let $\kappa = \kappa^{\aleph_0}$ be regular. Then it is consistent that $s_\mathbb{Q} = c = \kappa$ and $h_\mathbb{Q} = \aleph_1$.

The proof uses the iteration expounded above. A standard counting-of-names argument shows $c \leq \kappa$. To see $s_\mathbb{Q} \geq \kappa$, it suffices to argue that for a maximal $\mathbb{Q}$-filter $\mathcal{F}$, $\text{ran}(\ell_\mathcal{F}) \in \text{Dense}(\mathbb{Q})$ is not $\mathbb{Q}$-split by any ground model dense set: let $A \in \text{Dense}(\mathbb{Q})$. If $A \in \mathcal{F}$, then $\text{ran}(\ell_\mathcal{F}) \subseteq^* A$ and we are done. If $A \notin \mathcal{F}$, then by maximality of $\mathcal{F}$ there is $B \in \mathcal{F}$ such that $A \cap B$ is not dense. Since $\text{ran}(\ell_\mathcal{F}) \subseteq^* B$, $A \cap \text{ran}(\ell_\mathcal{F})$ is not dense either.

Finally, the proof of $h_\mathbb{Q} \leq \aleph_1$ is a standard argument which has been used for $h$ and its relatives like $g$ and $\text{cof}(\text{Sym}(\omega))$ a number of times (see, e.g., [6] and [25]). Recall that, by construction (iv), $V_\kappa \cap 2^{\omega_1} = \bigcup_{\gamma} (V_\kappa^\gamma \cap 2^{\omega_1})$ where $V_\kappa$ ($V_\kappa^\gamma$, respectively) is the forcing extension by $P_\kappa$ ($P_\kappa^\gamma$, resp.). Also, for $\gamma < \delta$, $V_\kappa^\gamma \cap 2^{\omega_1}$ is a proper subset of $V_\kappa^\delta \cap 2^{\omega_1}$ by (v) above. Let $D^\gamma$ be the collection of all $A \in \text{Dense}(\mathbb{Q})$ such that there is no dense $B \subseteq_{\text{nwd}} A$ which belongs to $V_\kappa^\gamma$. $D^\gamma$ is clearly open. It is also $\mathbb{Q}$-dense because of (v) and

**Lemma 2.3.** Assume $V \subseteq W$ are models of ZFC and $W$ contains reals which do not belong to $V$. Given a dense $C \in V$, there is a dense $A \subseteq C$ in $W$ such that no dense $B \subseteq_{\text{nwd}} A$ belongs to $V$.

**Proof.** Let $U_\eta$ enumerate all basic open subsets of $\mathbb{Q}$. In $V$ choose distinct $c_\sigma \in C \cap U_{|\sigma|}$ for $\sigma \in 2^{<\omega}$. Then, for any $f \in 2^{<\omega}$, $A_f = \{c_{f|n} : n \in \omega\} \subseteq C$ is dense. Let $f \in 2^{<\omega}$ be a real from $W \setminus V$. We need to check $A = A_f$ works. Assume $B \subseteq_{\text{nwd}} A$ is dense. Then there is a basic open set $U$ such that $B \cap U \subseteq A$. Consider $X = \{n : U_n \subseteq U\}$. Then $X \in V$, and for any $n \in X$, $B \cap \{c_\sigma : |\sigma| = n\} \subseteq A \cap \{c_\sigma : |\sigma| = n\} = \{c_{f|n}\}$. Furthermore, by density of $B$, there are infinitely many $n \in X$ such that $B \cap \{c_\sigma : |\sigma| = n\}$ is non-empty. This means that $f$ can be reconstructed from $B$. Therefore $B \notin V$. \[\square\]

On the other hand, (iv) gives $\bigcap_{\gamma} D^\gamma = \emptyset$. Therefore $h_\mathbb{Q} \leq \aleph_1$. This completes the proof of Theorem 2.2.

A similar argument shows $h = \aleph_1$ in our model. In particular, we get an alternative proof of the consistency of $h < \aleph_1$ in our model. A result originally obtained by Shelah [19] (see also [20, Theorem VI.8.2]).

3. The framework of the consistency of $h < h_\mathbb{Q}$

This section presents an outline of the proof of

**Theorem 3.1.** It is consistent that $h = \aleph_1$ and $h_\mathbb{Q} = c = \aleph_2$.

The main technical result, the preservation of $(\ast_\alpha)$ (defined below) in the successor step $\alpha$, will be given in Section 4.

Assume $CH$ and $\diamondsuit^\omega_{\aleph_1}$ hold in the ground model. The latter means there is a sequence $\{Z_\alpha : cf(\alpha) = \omega_1$ and $\alpha < \omega_2\}$ such that for all $Z \subseteq \omega_2$, the set $\{\alpha < \omega_2 : cf(\alpha) = \omega_1$ and $Z \cap \alpha = Z_\alpha\}$ is stationary. We shall perform a finite support iteration $(P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2)$ such that

(A) if $cf(\alpha) = \omega_1$, then $\dot{Q}_\alpha$ is Laver forcing $\mathbb{L}_{\mathcal{F}_\alpha}$ with a maximal $\mathbb{Q}$-filter $\mathcal{F}_\alpha$,

(B) if $\alpha$ is an even successor ordinal, then $\dot{Q}_\alpha$ is either trivial or Laver forcing $\mathbb{L}_{\mathcal{F}_\alpha}$ with a Ramsey ultrafilter $\dot{F}_\alpha$,

(C) if $\alpha$ is an odd successor or $cf(\alpha) = \omega$ or $\alpha = 0$, then $\dot{Q}_\alpha$ is Hechler forcing $\mathbb{D}$.

The combinatorial properties the $\dot{F}_\alpha$ have to satisfy are listed and discussed at the end of this section (see Lemmata 3.7 and 3.9).

The basic idea is this: at limit stages of cofinality $\omega_1$ we use forcing (A) to kill (an initial segment of) a potential witness for $h_\mathbb{Q} = \aleph_1$. $\diamondsuit^\omega_{\aleph_1}$ guarantees that (an initial segment of) every witness is guessed at some point where it will be destroyed.

Forcing (B) is designed to build up the mad families $A_\beta$, $\beta < \omega_1$, which will witness $h = \aleph_1$. For $\beta < \gamma$, $A_\gamma$ refines $A_\beta$. This means that any member of $A_\gamma$ is almost contained in a member of $A_\beta$. At each even successor stage $\alpha$, at most one of the $A_\beta$ will get a new member $A_\beta^\alpha$ which is generically added by $Q_\alpha$. Put $A^\leq_\beta := A_\beta \cap V_\alpha \in V_\alpha$ where $V_\alpha$ denotes the generic extension via $P_\alpha$. Notice that $A^\leq_\beta = \bigcup_{\gamma < \alpha} A^\leq_\beta^\gamma$ for limit ordinals $\alpha$, and that $A^\leq_\beta^\alpha = A^\leq_{\beta+1}$ unless $\alpha$ is an even successor ordinal. By a standard book-keeping argument, we guarantee that

(I) given $\alpha_0 < \omega_2$, $\beta_0 < \omega_1$ and $X_0 \in [\omega_0]^{<\omega} \cap V_{\alpha_0}$, there are an even successor $\alpha \geq \alpha_0$ and $Y_0 \in [X_0]^{<\omega} \cap V_\alpha$ which is almost contained in members of $A^\leq_\gamma$ for all $\gamma < \beta_0$ such that either $Y_0 \cap A$ is infinite for some $A \in A^\leq_{\beta_0}$ or $A^\leq_{\beta_0}$ is a generic subset of $Y_0$. 

Forcing (C) has auxiliary character and is intended to facilitate the proof of \((\dagger_\omega)\) (defined below) for even successors \(\alpha\). The main point is Lemma 3.5. For technical purposes (see Lemma 4.7 below), we think of Hechler forcing \(\mathbb{D}\) as forcing with \(\mathbb{L}_\mathcal{F}\) where \(\mathcal{F}\) is the Fréchet filter (the filter of cofinite sets). This is not the standard representation of \(\mathbb{D}\).

It is unclear whether (B) and (C) are needed. It may well be that the \(A_\beta\) can be built up without forcing; using \(CH\) this would be the case if at each stage we had to meet at most countably many requirements; however, in our construction, \(\mathbb{N}_1\) many requirements must be met at each stage, and this is why we use forcing (B).

\[\text{\textbullet \textbullet \textbullet} \]

For an almost disjoint family \(\mathcal{A}\), let \(\mathcal{I}(\mathcal{A})\) denote the ideal generated by \(\mathcal{A}\), that is, the collection of all \(X \in [\omega]^{\omega}\) such that there is \(Y \subseteq \mathcal{A}\) finite with \(X \subseteq \bigcup Y\). Say \(X\) is fat with respect to \(\mathcal{A}\) if \(X \cap A\) is infinite for infinitely many \(A \in \mathcal{A}\). If \(X\) is fat, then \(X \notin \mathcal{I}(\mathcal{A})\), and, if \(\mathcal{A}\) is a mad family, the converse holds.

The main property we want to preserve is that for all \(\alpha\),

\[
\forall X \in V_\alpha \cap [\omega]^{\omega} \exists \beta < \omega_1 \quad X \notin \mathcal{I}(\mathcal{A}_\beta) . \\
\tag{+\alpha}
\]

Clearly this is sufficient to guarantee \(\mathfrak{b} = \mathfrak{N}_1\) (Corollary 4.9).

The proof of \((+\alpha)\) comes in two stages. First we need to show that if a new \(X\) arises at stage \(\alpha\), there is some \(\beta\) such that it does not belong to \(\mathcal{I}(\mathcal{A}_\beta)\). Then we have to construct later \(A \in A_\beta\) in such a way that this property is preserved. This means we want: for all \(\alpha\),

\[
\forall X \in V_\alpha \cap [\omega]^{\omega} \exists \beta < \omega_1 \quad X \notin \mathcal{I}(\mathcal{A}_\beta) \\
\tag{\ast\alpha}
\]

and: for all \(\alpha\),

\[
\forall X \in V_\alpha \cap [\omega]^{\omega} \forall \beta < \omega_1 : \text{if } X \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha}) \text{ then } X \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha+1}) \tag{\ast\ast\alpha+1}
\]

By our set-up, \((\ast\ast\alpha+1)\) only needs proof if \(\alpha\) is an even successor ordinal. This will be built into the construction of the \(\mathcal{F}_\alpha\), see Lemma 3.3 and Corollary 3.10.

Obviously, \((\ast\alpha)\) and \((\ast\ast\alpha+1)\) for \(\gamma \geq \alpha\) together imply that \((+\alpha)\) holds. We also get the preservation of \((\ast\alpha)\) in limit stages of the iteration:

**Lemma 3.2.** Suppose \(\alpha < \omega_2\) is a limit ordinal and both \((\ast\gamma)\) and \((\ast\ast\gamma+1)\) hold for \(\gamma < \alpha\). Then \((\ast\alpha)\) holds.

**Proof.** Consider first the case \(cf(\alpha) = \omega_1\), and let \(X \in V_\alpha \cap [\omega]^{\omega}\). By ccc-ness, there is \(\gamma < \alpha\) such that \(X \in V_\gamma \cap [\omega]^{\omega}\). Since \((\ast\gamma)\) holds, we find \(\beta\) such that \(X \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \gamma})\). By \((\ast\ast\delta+1)\) for all \(\gamma \leq \delta < \alpha\), we see that \(X \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha})\), as required.

If \(\alpha\) has countable cofinality, fix strictly increasing \(\{\alpha_n : n \in \omega\}\) with \(\bigcup_n \alpha_n = \alpha\). Let \(\check{X} \in [\omega]^{\omega}\) be a \(\mathbb{P}_\alpha\)-name. For each \(n\) find a \(\mathbb{P}_\alpha\)-name \(\check{X}_n \in [\omega]^{\omega}\) such that, in \(V^{\mathbb{P}_{\alpha_n}}\), for all \(k\), the trivial condition of \(\check{1}_{[\alpha_n, \alpha]}\) does not force \(\check{X} \cap k \neq \check{X}_n \cap k\). By \((\ast\alpha_n)\) and ccc-ness, find \(\beta_n\) in \(V\) such that \(\check{X}_n\) is forced not to belong to \(\mathcal{I}(\mathcal{A}^{<\alpha}_\beta)\). Let \(\beta = \bigcup_n \beta_n\).

We claim that \(\check{X}\) is forced not to belong to \(\mathcal{I}(\mathcal{A}^{<\alpha}_\beta) = \bigcup_n \mathcal{I}(\mathcal{A}^{<\alpha}_\beta)\).

Let \(p \in \mathbb{P}_\alpha\), \(k_0, \ell \in \omega\) and \(\check{A}_i \in \mathcal{A}^{<\alpha}_\beta\), \(i < \ell\). We need to find \(q \leq p\) and \(k \geq k_0\) such that \(q\) forces \(k \in \check{X} \setminus \bigcup_i \check{A}_i\).

Find \(n\) such that \(p \in \mathbb{P}_{\alpha_n}\) and \(A_i = \check{A}_i[G_{\alpha_n}] \in \mathcal{A}^{<\alpha}_\beta\) where \(G_{\alpha_n}\) is \(\mathbb{P}_{\alpha_n}\)-generic containing \(p\). Work in \(V_{\alpha_n} = V[G_{\alpha_n}]\).

By construction, \(X_n = \dot{X}_n[G_{\alpha_n}] \notin \mathcal{I}(\mathcal{A}^{<\alpha}_\beta)\). Thus there is \(k \geq k_0\) belonging to \(X_n \setminus \bigcup_i A_i\). By construction, there is \(q \in \mathbb{P}_{[\alpha_n, \alpha]}\) forcing \(k \in \check{X}\).

This completes the proof of the preservation of \((\ast\alpha)\) in the limit step \(\alpha\). \(\square\)

The crux of the argument is the preservation of \((\ast\ast\alpha+1)\) in the successor step \(\alpha + 1\). This will be done in Section 4. If \(cf(\alpha) = \omega_1\), this will be particularly cumbersome (see Lemmata 3.7 and 4.3).\(^3\) For this purpose we need an auxiliary property, namely: for all \(\alpha\) which are either of cofinality \(\omega_1\) or even successors,

\[
\forall \alpha \text{ open } U \neq \emptyset \quad \forall \text{ dense } D \subseteq U \quad \exists \text{ one-to-one } f : D \to \omega \text{ all from } V_\alpha \\
\exists \beta < \omega_1 \quad \exists \text{ open } \emptyset \neq U' \subseteq U \quad \exists \text{ dense } D' \subseteq U' \cap D \\
\forall A \in \mathcal{A}^{<\alpha}_\beta \quad f^{-1}(A) \cap D' \text{ is nowhere dense.} \\
\tag{\dagger_\alpha}
\]

\(^3\) For even successors \(\alpha\), this will be easier, basically because of the Hechler reals added in the preceding stage (see Lemmata 3.9 and 4.5).
Notice that (†α) is stronger than (∗α): indeed, given \( X \in V_\alpha \cap [\omega]^{\omega} \), let \( f : \mathbb{Q} \to X \) be a bijection. If \( \beta \) is as stipulated by (†α), then \( X \not\in I(\mathcal{F}_\beta^{\leq \alpha}) \) follows. We shall see below (Corollary 3.6) under which circumstances the two properties are equivalent.

It may sound weird to require (†α) also for even successors. The reason for this is that if we also guarantee (††α+1) below, then we get (†α) for free in the limit stages of cofinality \( \omega_1 \).

\[
\forall \text{ open } U \neq \emptyset \ \forall \text{ dense } D \subseteq U \ \forall \text{ one-to-one } f : D \to \omega \text{ all from } V_\alpha
\]

\[
\forall \beta < \omega_1 \ \ f^{-1}(A_\beta^\alpha) \text{ is nowhere dense (if } A_\beta^\alpha \text{ is defined)} \quad (\dagger \dagger \alpha+1)
\]

Again, (††α+1) is stronger than (∗∗α+1):

**Lemma 3.3.** Suppose (††α+1) holds. Then (∗∗α+1) holds as well.

**Proof.** Let \( X \in V_\alpha \cap [\omega]^{\omega} \) and \( \beta < \omega_1 \) such that \( X \not\in I(\mathcal{F}_\beta^{\leq \alpha}) \). If \( X \) is fat with respect to \( \mathcal{F}_\beta^{\leq \alpha} \), it is still fat with respect to \( \mathcal{F}_\beta^{< \alpha} \) and, consequently, \( X \not\in I(\mathcal{F}_\beta^{< \alpha}) \). So assume \( X \) is not fat with respect to \( \mathcal{F}_\beta^{< \alpha} \). Choose \( Y \subseteq X \) in \( V_\alpha \) almost disjoint from all members of \( \mathcal{F}_\beta^{< \alpha} \) and a bijection \( f : \mathbb{Q} \to Y \). (††α+1) immediately implies \( Y \not\in I(\mathcal{F}_\beta^{< \alpha}) \) and, consequently, \( X \not\in I(\mathcal{F}_\beta^{< \alpha}) \). \( \square \)

**Lemma 3.4.** Suppose \( \alpha < \omega_2 \) is a limit ordinal of cofinality \( \omega_1 \), (†γ) holds for cofinally many \( \gamma < \alpha \), and (††γ+1) holds for all \( \gamma < \alpha \). Then (†α) holds.

**Proof.** Given \( U, D \), \( f \in V_\alpha \) as required, there is \( \gamma < \alpha \) such that \( U, D, f \in V_\gamma \). Without loss of generality, (†γ) holds. So we can find the required \( \beta, U', \) and \( D' \). For \( \delta \) with \( \gamma \leq \delta < \alpha \), apply (††δ+1) to \( U', D' \), \( f' := f|D' \) and \( \beta \). Then, indeed, \( f^{-1}(A) \cap D' \) is nowhere dense for all \( A \in \mathcal{F}_\beta^{< \alpha} \), as required. \( \square \)

As for (∗∗α+1), (††α+1) only needs proof if \( \alpha \) is an even successor ordinal. This is built into the construction of the \( \mathcal{F}_\alpha \), see Corollary 3.10 at the end of this section.

Thus it remains to be proved that (†α) holds at even successor stages \( \alpha \). This is where the Hechler reals come in. First, we prove a preliminary lemma which we shall also need for creating the \( \mathcal{F}_\alpha \) in the even successor step (see property (a) and Lemma 3.9 below).

**Lemma 3.5.** Assume \( V \subseteq W \) are models of ZFC, and \( W \) is a ccc extension of \( V \) containing a dominating real over \( V \). Also assume \( A \in V \), \( A \subseteq V \), is an almost disjoint family. Then: whenever \( \{X_n : n \in \omega\} \in W \) with all \( X_n \) being fat with respect to \( A \), there is \( \{Y_n : n \in [\omega]^{\omega} \cap n \in \omega\} \in W \) such that \( |(\bigcup_n Y_n) \cap A| < \aleph_0 \) for all \( A \in \mathcal{F}_\alpha \).

**Proof.** Let \( P \) be the forcing leading from \( V \) to \( W \). Let \( \check{X}_n \) be \( P \)-names for the \( X_n \). There are names \( \dot{A}_{mn}, m \in \omega \), for distinct members \( A \) such that \( \check{X}_n \) is forced to have infinite intersection with all \( \dot{A}_{mn} \). By ccc-ness, we may find a collection of distinct \( A_{kn}, k \in \omega \), from \( A \) which belongs to \( V \) such that each \( \dot{A}_{mn} \) is forced to be among the \( A_{kn} \). Note that, by assumption, for each \( k \), there is \( k' \geq k \) such that \( X_n \) has infinite intersection with \( A_{k'n} \).

Let \( f : \omega^2 \to \omega \) be a dominating real over \( V \) in \( W \). Let \( \prec \) be a well-ordering of \( \omega^2 \) of order type \( \omega \). Recursively define \( h : \omega \to \omega \) in \( W \) such that \( h(k) \) is the least \( k' \geq k \) such that \( X_n \cap A_{k'n} \) is infinite and \( A_{k'n} \) is distinct from all \( A_{h(k),n}, (k_0, n_0) < (k, n) \). Define \( g : \omega^2 \to \omega \) such that \( g(k, n) \) is the least \( \ell \in X_n \cap A_{h(k),n} \) larger than \( \max\{f(k, n), f(h(k), n)\} \). By construction, this is well-defined, and \( g \) clearly dominates \( f \). Also \( Y_n := \text{ran}(g(\{\omega \times \{n\}\})) \subseteq X_n \) is infinite. Put \( Y := \bigcup_n Y_n \). We need to prove that \( Y \cap A \) is finite for all \( A \in \mathcal{A} \).

Back in \( V \), for each \( A \in \mathcal{A} \), define \( f_A : \omega^2 \to \omega \) by \( f_A(k, n) = \max(A_{h(k),n} \cap A) \) if \( A \neq A_{h(k)} \); otherwise \( f_A(k, n) = 0 \).

Fix \( A \in \mathcal{A} \). Since \( f \) eventually dominates \( f_A \), we have that, for almost all \( (k, n) \), \( f_A(h(k), n) < f(h(k), n) < g(k, n) \). As \( g(k, n) \in A_{h(k),n} \), we get \( g(k, n) \notin A \) for such \((k, n)\) in case \( A \neq A_{h(k),n} \). By definition of \( h \), there is at most one \((k, n)\) such that \( A = A_{h(k),n} \). Therefore \( Y \cap A \) is finite. \( \square \)

**Corollary 3.6.** Let \( \alpha \) be an even successor ordinal and assume (∗α) holds. Then (†α) holds as well.

**Proof.** Let \( \alpha = \gamma + 1 \). We apply the lemma with \( V = V_\gamma \) and \( W = V_\alpha \). This is possible because \( V_\alpha \) is a Hechler extension of \( V_\gamma \). Recall that by construction \( \mathcal{F}_\beta^{\leq \alpha} = \mathcal{F}_\beta^{< \alpha} \) for all \( \beta \). In particular, all \( \mathcal{F}_\beta^{< \alpha} \) belong to \( V_\gamma \).

Let \( U, D, f \in V_\alpha \) be as in (†α). By (∗α), we find \( \beta \) such that for all non-empty basic open \( U_0 \subseteq U \), \( f(U_0 \cap D) \not\in I(\mathcal{F}_\beta^{< \alpha}) \). Now consider two cases.
Case 1. For some non-empty basic open $U' \subseteq U$, $f(U' \cap D)$ is not fat with respect to $\mathcal{A}_\beta^{\leq \alpha}$. Thus there are $\ell$ and $A_i \in \mathcal{A}_\beta^{\leq \alpha}, i < \ell$, such that $f(U' \cap D) \cap A$ is finite for all $A \in \mathcal{A}_\beta^{\leq \alpha} \setminus \{A_i : i < \ell\}$. Let $D' = (U' \cap D) \setminus \bigcup_i f^{-1}(A_i)$. Since $f(U_0 \cap D) \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha})$ for any non-empty basic open $U_0 \subseteq U'$, we see that $D'$ is dense in $U'$. Also $f^{-1}(A) \cap D'$ is finite for all $A \in \mathcal{A}_\beta^{\leq \alpha}$, as required.

Case 2. For all non-empty basic open $U_0 \subseteq U$, $f(U_0 \cap D)$ is fat with respect to $\mathcal{A}_\beta^{\leq \alpha}$. Put $U' = U$. Let $U_0' list the non-empty basic open subsets of $U$. By Lemma 3.5 applied to the $f(U_0' \cap D)$, we find infinite sets $D^i \subseteq U_0' \cap D$ such that $(\bigcup_i f(D^i)) \cap A$ is finite for all $A \in \mathcal{A}_\beta^{\leq \alpha}$. Put $D' := \bigcup_i D^i$. Then $D'$ is dense in $U'$ and $f^{-1}(A) \cap D'$ is finite for all $A \in \mathcal{A}_\beta^{\leq \alpha}$, and we are done. \[\square\]

**3**

In the remainder of this section, we shall explain the properties the filters $\mathcal{F}_\alpha$ ($\alpha$ either an even successor or a limit of uncountable cofinality) must satisfy. First consider the case where $\alpha$ is a limit of uncountable cofinality. Then $\mathcal{F}_\alpha$ is a maximal $\mathbb{Q}$-filter such that

(i) $\mathcal{F}_\alpha$ is a $p$-filter,

(ii) $\mathcal{F}_\alpha$ is a $q$-filter,

(iii) for all open $U \neq \emptyset$, all $F \in \mathcal{F}_\alpha$ and all one-to-one $f : F \cap U \rightarrow \omega$, there are $\beta < \omega_1, \emptyset \neq U' \subseteq U$ open and $G \in \mathcal{F}_\alpha, G \subseteq F$, such that $G \cap U' \cap f^{-1}(A)$ is nowhere dense for all $A \in \mathcal{A}_\beta^{\leq \alpha}$,

(iv) if $\mathcal{B} = \{B_\beta : \beta < \omega_1\} \in \mathcal{V}_\alpha$ is an initial segment of a potential witness for $h_\mathbb{Q} = \aleph_1$ handed down by $\Diamond_{\beta^2}$, then $\mathcal{F}_\alpha$ diagonalizes it.

A few comments about these properties are in order. In first recall that $\mathcal{F}$ is a $p$-filter if any countable $\mathcal{X} \subseteq \mathcal{F}$ has a pseudointersection belonging to $\mathcal{F}$. This means there is $G \in \mathcal{F}$ with $G \subseteq^* \mathcal{F}$ for all $\mathcal{X} \subseteq \mathcal{F}$. $\mathcal{F}$ is a $q$-filter if all partitions into finite pieces have a selector in $\mathcal{F}$ iff for all partitions $X_\alpha \in [\omega]^{<\omega} : n \in \omega$ of $\alpha$ there is $F \in \mathcal{F}$ with $|F \cap X_\alpha| \leq 1$ for all $n$. Property (iii) is needed to prove $\left( \ast_{\alpha+1} \right)$ (Lemma 4.3).

To appreciate (iv), recall that a witness for $h_\mathbb{Q} = \aleph_1$ would be a family $\mathcal{D} = \{D_\beta : \beta < \omega_1\}$ such that all $D_\beta$ are open $\mathbb{Q}$-dense, $D_\gamma \subseteq D_\beta$ for $\beta < \gamma$, and $\bigcap_\beta D_\beta = \emptyset$. By ccc-ness and $||\mathbb{P}_{\omega_2}|| = \aleph_2$, we may think of $\mathbb{P}_{\omega_2}$-names $\mathcal{D} = \{f_\beta : \beta < \omega_1\}$ of such witnesses as being coded into subsets of $\omega_2$. Let $\{Z_\alpha : cf(\alpha) = \omega_1$ and $\alpha < \omega_2\}$ be the $\Diamond_{\beta^2}$-sequence. (iv) now means that if the decoded $Z_\alpha$ is a $\mathbb{P}_\alpha$-name $\mathcal{B} = \{B_\beta : \beta < \omega_1\}$ such that the trivial condition of $\mathbb{P}_\alpha$ forces all $B_\beta$ are open $\mathbb{Q}$-dense, $B_\gamma \subseteq B_\beta$ for $\beta < \gamma$, and $\bigcap_\beta B_\beta = \emptyset$, then $\mathcal{F}_\alpha$ is forced to be a filter such that $\mathcal{F}_\alpha \cap \mathcal{B}_\beta \neq \emptyset$ for all $\beta < \omega_1$. If the decoded $Z_\alpha$ is not of this form, then condition (iv) is void.

**Lemma 3.7.** Let $\alpha$ be a limit ordinal of uncountable cofinality, and assume that $(\ast_{\alpha})$ holds. Then there is a maximal $\mathbb{Q}$-filter $\mathcal{F}_\alpha$ satisfying (i) through (iv) above.

**Proof.** We construct a $\subseteq^*$-decreasing chain $\{F_\alpha^\beta : \beta < \omega_1\}$ of dense subsets of $\mathbb{Q}$. In limit stages $\beta$, $F_\alpha^\beta$ is a dense pseudointersection of the $F_\alpha^\gamma$, $\gamma < \beta$. It is well-known this can be done [1, proof of Theorem 2.4]. This guarantees the $F_\alpha^\beta$ generate a $p$-filter $\mathcal{F}_\alpha$ (property (i) above).

For maximality and the properties (ii) through (iv), use $C H$ in the ground model to list all relevant objects in order type $\omega_1$. If $\beta \equiv 1 \mod 4$, choose dense $F_\alpha^\beta \subseteq F_\alpha^\beta - 1$ so as to guarantee maximality. If $\beta \equiv 2 \mod 4$, $F_\alpha^\beta \subseteq F_\alpha^\beta - 1$ is a dense selector of the partition handed down at that stage so that we get (ii).

If $\beta \equiv 3 \mod 4$, take care of (iii). Let $U_\beta \neq \emptyset$ be open and $f_\beta : F_\alpha^\beta - 1 \cap U_\beta \rightarrow \omega$ be one to one. Put $D_\beta := F_\alpha^\beta - 1 \cap U_\beta$. Clearly, $D_\beta$ is dense in $U_\beta$. Apply $(\ast_{\alpha})$ to $U_\beta, D_\beta$ and $f_\beta$ to get $\beta_0$, open $\emptyset \neq U_\beta' \subseteq U_\beta$ and dense $D_\beta' \subseteq U_\beta' \cap D_\beta$ such that $f_\beta^{-1}(A) \cap D_\beta'$ is nowhere dense for all $A \in \mathcal{A}_\beta^{\leq \alpha}$. Put $F_\alpha^\beta := D_\beta' \cup (F_\alpha^\beta - 1 \setminus U_\beta')$. Then, indeed, $F_\alpha^\beta \cap U_\beta' \cap f_\beta^{-1}(A)$ is nowhere dense for all $A \in \mathcal{A}_\beta^{\leq \alpha}$.

If $\beta \equiv 0 \mod 4$ is a successor, guarantee (iv) if it applies. Then all members of $\mathcal{B}$ are open $\mathbb{Q}$-dense where $\mathcal{B}$ is the $\alpha$-th member of the $\Diamond_{\beta^2}$-sequence. Therefore we can stipulate that $F_\alpha^\beta \subseteq F_\alpha^\beta - 1$ belongs to $\mathcal{B}_\beta$. \[\square\]

**Corollary 3.8.** $\mathbb{P}_{\omega_2}$ forces $h_\mathbb{Q} = \aleph_1 = \omega_2$. 
Proof. \( \epsilon \leq \aleph_2 \) is straightforward. So it suffices to argue that \( h_{\aleph} \geq \aleph_2 \). Let \( \hat{\mathcal{D}} = \{ \hat{\beta} : \beta < \omega_1 \} \) be a \( \mathbb{P}_{\omega_2} \)-name for a potential witness for \( h_{\aleph} = \aleph_1 \). By the coding explained above, think of \( \hat{\mathcal{D}} \) as a subset of \( \omega_2 \). By standard arguments using the ccc-ness of the forcing, the collection of \( \alpha < \omega_2 \) such that the trivial condition (of \( \mathbb{P}_\alpha \)) forces that all \( \hat{\mathcal{D}} \cap V[G_\alpha] \) are open \( \mathbb{Q} \)-dense in \( V[G_\alpha] \) contains an \( \omega_1 \)-club \( C \). This means that \( C = D \cap S^\alpha \) where \( D \) is club in \( \omega_2 \).

By \( \diamond S^\alpha \), there is \( \alpha < \omega_2 \) of cofinality \( \omega_1 \) such that \( \hat{\mathcal{B}} := \hat{\mathcal{D}} \cap V[G_\alpha] \) is considered at stage \( \alpha \). By (iv) above, \( \hat{\mathcal{F}}_\alpha \) is forced to diagonalize \( \hat{\mathcal{B}} \). A fortiori, \( \mathbb{P}_{\alpha+1} \) forces ran(\( \ell_{\mathcal{F}_\alpha} \)) \( \in \bigcap_{\beta < \omega_1} \hat{\mathcal{D}}_\beta \). Thus \( \hat{\mathcal{D}} \) is forced not to be a witness for \( h_{\aleph} \), as required. \( \square \)

If \( \alpha \) is an even successor, the book-keeping (I) hands us down \( \alpha_0 \leq \alpha, \beta_0 < \omega_1, X_0 \in [\omega]^{\omega_0} \cap V_{\alpha_0} \) and \( Y_0 \in [X_\alpha]^{\omega_0} \cap V_\alpha \) which is almost contained in members of \( \mathcal{A}_\gamma^{\alpha} \) for all \( \gamma < \beta_0 \). If \( Y_0 \cap A \) is infinite for some \( A \in \mathcal{A}_\beta^{\alpha} \), \( Q_\alpha \) is the trivial forcing. Otherwise \( \mathcal{F}_\alpha \subseteq [\omega]^{\omega_0} \) is a Ramsey ultrafilter containing \( Y_0 \) satisfying additionally

(a) for all \( F \in \mathcal{F}_\alpha \) and all one-to-one \( f : F \to \omega \) there are \( \beta < \omega_1 \) and \( G \in \mathcal{F}_\alpha, G \subseteq F \), such that \( f^{-1}(A) \cap G \) is finite for all \( A \in \mathcal{A}_\beta^{\alpha} \).

Recall a Ramsey ultrafilter is an ultrafilter which is both

(i) a \( p \)-filter and
(ii) a \( q \)-filter.

\( Q_\alpha = L_{\mathcal{F}_\alpha} \) generically adds \( \ell_{\mathcal{F}_\alpha} \) such that \( A_{\beta_0}^\alpha := \text{ran}(\ell_{\mathcal{F}_\alpha}) \) is almost contained in all members of \( \mathcal{F}_\alpha \) and almost disjoint from all members of \( \mathcal{A}_\gamma^{\alpha} \). Thus \( \mathcal{A}_{\beta_0+1}^{\alpha} = \mathcal{A}_{\beta_0}^{\alpha} \cup \{ A_{\beta_0}^\alpha \} \) is still an almost disjoint family. Property (a) is used to prove \( (\star_{\alpha+1}) \) (Lemma 4.5).

Lemma 3.9. Let \( \alpha \) be an even successor ordinal, and assume that \( (\star_\alpha) \) holds. Then there is a Ramsey ultrafilter \( \mathcal{F}_\alpha \subseteq [\omega]^{\omega_0} \) satisfying (a) above.

Proof. This is exactly like the proof of Lemma 3.7 above, only easier. Using CH we construct a \( \subseteq^* \)-decreasing chain \( \{ F_\alpha^\beta : \beta < \omega_1 \} \) of infinite subsets of \( \omega \) with \( F_0^\alpha = Y_0 \) which generates a Ramsey ultrafilter. The only thing we have to worry about is property (a). For this, we again list the relevant objects in order type \( \omega_1 \).

Suppose we are at successor stage \( \beta, F_{\alpha}^{\beta-1} \) and one-to-one \( f_\beta : F_{\alpha}^{\beta-1} \to \omega \) are given. By \( (\star_\alpha) \) we find \( \beta_1 \) such that \( \text{ran}(f_\beta) \notin \mathcal{I}(\mathcal{A}_{\beta_1}^{\alpha}) \). If \( \text{ran}(f_\beta) \) is not fat with respect to \( \mathcal{A}_{\beta_1}^{\alpha} \), we easily find \( F_\alpha^\beta \subseteq F_{\alpha}^{\beta-1} \) such that \( f^{-1}(A) \cap F_\alpha^\beta \) is finite for all \( A \in \mathcal{A}_{\beta_1}^{\alpha} \). Otherwise let \( \gamma = \alpha - 1 \) and recall that, by construction, \( \mathcal{A}_{\beta_1}^{\alpha} = \mathcal{A}_{\beta_1}^{\gamma} \) and \( V_\alpha \) contains a dominating real over \( V_\gamma \). Thus we may apply Lemma 3.5 to \( \text{ran}(f_\beta) \) and find \( F_\alpha^\beta \subseteq F_{\alpha}^{\beta-1} \) such that \( \text{ran}(f_\beta |_{F_\alpha^\beta}) \cap A \) is finite for all \( A \in \mathcal{A}_{\beta_1}^{\alpha} \). Hence so is \( f_\beta^{-1}(A) \cap F_\alpha^\beta \). \( \square \)

Corollary 3.10. The properties \( (\star_{\alpha+1}) \) and \( (\dagger_{\alpha+1}) \) hold for all \( \alpha \).

Proof. Since \( (\star_{\alpha+1}) \) follows from \( (\dagger_{\alpha+1}) \) by Lemma 3.3, it suffices to prove the latter.

Fix non-empty open \( U, \) dense \( D \subseteq U \) and one-to-one \( f : D \to \omega \) in \( V_\alpha \). Also fix \( \beta < \omega_1 \). Again we may assume \( Q_\alpha \) is non-trivial and \( \beta = \beta_0 \). Note that \( \mathcal{F}_\alpha, \) being a Ramsey ultrafilter, is in particular a \( p \)-point which is, in particular, a nowhere dense ultrafilter (see, e.g., [4] for these notions and the implications). Since \( f^{-1} : \text{ran}(f) \to D \) is one to one, this gives us \( F \in \mathcal{F}_\alpha \) such that \( f^{-1}(F) \) is nowhere dense. Thus \( f^{-1}(A_{\beta_0}^\alpha) \) is nowhere dense as well. \( \square \)

4. A rank analysis of \( L_{\mathcal{F}} \)-names

A detailed investigation of \( L_{\mathcal{F}} \)-names will complete the proof of Theorem 3.1. This investigation is based on a Cantor–Bendixson style rank analysis which has been first used in a forcing setting by Baumgartner and Dordal [5], and has since become a standard tool when dealing with forcing notions adjoining dominating reals.

Let \( \mathcal{F} \) be a filter on \( \omega \). As mentioned in the previous section, we assume that

(i) \( \mathcal{F} \) is a \( p \)-filter,
(ii) \( \mathcal{F} \) is a \( q \)-filter.
Assume $\dot{X} \in [\omega]^{\omega}$ is an $L_\mathcal{F}$-name. Say $\sigma \in \omega^{<\omega}$ forces $n \in \dot{X}$ if there is $T \in L_\mathcal{F}$ with stem $\sigma$ such that $T \models n \in \dot{X}$.

Next, recursively define when $\sigma \in \omega^{<\omega}$ favors $n \in \dot{X}$. Say

- $rk_n(\sigma) = 0$ iff $\sigma$ forces $n \in \dot{X}$,
- for $\alpha > 0$, $rk_n(\sigma) = \alpha$ if $rk_n(\sigma)$ is not less than $\alpha$ and $\{m \in \omega : rk_n(\sigma \upharpoonright m) < \alpha\}$ is $\mathcal{F}$-positive.

Since the second clause is the same for all rank functions, we shall omit it in future. If there is no $\alpha < \omega_1$ with $rk_n(\sigma) = \alpha$, we write $rk_n(\sigma) = \infty$. Say $\sigma$ favors $n \in \dot{X}$ if $rk_n(\sigma) < \omega_1$ (iff $rk_n(\sigma) \neq \infty$).

Call $\sigma \in \omega^{<\omega}$ good if there are an $\mathcal{F}$-positive set $E$ and $f : E \to \omega$ such that

- $\sigma \upharpoonright m$ favors $f(m) \in \dot{X}$ for all $m \in E$
- $f^{-1}(\{n\})$ belongs to the dual ideal of $\mathcal{F}$ for all $n \in \omega$.

By shrinking $E$ to a smaller $\mathcal{F}$-positive set, we may assume that $f$ is one to one in this definition. For indeed, since $\mathcal{F}$ is a $p$-filter, the sets $\omega \setminus f^{-1}(\{n\})$ from above have a pseudo-intersection $F \in \mathcal{F}$. As $\mathcal{F}$ is a $q$-filter, we may find $G \subseteq F$ such that $G \cap f^{-1}(\{n\})$ has at most one element for each $n \in \omega$. Then simply replace $E$ by $E \cap G$ to get $f$ one to one.

We proceed to argue that there are plenty of good nodes. To do this we introduce the rank $\rho$ by recursion on the ordinals as follows.

- $\rho(\sigma) = 0$ if $\sigma$ is good.

**Lemma 4.1.** $\rho(\sigma) < \omega_1$ for all $\sigma \in \omega^{<\omega}$.

**Proof.** Fix $\sigma \in \omega^{<\omega}$. For each $m \in \omega$ and $\tau \supseteq \sigma \upharpoonright m$ define an auxiliary rank $\rho_m(\tau)$ recursively.

- $\rho_m(\tau) = 0$ if there is $\ell_\tau$ such that $\tau$ favors $\ell_\tau \in \dot{X} \setminus m$.

A standard rank argument (see also the proof of Lemma 2.1) shows that $\rho_m(\sigma \upharpoonright m) < \omega_1$ for all $m$.

**Case 1.** $E = \{m : \rho_m(\sigma \upharpoonright m) = 0\}$ is $\mathcal{F}$-positive. This means we may find $f(m) = \ell_{\sigma \upharpoonright m}$ such that $\sigma \upharpoonright m$ favors $\ell_{\sigma \upharpoonright m} \in \dot{X} \setminus m$ for all $m \in E$. Clearly, all $f^{-1}(\{n\})$ are finite, so $f$ is as required. This shows $\sigma$ is good. Thus $\rho(\sigma) = 0$.

**Case 2.** $\{m : \rho_m(\sigma \upharpoonright m) \geq 1\}$ belongs to $\mathcal{F}$. We first prove that if $\rho_m(\tau) = 1$, then $\tau$ is good, i.e. $\rho(\tau) = 0$.

Indeed, $E = \{k : \rho_m(\tau \upharpoonright k) = 0\}$ is $\mathcal{F}$-positive. This means we may find $f(k) = \ell_{\tau \upharpoonright k}$ such that $\tau \upharpoonright k$ favors $\ell_{\tau \upharpoonright k} \in \dot{X} \setminus m$ for all $k \in E$. All $f^{-1}(\{n\})$ must belong to the dual ideal of $\mathcal{F}$, for if $f^{-1}(\{n\})$ was $\mathcal{F}$-positive, then $\tau$ would favor $n \in \dot{X} \setminus m$ (by the recursive definition of “favoring”). So $\rho_m(\tau) = 0$, a contradiction.

By induction on $\rho_m$, we now see that $\rho(\tau) \leq \rho_m(\tau)$ for all $\tau \supseteq \sigma \upharpoonright m$ with $\rho_m(\tau) \geq 1$. So the assumption of Case 2 in fact gives us that $\{m : \rho(\sigma \upharpoonright m) < \infty\}$ belongs to $\mathcal{F}$. Therefore $\rho(\sigma) < \infty$. This completes the proof of the lemma. \(\square\)

For each good node $\sigma \in \omega^{<\omega}$ let us fix (once and for all) a function $f_\sigma : E_\sigma \to \omega$ witnessing goodness. So $E_\sigma = \text{dom}(f_\sigma)$ is $\mathcal{F}$-positive, $\sigma \upharpoonright m$ favors $f_\sigma(m) \in \dot{X}$ for $m \in E_\sigma$ and, without loss of generality, $f_\sigma$ is one to one.

If $\mathcal{F}$ is a maximal $\mathcal{Q}$-filter, there is a further reduction.

**Lemma 4.2.** Assume $\mathcal{F}$ is a maximal $\mathcal{Q}$-filter. Then

(v) given $E \subseteq \mathcal{Q}$, there are $G \in \mathcal{F}$ and an open $U$ such that $G \cap U \subseteq E$ and $(G \setminus U) \cap E = \emptyset$.

**Proof.** Put $U := \bigcup\{U_0 \text{ basic open: } U_0 \cap E \cap F \text{ dense in } U_0 \text{ for all } F \in \mathcal{F}\}$ and $V := \bigcup\{U_0 \text{ basic open: } U_0 \cap E \cap F \text{ is nowhere dense for some } F \in \mathcal{F}\}$. Clearly $U$ and $V$ are disjoint open sets and $U \cup V$ is dense. Let $G := (U \cap E) \cup (V \setminus E)$. Obviously $G \cap E \subseteq E$ and $G \setminus U = G \cap V = V \setminus E$ so that $(G \setminus U) \cap E = \emptyset$.

We need to check $G \in \mathcal{F}$. This will follow by maximality once we know $G \cap F$ is dense for all $F \in \mathcal{F}$.

Assume there are $F_0 \in \mathcal{F}$ and $U_0$ basic open such that $U_0 \cap G \cap F_0$ is nowhere dense. Without loss of generality, either $U_0 \cap E \cap F$ is dense in $U_0$ for all $F \in \mathcal{F}$ in which case $U_0 \subseteq V$ or $U_0 \cap E \cap F$ is nowhere dense for some $F \in \mathcal{F}$ in which case $U_0 \subseteq V$. In the first case, $U_0 \cap G \cap F_0 = U_0 \cap E \cap F_0$, and we reach a contradiction. In the second case, $U_0 \cap G \cap F_0 = (U_0 \setminus E) \cap F_0$. Since $U_0 \cap (F_0 \cap F)$ is dense in $U_0$, $(U_0 \setminus E) \cap (F_0 \cap F)$ is also dense in $U_0$, and we reach again a contradiction. \(\square\)
By property (v), we may assume that for each good \( \sigma \), there are \( F_\sigma \in \mathcal{F} \) and \( U_\sigma \neq \emptyset \) open such that \( E_\sigma = F_\sigma \cap U_\sigma \).

\[ * * * \]

We are ready to deal with the preservation of \((\star_{\alpha+1})\) (see section Section 3) at the successor step \( \alpha + 1 \). That is, we assume that the model \( V_\alpha \) has been produced, \((\star_\alpha)\) holds, and we know the mad families \( \mathcal{A}_\beta^{\leq \alpha} \) as well as the filter \( \mathcal{F}_\alpha \).

We now force with \( Q_\alpha = \mathbb{L}_{\mathcal{F}_\alpha} \) to obtain the generic extension \( V_{\alpha+1} \).

We need to show that \((\star_{\alpha+1})\) still holds in \( V_{\alpha+1} \). That is, given a \( \mathbb{L}_{\mathcal{F}_\alpha} \)-name \( \hat{X} \in [\omega]^\omega \), we need to find \( \beta < \omega_1 \) such that the trivial condition forces that \( \hat{X} \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha+1}) \). Since at most one \( \mathcal{A}_\beta^{\leq \alpha+1} \) gets a new set at stage \( \alpha + 1 \), it clearly suffices to find \( \beta \) such that \( \hat{X} \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha}) \) is forced.

First consider the case \( c(f(\alpha)) = \omega_1 \).

**Lemma 4.3.** Assume \( \mathcal{F}_\alpha \) is a maximal \( \mathbb{Q} \)-filter satisfying (i) through (iii). Then there is \( \beta < \omega_1 \) such that \( \Vdash_{\mathbb{L}_{\mathcal{F}_\alpha}} \hat{X} \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha}) \).

**Proof.** Recall \( \mathcal{F}_\alpha \) satisfies

(iii) for all open \( U \neq \emptyset \), all \( F \in \mathcal{F}_\alpha \) and all one-to-one \( f : F \cap U \to \omega \), there are \( \beta < \omega_1 \), \( \emptyset \neq U' \subseteq U \) open and

\[ G \in \mathcal{F}_\alpha, \ G \subseteq F, \text{ such that } G \cap U' \cap f^{-1}(A) \text{ is nowhere dense for all } A \in \mathcal{A}_\beta^{\leq \alpha}. \]

We apply this to all \( U_\sigma, \ F_\sigma, \ \alpha \sigma \) where \( \sigma \) is good. Thus we get \( \beta_\sigma, \ U'_\sigma, \ \alpha \sigma \) satisfying the conclusion of (iii). Let \( \beta = \sup(\beta_\sigma : \sigma \text{ good}) \) and let \( G \in \mathcal{F}_\alpha \) be a pseudointersection of the \( \alpha \sigma \). Then it is easy to see that for all good \( \sigma \) there is an open \( \emptyset \neq U \subseteq U_\sigma \) such that

- \( G \cap U \cap f^{-1}\sigma(A) \) is nowhere dense for all \( A \in \mathcal{A}_\beta^{\leq \alpha} \).

**Claim 4.4.** \( \Vdash_{\mathbb{L}_{\mathcal{F}_\alpha}} \hat{X} \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha}) \).

**Proof.** Assume \( j \in \omega, \ A_i \in \mathcal{A}_\beta^{\leq \alpha} \) for \( i < j \), and \( T \in \mathcal{F}_\alpha \) is such that \( T \models \hat{X} \setminus k \subseteq \bigcup_{i < j} A_i \). We need to reach a contradiction.

Let \( \sigma \in T \) extending the stem of \( T \) be good. Such \( \sigma \) exists by **Lemma 4.1**. By (iii) and the discussion above, we find a basic open set \( U \subseteq U_\sigma \) such that \( G \cap U \cap f^{-1}(A) \) is nowhere dense for all \( A \in \mathcal{A}_\beta^{\leq \alpha} \). Thus there is \( H \subseteq G \) belonging to \( \mathcal{F}_\alpha \) such that \( f_\sigma(H \cap U) \cap A_i = \emptyset \) for all \( i < j \). Let \( m \in H \cap U \) such that \( f_\sigma(m) \geq k \) and \( \sigma^{-1}m \in T \). Such \( m \) clearly exists because \( \text{succ}_T(\sigma) \cap H \cap U \) is still dense below \( U \). By construction, we have \( f_\sigma(m) \notin \bigcup_{i < j} A_i \).

By definition of goodness, \( \sigma^{-1}m \) favors \( f_\sigma(m) \in \hat{X} \). Thus \( \text{rk}_{f_\sigma(m)}(\sigma^{-1}m) < \omega_1 \). A standard induction on the latter rank (as in the proof of **Lemma 2.1**) now shows that there is \( S \in \mathbb{L}_{\mathcal{F}_\alpha} \) extending \( T_{\sigma^{-1}m} \) which forces \( f_\sigma(m) \in \hat{X} \). This contradiction finishes the proof of the claim and the lemma. \( \square \)

We next turn to the case where \( \sigma \) is an even successor.

**Lemma 4.5.** Assume \( \mathcal{F}_\alpha \) is a Ramsey ultrafilter satisfying (a). Then there is \( \beta < \omega_1 \) such that \( \Vdash_{\mathbb{L}_{\mathcal{F}_\alpha}} \hat{X} \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha}) \).

**Proof.** Recall

(a) for all \( F \in \mathcal{F}_\alpha \) and all one-to-one \( f : F \to \omega \) there are \( \beta < \omega_1 \) and \( G \in \mathcal{F}_\alpha, \ G \subseteq F, \) such that \( f^{-1}(A) \cap G \) is finite for all \( A \in \mathcal{A}_\beta^{\leq \alpha} \).

Since \( \mathcal{F}_\alpha \) is an ultrafilter, any \( \mathcal{F}_\alpha \)-positive set belongs to \( \mathcal{F}_\alpha \). Therefore we can apply (a) to all \( E_\sigma \) and \( f_\sigma \) where \( \sigma \) is good to get \( \beta < \omega_1 \) and \( G \in \mathcal{F}_\alpha \) such that for all good \( \sigma 

- \( f_\sigma^{-1}(A) \cap G \) is finite for all \( A \in \mathcal{A}_\beta^{\leq \alpha} \).

The proof of the lemma is complete once we show

**Claim 4.6.** \( \Vdash_{\mathbb{L}_{\mathcal{F}_\alpha}} \hat{X} \notin \mathcal{I}(\mathcal{A}_\beta^{\leq \alpha}) \).

**Proof.** This is like the proof of **Claim 4.4**, only easier. We leave the details to the reader. \( \square \)
Finally, consider the case where \( \alpha \) is an odd successor or \( cf(\alpha) = \omega \) or \( \alpha = 0 \). In this case, \( Q_\alpha = \mathbb{D} \) is Hechler forcing, and \((\star_{\alpha+1})\) follows from the classical work of Baumgartner and Dordal [5]. To make our exposition self-contained, we sketch the argument. As mentioned earlier, we think of \( \mathbb{D} \) as forcing with \( \mathbb{L}_\mathcal{F} \) where \( \mathcal{F} \) is the Fréchet filter. The advantage of this is that the forcing fits into the framework discussed at the beginning of this section. Note that while \( \mathcal{F} \) is a \( p \)-filter it is not a \( q \)-filter. However, the only place where we used the latter property was to get the functions \( f_\sigma \) one to one for good \( \sigma \). This is still possible here: first notice they must be finite to one by definition of goodness, then replace \( \text{dom}(f_\sigma) \) by an infinite selector \( E_\sigma \subseteq \text{dom}(f_\sigma) \) for the \( f_\sigma^{-1}(\{n\}) \). Since \( \mathcal{F} \) is the cofinite filter, \( E_\sigma \) is still \( \mathcal{F} \)-positive.

**Lemma 4.7.** Assume \((\star_\alpha)\) holds. Also assume \( Q_\alpha = \mathbb{D} = \mathbb{L}_\mathcal{F} \) where \( \mathcal{F} \) is the cofinite filter. Then there is \( \beta < \omega_1 \) such that \( \Vdash_{\mathbb{D}} \dot{X} \notin \mathcal{I}(\mathcal{A}_{\mathbb{D}}^\leq \beta) \).

**Proof.** By \((\star_\alpha)\) applied to all \( E_\sigma \) and \( f_\sigma \) where \( \sigma \) is good, we find \( \beta < \omega_1 \) such that for all good \( \sigma \)

- \( \text{ran}(f_\sigma) \notin \mathcal{I}(\mathcal{A}_{\mathbb{D}}^\leq \beta) \).

We finish the proof of the lemma by showing

**Claim 4.8.** \( \Vdash_{\mathbb{D}} \dot{X} \notin \mathcal{I}(\mathcal{A}_{\mathbb{D}}^\leq \beta) \).

**Proof.** Again, this is an easier version of Claim 4.4, so we omit the details. \( \square \)

**Corollary 4.9.** \( P_{\omega_2} \) forces \( h = \aleph_1 \).

**Proof.** We need to check the \( \mathcal{A}_\beta, \beta < \omega_1 \), are indeed witnesses for \( h = \aleph_1 \). By the book-keeping argument (I), they must be maximal almost disjoint families in \([\omega]^\omega\) in the model \( V_{\omega_2} \). Therefore it suffices to argue that for all \( X \in [\omega]^\omega \) there is \( \beta \) such that \( X \notin \mathcal{I}(\mathcal{A}_\beta) \).

Since, by Corollary 3.10 (see also Lemma 3.3), we do have \((\star \star_{\alpha+1})\) and \((\dagger_{\alpha+1})\) for all \( \alpha \), we know that it suffices to prove \((\star_\alpha)\) holds for all \( \alpha \). This is done by induction on \( \alpha \). The case \( \alpha = 0 \) is trivial because all \( \mathcal{A}_{\mathbb{D}}^\leq \beta \) are empty. If \( \alpha \) is a limit ordinal, we are done by Lemma 3.2. If \( \alpha = \gamma + 1 \) and either \( \gamma \) is odd or \( cf(\gamma) = \omega \) or \( \gamma = 0 \), we are done by Lemma 4.7. If \( \gamma \) is an even successor, use Lemmata 3.9 and 4.5. If \( cf(\gamma) = \omega_1 \), first use Corollary 3.6 and the induction hypothesis to show \((\dagger_\beta)\) holds cofinally below \( \gamma \). By Lemma 3.4, \((\dagger_\gamma)\) holds. This means that, according to Lemma 3.7, we constructed the maximal \( \mathcal{Q} \)-filter \( \mathcal{F}_\gamma \) satisfying properties (i) through (iv). By Lemma 4.3, \((\star_\alpha)\) still holds. \( \square \)

The proof of Theorem 3.1 is complete by 3.8 and 4.9.

5. Questions

The following is the most interesting question which is still open about the cardinals discussed in this work:

**Question 5.1.** (1) Is \( s_{\mathcal{Q}} = \min(\text{add}(\mathcal{M}), s) \) or \( s_{\mathcal{Q}} < \min(\text{add}(\mathcal{M}), s) \) consistent?

(2) (Balcar–Hrušák–Hernández, [1, Questions 3.11 (6)])

Is \( \tau_{\mathcal{Q}} = \max(\text{cof}(\mathcal{M}), \tau) \) or \( \tau_{\mathcal{Q}} > \max(\text{cof}(\mathcal{M}), \tau) \) consistent?

Concerning the proof of Theorem 3.1: while the approach seems natural to us and things are quite a bit simplified by the fact that we are using a finite support iteration of ccc forcing notions, the model can hardly be called a “canonical model” for \( h < h_{\mathcal{Q}} \). Zapletal (see [28] and [29]) has singled out the Mathias model (the \( \omega_2 \)-stage countable support iteration of Mathias forcing) as the canonical model for \( h = \epsilon = \aleph_2 \) in the sense that every tame invariant (very roughly speaking, an invariant defined by a projective formula) which is consistently less than \( h \) has value \( \aleph_1 \) in this model. It is well-known that Mathias forcing decomposes as a two-step iteration of first forcing with \( P(\omega)/\text{fin} \) and then with \( \mathbb{L}_\mathcal{F} \) where \( \mathcal{F} \) is the generic Ramsey ultrafilter added by the first forcing. Similarly, forcing with \( \text{Dense}(\mathbb{Q})/\text{nwd} \) (see [1, Section 2] for this forcing notion) generically adjoins a maximal \( \mathcal{Q} \)-filter \( \mathcal{F} \) which is both a \( p \)-filter and a \( q \)-filter, and the \( \omega_2 \)-stage countable support iteration of \( \mathbb{P} := \text{Dense}(\mathbb{Q})/\text{nwd} \star \mathbb{L}_\mathcal{F} \) should be the canonical model for \( h_{\mathcal{Q}} = \epsilon = \aleph_2 \). Thus:

**Conjecture 5.2.** \( h = \aleph_1 \) in this model.
Note this does not follow from Zapletal’s work because \( h \) is not tame.

Showing \( h = \aleph_1 \) in a countable support iteration may involve quite some work. Something closely related has been done by Shelah and Spinas [22] (see also [23]) who proved that \( h_2 \), the distributivity number of the square of \( \mathcal{P}(\omega) / \text{fin} \), is \( \aleph_1 \) in the iterated Mathias model.

In recent work, Hrušák and Zapletal [14] have found a canonical way of associating an ideal \( \text{tr}(\mathcal{I}) \) on \( \omega \) with a \( \sigma \)-ideal \( \mathcal{I} \) on \( 2^\omega \) such that for many proper forcing notions of the form \( \mathbb{P} = \text{Borel}/\mathcal{I}, \mathcal{P}(\omega)/\text{tr}(\mathcal{I}) \) is proper and forcing equivalent to the two-step iteration of \( \mathbb{P} \) and an \( \aleph_0 \)-distributive forcing \( \mathcal{Q} \). In a number of cases \( \mathcal{Q} \) is \( \mathcal{P}(\omega)/\text{fin} \) in the sense of \( V^\mathbb{P} \). Theorem 3.1 suggests this is, in general, not the case if \( \mathcal{I} \) is the meager ideal \( \mathcal{M} \).

**Conjecture 5.3** (See [14, Question 5.1]). If \( \mathcal{I} = \mathcal{M}, \mathcal{Q} \) is consistently distinct from \( \mathcal{P}(\omega)/\text{fin} \) (in the sense of the Cohen extension \( V^\mathbb{C} \)).

A scenario of the proof is as follows. If \( \mathcal{I} = \mathcal{M}, \mathcal{P}(\omega)/\text{tr}(\mathcal{I}) \) is equivalent to \( \mathcal{P}(\mathcal{Q})/\text{nwd} \) which decomposes as a two-step iteration \( \mathcal{C} \star \mathcal{Q} \) [1, Theorem 4.1] where \( \mathcal{Q} \) is forced to be \( h_0^\mathcal{C} \)-distributive (an unpublished result of the author). Assume \( V \) satisfies \( h < h_\mathcal{Q} \). We conjecture \( \mathcal{C} \) cannot increase the value of \( h \). This would mean that in \( V^\mathcal{C} \), the distributivity of \( \mathcal{P}(\omega)/\text{fin} \) is smaller than the one of \( \mathcal{Q} \). Hence the forcing notions are not equivalent.

As mentioned in the introduction, \( p_\mathcal{Q} = p, t_\mathcal{Q} = t \) and \( t_\mathcal{Q} = i \) [1], and we have a fairly clear picture of \( h_\mathcal{Q}, s_\mathcal{Q} \) and \( \tau_\mathcal{Q} \). But what about \( a_\mathcal{Q} \) and \( u_\mathcal{Q} \) ? The former is the least size of a maximal family of dense subsets of \( \mathcal{Q} \) whose pairwise intersection is not dense, and the latter is the minimal cardinality of a base of a maximal \( \mathcal{Q} \)-filter.

**Problem 5.4.** Investigate \( a_\mathcal{Q} \) and \( u_\mathcal{Q} \).

The almost disjointness number of \( \mathcal{P}(\mathcal{Q})/\text{nwd} \) has been studied by Steprāns [24], but it is unclear how it is related to \( a_\mathcal{Q} \).

**Acknowledgements**

Research on this work was started, Theorem 2.2 and the consistency of \( s_\mathcal{Q} < s \) were proved, and the basic ideas for the proof of Theorem 3.1 were obtained, while the author was visiting the Instituto de Matemáticas of the UNAM, Unidad Morelia, in September 2003. I would like to thank Michael Hrušák for inviting me, for introducing me to [1] and for asking the questions which led to the results in this paper. I am grateful to Michael and everybody at the Instituto for their generous hospitality which made my stay so pleasant I am bound to visit more often. Theorem 1.4 was proved, and the details of Theorem 3.1 were worked out, during fall and winter 2004/2005. Finally let me thank Yi Zhang for inviting me to the 2004 Logic, Algebra and Geometry conference in St. Petersburg and thus providing an opportunity to present my results.

The author was partially supported by Grant-in-Aid for Scientific Research (C)(2) 15540120, Japan Society for the Promotion of Science.

**References**


